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Free vibration analysis of axially moving laminated beams with axial tension based on 1D refined theories using Carrera unified formulation

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Abstract. In this paper, free vibration finite element analysis of axially moving laminated composite beams subjected to axial tension is studied. It is assumed that the beam has a constant axial velocity and is subject to uniform axial tension. The analysis is based on higher-order theories that have been presented by Carrera Unified Formulation (CUF). In the CUF technique, the three dimensional (3D) displacement fields are expressed as the approximation of the arbitrary order of the displacement unknowns over the cross-section. This higher-order expansion is considered in equivalent single layer (ESL) model. The governing equations of motion are obtained via Hamilton's principle. Finally, several numerical examples are presented and the effect of the ply-angle, travelling speed and axial tension on the natural frequencies and beam stability are demonstrated.

Keywords: axially moving; laminated composite beams; critical speeds; free vibration; finite element method; refined beam theory; carrera unified formulation

1. Introduction

Axially moving continua are involved in many engineering applications. Power transmission chains, band saws blades, robot arms, aerial cable tramways, conveyor belts, paper sheets in the process, magnetic tapes and textile fibers are just a few of many technological examples. Since many vehicles such as automobiles and aircraft can be subjected to an axial speed, many structures inside their frames can be classified as axially moving beams. Fibre-reinforced composite materials have many characteristics such as high strength-to-mass ratio and high corrosion resistance. Therefore, the materials of these axially moving beams can be replaced by fibre-reinforced composite in achieving better performance. In axially moving beams, the vibration analysis is important in obtaining better quality. The natural frequencies may be significantly affected by the axial motion even at low velocity. In the critical speed, the first natural frequency becomes zero and in the speeds higher than this speed divergence instability can occur. Thus, precise prediction of dynamic characteristics is required to avoid the violent vibration and optimal design.

As reviewed by Wickert and Mote (1988) the vibrations of axially moving structures made of isotropic materials have been studied for many years up to 1988. Wickert and Mote (1990) presented a classical vibration theory for the response of axially moving strings and beams using an eigenfunction method. Wickert (1992) investigated free non-linear vibration of an axially moving, elastic, tensioned beam in the sub- and supercritical speed ranges using the asymptotic method of Krylov, Bogoliubov, and Mitropolsky. Hwang and Perkins (1992) studied the stability of axially moving beams in the supercritical speeds. Styliano and Tabarrok (1994) examined the effects of physical damping, tip mass, tip support and wall flexibility on the stability characteristics of the axially moving beams. Al-Bedoor and Khulief (1996) introduced a systematic approach to obtain an approximate analytical solution for the vibrational motion of an elastic beam during axial deployment. Pakdemirli and Özkaya (1998) studied transverse vibrations of a simply supported traveling beam with constant speed, and the equations of motion solved approximately using the method of multiple scales. Öz and Pakdemirli (1999) and Öz (2001) calculated the natural frequencies of an axially traveling beam with variable velocity on simply supports and fixed supports, respectively. Chakraborty *et al*. (1999) investigated the free and forced responses of a slender traveling beam including the non-linear terms. Özkaya and Öz (2002) determined natural frequencies of axially traveling beams using artificial neural networks. Kong and Parker (2004) presented eigensolutions of axially moving beams with small flexural stiffness using a different perturbation method. Lee *et al*. (2004) studied transverse vibration of an axially moving Timoshenko beam using exact dynamic stiffness matrix in structural dynamics. Lee and Jang (2007) investigated the stability of axially moving beams using the spectral element method. Ghayesh and Khadem (2008) formulated free non-linear transverse vibration of an axially moving beam in which rotary inertia and temperature variation effects have been considered. Ghayesh and Balar (2010) investigated the nonlinear parametric vibrations and stability of an axially moving Timoshenko beam considering two dynamic models. Ghayesh and Amabili (2013) investigated the non-linear dynamics of an axially moving beam with time-dependent axial speed. Rezaee and

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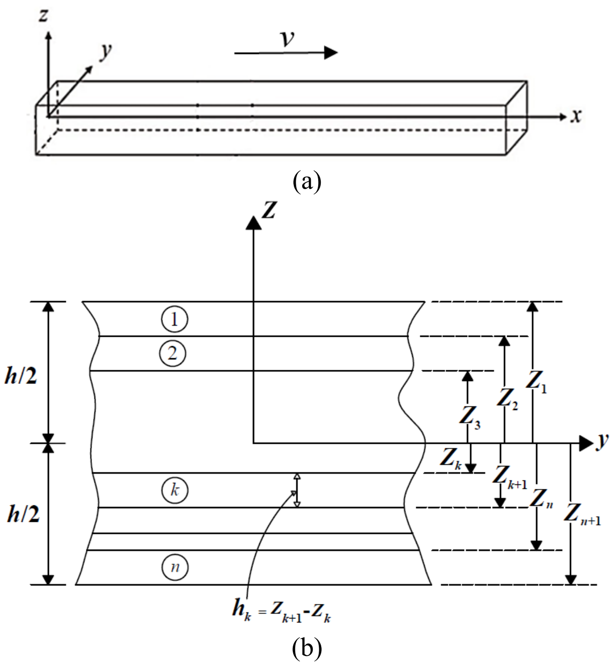

Fig. 1 (a) Adopted Cartesian coordinate system and (b) cross-section of the beam

Lotfan (2015) studied the stability and non-linear vibrations of an axially moving nanoscale visco-elastic Rayleigh beam by applying the nonlocal theory and considering small fluctuations in the axial velocity. Mokhtari and Mirdamadi (2018) scrutinized vibration and stability of an axially translating viscoelastic Timoshenko beam constrained by simple supports and subjected to axial pretension. Ghorbanpour Arani *et al*. (2017) investigated the vibration of functionally graded nanocomposite plates moving in two directions.

All of the studies mentioned above considered only classical models based on the Euler-Bernoulli and Timoshenko theories. Euler-Bernoulli theory does not consider transverse shear deformation. The Timoshenko model considers a constant shear deformation along the cross-section of the beam. However, both models do not adequately consider warping such as out-plane deformations and in-plane deformations and bending-torsion coupling. Higher-order models are presented in the Carrera Unified Formulation (CUF) framework which was developed for plates and shells by Carrera (1995), Carrera (2002), Carrera (2003) and Carrera *et al*. (2008) and it was extended to beam modelling by Carrera and Giunta (2010). This formulation has also been utilized to analyze the composite laminated beams with straight fibers (Catapano *et al*. 2011, Pagani *et al*. 2014, Carrera *et al*. 2016, Carrera *et al*. 2015, Carrera *et al*. 2016a, b), composite laminated beams with curvilinear fibers (Daraei *et al*. 2020) and sandwich beams (Giunta *et al*. 2015, Liu *et al*. 2017, Hui *et al*. 2017, Giunta *et al*. 2013). Resently some authors incorporated CUF with B-spline basis functions (Yan *et al*. 2020, Alesadi *et al*. 2017a, b, Alesadi *et al*. 2019, Ghazanfari *et al*. 2019, radial basis functions (Pagani *et al*. 2016) and spherical hankel-basis functions (Alesadi *et al*. 2020) for analysis of structures. Also, Pagani and Carrera (2018) could recently introduce CUF including geometrical nonlinearities to analysis of beam structures.

Although there is much work devoted on different approaches for various problems on nonmoving (for example, Kahya and Turan (2018), Ho-Huu *et al*. (2018), Kahya *et al*. (2019)) and moving beams, there is no higher-order model to analyze the moving laminated beams. In this paper, the natural frequencies of axially moving and tensioned laminated composite beams will be investigated via refined beam theory based on the CUF. Also, equivalent single layer (ESL) model is used as a solution field description approach that considers the whole laminates as a single layer. In this model, the number of unknown variables is independent of the number of constitutive layers.

2. Formulation

Consider an n-ply laminated composite beam with length L, width b and total thickness h which is comprised of layers of orthotropic materials. The beam is assumed to be moving with constant velocity v in the x-direction and also with initial axial tension σ^0. The subscript k is used for denoting the number of a generic layer. Cross-section Ω is defined on the yz-plane and x-axis is coincident to the longitudinal axis of the beam (Fig. 1).

2.1 Displacement field

In the framework of the CUF, the 3D displacement field is defined in a compact form as follow (Carrera *et al*. 2011)

$$\mathbf{u}(x,y,z) = F_\tau(y,z)\mathbf{u}_\tau(x) \qquad \tau = 1,2,\ldots,M \tag{1}$$

Table 1 Taylor-like polynomials

N	M	F_τ
0	1	$F_1=1$
1	3	$F_2=y$ $F_3=z$
2	6	$F_4=y^2$ $F_5=yz$ $F_6=z^2$
3	10	$F_7=y^3$ $F_8=y^2z$ $F_9=yz^2$ $F_{10}=z^3$
.	.	.
.	.	.
.	.	.
N	$(N+1)(N+2)/2$	$F_{(N^2+N+2)/2}=y^N$ … $F_{(N+1)(N+2)/2}=z^N$

where F_τ are expansion functions of the y and z coordinates on the cross-section, $\mathbf{u}_\tau$ are the generalized displacement vector with respect to axial coordinate x, and M stands for the number of the expansion terms. According to the Einstein notation, the repeated subscript τ stands for summation. Taylor-like polynomials are employed as expansion functions F_τ that consist of the 2D base $y^i z^j$ (i and j are positive integers). N is the order of the beam model, and M and F_τ are functions of N which can be obtained via Taylor-like expansions, as shown in Table 1. Therefore, the N-order displacement field is

$$\begin{aligned} u_x &= u_{x1}+u_{x2}y+u_{x3}z+\cdots+u_{x\frac{(N^2+N+2)}{2}}y^N+\cdots + u_{x\frac{(N+1)(N+2)}{2}}z^N \\ u_y &= u_{y1}+u_{y2}y+u_{y3}z+\cdots+u_{y\frac{(N^2+N+2)}{2}}y^N+\cdots + u_{y\frac{(N+1)(N+2)}{2}}z^N \\ u_z &= u_{z1}+u_{z2}y+u_{z3}z+\cdots+u_{z\frac{(N^2+N+2)}{2}}y^N+\cdots + u_{z\frac{(N+1)(N+2)}{2}}z^N \end{aligned} \tag{2}$$

The Finite Element Method (FEM) is used to discretize the beam along the x-axis and the beam elements with two-nodes are formulated here. Therefore, linear approximations along the longitudinal axis are adopted. The nodal displacement vector for a two-node beam element is defined as (Carrera *et al.* 2011)

$$\mathbf{u}_{\tau i}=\{u_{x\tau i}\quad u_{y\tau i}\quad u_{z\tau i}\}^T \qquad i=1,2 \tag{3}$$

The superscript "T" respects the transposition operator. The displacement variables can be approximated along the x-axis by using the shape functions N_i as follow

$$\mathbf{u}=N_i F_\tau \mathbf{u}_{\tau i} \tag{4}$$

For the sake of brevity, the linear shape functions N_i are not presented here and can be found in Bathe (1996).

2.2 Strain and stress field

The strain vector is expressed as follow

$$\boldsymbol{\varepsilon}=\{\varepsilon_{xx}\quad \varepsilon_{yy}\quad \varepsilon_{zz}\quad \varepsilon_{yz}\quad \varepsilon_{xz}\quad \varepsilon_{xy}\}^T \tag{5}$$

Therefore, the following linear strain-displacement relations are employed

$$\boldsymbol{\varepsilon}=\begin{Bmatrix} F_\tau \dfrac{\partial N_i}{\partial x}u_{x\tau i} \\ \dfrac{\partial F_\tau}{\partial y}N_i u_{y\tau i} \\ \dfrac{\partial F_\tau}{\partial z}N_i u_{z\tau i} \\ \dfrac{\partial F_\tau}{\partial z}N_i u_{y\tau i}+\dfrac{\partial F_\tau}{\partial y}N_i u_{z\tau i} \\ \dfrac{\partial F_\tau}{\partial z}N_i u_{x\tau i}+F_\tau\dfrac{\partial N_i}{\partial x}u_{z\tau i} \\ \dfrac{\partial F_\tau}{\partial y}N_i u_{x\tau i}+F_\tau\dfrac{\partial N_i}{\partial x}u_{y\tau i} \end{Bmatrix} \tag{6}$$

The stress-strain relations for kth composite layer in the x, y and z directions can be written as follow (Reddy 2004)

$$\boldsymbol{\sigma}=\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{Bmatrix}^{(k)} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16}\\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26}\\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36}\\ 0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0\\ 0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0\\ \bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66} \end{bmatrix}^{(k)} \begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\\ \varepsilon_{xy}\end{Bmatrix}^{(k)} \tag{7}$$

where the $\bar{C}_{ij}$ are the transformed elastic coefficients referred to the (x,y,z) coordinate system. These coefficients are presented in Appendix A and details can be found in Reddy (2004).

2.3 Equation of motion

Here, the equation of motion is derived by using Hamilton's principle for the piece of the beam in time t. Hamilton's principle can be expressed in the familiar form as follow

$$\delta\int_{t_\mathrm{i}}^{t_\mathrm{f}}(U-T)dt=0 \tag{8}$$

where U is the total potential energy and T is the kinetic energy. U includes the strain energy U_s and the energy due to initial axial tension U_g.

$$U=U_s+U_g \tag{9}$$

The variation of the strain energy U_s for each k layer can be written as

$$\delta {U_s}^k=\int_{\Omega_k}\int_{l_k}\delta\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}\,dx\,d\Omega \tag{10}$$

where Ω indicates integration over y and z. Substituting CUF displacement field in Eq. (4), the geometrical relations in Eq. (6) and the constitutive equations in Eq. (7) into Eq. (10), it will be changed to the shape presented in Appendix B. However, the final form of the Eq. (10) is written as in the following

$$\delta U_s^k = \mathbf{u}_{\tau i}{}^T \left[\overline{\mathbf{k}}\right] \delta \mathbf{u}_{sj} \tag{11}$$

where $\left[\overline{\mathbf{k}}\right]$ is the stiffness matrix that is written in a 3×3 array of the fundamental nucleus (FN) as

$$\left[\overline{\mathbf{k}}\right] = \begin{bmatrix} \bar{k}_{xx}^{kij\tau s} & \bar{k}_{xy}^{kij\tau s} & \bar{k}_{xz}^{kij\tau s} \\ \bar{k}_{yx}^{kij\tau s} & \bar{k}_{yy}^{kij\tau s} & \bar{k}_{yz}^{kij\tau s} \\ \bar{k}_{zx}^{kij\tau s} & \bar{k}_{zy}^{kij\tau s} & \bar{k}_{zz}^{kij\tau s} \end{bmatrix} \tag{12}$$

and the he explicit form of this nucleus can be written as

$$\begin{aligned}
\bar{k}_{xx}^{kij\tau s} &= \bar{C}_{11}^k A^{kij\tau s} + \bar{C}_{16}^k\left(B^{kij\tau s} + C^{kij\tau s}\right) + \bar{C}_{66}^k D^{kij\tau s} \\ &\quad + \bar{C}_{55}^k E^{kij\tau s} \\
\bar{k}_{xy}^{kij\tau s} &= \bar{C}_{12}^k C^{kij\tau s} + \bar{C}_{26}^k D^{kij\tau s} + \bar{C}_{45}^k E^{kij\tau s} + \bar{C}_{16}^k A^{kij\tau s} \\ &\quad + \bar{C}_{66}^k B^{kij\tau s} \\
\bar{k}_{xz}^{kij\tau s} &= \bar{C}_{13}^k F^{kij\tau s} + \bar{C}_{36}^k G^{kij\tau s} + \bar{C}_{45}^k H^{kij\tau s} + \bar{C}_{55}^k I^{kij\tau s} \\
\bar{k}_{yx}^{kij\tau s} &= \bar{C}_{12}^k B^{kij\tau s} + \bar{C}_{16}^k A^{kij\tau s} + \bar{C}_{26}^k D^{kij\tau s} + \bar{C}_{66}^k C^{kij\tau s} \\ &\quad + \bar{C}_{45}^k E^{kij\tau s} \\
\bar{k}_{yy}^{kij\tau s} &= \bar{C}_{22}^k D^{kij\tau s} + \bar{C}_{26}^k\left(B^{kij\tau s} + C^{kij\tau s}\right) + \bar{C}_{44}^k E^{kij\tau s} \\ &\quad + \bar{C}_{66}^k A^{kij\tau s} \\
\bar{k}_{yz}^{kij\tau s} &= \bar{C}_{23}^k G^{kij\tau s} + \bar{C}_{36}^k F^{kij\tau s} + \bar{C}_{44}^k H^{kij\tau s} + \bar{C}_{45}^k I^{kij\tau s} \\
\bar{k}_{zx}^{kij\tau s} &= \bar{C}_{13}^k I^{kij\tau s} + \bar{C}_{36}^k H^{kij\tau s} + \bar{C}_{45}^k G^{kij\tau s} + \bar{C}_{55}^k F^{kij\tau s} \\
\bar{k}_{zy}^{kij\tau s} &= \bar{C}_{23}^k H^{kij\tau s} + \bar{C}_{44}^k G^{kij\tau s} + \bar{C}_{45}^k F^{kij\tau s} + \bar{C}_{36}^k I^{kij\tau s} \\
\bar{k}_{zz}^{kij\tau s} &= \bar{C}_{33}^k E^{kij\tau s} + \bar{C}_{44}^k D^{kij\tau s} + \bar{C}_{45}^k\left(B^{kij\tau s} + C^{kij\tau s}\right) \\ &\quad + \bar{C}_{55}^k A^{kij\tau s}
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
A^{kij\tau s} &= \int_{l_k} \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx \int_{\Omega_k} F_\tau F_s \, dydz \\
B^{kij\tau s} &= \int_{l_k} N_i \frac{\partial N_j}{\partial x} dx \int_{\Omega_k} \frac{\partial F_\tau}{\partial y} F_s \, dydz \\
C^{kij\tau s} &= \int_{l_k} \frac{\partial N_i}{\partial x} N_j dx \int_{\Omega_k} F_\tau \frac{\partial F_s}{\partial y} dydz \\
D^{kij\tau s} &= \int_{l_k} N_i N_j dx \int_{\Omega_k} \frac{\partial F_\tau}{\partial y}\frac{\partial F_s}{\partial y} dydz \\
E^{kij\tau s} &= \int_{l_k} N_i N_j dx \int_{\Omega_k} \frac{\partial F_\tau}{\partial z}\frac{\partial F_s}{\partial z} dydz \\
F^{kij\tau s} &= \int_{l_k} \frac{\partial N_i}{\partial x} N_j dx \int_{\Omega_k} F_\tau \frac{\partial F_s}{\partial z} dydz \\
G^{kij\tau s} &= \int_{l_k} N_i N_j dx \int_{\Omega_k} \frac{\partial F_\tau}{\partial y}\frac{\partial F_s}{\partial z} dydz \\
H^{kij\tau s} &= \int_{l_k} N_i N_j dx \int_{\Omega_k} \frac{\partial F_\tau}{\partial z}\frac{\partial F_s}{\partial y} dydz \\
I^{kij\tau s} &= \int_{l_k} N_i \frac{\partial N_j}{\partial x} dx \int_{\Omega_k} \frac{\partial F_\tau}{\partial z} F_s \, dydz
\end{aligned} \tag{14}$$

The variation of the energy due to initial axial tension U_g can be written as

$$\delta U_g = \int_\Omega \int_l \delta \varepsilon_{xx}^{nl} \sigma_{xx}^0 dx d\Omega \tag{15}$$

where σ_{xx}^0 is equal to initial axial tension σ^0, and ε_{xx}^{nl} is the nonlinear strain in the x-direction and can be expressed as

$$\varepsilon_{xx}^{nl} = \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right] \tag{16}$$

Therefore

$$\delta \varepsilon_{xx}^{nl} = \frac{\partial \delta u_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial \delta u_y}{\partial x}\frac{\partial u_y}{\partial x} + \frac{\partial \delta u_z}{\partial x}\frac{\partial u_z}{\partial x} \tag{17}$$

Substituting Eqs. (17) and (4) in Eq. (15), it gives the form presented in Appendix B and its final form can be written as

$$\delta U_g = \mathbf{u}_{\tau i}{}^T \left[\overline{\overline{\mathbf{k}}}\right] \delta \mathbf{u}_{sj} \tag{18}$$

where $\left[\overline{\overline{\mathbf{k}}}\right]$ is the geometric stiffness matrix for a stationary beam subjected an initial axial tension σ^0 as the form

$$\left[\overline{\overline{\mathbf{k}}}\right] = \begin{bmatrix} \bar{\bar{k}}_{xx}^{ij\tau s} & \bar{\bar{k}}_{xy}^{ij\tau s} & \bar{\bar{k}}_{xz}^{ij\tau s} \\ \bar{\bar{k}}_{yx}^{ij\tau s} & \bar{\bar{k}}_{yy}^{ij\tau s} & \bar{\bar{k}}_{yz}^{ij\tau s} \\ \bar{\bar{k}}_{zx}^{ij\tau s} & \bar{\bar{k}}_{zy}^{ij\tau s} & \bar{\bar{k}}_{zz}^{ij\tau s} \end{bmatrix} \tag{19}$$

and explicit form of non-zero terms of the FN of this matrix is given as follows

$$\bar{\bar{k}}_{xx}^{ij\tau s} = \bar{\bar{k}}_{yy}^{ij\tau s} = \bar{\bar{k}}_{zz}^{ij\tau s} = \sigma^0 \int_l \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx \int_\Omega F_\tau F_s \, d\Omega \tag{20}$$

Thus, we can write

$$\delta \int_{t_i}^{t_f} U dt = \int_{t_i}^{t_f} \mathbf{u}_{\tau i}{}^T \left(\left[\overline{\mathbf{k}}\right] + \left[\overline{\overline{\mathbf{k}}}\right]\right) \delta \mathbf{u}_{sj} dt \tag{21}$$

When arbitrary function, ϕ, is known in the spatial description, its total time derivative, D/Dt, for a given point, known as the material derivative, is (Reddy 2013)

$$\frac{D}{Dt}[\phi(x, y, z, t)] = \frac{\partial \phi}{\partial t} + \mathbf{v}.\nabla\phi \tag{22}$$

where $\mathbf{v}$ is the velocity vector, the dot is the scalar product and ∇ is the gradient operator. Moreover, measured by a stationary observer, the longitudinal velocity of a point on the beam consists of two parts, the axial transport velocity of the beam, v, and the local velocity caused by variations in the longitudinal displacement. Therefore, by using this fact and also the meaning of the total time derivative in Eq. (22), the velocity components in the x, y and z directions of a point of the axially moving beam can be written as

$$\begin{aligned}
v_x^* &= v + \frac{Du_x}{Dt} = v + \frac{\partial u_x}{\partial t} + v\frac{\partial u_x}{\partial x} \\
v_y^* &= \frac{Du_y}{Dt} = \frac{\partial u_y}{\partial t} + v\frac{\partial u_y}{\partial x} \\
v_z^* &= \frac{Du_z}{Dt} = \frac{\partial u_z}{\partial t} + v\frac{\partial u_z}{\partial x}
\end{aligned} \tag{23}$$

Subsequently, the kinetic energy T can be written as

$$\begin{aligned}
T = \frac{1}{2}\int_\Omega \int_l \rho \Bigg[&\left(v + \frac{\partial u_x}{\partial t} + v\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial t} + v\frac{\partial u_y}{\partial x}\right)^2 \\ &+ \left(\frac{\partial u_z}{\partial t} + v\frac{\partial u_z}{\partial x}\right)^2\Bigg] dx d\Omega
\end{aligned} \tag{24}$$

or

$$T = T_1 + T_2 + T_3 + T_4 \tag{25}$$

where

$$\begin{gathered}T_1 = \frac{1}{2}\int_V \rho\left[\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_y}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right] dV \\ T_2 = \frac{1}{2}\int_V \rho v^2\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right] dV \\ T_3 = \int_V \rho v\left[\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial t} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial t} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial t} + \frac{\partial u_x}{\partial t} + v\frac{\partial u_x}{\partial x}\right] dV \\ T_4 = \frac{1}{2}\rho v^2 AL\end{gathered} \tag{26}$$

where ρ is the mass density, V is the volume domain of the body, and A is the cross-section area of the beam.

The first term of Eq. (25) T_1 is the kinetic energy for the nonmoving beam. Taking its variation and integrating it with respect to the time between initial time t_{i} and the final time t_{f}, we obtain

$$\begin{aligned}\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_1\ dt = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\int_V \rho\Big[&\frac{\partial\delta u_x}{\partial t}\frac{\partial u_x}{\partial t} + \frac{\partial\delta u_y}{\partial t}\frac{\partial u_y}{\partial t} \\ &+ \frac{\partial\delta u_z}{\partial t}\frac{\partial u_z}{\partial t}\Big] dV dt\end{aligned} \tag{27}$$

By using integration by parts

$$\begin{aligned}\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_1\ dt = &\int_V \rho\left(\delta u_x\frac{\partial u_x}{\partial t}\right) dV\Big|_{t_i}^{t_f} - \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\int_V \rho\left(\delta u_x\frac{\partial^2 u_x}{\partial t^2}\right) dV dt \\ &+ \int_V \rho\left(\delta u_y\frac{\partial u_y}{\partial t}\right) dV\Big|_{t_i}^{t_f} \\ &- \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\int_V \rho\left(\delta u_y\frac{\partial^2 u_y}{\partial t^2}\right) dV dt \\ &+ \int_V \rho\left(\delta u_z\frac{\partial u_z}{\partial t}\right) dV\Big|_{t_i}^{t_f} \\ &- \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\int_V \rho\left(\delta u_z\frac{\partial^2 u_z}{\partial t^2}\right) dV dt\end{aligned} \tag{28}$$

A virtual displacement $\delta\mathbf{u}$ satisfies the conditions (Reddy 2013

$$\begin{gathered}\delta\mathbf{u}(x,y,z,t_{\mathrm{i}}) = \delta\mathbf{u}(x,y,z,t_{\mathrm{f}}) = 0 \ \text{ for all } \ x,y,z \\ \delta\mathbf{u}(x,y,z,t) = 0 \ \text{ on } \ \Gamma_u \ \text{ for all } \ t\end{gathered} \tag{29}$$

in which Γ_u is the segment of the boundary Γ where the displacement vector $\mathbf{u}$ is specified. Therefore, in Eq. (28), the first, third and fifth terms vanish. Finally, substituting CUF displacement field (Eq. (4)) into Eq. (28), it gives the form presented in Appendix B and its final form can be written as

$$\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_1\ dt = -\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \delta\mathbf{u}_{sj}{}^T[\mathbf{m}]\ddot{\mathbf{u}}_{\tau i}\, dt \tag{30}$$

where a dot means differentiation with respect to t and $[\mathbf{m}]$ is the mass matrix and has the form

$$[\boldsymbol{m}] = \begin{bmatrix} m_{xx}^{ij\tau s} & m_{xy}^{ij\tau s} & m_{xz}^{ij\tau s} \\ m_{yx}^{ij\tau s} & m_{yy}^{ij\tau s} & m_{yz}^{ij\tau s} \\ m_{zx}^{ij\tau s} & m_{zy}^{ij\tau s} & m_{zz}^{ij\tau s}\end{bmatrix} \tag{31}$$

and the explicit form of the non-zero terms of the FN of this matrix can be written as (Carrera *et al*. 2011)

$$m_{xx}^{ij\tau s} = m_{yy}^{ij\tau s} = m_{zz}^{ij\tau s} = \rho\int_l N_i N_j dx\int_\Omega F_\tau F_s dy dz \tag{32}$$

The second term of Eq. (25) T_2 is related to the effect of axial speed on the displacements. We can write

$$\begin{aligned}\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_2\ dt = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\int_V \rho v^2\Big[&\frac{\partial\delta u_x}{\partial x}\frac{\partial u_x}{\partial x} + \frac{\partial\delta u_y}{\partial x}\frac{\partial u_y}{\partial x} \\ &+ \frac{\partial\delta u_z}{\partial x}\frac{\partial u_z}{\partial x}\Big] dV dt\end{aligned} \tag{33}$$

Substituting CUF displacement field (Eq. (4)) into Eq. (33), it gives the form presented in Appendix B and its final form can be written as

$$\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_2\ dt = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} \delta\mathbf{u}_{sj}{}^T[\underline{\mathbf{k}}]\mathbf{u}_{\tau i}\, dt \tag{34}$$

where $[\underline{\mathbf{k}}]$ is the geometric stiffness matrix for an axially moving beam as the form

$$[\underline{\mathbf{k}}] = \begin{bmatrix} \underline{k}_{xx}^{ij\tau s} & \underline{k}_{xy}^{ij\tau s} & \underline{k}_{xz}^{ij\tau s} \\ \underline{k}_{yx}^{ij\tau s} & \underline{k}_{yy}^{ij\tau s} & \underline{k}_{yz}^{ij\tau s} \\ \underline{k}_{zx}^{ij\tau s} & \underline{k}_{zy}^{ij\tau s} & \underline{k}_{zz}^{ij\tau s}\end{bmatrix} \tag{35}$$

and explicit form of non-zero terms of the FN of this matrix is found by

$$\begin{aligned}\underline{k}_{xx}^{ij\tau s} &= \underline{k}_{yy}^{ij\tau s} = \underline{k}_{zz}^{ij\tau s} \\ &= \rho v^2\int_l \frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} dx\int_\Omega F_\tau F_s dy dz\end{aligned} \tag{36}$$

For the third term T_3, we can write

$$\begin{aligned}\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_3\ dt = \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\int_V \rho v\Big(&\frac{\partial\delta u_x}{\partial x}\frac{\partial u_x}{\partial t} + \frac{\partial u_x}{\partial x}\frac{\partial\delta u_x}{\partial t} \\ &+ \frac{\partial\delta u_y}{\partial x}\frac{\partial u_y}{\partial t} + \frac{\partial u_y}{\partial x}\frac{\partial\delta u_y}{\partial t} + \frac{\partial\delta u_z}{\partial x}\frac{\partial u_z}{\partial t} \\ &+ \frac{\partial u_z}{\partial x}\frac{\partial\delta u_z}{\partial t} + \frac{\partial\delta u_x}{\partial t} + v\frac{\partial\delta u_x}{\partial x}\Big) dV dt\end{aligned} \tag{37}$$

By using integration by parts

$$\begin{aligned}\delta\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} T_3\ dt = &\int_{t_i}^{t_f}\int_V \rho v\left(\frac{\partial\delta u_x}{\partial x}\frac{\partial u_x}{\partial t}\right) dV dt \\ &+ \int_V \rho v\left(\delta u_x\frac{\partial u_x}{\partial x}\right) dV\Big|_{t_i}^{t_f} \\ &- \int_{t_i}^{t_f}\int_V \rho v\left(\delta u_x\frac{\partial^2 u_x}{\partial x\partial t}\right) dV dt \\ &+ \int_{t_i}^{t_f}\int_V \rho v\left(\frac{\partial\delta u_y}{\partial x}\frac{\partial u_y}{\partial t}\right) dV dt \\ &+ \int_V \rho v\left(\delta u_y\frac{\partial u_y}{\partial x}\right) dV\Big|_{t_i}^{t_f} \\ &- \int_{t_i}^{t_f}\int_V \rho v\left(\delta u_y\frac{\partial^2 u_y}{\partial x\partial t}\right) dV dt \\ &+ \int_{t_i}^{t_f}\int_V \rho v\left(\frac{\partial\delta u_z}{\partial x}\frac{\partial u_z}{\partial t}\right) dV dt\end{aligned} \tag{38}$$

$$+\int_V \rho v\left(\delta u_z \frac{\partial u_z}{\partial x}\right) dV \Big|_{t_i}^{t_f} - \int_{t_i}^{t_f}\int_V \rho v\left(\delta u_z \frac{\partial^2 u_z}{\partial x \partial t}\right) dV dt$$
$$+\int_V \rho v(\delta u_x) dV \Big|_{t_i}^{t_f}$$
$$+\int_{t_i}^{t_f}\int_V \rho v^2 \left(\frac{\partial \delta u_x}{\partial x}\right) dV dt$$

By satisfying the conditions presented in Eq. (29), the second, fifth, eighth, tenth and eleventh terms of above equation vanish. Finally, substituting Eq. (4) into Eq. (38), it gives the form presented in Appendix B and its final form can be written as

$$\delta \int_{t_i}^{t_f} T_3 \; dt = -\int_{t_i}^{t_f} \delta \mathbf{u}_{sj}{}^T [\mathbf{g}] \dot{\mathbf{u}}_{\tau i} \, dt \tag{39}$$

where $[\mathbf{g}]$ is the gyroscopic matrix as the form

$$[\mathbf{g}] = \begin{bmatrix} g_{xx}^{ij\tau s} & g_{xy}^{ij\tau s} & g_{xz}^{ij\tau s} \\ g_{yx}^{ij\tau s} & g_{yy}^{ij\tau s} & g_{yz}^{ij\tau s} \\ g_{zx}^{ij\tau s} & g_{zy}^{ij\tau s} & g_{zz}^{ij\tau s} \end{bmatrix} \tag{40}$$

and explicit form of non-zero terms of the FN of this matrix is given as follows

$$g_{xx}^{ij\tau s} = g_{yy}^{ij\tau s} = g_{zz}^{ij\tau s} = \int_\Omega \int_l \rho v \left[F_\tau F_s \left(\frac{\partial N_i}{\partial x} N_j - N_i \frac{\partial N_j}{\partial x} \right) \right] dx d\Omega \tag{41}$$

The fourth term T_4 is a constant value. Therefore, it is omitted by variation. Finally, by using Hamilton's principle in Eq. (8), the governing equation of motion for free vibration analysis the axially moving beam can be formulated as

$$[\mathbf{m}]\{\ddot{\mathbf{u}}\} + [\mathbf{g}]\{\dot{\mathbf{u}}\} + [\mathbf{k}]\{\mathbf{u}\} = \{0\} \tag{42}$$

where the total stiffness matrix $[\mathbf{k}]$ is

$$[\mathbf{k}] = [\overline{\mathbf{k}}] + [\overline{\overline{\mathbf{k}}}] - [\underline{\mathbf{k}}] \tag{43}$$

After CUF assembly technique for the fundamental nuclei (this technique is presented in Sec.(2.4)), the mass matrix, gyroscopic matrix and stiffness matrix of the beam can be obtained by conventional methods of assembling matrices of all elements of the moving beam. Therefore, Eq. (42) for the whole beam is given as

$$[\mathbf{M}]\{\ddot{\mathbf{U}}\} + [\mathbf{G}]\{\dot{\mathbf{U}}\} + [\mathbf{K}]\{\mathbf{U}\} = \{0\} \tag{44}$$

If $\{\mathbf{U}(t)\}$ defined as

$$\{\mathbf{U}(t)\} = \{\widehat{\mathbf{U}}(t)\} e^{\lambda t} \tag{45}$$

Eq. (44) yields

$$\lambda^2[\mathbf{M}]\{\widehat{\mathbf{U}}\} + \lambda[\mathbf{G}]\{\widehat{\mathbf{U}}\} + [\mathbf{K}]\{\widehat{\mathbf{U}}\} = \{0\} \tag{46}$$

To find the eigenvalues λ, Eq. (46) can be rewritten in the form as

$$\lambda[\mathbf{A}]\{\overline{\mathbf{U}}\} + [\mathbf{B}]\{\overline{\mathbf{U}}\} = \{0\} \tag{47}$$

where

$$\{\overline{\mathbf{U}}\} = \begin{Bmatrix} \lambda\{\widehat{\mathbf{U}}\} \\ \{\widehat{\mathbf{U}}\} \end{Bmatrix}, \qquad [\mathbf{A}] = \begin{bmatrix} [\mathbf{M}] & [0] \\ [0] & [\mathbf{K}] \end{bmatrix}, \qquad [\mathbf{B}] = \begin{bmatrix} [\mathbf{G}] & [\mathbf{K}] \\ -[\mathbf{K}] & [0] \end{bmatrix} \tag{48}$$

In eigenvalue problem presented in Eq. (47), $[\mathbf{A}]$ and $[\mathbf{B}]$ are symmetric and skew-symmetric matrix, respectively. The eigenvalues λ are in general complex number in the form as $\lambda = \alpha + i\omega$. While the real parts of all eigenvalues λ are zero ($\lambda = i\omega$), the moving beam is stable and the values of ω are the natural frequencies of the beam. The values ω decrease with increasing the transport speed until ω become zero in the critical speed. In the supercritical transport speed, violent vibration and divergence instability can occur. In these conditions, the real parts of some of the eigenvalues λ are non-zero ($\alpha \neq 0$).

2.4 CUF assembly technique

The assembly of the fundamental nuclei for each layer consists of four loops on indexes i, j, τ and s (Carrera *et al.* 2011). The loops on τ and s build the matrix for the given pair of *i* and *j*, and the loops on *i* and *j* give the matrix of an element. In the case of equivalent single layer (ESL), the variables are the same for each layer. Therefore, for assembling fundamental nuclei from layer to multilayer, they can be simply summed on the loop *k*. For example, assemblage procedure related to a two-node beam element is depicted in Fig. 2 (a two-layer beam has been considered in this figure).

3. Numerical results

To give the natural frequencies of axially moving laminated beams, Mathematica computer program is used based on the formulation presented above.

First, we performed a convergence analysis for a clamped three-layer moving beam by considering both the number of beam elements and the order of the beam model. The geometry is taken L/h=10 and the material properties are assumed: E_1=144.8 Gpa, E_2= E_3=9.65 Gpa, G_{12}= G_{13}=4.14 Gpa, G_{23}=3.45 Gpa, $\nu_{12} = \nu_{13} = \nu_{23}$ =0.3, ρ=1389.23 kg/m^3. The results of this convergence study are tabulated in Table 2. Rows are related to different number of elements while columns are related to increasing order of the beam.

Table 3, shows the comparison of the nondimensional natural frequencies of an axially moving isotropic beam with fixed-fixed boundary conditions and length-to-height ratio $L/h = 15$. The results by the present method are compared with Euler beam model from Simpson (1973). In this comparison, the sides of the cross section are assumed equal and Poisson's ratio is taken 0.33. The number of elements and the order of the beam model are equal 80 and 5, respectively.

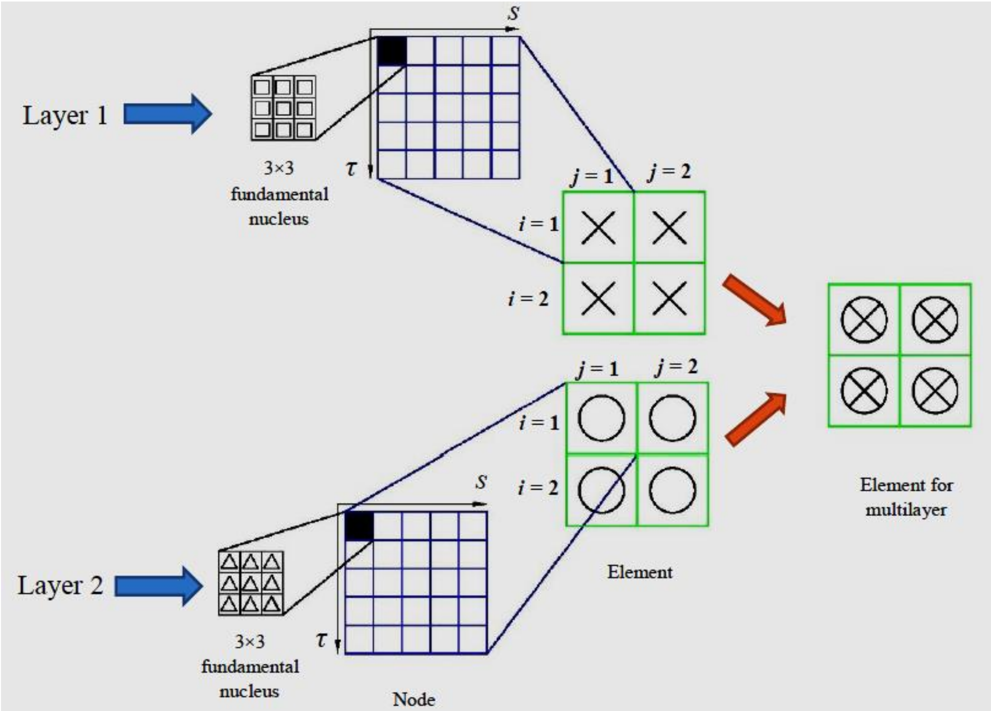

Fig. 2 CUF assembly technique for the fundamental nuclei of a two-node beam element in a two-layer beam

Table 2 Natural frequency parameters $\bar{\omega} = (\omega\, L^2/\pi h)\sqrt{\rho/E_2}$ for a three-layer moving beam with stacking sequence [45°/-45°/45°], axial velocity parameter $(\bar{v} = v\, L/h)\sqrt{\rho/E_2} = 2$ and axial tension parameter $\bar{\sigma}^0 = \sigma^0\, L^3/\pi^2 E_2 h^3 = 10$

No of elements	N				
	1	2	3	4	5
5	4.236	4.149	4.119	4.107	4.103
10	3.581	3.386	3.337	3.322	3.320
20	3.406	3.158	3.094	3.078	3.076
30	3.373	3.112	3.043	3.026	3.025
40	3.362	3.096	3.024	3.006	3.005
50	3.356	3.088	3.015	2.997	2.994
60	3.353	3.084	3.010	2.991	2.990
70	3.351	3.081	3.006	2.988	2.986
80	3.351	3.081	3.005	2.988	2.986

Table 3 Comparison of nondimensional natural frequency of a fixed-fixed axially moving isotropic beam

	$\bar{v}$									
	0	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4	2.7
Present	22.070	21.909	21.427	20.623	19.495	18.037	16.240	14.079	11.497	8.335
Simpson (1973)	22.373	22.219	21.758	20.987	19.904	18.504	16.775	14.693	12.201	9.158

Table 4 Material properties of the beam

E_1(Gpa)	E_2(Gpa)	E_3(Gpa)	G_{12}(Gpa)	G_{13}(Gpa)	G_{23}(Gpa)	ν_{12}	ν_{13}	ν_{23}	ρ(kg/m^3)
144.8	9.65	9.65	4.14	4.14	3.45	0.30	0.30	0.30	1389.23

Table 5 Natural frequency parameters $\bar{\omega}$ for a five-layer moving beam with stacking sequence [θ/-θ/θ/-θ/θ] (the number of beam elements is taken *NE*=80)

	$\bar{v}=1,\ \bar{\sigma}^0=5$					$\bar{v}=2,\ \bar{\sigma}^0=10$				
θ	*N*=1 (*M*=3)	*N*=2 (*M*=6)	*N*=3 (*M*=10)	*N*=4 (*M*=15)	*N*=5 (*M*=21)	*N*=1 (*M*=3)	*N*=2 (*M*=6)	*N*=3 (*M*=10)	*N*=4 (*M*=15)	*N*=5 (*M*=21)
0°	6.924	6.884	6.727	6.727	6.726	6.701	6.660	6.495	6.494	6.493
15°	6.186	5.906	5.762	5.638	5.631	5.941	5.654	5.499	5.370	5.362
30°	5.129	4.239	4.138	3.970	3.961	4.853	3.917	3.806	3.628	3.619
45°	3.932	2.833	2.776	2.728	2.724	3.601	2.412	2.350	2.298	2.294
60°	2.905	2.354	2.337	2.334	2.332	2.503	1.895	1.877	1.873	1.872
90°	2.388	2.297	2.292	2.292	2.292	1.938	1.838	1.832	1.832	1.832

Table 6 Natural frequency parameters $\bar{\omega}$ for a three-layer moving beam with stacking sequence [30°/-30°/30°] and an axial initial tension $\bar{\sigma}^0=10$ (the number of beam elements is taken *NE*=80)

		N				
Boundary conditions	Velocity parameter $\bar{v}$	1	2	3	4	5
S-S	0	3.899	3.478	3.469	3.466	3.466
	1	3.666	3.239	3.227	3.222	3.222
	2	2.971	2.520	2.498	2.489	2.488
	3	1.813	1.263	1.228	1.215	1.214
C-C	0	4.804	4.607	4.506	4.478	4.472
	1	4.573	4.367	4.259	4.229	4.223
	2	3.883	3.647	3.522	3.486	3.480
	3	2.740	2.452	2.297	2.249	2.243
C-F	0	1.901	1.860	1.847	1.842	1.841
	1	1.755	1.710	1.696	1.691	1.690
	2	1.312	1.254	1.240	1.234	1.233
	3	0.511	0.436	0.425	0.419	0.418

The other parameters used in these results are considered as follows

$$\bar{\omega}=\omega\frac{L^2}{h}\sqrt{\frac{12\rho}{E}},\qquad \bar{v}=v\frac{L}{2h}\sqrt{\frac{12\rho}{E}} \tag{49}$$

The results of the free vibration analysis of axially moving laminated beams are presented in the following. All layers in the laminates are assumed to have the same material properties. The material properties are defined in Table 4. In this study, *NE* stands the total number of beam elements. The parameters used in the results are presented as follows

$$\bar{\omega}=\omega\frac{L^2}{\pi h}\sqrt{\frac{\rho}{E_2}},\ \bar{v}=v\frac{L}{h}\sqrt{\frac{\rho}{E_2}},\qquad \bar{\sigma}^0=\sigma^0\frac{L^3}{\pi^2E_2h^3} \tag{50}$$

where $\bar{\omega}$, $\bar{v}$ and $\bar{\sigma}^0$ are dimensionless natural frequency, traveling speed and axial tension (positive when tensile), respectively.

Natural frequency parameters of a fixed-fixed five-layer moving beam with stacking sequence [θ/-θ/θ/-θ/θ] for θ=0°, 15°, 30°, 45°, 60° and 90° are tabulated in Table 5. They have been computed for different orders of Taylor expansion. In this analysis, two different axial velocities and axial tensions have been assumed for the moving beam. The other parameters are L/h=20 and b=h. It can be seen that the laminated beams with less angle-ply θ have greater natural frequencies due to effect of the angle-ply θ on the flexural stiffness.

In Table 6, natural frequency parameters of a three-layer moving beam with stacking sequence [30°/-30°/30°] are presented. The analysis consider different expansion orders (N=1 to N=5), velocities ($\overline{v} = 0$ to $\overline{v} = 3$) and boundary conditions (clamped-clamped C-C, simply supported S-S and clamped-free C-F). Also, The geometry is taken L/h=10 and b=h, and it is assumed that this beam is subjected to an axial tension $\bar{\sigma}^0 = 10$. It can be seen that the frequency of vibration may be significatly affected by the boundary conditions.

For a fixed-fixed three-layer moving beam with stacking sequence [0°/90°/0°], the variation of the dimensionless natural frequency $\bar{\omega}$ against dimensionless velocity $\bar{v}$, for different axial tension values is shown in Fig. 3. In this analysis, the geometry is taken L/h=15 and b=h. As seen in the Fig. 3, the natural frequencies decrease with increasing velocity till the velocity reach the critical velocity that in this value of velocity, natural frequency become zero. It can also be seen that the critical velocity and the natural frequencies in a certain velocity have higher values for greater axial tension.

The dimensionless natural frequency of a fixed-fixed moving beam with an axial tension $\bar{\sigma}^0 = 20$ against the dimensionless velocity for four different angle-plies θ is shown in Fig. 4. The stacking sequence is assumed [θ/-θ/θ] and the other parameters are L/h=15 and b=h. For each angle-ply, increasing the axial speed causes the fundamental frequencies to decrease and the beam loses stability at higher speeds. Moreover, when the angle-ply decreases, the fundamental frequencies increase as shown in the Figure. This is due to the fact that reduction the angle-ply θ causes the flexural rigidity to increase.

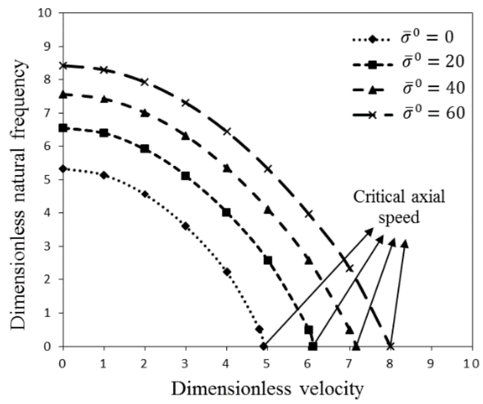

Fig. 3 The effect of transport speed on the natural frequency of a fixed-fixed laminated beam with various axial tensions (NE and N are taken 80 and 5, respectively)

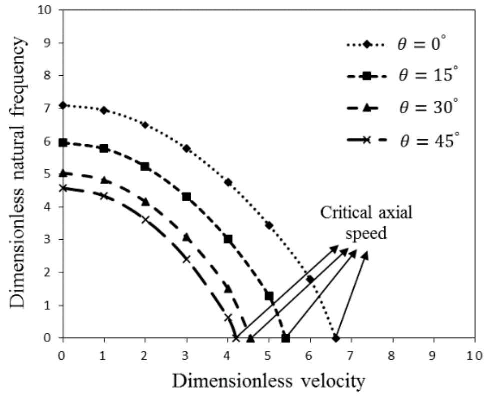

Fig. 4 The effect of transport speed on the natural frequency of a fixed-fixed laminated beam with various angle-plies (NE and N are taken 80 and 5, respectively)

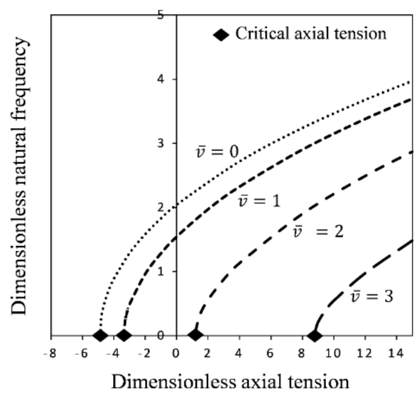

Fig. 5 Variation of the natural frequency against axial tension of a fixed-fixed laminated moving beam with various velocities (NE and N are taken 80 and 5, respectively)

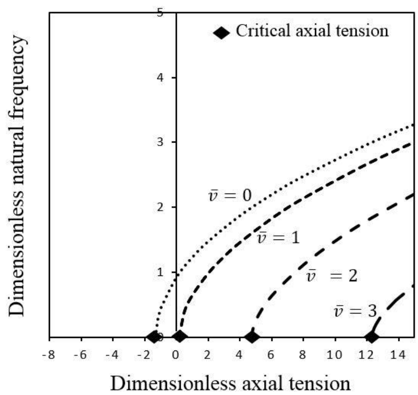

Fig. 6 Variation of the natural frequency against axial tension of a simply supported laminated moving beam with various velocities (NE and N are taken 80 and 5, respectively)

It can also be seen that the difference between natural frequencies for two different angle-pies is more in higher velocities.

Variation of the dimensionless natural frequency $\bar{\omega}$ against dimensionless axial tension $\bar{\sigma}^0$ for four different values of axial speed $\bar{v}$ is reported here for C-C and S-S boundary conditions in Figs. 5 and 6, respectively. The stacking sequence is assumed [60°/-60°/60°] and the geometry is taken L/h=15 and b=h. In the Figs. 5 and 6, the critical axial tensions are shown. In the axial tensions lower than critical axial tensions, divergence instability can occur. As seen in these figures, for the moving beam ($\bar{v} \neq 0$), the critical axial force can be positive or negative. But, for the stationary beam ($\bar{v} = 0$), critical axial force is negative and this value is equal to the buckling load of the beam.

4. Conclusions

In this paper, a finite element analysis based on higher-order theories within the framework of Carrera Unified Formulation is formulated for free vibration analysis of axially traveling and tensioned laminated beams. Hamilton's principle is used to obtain the governing equation of motion. Some examples are presented to investigate the effects of some parameters such as ply-angle, traveling speed and axial tension on the natural frequencies and beam stability. The numerical results have shown that the natural frequencies decrease with increasing traveling speed till the speed reach the critical speed and the beam loses stability at higher this speed. Also, natural frequencies increase with increasing axial tension, as well as with decreasing angle-ply.

References

AL-Bedoor, B.O. and Khulief, Y.A. (1996), “An approximate analytical solution of beam vibrations during axial motion”, *J. Sound Vib.*, **192**(1), 159-171. https://doi.org/10.1006/jsvi.1996.0181.

Alesadi, A., Galehdari, M. and Shojaee, S. (2017a), “Free vibration and buckling analysis of cross-ply laminated composite plates using Carrera’s unified formulation based on Isogeometric approach”, *Comput. Struct.*, **183**, 38-47. https://doi.org/10.1016/j.compstruc.2017.01.013.

Alesadi, A., Galehdari, M. and Shojaee, S. (2017b), “Free vibration and buckling analysis of composite laminated plates using layerwise models based on isogeometric approach and Carrera unified formulation”, *Mech. Adv. Mater. Struct.*, **25**, 1018-1032. https://doi.org/10.1080/15376494.2017.1342883.

Alesadi, A., Ghazanfari, S. and Shojaee, S. (2019), “B-spline finite element approach for the analysis of thin-walled beam structures based on 1D refined theories using carrera unified formulation”, *Thin-Wall. Struct.*, **130**, 313-320. https://doi.org/10.1016/j.tws.2018.05.016

Alesadi, A., Shojaee, S. and Hamzehei-Javaran, S. (2020), “Spherical Hankel-based free vibration analysis of cross-ply laminated plates using refined finite element theories”, *Iran. J. Sci. Technol. Trans. Civ. Eng.*, **44**, 127-137. https://doi.org/10.1007/s40996-019-00242-6

Bathe, K.J. (1996), *Finite Element Procedure*, Prentice Hall.

Carrera, E. (1995), “A class of two dimensional theories for multilayered plates analysis”, *Atti della accad, Delle Sci. Di Torino. Cl. Di Sci. Fis. Mat. e Nat*, **19**, 1-39.

Carrera, E., Brischetto, S. and Robaldo, A. (2008), “Variable kinematic model for the analysis of functionally graded material plates”, *AIAA J.*, **46**, 194-203. https://doi.org/10.2514/1.32490.

Carrera, E., Filippi, M., Mahato, P.K.R. and Pagani, A. (2015), “Advanced models for free vibration analysis of laminated beams with compact and thin-walled open/closed sections”, *J. Comp. Mater.*, **49**(17), 2085-2101. https://doi.org/10.1177/0021998314541570.

Carrera, E., Filippi, M., Mahato, P.K.R. and Pagani, A. (2016a), “Accurate static response of single- and multi-cell laminated box beams”, *Compos. Struct.*, **136**, 372-383. https://doi.org/10.1016/j.compstruct.2015.10.020.

Carrera, E., Filippi, M., Mahato, P.K.R. and Pagani, A. (2016b), “Free-vibration tailoring of single- and multi-bay laminated box structures by refined beam theories”, *Thin-Wall. Struct.*, **109**, 40-49. https://doi.org/10.1016/j.tws.2016.09.014.

Carrera, E. and Giunta, g. (2010), “Refined beam theories based on a unified formulation”, *Int. J. Appl. Mech.*, **2**, 117-143. https://doi.org/10.1142/S1758825110000500.

Carrera, E., Giunta, G. and Petrolo, M. (2011), *Beam Structures: Classical and Advanced Theories*, John Wiley & Sons Ltd, Chichester, West Sussex, United Kingdom. https://doi.org/10.1002/9781119978565.

Carrera, E., Pagani, A. and Banerjee J.R. (2016), “Linearized buckling analysis of isotropic and composite beam-columns by carrera unified formulation and dynamic stiffness method”, *Mech. Adv. Mater. Struct.*, **23**(9), 1092-1103. https://doi.org/10.1080/15376494.2015.1121524.

Carrera, E. (2002), “Theories and finite elements for multilayered, anisotropic, composite plates and shells”, *Arch. Comput. Meth. Eng.*, **9**, 87-140. https://doi.org/10.1007/BF02736649.

Carrera, E. (2003), “Theories and finite elements for multilayered plates and shells: a unified compact formulation with numerical assessment and benchmarking”, *Arch. Comput. Meth. Eng.*, **10**, 215-296. https://doi.org/10.1007/BF02736224.

Catapano, A., Giunta, G., Belouettar, S. and Carrera, E. (2011), “Static analysis of laminated beams via a unified formulation”, *Compos. Struct.*, **94**, 75-83. https://doi.org/10.1016/j.compstruct.2011.07.015

Chakraborty, G., Mallik, A.K. and Hatwal, H. (1999), “Non-linear vibration of a traveling beam”, *Int. J. Nonlinear Mech.*, **34**(4), 655-670. https://doi.org/10.1016/S0020-7462(98)00017-1.

Daraei, B., Shojaee, S. and Hamzehei-Javaran, S. (2020), “Free vibration analysis of composite laminated beams with curvilinear fibers via refined theories”, *Mech. Adv. Mater. Struct.,* https://doi.org/10.1080/15376494.2020.1797959.

Ghayesh, M.H. and Amabili, M. (2013), “Steady-state transverse response of an axially moving beam with time-dependent axial speed”, *Int. J. Non-Linear Mech.*, **49**, 40-49. https://doi.org/10.1016/j.ijnonlinmec.2012.08.003.

Ghayesh, M.H. and Balar, S. (2010), “Non-linear parametric vibration and stability analysis for two dynamic models of axially moving Timoshenko beams”, *Appl. Math. Model.*, **34**(10), 2850-2859. https://doi.org/10.1016/j.apm.2009.12.019.

Ghayesh, M.H. and Khadem, S.E. (2008), “Rotary inertia and temperature effects on non-linear vibration, steady-state response and stability of an axially moving beam with time-dependent velocity”, *Int. J. Mech. Sci.*, **50**(3), 389-404. https://doi.org/10.1016/j.ijmecsci.2007.10.006.

Ghazanfari, S., Hamzehei-Javaran, S., Alesadi, A. and Shojaee, S. (2019), “Free vibration analysis of cross-ply laminated beam structures using refined beam theories and B-spline basis functions”, *Mech. Adv. Mater. Struct.*, https://doi.org/10.1080/15376494.2019.1574939.

Ghorbanpour Arani, A., Haghparast, E. and BabaAkbar Zarei, H.

(2017), “Vibration analysis of functionally graded nanocomposite plate moving in two directions”, *Steel Compos. Struct.*, **23**(5), 529-541. https://doi.org/10.12989/scs.2017.23.5.529.

Giunta, G., Belouettar, S., Nasser, H., Kiefer-Kamal, E.H. and Thielen, T. (2015), “Hierarchical models for the static analysis of three-dimensional sandwich beam structures”, *Compos. Struct.*, **133**, 1284-1301. https://doi.org/10.1016/j.compstruct.2015.08.049.

Giunta, G., Metla, N., Koutsawa, Y. and Belouettar, S. (2013), “Free vibration and stability analysis of three-dimensional sandwich beams via hierarchical models”, *Compos. Part B Eng.*, **47**, 326-338. https://doi.org/10.1016/j.compositesb.2012.11.017.

Ho-Huu, V., Vo-Duy, T., Duong-Gia, D. and Nguyen-Thoi, T. (2018), “An efficient procedure for lightweight optimal design of composite laminated beams”, *Steel Compos. Struct.*, **27**(3), 297-310. https://doi.org/10.12989/scs.2018.27.3.297.

Hui, Y., Giunta, G., Belouettar, S., Huang, Q., Hu, H. and Carrera, E. (2017), “A free vibration analysis of three-dimensional sandwich beams using hierarchical one-dimensional finite elements”, *Compos. Part B Eng.*, **110**, 7-19. https://doi.org/10.1016/j.compositesb.2016.10.065.

Hwang, S.J. and Perkins, N.C. (1992), “Supercritical stability of an axially moving beam Part II: Vibration and stability analysis”, *J. Sound Vib.*, **154**, 397-409. https://doi.org/10.1016/0022-460X(92)90775-S.

Kahya, V., Karaca, S. and Vo, T. (2019), “Shear-deformable finite element for free vibrations of laminated composite beams with arbitrary lay-up”, *Steel Compos. Struct.*, **33**, 473-487. https://doi.org/10.12989/scs.2019.33.4.473.

Kahya, V. and Turan, M. (2018), “Vibration and buckling of laminated beams by a multi-layer finite element model”, *Steel Comp. Struct.*, **28**(4), 415-426. https://doi.org/10.12989/scs.2018.28.4.415

Kong, L. and Parker, R.G. (2004), “Approximate eigensolutions of axially moving beams with small flexural stiffness”, *J. Sound Vib.*, **276**(1-2), 459-469. https://doi.org/10.1016/j.jsv.2003.11.027

Lee, U. and Jang, I. (2007), “On the boundary conditions for axially moving beams”, *J. Sound Vib.*, **306**(3-5), 675-690. https://doi.org/10.1016/j.jsv.2007.06.039

Lee, U., Kim, J. and Oh, H. (2004), “Spectral analysis for the transverse vibration of an axially moving Timoshenko beam”, *J. Sound Vib.*, **271**(3-5), 685-703. https://doi.org/10.1016/S0022-460X(03)00300-6.

Liu, B., Zhao, L., Ferreira, A.J.M., Xing, Y.F., Neves, A.M.A. and Wang, J. (2017), “Analysis of viscoelastic sandwich laminates using a unified formulation and a differential quadrature hierarchical finite element method”, *Compos. Part B Eng.*, **110**, 185-192. https://doi.org/10.1016/j.compositesb.2016.11.028.

Mokhtari, A. and Mirdamadi, H.R. (2018), “Study on vibration and stability of an axially translating viscoelastic Timoshenko beam: Non-transforming spectral element analysis”, *Appl. Math. Model.*, **56**, 342-358. https://doi.org/10.1016/j.apm.2017.12.007.

Öz, H.R. (2001), “On the vibrations of an axially traveling beam on fixed supports with variable velocity”, *J. Sound Vib.*, **239**(3), 556-564. https://doi.org/10.1006/jsvi.2000.3077

Öz, H.R. and Pakdemirli, M. (1999), “Vibrations of an axially moving beam with time dependent velocity”, *J. Sound Vib.*, **227**(2), 239-257. https://doi.org/10.1006/jsvi.1999.2247.

Özkaya, E. and Öz, H.R. (2002), “Determination of natural frequencies and stability regions of axially moving beams using artificial neural networks method”, *J. Sound Vib.*, **252**(4), 782-789. https://doi.org/10.1006/jsvi.2001.3991.

Pagani, A., Carrera, E., Boscolo, M. and Banerjee, J.R. (2014), “Refined dynamic stiffness elements applied to free vibration analysis of generally laminated composite beams with arbitrary boundary conditions”, *Compos. Struct.*, **110**, 305-316. https://doi.org/10.1016/j.compstruct.2013.12.010.

Pagani, A., Carrera, E. and Ferreira, A.J.M. (2016), “Higher-order theories and radial basis functions applied to free vibration analysis of thin-walled beams”, *Mech. Adv. Mater. Struct.*, **23**, 1080-1091. https://doi.org/10.1080/15376494.2015.1121555.

Pagani, A. and Carrera, E. (2018), “Unified formulation of geometrically nonlinear refined beam theories”, *Mech. Adv. Mater. Struct.*, **25**, 15-31. https://doi.org/10.1080/15376494.2016.1232458.

Pakdemirli, M. and Özkaya, E. (1998), “Approximate boundary layer solution of a moving beam problem”, *Math. Comput. Appl.*, **3**(2), 93-100. https://doi.org/10.3390/mca3020093.

Reddy, J.N. (2013), *An Introduction to Continuum Mechanics*, Cambridge University Press, Second Edition.

Reddy, J.N. (2004), *Mechanics of Laminated Composite Plates and Shells: Theory and Analysis*, CRC Press, Boca Raton, FL, Second Edition.

Rezaee, M. and Lotfan, S. (2015), “Non-linear nonlocal vibration and stability analysis of axially moving nanoscale beams with time-dependent velocity”, *Int. J. Mech. Sci.*, **96**, 36-46. https://doi.org/10.1016/j.ijmecsci.2015.03.017.

Simpson, A. (1973), “Transverse modes and frequencies of beams translating between fixed end supports”, *J. Mech. Eng. Sci.*, **15**(3), 159-164. https://doi.org/10.1243/JMES_JOUR_1973_015_031_02.

Siylianou, M. and Tabarrok, B. (1994), “Finite element analysis of an axially-moving beam, part II: Stability analysis”, *J. Sound Vib.*, **178**(4), 455-481. https://doi.org/10.1006/jsvi.1994.1498.

Wickert, J.A. and Mote, C.D. (1990), “Classical vibration analysis of axially moving continua”, *J. Appl. Mech.*, **57**(3), 738-744. https://doi.org/10.1115/1.2897085

Wickert, J.A. and Mote, C.D. (1988), “Current research on the vibration and stability of axially moving materials”, *Shock Vib. Dig.*, **20**, 3-13.

Wickert, J.A. (1992), “Non-linear vibration of a traveling tensioned beam”, *Int. J. Non-Linear Mech.*, **27**(3), 503-517. https://doi.org/10.1016/0020-7462(92)90016-Z.

Yan, Y., Carrera, E., Pagani, A., Kaleel, I. and Miguel, A.G.d. (2020), “Isogeometric analysis of 3D straight beam-type structures by Carrera Unified Formulation”, *Appl. Math. Model.*, **79**, 768-792. https://doi.org/10.1016/j.apm.2019.11.003

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Appendix A

The transformed elastic coefficients $\bar{C}_{ij}$ for an orthotropic layer can be written as follows (Reddy 2004)

$$\bar{C}_{11} = C_{11}\cos^4\theta + 2(C_{12} + 2C_{66})\sin^2\theta\cos^2\theta + C_{22}\sin^4\theta$$
$$\bar{C}_{12} = (C_{11} + C_{22} - 4C_{66})\, sin^2\theta\, cos^2\theta + C_{12}(\sin^4\theta + \cos^4\theta)$$
$$\bar{C}_{13} = C_{13}\cos^2\theta + C_{23}\sin^2\theta$$
$$\bar{C}_{16} = (C_{11} - C_{12} - 2C_{66})\, sin\,\theta\, cos^3\theta + (C_{12} - C_{22} + 2C_{66})\sin^3\theta\cos\theta$$
$$\bar{C}_{22} = C_{11}\sin^4\theta + 2(C_{12} + 2C_{66})\sin^2\theta\cos^2\theta + C_{22}\cos^4\theta$$
$$\bar{C}_{23} = C_{23}\cos^2\theta + C_{13}\sin^2\theta$$
$$\bar{C}_{26} = (C_{11} - C_{12} - 2C_{66})\, sin^3\theta\, cos\,\theta + (C_{12} - C_{22} + 2C_{66})\sin\theta\cos^3\theta$$
$$\bar{C}_{33} = C_{33}$$
$$\bar{C}_{36} = (C_{13} - C_{23})\sin\theta\cos\theta$$
$$\bar{C}_{66} = (C_{11} + C_{22} - 2C_{12} - 2C_{66})\, sin^2\theta\, cos^2\theta + C_{66}(\sin^4\theta + \cos^4\theta)$$
$$\bar{C}_{44} = C_{44}\cos^2\theta + C_{55}\sin^2\theta$$
$$\bar{C}_{45} = (C_{55} - C_{44})\sin\theta\cos\theta$$
$$\bar{C}_{55} = C_{55}\cos^2\theta + C_{44}\sin^2\theta$$

where the ply-angle θ is the angle between the x and x_1 axis. Here, the material coordinate axes x_1, x_2 and x_3 are taken to be the fiber orientation, the *xy*-plane direction perpendicular to the fiber orientation and the transverse direction, respectively. The stiffness coefficient C_{ij} are

$$C_{11} = \frac{1 - \nu_{23}\nu_{23}}{E_2E_3\Delta}, C_{12} = \frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2E_3\Delta}$$
$$C_{13} = \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2E_3\Delta}$$
$$C_{22} = \frac{1 - \nu_{13}\nu_{31}}{E_1E_3\Delta}, C_{23} = \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1E_3\Delta}, C_{33} = \frac{1 - \nu_{12}\nu_{21}}{E_1E_2\Delta}$$
$$C_{44} = G_{23}, C_{55} = G_{13}, C_{66} = G_{12}$$
$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1E_2E_3}$$

in which E_1, E_2 and E_3 are Young's moduli in 1, 2 and 3 material coordinates, respectively. Also, G_{12}, G_{13} and G_{23} are shear moduli in the 1-2, 1-3 and 2-3 planes, respectively, and ν_{ij} is Poisson's ratio that the following reciprocal relations hold

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}, \frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2}$$

Appendix B

The variation of the strain energy U_s, energy due to initial axial tension U_g, kinetic energy for the nonmoving beam T_1 and kinetic energy for the moving beam T_2 and T_3 are as follows

$$\begin{aligned}\delta U_s^k = \int_{\Omega_k}\int_{l_k} & N_j \delta u_{ysj}\frac{\partial F_s}{\partial y}\Bigg(N_i u_{y\tau i}\bar{C}_{22}^k\frac{\partial F_\tau}{\partial x} + N_i u_{z\tau i}\bar{C}_{23}^k\frac{\partial F_\tau}{\partial z} \\ & + F_\tau u_{x\tau i}\bar{C}_{12}^k\frac{\partial N_i}{\partial x} \\ & + \bar{C}_{26}^k\left(N_i u_{x\tau i}\frac{\partial F_\tau}{\partial y} + F_\tau u_{y\tau i}\frac{\partial N_i}{\partial x}\right)\Bigg) \\ & + N_j \delta u_{zsj}\frac{\partial F_s}{\partial z}\Bigg(N_i u_{y\tau i}\bar{C}_{23}^k\frac{\partial F_\tau}{\partial y} \\ & + N_i u_{z\tau i}\bar{C}_{23}^k\frac{\partial F_\tau}{\partial z} + F_\tau u_{x\tau i}\bar{C}_{13}^k\frac{\partial N_i}{\partial x} \\ & + \bar{C}_{36}^k\left(N_i u_{x\tau i}\frac{\partial F_\tau}{\partial y} + F_\tau u_{y\tau i}\frac{\partial N_i}{\partial x}\right)\Bigg) \\ & + \Bigg(N_j \delta u_{zsj}\frac{\partial F_s}{\partial y} \\ & + N_j \delta u_{ysj}\frac{\partial F_s}{\partial z}\Bigg)\Bigg(\bar{C}_{44}^k\Bigg(N_i u_{z\tau i}\frac{\partial F_\tau}{\partial y} \\ & + N_i u_{y\tau i}\frac{\partial F_\tau}{\partial z}\Bigg) \\ & + \bar{C}_{45}^k\left(N_i u_{x\tau i}\frac{\partial F_\tau}{\partial z} + F_\tau u_{z\tau i}\frac{\partial N_i}{\partial x}\right)\Bigg) \\ & + F_s \delta u_{xsj}\frac{\partial N_j}{\partial x}\Bigg(N_i u_{y\tau i}\bar{C}_{12}^k\frac{\partial F_\tau}{\partial y} \\ & + N_i u_{z\tau i}\bar{C}_{13}^k\frac{\partial F_\tau}{\partial z} + F_\tau u_{x\tau i}\bar{C}_{11}^k\frac{\partial N_i}{\partial x} \\ & + \bar{C}_{16}^k\left(N_i u_{x\tau i}\frac{\partial F_\tau}{\partial y} + F_\tau u_{y\tau i}\frac{\partial N_i}{\partial x}\right)\Bigg) \\ & + \Bigg(N_j \delta u_{xsj}\frac{\partial F_s}{\partial y} \\ & + F_s \delta u_{ysj}\frac{\partial N_j}{\partial x}\Bigg)\Bigg(N_i u_{y\tau i}\bar{C}_{26}^k\frac{\partial F_\tau}{\partial y} \\ & + N_i u_{z\tau i}\bar{C}_{36}^k\frac{\partial F_\tau}{\partial y} + F_\tau u_{x\tau i}\bar{C}_{16}^k\frac{\partial N_i}{\partial x} \\ & + \bar{C}_{66}^k\left(N_i u_{x\tau i}\frac{\partial F_\tau}{\partial y} + F_\tau u_{y\tau i}\frac{\partial N_i}{\partial x}\right)\Bigg) \\ & + \Bigg(N_j \delta u_{xsj}\frac{\partial F_\tau}{\partial z} \\ & + F_s \delta u_{zsj}\frac{\partial N_j}{\partial x}\Bigg)\Bigg(\bar{C}_{45}^k\Bigg(N_i u_{z\tau i}\frac{\partial F_\tau}{\partial y} \\ & + N_i u_{y\tau i}\frac{\partial F_\tau}{\partial z}\Bigg) \\ & + \bar{C}_{55}^k\left(N_i u_{x\tau i}\frac{\partial F_\tau}{\partial z} + F_\tau u_{z\tau i}\frac{\partial N_i}{\partial x}\right)\Bigg) dx d\Omega\end{aligned}$$

$$\begin{aligned}\delta U_g = \int_\Omega\int_l \sigma^0 \Bigg[F_\tau F_s\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\big(\delta u_{xsj}u_{x\tau i} + \delta u_{ysj}u_{y\tau i} \\ + \delta u_{zsj}u_{z\tau i}\big)\Bigg] dx d\Omega\end{aligned}$$

$$\begin{aligned}\delta\int_{t_\mathrm{i}}^{t_\mathrm{f}} T_1\ dt = -\int_{t_\mathrm{i}}^{t_\mathrm{f}}\int_\Omega\int_l \rho\big[F_\tau F_s N_i N_j\big(\delta u_{xsj}\ddot{u}_{x\tau i} \\ + \delta u_{ysj}\ddot{u}_{y\tau i} + \delta u_{zsj}\ddot{u}_{z\tau i}\big)\big] dx d\Omega dt\end{aligned}$$

$$\begin{aligned}\delta\int_{t_\mathrm{i}}^{t_\mathrm{f}} T_2\ dt = \int_{t_\mathrm{i}}^{t_\mathrm{f}}\int_\Omega\int_l \rho v^2\Bigg[F_\tau F_s\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}\big(\delta u_{xsj}u_{x\tau i} \\ + \delta u_{ysj}u_{y\tau i} + \delta u_{zsj}u_{z\tau i}\big)\Bigg] dx d\Omega dt\end{aligned}$$

$$\begin{aligned}\delta\int_{t_\mathrm{i}}^{t_\mathrm{f}} T_3\ dt = -\int_{t_\mathrm{i}}^{t_\mathrm{f}}\int_\Omega\int_l \rho v\Bigg[(F_\tau F_s)\Bigg(\frac{\partial N_i}{\partial x}N_j \\ - N_i\frac{\partial N_j}{\partial x}\Bigg)\big(\dot{u}_{x\tau i}\delta u_{xsj} + \dot{u}_{y\tau i}\delta u_{ysj} \\ + \dot{u}_{z\tau i}\delta u_{zsj}\big)\Bigg] dx d\Omega dt\end{aligned}$$