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Completeness requirements of shape functions for higher order finite elements

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Abstract. An alternative interpretation of the completeness requirements for the higher order elements is presented. Apart from the familiar condition, $\Sigma_i N_i=1$, some additional conditions to be satisfied by the shape functions of higher order elements are identified. Elements with their geometry in the natural form, i.e., without any geometrical distortion, satisfy most of these additional conditions inherently. However, the geometrically distorted elements satisfy only fewer conditions. The practical implications of the satisfaction or non-satisfaction of these additional conditions are investigated with respect to a 3-node bar element, and 8- and 9-node quadrilateral elements. The results suggest that non-satisfaction of these additional conditions results in poorer performance of the element when the element is geometrically distorted. Based on the new interpretation of completeness requirements, a 3-node element and an 8-node rectangular element that are insensitive to mid-node distortion under a quadratic displacement field have been developed.

Key words: finite element; completeness requirements; distortion sensitivity; shape functions; geometric distortion; higher order elements.

1. Introduction

Convergence of finite element results with the mesh refinement is an important requirement to be satisfied by any successful element. The *continuity* and *completeness* requirements (e.g., see, Zienkiewicz 1977, Bathe 1996) must be satisfied so as to ensure convergence. The continuity requirement demands that the shape functions must be so chosen that the assumed interpolation function of the field variable, viz., the displacement in structural analysis, is sufficiently continuous within and across the boundary of the elements. The completeness requirement demands that the shape functions must be such that the interpolation function is capable of representing an arbitrary linear polynomial displacement field exactly. In other words, the shape functions must ensure that the element is capable of representing rigid body motion and constant strain state correctly.

The continuity requirement is usually satisfied by ensuring that the shape functions (and hence the displacement model) are polynomials of an appropriate order. Typically, if the stiffness integrands involve derivatives of order m, the shape functions must be at least C^m continuous within the element and C^{m-1} continuous across the element interface. The completeness requirements are satisfied for a 2-D isoparametric element, for example, if the element shape functions, N_i, satisfy the

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conditions,

$$\sum_i N_i = 1 \tag{1}$$

$$\sum_i N_i x_i = x \quad \text{and} \tag{2}$$

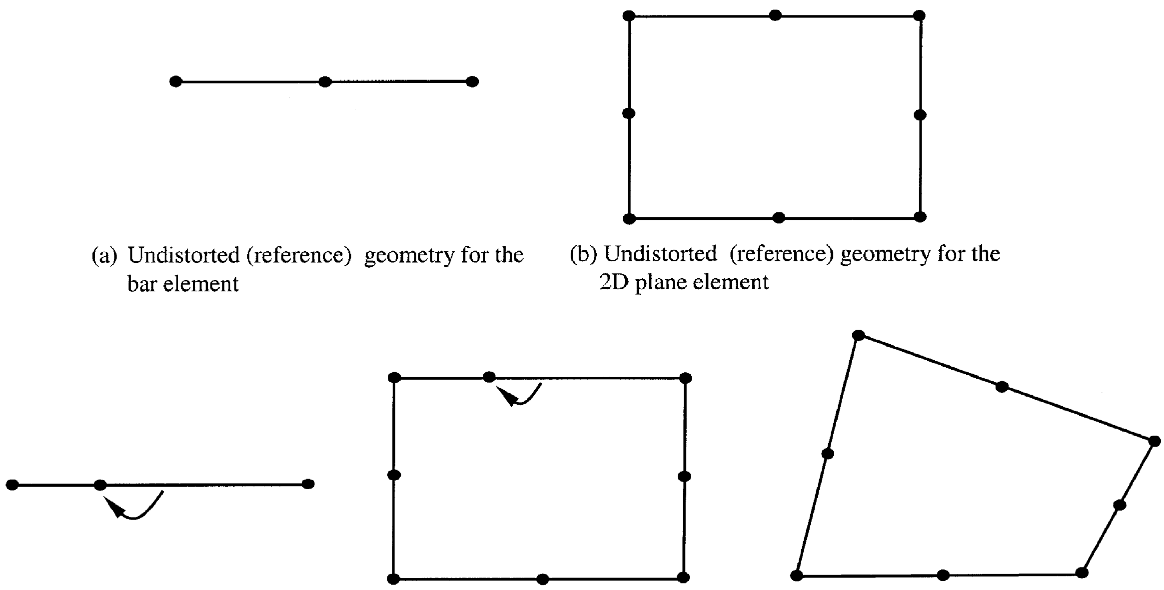
$$\sum_i N_i y_i = y \tag{3}$$

In the literature, the completeness requirements are frequently associated with the ability of the element to represent an arbitrary linear polynomial displacement field. For this reason, we will refer to these completeness requirements as the *linear completeness requirements*. In the limit of mesh refinement, the element size becomes so small that the displacement field within each element can be considered to be linear, and hence an element satisfying the linear completeness requirements is expected to converge to an exact solution without any problem.

It was once believed that elements satisfying the continuity and completeness requirements would indeed, in the limit of mesh refinement, converge to the actual solution without any difficulty. However, there have been instances which have contradicted this belief. The host of locking problems (e.g. see, Prathap 1993, Wilson *et al*. 1973) and the deterioration of solution accuracy with geometric distortion are two examples. To avoid the problem of locking, in addition to the continuity and completeness requirements, certain other conditions have to be satisfied by the displacement model (Prathap 1993). Non-conforming elements (e.g. see, Zienkiewicz 1977, Wilson *et al*. 1973, Taylor *et al*. 1976) which violate the inter-element continuity requirements have been used successfully. However, such formulations are highly sensitive to geometric distortion of the element, and the patch test (e.g. see, Irons *et al.* 1972) has been introduced to check the ill effects of the violation of inter-element continuity and completeness requirements in a global sense, *i.e.*, for a patch. These observations are the source of motivation for several investigations into the continuity and completeness requirements.

It is frequently inconvenient to refine the mesh successively in order to check for convergence. As an alternative to successive mesh refinement, higher order elements are sometimes used. In the context of the higher order elements, however, the completeness requirements need to be interpreted carefully. These elements use a higher order interpolation model, and attempt to obtain convergence using fewer elements. With fewer elements, the elements are larger, and hence, the linear completeness requirements are no longer sufficient as they ensure convergence only in the limit of mesh refinement. Thus, for the effective use of higher order elements, it is necessary to investigate the higher order completeness requirements of shape functions. Higher order patch tests (e.g., Taylor *et al.* 1986) do address the problem of representation of higher order strain states. However, in this paper, the problem is looked at from the viewpoint of an individual element instead of a patch. This yields some interesting insights with regard to the additional conditions to be satisfied by shape functions apart from the usual ones given by Eqs. (1)-(3). An important related reference is Lee and Bathe (1993) wherein the effect of element distortion and is relation to completeness requirement is discussed for isoparametric elements.

The concept of completeness is closely related to the geometric distortion of the element. The later part of this paper deals with the geometric distortion of a quadratic bar element and a quadratic 2D plane element, and hence a definition of the type of distortion that will be discussed in this paper would be in order. Two types of distortion, viz., *mid-node distortion* and *angular distortion* as



(a) Undistorted (reference) geometry for the bar element
(b) Undistorted (reference) geometry for the 2D plane element
(c) Typical mid-node distortion
(d) Typical angular distortion

Fig. 1 Undistorted and distorted geometries for the quadratic bar and quadratic plane 2D elements

shown in Fig. 1, are considered. In the case of a bar element, a geometry with the mid-node equi-spaced from the end nodes is referred to as the undistorted geometry. In the case of a 2D element, a rectangular/square geometry with the mid-nodes equi-spaced from their corresponding adjacent corner nodes is referred to as the undistorted geometry. For a bar element, only the mid-node distortion applies. A 2D element can have either an angular distortion or a mid-node distortion or both. In the case of the angular distortion, the element is of a quadrilateral shape with the mid-nodes equi-spaced from the adjacent corner nodes, whereas in the case of the mid-node distortion, the mid-node is not equi-spaced although the element is of rectangular shape. Only one type of distortion is considered at a time. The case of simultaneous presence of both distortion types is beyond the scope of this paper.

Angular distorted elements usually appear in the mesh generation of curved geometries and in transition regions from coarse to fine mesh, and also in the nonlinear problems involving large deformations. For fracture mechanics application of rectangular elements, the mid-node distortion is sometimes introduced intentionally in order to simulate the $1/\sqrt{r}$ stress singularity. The mid-node distortion also occurs in nonlinear problems involving large deformations.

2. Conditions to be satisfied by shape functions

In this section, a complete list of conditions under which the shape functions satisfy completeness requirements is derived for a C^0 continuous higher order 2-D element.

Consider a polynomial displacement field in the global Cartesian coordinate system of the form

$$u=a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+\cdots \tag{4}$$

$$v=b_0+b_1x+b_2y+b_3x^2+b_4xy+b_5y^2+\cdots \tag{5}$$

where a_j and b_j ($j=1, 2, 3, ...$) are undetermined constants. The nodal displacement values corresponding to this field are given by

$$u_i=a_0+a_1x_i+a_2y_i+a_3x_i^2+a_4x_iy_i+a_5y_i^2+\cdots \tag{6}$$

$$v_i=b_0+b_1x_i+b_2y_i+b_3x_i^2+b_4x_iy_i+b_5y_i^2+\cdots \tag{7}$$

where i refers to a typical node number. Using the shape functions, the assumed displacement field is written as

$$u=\sum_i N_iu_i \tag{8}$$

$$v=\sum_i N_iv_i \tag{9}$$

The shape functions may be assumed to be defined either in terms of the global Cartesian coordinates (x, y) or the local coordinates (ξ, η).

We look for an element that can represent exactly the field given by Eqs. (4) and (5). Therefore, the shape functions need to be chosen in such a way that the element reproduces the field given by Eq. (4) and (5) exactly at any point inside the element, when the nodes are assigned displacement values in accordance with Eq. (6) and (7). Substituting for u, v, u_i and v_i from Eqs. (4)-(7), Eqs. (8) and (9) may be written as

$$a_0+a_1x+a_2y+a_3x^2+a_4xy+\cdots=a_0\left(\sum_i N_i\right)+a_1\left(\sum_i N_ix_i\right)+a_2\left(\sum_i N_iy_i\right)+ \\ a_3\left(\sum_i N_ix_i^2\right)+a_4\left(\sum_i N_ix_iy_i\right)+\cdots \tag{10}$$

$$b_0+b_1x+b_2y+b_3x^2+b_4xy+\cdots=b_0\left(\sum_i N_i\right)+b_1\left(\sum_i N_ix_i\right)+b_2\left(\sum_i N_iy_i\right)+ \\ b_3\left(\sum_i N_ix_i^2\right)+b_4\left(\sum_i N_ix_iy_i\right)+\cdots \tag{11}$$

Since the constants a_i and b_i are arbitrary, Eqs. (10) and (11) are satisfied only if the following conditions are satisfied:

$$x^py^q=\sum_i N_ix_i^py_i^q, \quad p, q=1, 2, 3, ... \tag{12}$$

Similarly, for a 3-D displacement field, the condition to be satisfied is of the form, $x^py^qz^r=\Sigma_iN_ix_i^py_i^qz_i^r$, $p, q, r=0, 1, 2, 3, ...$ Eq. (12) is now interpreted for a particular element. For an 8-node quadrilateral element, eight arbitrary constants can be accommodated in the displacement model, and hence we may ideally want the following displacement field with eight unknown coefficients to be represented by the element exactly:

$$u=a_0+a_1x+a_2y+a_3x^2+a_4xy+a_5y^2+a_6x^2y+a_7xy^2 \tag{13}$$

$$v=b_0+b_1x+b_2y+b_3x^2+b_4xy+b_5y^2+b_6x^2y+b_7xy^2 \tag{14}$$

For this to be true, Eq. (12) yield the following eight conditions.

$$\sum_i N_i = 1 \tag{15}$$

$$\sum_i N_i x_i = x \tag{16}$$

$$\sum_i N_i y_i = y \tag{17}$$

$$\sum_i N_i x_i^2 = x^2 \tag{18}$$

$$\sum_i N_i x_i y_i = xy \tag{19}$$

$$\sum_i N_i y_i^2 = y^2 \tag{20}$$

$$\sum_i N_i x_i^2 y_i = x^2 y \tag{21}$$

$$\sum_i N_i x_i y_i^2 = xy^2 \tag{22}$$

2.1. Interpretation of the conditions represented by Eqs. (15)-(22)

The condition given by Eq. (15) is well known in the literature. The conditions given by Eqs. (16) and (17) are also familiar although their interpretation here is rather different; these are the conditions that need to be satisfied by the shape functions for the element to represent correctly the rigid body motions and constant strain conditions. In the literature, Eqs. (16) and (17) are usually presented as the basis of geometric interpolation for isoparametric elements. Thus, isoparametric formulations inherently satisfy these equations (Eqs. 16 and 17). Satisfaction of Eqs. (15)-(17) ensures convergence in the limit of mesh refinement. Thus, the conditions, Eqs. (15)-(17), are already known in the literature. However, the conditions given by Eqs. (18)-(22) are rarely discussed and hence form the focus of this paper.

When the elements have no geometrical distortion, all the conditions (Eqs. 15-22) are usually satisfied. In distorted situations, however, Eqs. (18)-(22) are not generally satisfied. Non-satisfaction of these additional conditions affects the performance of the element. This is demonstrated for a few typical elements in later sections. For ease of reference, Eqs. (15)-(17) and Eqs. (18)-(22) will, hereinafter, be referred to as the *linear or lower order completeness requirements* and the *higher order completeness requirements*, respectively.

What do the higher order completeness requirements signify? Satisfaction of Eq. (18), for example, would mean that the x^2 term of the displacement field can be exactly represented by the shape functions. The other conditions can be interpreted in a similar manner.

Eqs. (15)-(22) represent eight conditions in eight unknown shape functions and hence can, in principle, be solved symbolically for shape functions. However, the numerical solution may not exist when the coefficient matrix on the left hand side is singular. For a non-singular case, solving Eqs. (15)-(22) appears to be an elegant approach to derive shape functions. However, the shape functions derived thus have their limitations. The numerical problems considered later will demonstrate this.

2.2. Conditions based on the derivatives of shape functions

Differentiation of Eqs. (15)-(22) with respect to x, yields the following equations,

$$\sum_i N_{i,x}=0 \tag{23}$$

$$\sum_i N_{i,x}x_i=1 \tag{24}$$

$$\sum_i N_{i,x}y_i=0 \tag{25}$$

$$\sum_i N_{i,x}x_i^2=2x \tag{26}$$

$$\sum_i N_{i,x}x_iy_i=y \tag{27}$$

$$\sum_i N_{i,x}y_i^2=0 \tag{28}$$

$$\sum_i N_{i,x}x_i^2y_i=2xy \tag{29}$$

$$\sum_i N_{i,x}x_iy_i^2=y^2 \tag{30}$$

Eqs. (23)-(30) may be viewed as completeness requirements stated in terms of the shape function derivatives. Since Eqs. (23)-(30) are obtained from Eqs. (15)-(22) by differentiation, satisfaction of Eqs. (15)-(22) would automatically mean satisfaction of Eqs. (23)-(30) while the converse is not true. Hence, Eqs. (15)-(22) represent a more stringent set of completeness requirements than Eqs. (23)-(30).

Incidentally, the solution of Eqs. (23)-(30) immediately yields the global shape function derivatives, $N_{i,x}$, i.e., the shape function derivatives with respect to x. No additional matrix inversion is required in solving these equations as the left hand side is the same as that of Eqs. (15)-(22). Similarly, the derivative with respect to y, $N_{i,y}$, can also be computed.

3. Demonstrative problems

In this section, the effect of non-satisfaction of the additional completeness requirements on the performance of the element is demonstrated for a 3-node bar element, and 8- and 9-node quadrilateral elements.

3.1. A cantilever bar modelled as a single 3-node bar element with mid-node distortion - Problem 1

A cantilever bar under uniformly distributed axial load modelled by a single 3-node bar element is considered (Fig. 2). The mid-node, i.e., node no. 2, is deliberately placed away from the centre of the element. The length, area of cross section, and Young's modulus of the material are taken as 10, 1 and 50, respectively. No specific units are assigned to these numbers. Consistent units may be assumed. The intensity of axial pull is taken as 10 per unit length. Although this is a trivial problem to be solved by the finite element method, it serves as a good example to demonstrate the effect of non-satisfaction of the additional conditions discussed in section 2.

The exact displacement, u, of the cantilever bar under the given uniformly distributed tensile load is expressed as $u=2x-(x^2/10)$ which is a quadratic polynomial of the form $u=a_0+a_1x+a_2x^2$. This involves three coefficients and hence we may ideally want a 3-node element to solve this problem exactly. However, whether or not a 3-node element can solve the problem exactly depends on the type of shape functions employed. For the quadratic displacement field to be represented exactly by the 3-node element, the shape functions need to satisfy the following three conditions.

$$\sum_i N_i=1 \tag{31}$$

$$\sum_i N_i x_i=x \tag{32}$$

$$\sum_i N_i x_i^2=x^2 \tag{33}$$

We now consider two different formulations to study the significance of the conditions (31)-(33).

Formulation 1: Standard isoparametric formulation of a quadratic bar element is considered. The shape functions are as follows:

$$N_1=-\frac{1}{2}\xi(1-\xi) \tag{34}$$

$$N_2=(1-\xi^2) \tag{35}$$

$$N_3=\frac{1}{2}\xi(1+\xi) \tag{36}$$

Formulation 2: This formulation is new, and is based on the concepts developed in this paper. The shape functions are derived by solving Eqs. (31)-(33). First, we substitute $x=N_1x_1+N_2x_2+N_3x_3$ (where N_1, N_2 and N_3 are given by Eqs. 34-36) on the right hand side of Eqs. (31)-(33), and then, solve for the new shape functions which are of the form,

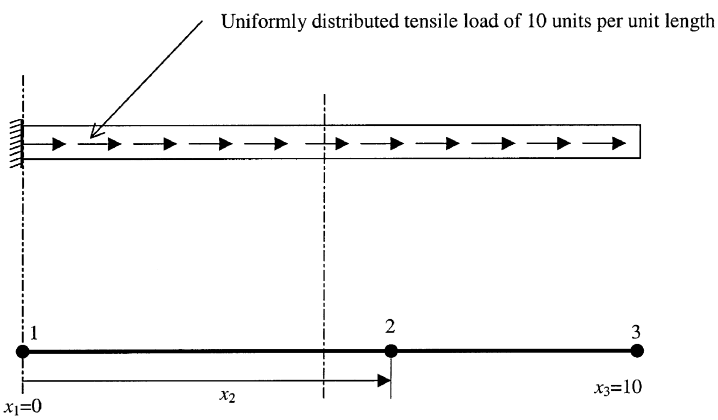


Fig. 2 A cantilever beam under a uniformly distributed tensile load modelled with a single 3-node bar element having mid-node distortion

$$N_1 = -\frac{1}{2}\xi(1-\xi)H_1H_2 \tag{37}$$

$$N_2 = (1-\xi^2)H_1H_3 \tag{38}$$

$$N_3 = \frac{1}{2}\xi(1+\xi)H_2H_3 \tag{39}$$

where

$$H_1 = \frac{(2\alpha-1)\xi+2(\alpha-1)}{2\alpha} \tag{40}$$

$$H_2 = (2\alpha-1)\xi-1 \tag{41}$$

$$H_3 = \frac{(2\alpha-1)\xi-2\alpha}{2(1-\alpha)} \tag{42}$$

$$\alpha = \frac{x_2-x_1}{x_3-x_1} \tag{43}$$

The quantity α, defined in Eq. (43), provides a measure of mid-node distortion. Let us now investigate if the conditions, Eqs. (31)-(33), are satisfied by Formulation 1. Using Eqs. (34)-(36), it is easy to verify that the first condition (Eq. 31) is satisfied identically. The second condition (Eq. 32) is satisfied inherently as we will be using the isoparametric formulation with Eq. (32) itself as the interpolation formula for x. Let us now check the third condition (Eq. 33).

Substituting for N_i's (from Eqs. 34-36) in Eq. (32), x may be written in terms of ξ as

$$x = c_0 + c_1\xi + c_2\xi^2 \tag{44}$$

where c_i, i=0, 1, 2 are constants. Using Eq. (40),

$$x^2 = (c_0+c_1\xi+c_2\xi^2)^2 = d_0+d_1\xi+d_2\xi^2+d_3\xi^3+d_4\xi^4 \tag{45}$$

where d_i, i=0, 1, 2, 3, 4 are constants. Thus, the expression for x^2 (Eq. 45), and hence the right hand side of Eq. (33), involve ξ^3 and ξ^4 terms which means that, for the satisfaction of Eq. (33), the left hand side of Eq. (33) also must involve ξ^3 and ξ^4 terms. However, using the expression for shape functions (Eqs. 34-36), the left hand side of Eq. (33) can be shown to be a quadratic function in ξ. Therefore, the third condition (Eq. 33) is not satisfied in general. However, when the mid-node is at the centre of the element, the relation between x and ξ is linear, in which case, the constant c_2 in Eq. (44) and the constants c_2, d_3 and d_4 in Eq. (45) vanish. Hence, for this special case, both the left and right hand side of Eq. (33) are quadratic functions of ξ, and Eq. (33) is satisfied.

Another way to check if Eq. (33) is satisfied is to rewrite Eq. (33) in the form of a residue, $R=\Sigma_i N_i x_i^2 - x^2$, and then substitute for x using Eq. (32) and N_i's using Eqs. (34)-(36). The resulting expression for the residue is simplified to yield

$$R = \frac{1}{4}(x_1-2x_2+x_3)\xi(1-\xi^2)\{x_1(\xi-2)-2\xi x_2+x_3(\xi+2)\} \tag{46}$$

It can be easily verified that the residue, R, reduces to zero when the mid-node is located at the centre of the element, i.e., $x_2=(x_1+x_3)/2$.

For Formulation 2, the conditions, Eqs. (31)-(33), are inherently satisfied whatever the position of the mid-node, as the shape functions themselves have been derived by solving Eqs. (31)-(33). Table 1

Table 1 A summary of the completeness requirements satisfied by Formulations 1 and 2 of Problem 1 under mid-node distortion

Completeness requirement		Formulation 1	Formulation 2 (new)
$\Sigma_i N_i=1$	Eq. (31)	Yes	Yes
$\Sigma_i N_i x_i=x$	Eq. (32)	Yes	Yes
$\Sigma_i N_i x_i^2=x^2$	Eq. (33)	No	Yes

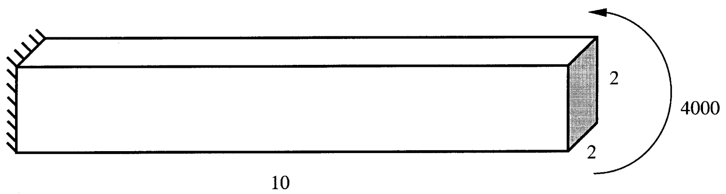
summarises our observations on the satisfaction/non-satisfaction of Eqs. (31)-(33).

For the numerical solution of the problem, consistent load vectors are used in both formulations. The stiffness matrix and the load vector are evaluated with 3 Gauss points, which is the exact numerical integration for both formulations. The numerical results of this demonstrative problem are presented in section 4.1.

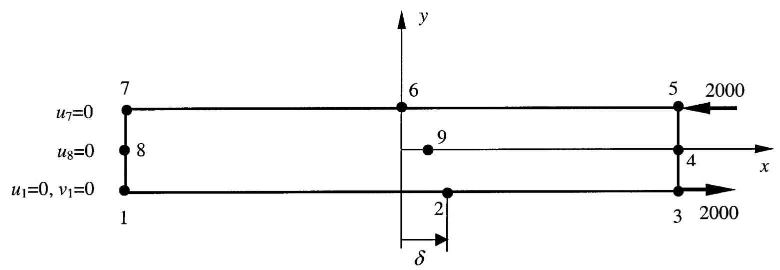
3.2. A cantilever beam under pure bending modelled with one 2D rectangular element with mid-node distortion - Problem 2

A cantilever beam of size $10\times2\times2$ is subject to a pure bending moment of 4000 units applied at the free end (Fig. 3). This problem is modelled using one 8-node/9-node rectangular element. One of the mid-nodes, i.e., node no. 6 (Fig. 3), is deliberately placed away from the mid-side position. Young's modulus of the material and Poisson's ratio are taken as 1500 and 0.25, respectively. Three different formulations are considered.

Formulation 1: Standard isoparametric formulation of the quadratic Serendipity element (8-node) is considered. The shape functions are as follows:



(a) A cantilever beam under pure bending



(b) An one-element finite element model

Fig. 3 A cantilever beam under a pure bending load modelled with one plane rectangular element (8-node/9-node) having mid-node distortion

$$N_i=\frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1),\ \text{for corner nodes} \tag{47}$$

$$N_i=\frac{\xi_i^2}{2}(1+\xi\xi_i)(1-\eta^2)+\frac{\eta_i^2}{2}(1+\eta\eta_i)(1-\xi^2),\ \text{for mid-nodes} \tag{48}$$

Formulation 2: The shape functions are obtained in principle by solving Eqs. (15)-(22). Analytical solution of these equations would yield expressions for the shape functions that are too long to handle, and hence is not of interest. For computing the stiffness matrix, it is sufficient to compute the numerical values of shape function derivative at the sampling points of numerical integration, and these values are directly obtained by solving Eqs. (23)-(30) numerically. Other details regarding this formulation are given in Appendix A.

Formulation 3: Standard isoparametric formulation of a 9-node Lagrangian element is considered. The shape functions are as follows:

$$N_9=(1-\xi^2)(1-\eta^2) \tag{49}$$

$$N_8=\frac{1}{2}(1+\xi)(1-\eta^2)-\frac{N_9}{2} \tag{50}$$

$$N_7=\frac{1}{2}(1-\xi^2)(1-\eta)-\frac{N_9}{2} \tag{51}$$

$$N_6=\frac{1}{2}(1-\eta^2)(1-\xi)-\frac{N_9}{2} \tag{52}$$

$$N_5=\frac{1}{2}(1-\xi^2)(1+\eta)-\frac{N_9}{2} \tag{53}$$

$$N_4=\frac{1}{4}(1+\xi)(1-\eta)-\frac{1}{2}(N_7+N_8)-\frac{N_9}{4} \tag{54}$$

$$N_3=\frac{1}{4}(1-\xi)(1-\eta)-\frac{1}{2}(N_6+N_7)-\frac{N_9}{4} \tag{55}$$

$$N_2=\frac{1}{4}(1-\xi)(1+\eta)-\frac{1}{2}(N_5+N_6)-\frac{N_9}{4} \tag{56}$$

$$N_1=\frac{1}{4}(1+\xi)(1+\eta)-\frac{1}{2}(N_8+N_5)-\frac{N_9}{4} \tag{57}$$

The pure bending moment is simulated by two equal and opposite forces of 2000 magnitude as shown in Fig. 3. In all the three formulations, the stiffness matrix is evaluated by the numerically exact 3-point Gaussian quadrature.

The solution to a cantilever beam under pure bending moment involves a quadratic polynomial in x and y. A quadratic displacement field in x and y involves six unknown coefficients, viz a_0, a_1, a_2, a_3, a_4 and a_5 of Eq. (13), and b_0, b_1, b_2, b_3, b_4 and b_5 of Eq. (14). An 8-node element can handle eight unknown coefficients. For a quadratic displacement field to be represented exactly by an 8-node rectangular element, the conditions to be satisfied are given by Eqs. (15)-(20). It is easy to verify that the condition, $\Sigma_i N_i=1$, is satisfied by Formulation 1; also the conditions, $\Sigma_i N_i x_i=x$ and $\Sigma_i N_i y_i=y$, are inherently satisfied because of the isoparametric formulation. Similarly, all these three

Table 2 A summary of the completeness requirements satisfied by Formulations 1, 2 and 3 of Problem 2 under mid-node distortion

Completeness requirement		Formulation 1 (8-node 'Serendipity')	Formulation 2 (new)	Formulation 3 (9-node Lagrangian)
$\Sigma_i N_i=1$	Eq. (15)	Yes	Yes	Yes
$\Sigma_i N_i x_i=x$	Eq. (16)	Yes	Yes	Yes
$\Sigma_i N_i y_i=y$	Eq. (17)	Yes	Yes	Yes
$\Sigma_i N_i x_i^2=x^2$	Eq. (18)	No	Yes	No
$\Sigma_i N_i x_i y_i=xy$	Eq. (19)	No	Yes	No
$\Sigma_i N_i y_i^2=y^2$	Eq. (20)	No	Yes	No
$\Sigma_i N_i x_i^2 y_i=x^2 y$	Eq. (21)	No	Yes	No
$\Sigma_i N_i x_i y_i^2=xy^2$	Eq. (22)	No	Yes	No

conditions are also satisfied by Formulations 2 and 3. The rest of the conditions were tested using *Mathematica* software. The results are summarised in Table 2.

It is seen from the table that Formulation 2 satisfies all the conditions, and Formulations 1 and 3 satisfy only the first three conditions. The numerical results of all the three formulations are discussed in section 4.2.

3.3. A cantilever beam under pure bending modelled with two 2D elements with angular distortion - Problem 3

We consider the same cantilever problem with all the three element formulations as in Problem 2. However, two elements are used here along the length, and the angular distortion as shown in Fig. 4 is considered instead of the mid-node distortion. As in Problem 2, it is easy to show that all the three formulations satisfy only Eqs. (15)-(17). *Mathematica* is again used to test the rest of the conditions, Eqs. (18)-(22). The results are summarised in Table 3. It is seen from the table that Formulation 1 satisfies only the first three conditions, Formulation 2 satisfies all the conditions, and Formulation 3 satisfies the first six conditions.

It is seen from Tables 2 and 3 that Formulation 2 satisfies all the completeness requirements. However, it is easy to verify that it does not satisfy the inter-element compatibility. Formulation 1,

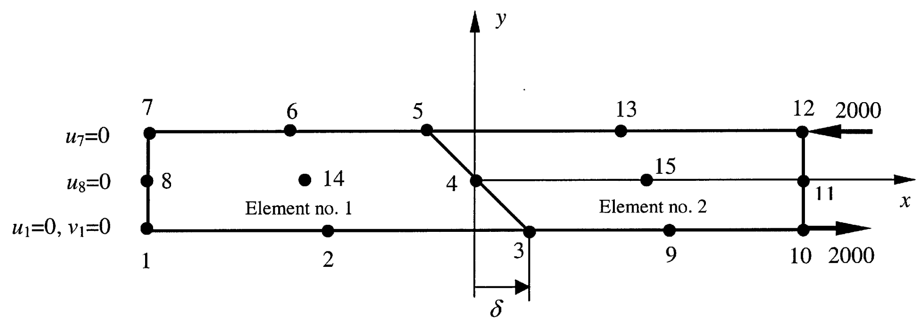


Fig. 4 A cantilever beam under a pure bending load modelled with two plane rectangular elements (8-node/ 9-node) having angular distortion

Table 3 A summary of the completeness requirements satisfied by Formulations 1, 2 and 3 of Problem 3 under angular distortion

Completeness requirement		Formulation 1 (8-node 'Serendipity')	Formulation 2 (new)	Formulation 3 (9-node Lagrangian)
$\Sigma_i N_i=1$	Eq. (15)	Yes	Yes	Yes
$\Sigma_i N_i x_i=x$	Eq. (16)	Yes	Yes	Yes
$\Sigma_i N_i y_i=y$	Eq. (17)	Yes	Yes	Yes
$\Sigma_i N_i x_i^2=x^2$	Eq. (18)	No	Yes	Yes
$\Sigma_i N_i x_i y_i=xy$	Eq. (19)	No	Yes	Yes
$\Sigma_i N_i y_i^2=y^2$	Eq. (20)	No	Yes	Yes
$\Sigma_i N_i x_i^2 y_i=x^2 y$	Eq. (21)	No	Yes	No
$\Sigma_i N_i x_i y_i^2=xy^2$	Eq. (22)	No	Yes	No

however, satisfies the inter-element compatibility, but only the first three completeness requirements. Formulation 3 satisfies the inter-element compatibility and the first six completeness requirements, viz. Eqs. (15)-(20) which are just required to capture exactly the quadratic displacement of the problem. The numerical results of the problem are presented in section 4.3.

4. Discussion of numerical results

4.1. Results for Problem 1

The displacement values computed by the two formulations are plotted in Fig. 5. The following are the main observations:

1. Formulation 2 gives exact displacements at every point in the element whatever the position of

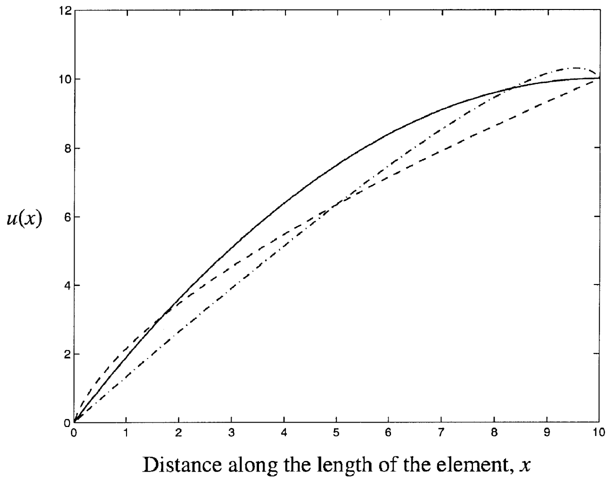


Fig. 5 Computed displacement values, $u(x)$, for Formulations 1 and 2 of Problem 1 (—— , Exact solution, Formulation 1 for x_2=5, Formulation 2 for any x_2: 0<x_2<10; ------ , Formulation 1 for x_2=3; –·–·– , Formulation 1 for x_2=7)

Table 4 Error in the displacement of the mid-node for Formulation 1 of Problem 1

x_2	$\alpha=\frac{x_2-x_1}{x_3-x_1}$	% error, $\frac{\Delta u_2}{u_2}\times 100$
1.0	0.1	−117.8947
3.0	0.3	−10.9804
5.0	0.5	0.0000
7.0	0.7	−6.1538
9.0	0.9	−22.6263

mid-node.

2. Formulation 1 gives exact displacements at every point in the element for x_2=5, i.e., only when the mid-node is at the centre of the element.
3. For other values of x_2, Formulation 1 generally gives inaccurate displacement distribution. However, at the end nodes it still gives the exact results. The error in the displacement of the mid-node for typical mid-node positions is shown in Table 4.

The poorer performance of Formulation 1 is a direct consequence of non-satisfaction of the completeness requirement given by Eq. (33). In order to prove this fact, an error model is developed now. Let us start with the error, E, introduced by non-satisfaction of Eq. (33).

$$E=x^2-\sum_i N_i x_i^2 \tag{58}$$

The error in the shape functions, ΔN_i, is given by the solution of the simultaneous equations,

$$\sum_i \Delta N_i=0 \tag{59}$$

$$\sum_i \Delta N_i x_i=0 \tag{60}$$

$$\sum_i \Delta N_i x_i^2=E \tag{61}$$

Let

$$M_i=N_i+\Delta N_i \tag{62}$$

Using the shape functions, M_i, the stiffness matrix and load vector are derived in the usual manner. The equations for the element may be written in the form

$$(\boldsymbol{K}+\Delta\boldsymbol{K})(\boldsymbol{u}+\Delta\boldsymbol{u})=\boldsymbol{P}+\Delta\boldsymbol{P} \tag{63}$$

where $\boldsymbol{K}$, $\boldsymbol{u}$ and $\boldsymbol{P}$ are the exact stiffness matrix, displacement vector and load vector, respectively, and $\Delta\boldsymbol{K}$, $\Delta\boldsymbol{u}$ and $\Delta\boldsymbol{P}$ are the corresponding quantities due to the error in shape functions.

Eq. (63) is re-written as

$$\Delta\boldsymbol{u}=(\boldsymbol{K}+\Delta\boldsymbol{K})^{-1}\,(\Delta\boldsymbol{P}-\Delta\boldsymbol{K}\boldsymbol{u}) \tag{64}$$

where

$$\boldsymbol{u}=\boldsymbol{K}^{-1}\,\boldsymbol{P} \tag{65}$$

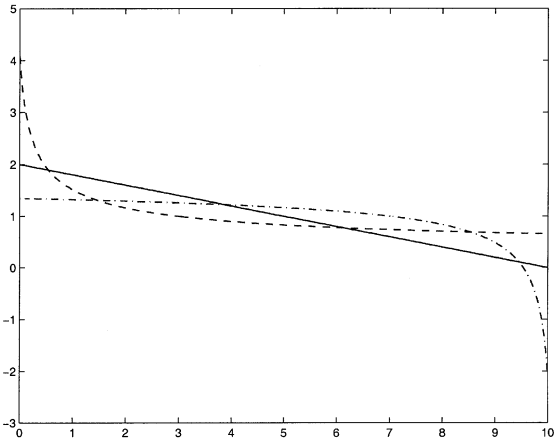


Fig. 6 Computed strain values, $\varepsilon(x)$, for Formulations 1 and 2 of Problem 1 (——, Exact solution, Formulation 1 for x_2=5, Formulation 2 for any x_2: $0<x_2<10$; ------, Formulation 1 for x_2=3; –·–·–, Formulation 1 for x_2=7)

After simplifying the expressions for $\Delta \boldsymbol{u}$ and $\boldsymbol{u}$, the expression for the error in the displacement of the mid-node is obtained as

$$\frac{\Delta u_2}{u_2} = \frac{7(1-2\alpha)^2}{20\alpha(\alpha-2)} \tag{66}$$

where α is defined in Eq. (43). It is easy to verify that Eq. (66) exactly predicts the errors shown in Table 4. This proves that the error in the finite element solution with Formulation 1 is entirely due to non-satisfaction of the completeness requirement, Eq. (33).

Fig. 6 shows the plot of strain distribution along the length of the element. It is seen from Fig. 6 that the distribution is linear (as it should be based on an exact solution) for Formulation 2 regardless of the position of mid-node in the range, $0 \leq x_2 \leq 10$. For Formulation 1, the distribution is generally nonlinear except for the case of x_2=5.

4.2. Results of Problem 2

The results for Problem no.2 are shown in Tables 5 and 6 which demonstrate the effect of mid-

Table 5 Computed vertical displacement at node no.5 (Fig. 3) for Problem 2

Mid-node off-set, δ	Formulation 1 (8-node 'Serendipity')	Formulation 2 (new)	Formulation 3 (9-node Lagrangian)
0.0	100.0	100	100.0
1.0	74.9	100	75.2
2.0	39.5	100	30.6
3.0	12.8	100	11.7
4.0	9.8	100	8.2
Exact solution		100	

Table 6 Computed σ_{xx} stress magnitude at node no. 6 (Fig. 3) for Problem 2

Mid-node off-set, δ	Formulation 1 (8-node 'Serendipity')	Formulation 2 (new)	Formulation 3 (9-node Lagrangian)
0.0	3000	3000	3000
1.0	2445	3000	2183
2.0	1350	3000	835
3.0	428	3000	353
4.0	293	3000	284
Exact solution		3000	

node distortion for the three formulations.

It is seen from Tables 5 and 6 that Formulation 2 gives exact displacement and stress values regardless of the position of the mid-node. Formulations 1 and 3 give exact values only when $\delta=0$, i.e., for the case of no mid-node distortion, and for $\delta \neq 0$, the computed values involve large errors. Thus, Formulations 1 and 3 are sensitive to mid-node distortion whereas Formulation 2 is not. The reason for the distortion insensitivity of Formulation 2 is that it satisfies (see Table 2) all the necessary completeness requirements, i.e., Eqs. (15)-(22) for the quadratic displacement field under consideration.

4.3. Results for Problem 3

The effect of angular distortion on the tip deflection is studied in this problem. The distortion is introduced by varying the x-coordinate of nodes 3 and 5 (Fig. 4) by equal distances (δ) in opposite directions. While doing so, the mid-nodes are always kept at the centre of the corresponding sides. The displacement and stress values at typical points are shown in Tables 7 and 8, respectively, for all the three formulations. The numerical integration of the stiffness matrix was carried out with 3x3 Gaussian quadrature. It is seen from Tables 7 and 8 that Formulations 1 and 2 are sensitive to angular distortion whereas Formulation 3 is not. Formulations 1 and 2 exhibit sensitivity to angular distortion for different reasons: Under angular-distorted geometry, Formulation 1 satisfies inter-element compatibility but not all the necessary completeness requirements, i.e., Eqs. (15)-(20) for representing the quadratic displacement field of the problem (see Table 3). On the other hand, Formulation 2 satisfies all the completeness requirements, but not the inter-element compatibility. However, Formulation 3 satisfies all the necessary completeness requirements, i.e., Eqs. (15)-(20),

Table 7 Computed vertical displacement of node no. 12 (Fig. 4) for Problem 3

Mid-node off-set, δ	Formulation 1 (8-node 'Serendipity')	Formulation 2 (new)	Formulation 3 (9-node Lagrangian)
0.0	100.0	100.0	100
1.0	99.4	100.8	100
2.0	89.4	108.1	100
3.0	59.7	128.7	100
4.0	32.0	169.5	100
Exact solution		100	

Table 8 Computed σ_{xx} stress magnitude at node no. 5 (Fig. 4) for Problem 3

Mid-node off-set, δ	Formulation 1 (8-node 'Serendipity')	Formulation 2 (new)	Formulation 3 (9-node Lagrangian)
0.0	3000	3000	3000
1.0	3058	3286	3000
2.0	2765	4114	3000
3.0	1627	5239	3000
4.0	629	6717	3000
Exact solution		3000	

as well as the inter-element compatibility requirement under angular-distorted geometries, and hence, this element is distortion-insensitive for the quadratic displacement field.

It is seen that Formulation 2 resulted in exact displacements for Problem 2 (see Tables 5 and 6) and not for problem 3 (see Tables 7 and 8). This is because of the fact that in Problem 2 the element geometry is rectangular although it has mid-node distortion. For rectangular geometries, the shape functions of Formulation 2 satisfy inter-element compatibility requirements in addition to all the completeness requirements.

It is well known in the literature (e.g., Lee and Bathe 1993) that under distorted geometry, the maximum order of polynomial up to which an element can preserve completeness generally decreases, and the Lagrangian elements are better than Serendipity in this respect. In this paper, the problem of degeneration of the completeness has been looked at from a different perspective. The conditions (Eq. 12) that the element shape functions need to satisfy in order to preserve the original completeness of the undistorted element have been presented. Further, these conditions have also been exploited successfully to develop a distortion insensitive 3-node bar element, and an 8-node rectangular element that is insensitive to mid-node distortion.

5. Conclusions

An alternative interpretation of the completeness requirements of higher order C^0 elements is presented in terms of certain conditions to be satisfied by the shape functions. These conditions are of the form,

$$\sum_i N_i x_i^p y_i^q z_i^r = x^p y^q z^r, \quad p, q, r = 0, 1, 2, 3, \ldots n \tag{67}$$

where n is the number of nodes in the element, and $x^p y^q z^r$ is a typical monomial term in the polynomial displacement model. All these conditions are usually satisfied by the existing elements in their undistorted geometry. However, under geometric distortions, not all the conditions are satisfied. The significance of non-satisfaction these conditions has been investigated for three typical elements. The following are some of the important observations:

1. Non-satisfaction of these conditions leads to poorer performance under geometric distortions of the element. For a 3-node bar element (Formulation 1 of Problem 1), an error model (Eq. 66) for the mid-node displacement has been derived based on the residue of that condition (Eq. 33) which is not satisfied.

2. A 3-node bar element that is insensitive to mid-node distortion has been developed by solving the set of equations representing the necessary completeness requirement of the problem given by Eq. (67). This approach can be directly extended to any higher order bar element.
3. Extension of the above method to develop distortion-insensitive 2-D elements is more challenging. For example, such an 8-node quadrilateral element works well only for mid-node distortions (Problem 2) and not for angular distortions (Problem 3). This is because the shape functions thus derived do not satisfy the inter-element compatibility under angular distortions.
4. The shape functions of the 9-node Lagrangian element for Problem 3 satisfy, in addition to satisfying inter-element compatibility across the element boundaries, all the necessary completeness requirements under angular distortions, and hence the element's insensitivity to angular distortions.

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Appendix A

Formulation of the 8-node plane element that satisfies all the completeness requirements

The starting point for this formulation is the set of completeness conditions represented by Eq. (12). For convenience we denote these conditions in the form,

$$\bar{\boldsymbol{P}}\boldsymbol{N}=\boldsymbol{P} \tag{68}$$

where

$$\bar{\boldsymbol{P}}=[\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3, \ldots, \boldsymbol{P}_8] \tag{69}$$

$$\boldsymbol{P}_i=[1,\ x_i, y_i, x_i^2, x_iy_i, y_i^2, x_i^2y_i, x_iy_i^2]^T \tag{70}$$

$$\boldsymbol{N}=[N_1, N_2, N_3, \ldots, N_8]^T \tag{71}$$

$$\boldsymbol{P}=[1,\ x, y, x^2, xy, y^2, x^2y, xy^2]^T \tag{72}$$

and the subscript i ($1 \leq i \leq 8$) refers to a typical node number. The shape functions, N_i, are obtained in principle by solving Eq. (68).

$$\boldsymbol{N}=\bar{\boldsymbol{P}}^{-1}\boldsymbol{P} \tag{73}$$

The strain-displacement matrix, $\boldsymbol{B}$, is obtained as,

$$\boldsymbol{B}=\begin{bmatrix} \boldsymbol{N}_x & 0 \\ 0 & \boldsymbol{N}_y \\ \boldsymbol{N}_y & \boldsymbol{N}_x \end{bmatrix}=\begin{bmatrix} (\bar{\boldsymbol{P}}^{-1}\boldsymbol{P}_x) & 0 \\ 0 & (\bar{\boldsymbol{P}}^{-1}\boldsymbol{P}_y) \\ (\bar{\boldsymbol{P}}^{-1}\boldsymbol{P}_y) & (\bar{\boldsymbol{P}}^{-1}\boldsymbol{P}_x) \end{bmatrix} \tag{74}$$

where the subscripts x and y refer to derivatives with respect to x and y, respectively. The stiffness matrix, $\boldsymbol{K}$, is obtained as,

$$\boldsymbol{K}=\int_V \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\,dv \tag{75}$$

The expression for $\boldsymbol{B}$ and hence $\boldsymbol{K}$ in Eq. (75) involves the global coordinates, x and y. In order to facilitate the numerical integration of Eq. (75), x and y are expressed in terms of the local coordinates ξ and η using the shape functions given by Eqs. (47) and (48), in which case, Eq. (75) takes the form,

$$\boldsymbol{K}=\int_{-1}^{1}\int_{-1}^{1} \boldsymbol{B}^T\boldsymbol{D}\boldsymbol{B}\; t\; Det[\boldsymbol{J}]\,d\xi d\eta \tag{76}$$

where $\boldsymbol{J}$ is the Jacobian matrix and t is the thickness of the element. Gaussian 3×3 integration rule is used to integrate Eq. (76).