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Influence of internal pores and graphene platelets on vibration of non-uniform functionally graded columns

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Abstract. This paper deals with free vibration analysis of non-uniform column resting on elastic foundations and subjected to follower force at its free end. The internal pores and graphene platelets (GPLs) are distributed in the matrix according to different patterns. The model is proposed with material parameters varying in the thickness of column to achieve graded distributions in both porosity and nanofillers. The elastic modulus of the nanocomposite is obtained by using Halpin-Tsai micromechanics model. The differential quadrature method as an efficient and accurate numerical approach is used to discretize the governing equations and to implement the boundary conditions. It is observed that the maximum vibration frequency obtained in the case of symmetric porosity and GPL distribution, while the minimum vibration frequency is obtained using uniform porosity distribution. Results show that for better understanding of mechanical behavior of nanocomposite column, it is crucial to consider porosities inside the material structure.

Keywords: columns; vibration; pores and graphene platelets; Halpin-Tsai micromechanics model; elastic foundation; Functionally Graded Materials (FGMs)

1. Introduction

Normally, Functionally graded materials (FGMs) are heterogeneous materials in which the elastic and thermal properties change from one surface to the other, gradually and continuously. The material is constructed by smoothly changing the volume fraction of its constituent materials. FGMs offer great promise in applications where the operating conditions are severe, including spacecraft heat shields, heat exchanger tubes, plasma facings for fusion reactors, engine components, and high-power electrical contacts or even magnets. For example, in a conventional thermal barrier coating for high-temperature applications, a discrete layer of ceramic material is bonded to a metallic structure. However, the abrupt transition in material properties across the interface between distinct materials can cause large interlaminar stresses and lead to plastic deformation or cracking (Finot and Suresh 1996). These adverse effects can be alleviated by functionally grading the material to have a smooth spatial variation of material composition. The concept of FGMs was first introduced in Japan in 1984. Since then it has gained considerable attention (Koizumi 1993). A lot of different applications of FGMs can be found in (Zhu and Meng 1995). Mahmoud *et al.* (2011) investigated free vibration analysis of a non-uniform column resting on an elastic foundation and subjected to follower force. Smith and Herrmann (1972) introduced a stability of a cantilevered beam on an elastic foundation subjected to a follower force at its free end. He found that the critical load for flutter is independent of the foundation modulus which characterizes the Winkler-type embedding. Sundararajan (1974) presented stability of columns on Winkler type elastic foundations subjected to stationary forces (conservative or non-conservative). Various cases were discussed and a theorem on the influence of the foundation on the critical load was derived. Hauger and Vetter (1976) discussed the influence of an elastic foundation on the stability of a tangentially loaded column. Celep (1980) presented the stability analysis of a beam on an elastic foundation subjected to a nonconservative load. Based on the Lagrange interpolation Chan (Quan and Chan, 1989) provided explicit formulations to compute the weighting coefficients of the DQ discretization of the first and second order derivatives. Application of DQM to flexural vibration analysis of a geometrically nonlinear beam was introduced by Yusheng Feng and Bert (1992). There are many types of grid distributions such as; uniform space grid distribution. It was introduced by Wang and Bert (1993) as a new approach in applying DQ to free vibration analysis of a beam and plates. Bert and Malik (1996) indicated an important fact that the preferred type of grid points changes with the problem of interest; and recommended to use Chebyshev-Gauss-Lobatto grid distributions for structural mechanics computation. Lee and Yang (1994) discussed the influence

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of a Winkler elastic foundation and the slenderness ratio on the non-conservative instability of cantilever non-uniform beams subjected to an end concentrated follower force. Du *et al*. (1996) applied the DQM to the buckling analysis of columns and plates. The numerical results obtained were compared with those from existing literature and achieved high accuracy. Also there are many types of implementation of boundary conditions such as δ-type a small distance δ from the boundary. It was developed in the DQM to apply more than one boundary condition at discretized point; these results often based on value of δ and may be get ill-conditioned matrices. The clamped and simply supported boundary conditions using generalized DQ were introduced by Shu and Du (1997a). This approach directly substitutes the boundary conditions into the governing equations, abbreviated as SBCGE. It was used to overcome the drawbacks of δ-type. Also Shu and Du (1997b) presented an implementation of the general boundary conditions in the free vibration analysis of rectangular plates which directly couples the boundary conditions with the governing equations, abbreviated as CBCGE. As shown in the book of Shu (2000) the DQ is a global method, which is equivalent to the highest-order finite difference scheme. As compared to the low order finite difference schemes and finite element methods, the DQM can obtain very accurate numerical results by using a considerably small number of grid points. Consequently, it requires less computational effort and virtual storage. In general, the DQM uses a non-uniform mesh for numerical discretization. Karami *et al.* (2003) discussed that the differential quadrature element method (DQEM) could be employed as an accurate method for practical beam applications. The DQEM was applied to a non-uniform or discontinuous cross section beam and a beam subjected to heavy concentrated masses resting on elastic foundation in comparison with the finite element method. Ebrahimi *et al.* (2019) proposed a new gusset plate passing through the HSS columns and beams, named as through gusset plate to study the force transfer mechanism in such gusset plates of SCBFs compared to the case with conventional gusset plates. Nguyen *et al*. (2019) investigate the static behavior of a novel RCS beam-column exterior joint. The studied joint detail is a through-column type in which an H steel profile totally embedded inside RC column is directly welded to the steel beam. Wang and Sun (2019) investigate on seismic behavior of out-of-code Q690 circular high-strength concrete-filled thin-walled steel tubular (HCFTST) columns made up of high-strength (HS) steel tubes (yield strength $f_y \geq 690$ MPa). Six shear-critical square tubed steel reinforced concrete (TSRC) columns using the high-strength concrete (fcu,150 = 86.6 MPa) were tested under constant axial and lateral cyclic loads (Li *et al*. 2019). Song *et al.* (2019) present a preliminary numerical study on stainless steel-concrete composite beam-to-column joints with bolted flush endplates. In order to ensure a consistent corrosion resistance within the whole structural system, all structural steel components were designed with austenitic stainless steel, including beams, columns, endplates, bolts, reinforcing bars and shear connectors. Lai *et al.* (2019) report additional test data, analytical and numerical studies leading to a new design method to predict the ultimate resistance of composite columns made of high strength steel and high strength concrete. Bambaeechee (2019) investigates free vibration of AFG and uniform beams with general elastic supports. An efficient and free of shear locking finite element model is developed and employed to study free vibration of tapered bidirectional functionally graded material (BFGM) beams by Nguyen and Tran (2018). Investigation on the thermal buckling resistance of simply supported FGM beams having parabolic-concave thickness variation and temperature dependent material properties is presented by Arioui *et al*. (2018). Hadji *et al*. (2014) study static and free vibration of functionally graded beams via a higher order shear deformation beam theory. Mirjavadi *et al*. (2017) investigate the thermo-mechanical vibration behavior of two dimensional functionally graded (2D-FG) porous nanobeam. Shafiei and Setoodeh (2017) study the nonlinear free vibration and post-buckling of functionally graded carbon nanotube reinforced composite (FG-CNTRC) beams resting on a nonlinear elastic foundation. Yaghoobi *et al.* (2014) investigate nonlinear vibration and post-buckling of beams made of functionally graded materials (FGMs) resting on nonlinear elastic foundation subjected to thermo-mechanical loading. Marin and Agarwal (2013), proved the uniqueness theorem and some continuous dependence theorems without recourse to any energy conservation law, or to any boundedness assumptions on the thermoelastic coefficients. Marin and Florea (2014) considered a porous thermoelastic body, including voidage time derivative among the independent constitutive variables. For the initial boundary value problem of such materials, they analyzed the temporal behaviour of the solutions. Marin (2010) considered with some basic theorems for microstretch thermoelastic materials. He avoided the use of positive definiteness assumptions on the thermoelastic coefficients. Marin and Nicaise (2016) studied thermoelastic dipolar bodies which have a double porosity structure. In contrast with previous papers dedicated to classical elastic bodies, in our context the double porosity structure of the body is influenced by the displacement field, which is consistent with real models. In another study (Marin *et al*. 2016) dedicated to some results in the thermodynamic theory of porous elastic bodies. Unlike other studies, their study included the voidage time derivative among the independent constitutive variables. In another study (Marin *et al.* 2017) was dedicated to the Saint-Venant’s problem in the context of the theory of porous dipolar bodies. They considered a right cylinder consisting of an inhomogeneous and anisotropic material. Bacciocchi *et al.* (2020) focused on the long-time behavior of concrete beams reinforced by Carbon Fiber Reinforced Polymer (CFRP) strips applied on their external surfaces. A simple nonlocal beam model was proposed to study buckling response of protein microtubules by Civalek and Cigdem (2016). A new size-dependent beam model was introduced on the basis of hyperbolic shear deformation beam and modified strain gradient theory. The governing differential equations and corresponding boundary conditions were obtained with the aid of minimum total potential energy principle (Akgöz and Civalek 2015). Trovalusci *et al.* (2010) investigated the dynamical behavior of composite microcracked solids. Erasmo *et al*. (2016) investigated the dynamic stability of cracked beams under conservative and non-conservative forces and for various boundary conditions. Tornabene et al. (2016) studied the effect of Carbon Nanotube (CNT) agglomeration on the free vibrations of laminated composite doubly-curved shells and panels reinforced by CNTs. The mechanical behavior of the pipe under pure tension was investigated by both theoretical and numerical analysis, then the contribution of the external pressure to the axial problem was examined using an analytical analysis (Cornacchia *et al.* 2019). Fantuzzi and Borgia (2019) investigated the limitations of current DNV standards for piston design in offshore technologies when compared to classical numerical approaches and reference results provided by the existing literature. Pingaro *et al.* (2019) used the numerical framework of the virtual element method for numerical simulations to reduce the computational burden. The computational strategies and the discretization adopted allowed them to efficiently solve the series (hundreds) of simulations and to rapidly converged to the RVE size detection. Sharma *et al*. (2005a, b) integrated an analytical approach with the Chebyshev polynomials technique to study the buckling and free vibration of isotropic and laminated composite sector plates based on the first-order shear deformation theory. Liu and Wang (2015) studied Thermal vibration of a single-walled carbon nanotube predicted by semiquantum molecular dynamics. Zhang and Wang (2018) investigated the nonlinear thermal vibrational behavior of single-layered BP (SLBP) via a nonlinear orthotropic plate model (OPM) and molecular dynamics (MD) simulations. Xu *et al.* (2016) studied the vibration of double-layered graphene sheets (DLGS) using A nonlocal Kirchhoff plate model with the van der Waals (vdW) interactions. Ahmed Houari *et al*. (2018) presented a closed-form solutions for exact critical buckling loads of nonlocal strain gradient functionally graded beams. Chen *et al.* (2017) investigated vibration and stability of initially stressed sandwich plates with FGM face sheets. Barka *et al.* (2016) studied thermal post-buckling behavior of imperfect temperature-dependent FG structures. Bouguenina *et al.* (2015) studied FG plates with variable thickness subjected to thermal buckling. Park *et al.* (2016) used modified couple stress based third-order shear deformation theory for dynamic analysis of sigmoid functionally graded materials (S-FGM) plates. Wu and Liu (2016) developed a state space differential reproducing kernel (DRK) method in order to study 3D analysis of FG circular plates. Arefi (2015) suggested an analytical solution of a curved beam with different shapes made of functionally graded materials (FGMs). Bennai *et al.* (2015) developed a new refined hyperbolic shear and normal deformation beam theory to study the free vibration and buckling of functionally graded (FG) sandwich beams under various boundary conditions. Bouchafa *et al*. (2015) used refined hyperbolic shear deformation theory (RHSDT) for the thermoelastic bending analysis of functionally graded sandwich plates. Tahouneh (2016) presented a 3-D elasticity solution for free vibration analysis of continuously graded carbon nanotube-reinforced (CGCNTR) rectangular plates resting on two-parameter elastic foundations. The volume fractions of oriented, straight single-walled carbon nanotubes (SWCNTs) were assumed to be graded in the thickness direction. Moradi-Dastjerdi and Momeni-Khabisi (2016) studied Free and forced vibration of plates reinforced by wavy carbon nanotube (CNT). The plates were resting on Winkler-Pasternak elastic foundation and subjected to periodic or impact loading.

Nowadays, the use of carbon nanotubes in polymer/carbon nanotube composites has attracted wide attention (Wagner *et al.* 1997). A high aspect ratio, low weight of CNTs and their extraordinary mechanical properties (strength and flexibility) provide the ultimate reinforcement for the next generation of extremely lightweight but highly elastic and very strong advanced composite materials. On the other hand, by using of the polymer/CNT composites in advanced composite materials, we can achieve structures with low weight, high strength and high stiffness in many structures of civil, mechanical and space engineering.

Many researchers have reported that mechanical properties of polymeric matrices can be drastically increased (Montazeri *et al*. 2010, Yeh *et al*. 2006) by adding a few weight percent (wt%) MWCNTs. Montazeri *et al*. (2010) showed that modified Halpin-Tsai equation with exponential Aspect ratio can be used to model the experimental result of MWNT composite samples. They also demonstrated that reduction in Aspect ratio (L/d) and nanotube length cause a decrease in aggregation and Above 1.5 wt%, nanotubes agglomerate causing a reduction in Young's modulus values. Thus, it is important to determine the effect Aspect ratio and arrangement of CNTs on the effective properties of carbon nanotube-reinforced composite (CNTRC). Yeh et al. (2006) used the Halpin-Tsai equation to shows the effect of MWNT shape factor (L/d) on the mechanical properties. They showed that the mechanical properties of nanocomposite samples with the higher shape factor (L/d) values were better than the ones with the lower shape factor. The reinforcement effect of MWCNTs with different aspect ratio in an epoxy matrix has been carried out by Martone *et al*. (2011). They showed that progressive reduction of the tubes effective aspect ratio occurs because of the increasing connectedness between tubes upon an increase in their concentration. Also they investigated on the effect of nanotube curvature on the average contacts number between tubes by means of the waviness that accounts for the deviation from the straight particles assumption. Tornabene *et al*. (2019) investigated free vibration analysis of arches and beams made of composite materials via a higher-order mathematical formulation. Tornabene *et al*. (2017) studied free vibration analysis of composite sandwich plates and doubly curved shells with variable stiffness. The reinforcing fibers were located in the external skins of the sandwich structures according to curved paths. Tornabene *et al*. (2018) studied free vibration of laminated nanocomposite plates and shells using first-order shear deformation theory and the Generalized Differential Quadrature (GDQ) method. Each layer of the laminate was modelled as a three-phase composite. A survey of several methods under the heading of strong formulation finite element method (SFEM) was presented by Tornabene *et al*. (2015).

The present work aims to investigate the free vibration of FG columns with GPLs either uniformly or non-uniformly dispersed in the metal matrix. The effects of GPL reinforcing nanofillers and the porosity distribution are studied in detail through a parametric study to find out the best GPL and porosity distributions to achieve the highest effective column stiffness.

2. Problem description

Consider a cantilevered column resting on elastic foundation K subjected to a follower force p as shown in Fig. 1. The structure has continuous grading of GPLs-reinforcement through thickness direction. Three different GPL dispersion patterns, denoted by A, B, and C, are considered for each porosity distribution (Fig. 2). The GPL volume content V_{GPL} is assumed to vary along the thickness smoothly with its peak values (S_{ij}, i,j=1, 2, 3) being determined based on the specific porosity distribution. To facilitate a direct and meaningful comparison, the total amount of GPLs is kept the same for three different GPL distribution patterns. This leads to $s_{1i} \neq s_{2i} \neq s_{3i}$ (i=1, 2, 3).

The mechanical properties of a porous structure with different types of porosity distributions can be expressed by

$$E(z) = E_1(1 - e_0\lambda(z)) \tag{1}$$

$$G(z) = E(z)/2(1+\upsilon(z)) \tag{2}$$

$$\rho(z) = \rho_1(1 - e_m\lambda(z)) \tag{3}$$

in which, for symmetric porosity distribution

$$\lambda(z) = \cos(\pi z/h) \tag{4}$$

and for uniform porosity distribution

$$\lambda(z) = \lambda \tag{5}$$

where E_1, G_1, and ρ_1 are the maximum values of elasticity moduli, shear moduli and mass density. Also, e_0 and e_m are the coefficients of porosity and mass density, respectively, defined by (Kitipornchai *et al*. 2017)

$$e_0 = 1 - \frac{E_2}{E_1} = 1 - \frac{G_2}{G_1}$$
$$e_m = \frac{1.121(1 - \sqrt[2.3]{1 - e_0\lambda(z)})}{\lambda(z)} \tag{6}$$

Also based on the closed-cell graphene-reinforcement scheme, Poisson's ratio (z) can be expressed by

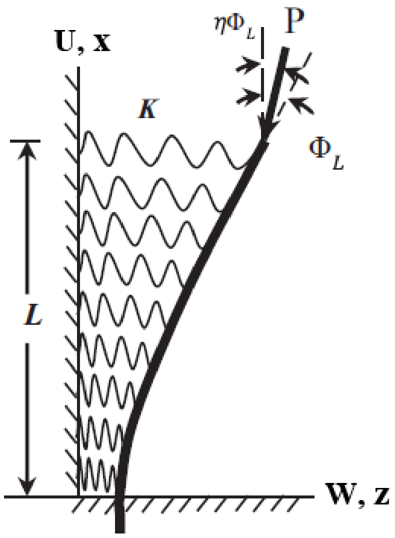

Fig. 1 The column resting on an elastic foundation subjected to follower force

(Kitipornchai *et al*. 2017)

$$\upsilon(z) = 0.221\tilde{p} + \upsilon_1(0.342\tilde{p}^2 - 1.21\tilde{p} + 1) \tag{7}$$

In which υ_1 is the Poisson's ratio of pure matrix materials without pores and

$$\tilde{p} = 1.121(1 - \sqrt[2.3]{1 - e_0\lambda(z)}) \tag{8}$$

Also, λ(z) for uniform porosity distribution can be expressed by

$$\lambda = \frac{1}{e_0} - \frac{1}{e_0}\left(\frac{\tilde{M}/h + 0.121}{1.121}\right)^{2.3} \tag{9}$$

In which

$$\tilde{M} = \int_{-h/2}^{h/2}(1 - \tilde{p})dz \tag{10}$$

According to the distribution patterns depicted in Fig. 2, the volume fraction of GPLs can be written as (i=1,2,3)

$$V_{GPL} = \begin{cases} S_{i1}\left[1 - \cos(\pi z/h)\right], \text{Pattern A} \\ S_{i2}\left[1 - \cos(\pi z/2h + \pi/4)\right], \text{Pattern B} \\ S_{i3}, \text{Pattern C} \end{cases} \tag{11}$$

The relation between the volume fraction of GPLs and their weight fraction W_{GPL} can be expressed by

$$\frac{W_{GPL}}{W_{GPL} + \frac{\rho_{GPL}}{\rho_M} - \frac{\rho_{GPL}}{\rho_M}W_{GPL}}\int_{-h/2}^{h/2}(1 - e_m\lambda(z))dz = \int_{-h/2}^{h/2}V_{GPL}(1 - e_m\lambda(z))dz \tag{12}$$

In which ρ_{GPL} and ρ_M are mass density of GPL and metal matrix, respectively. Based on Halpin-Tsai micromechanical model, it is possible to obtain material properties of GPL-reinforced metal matrix structures

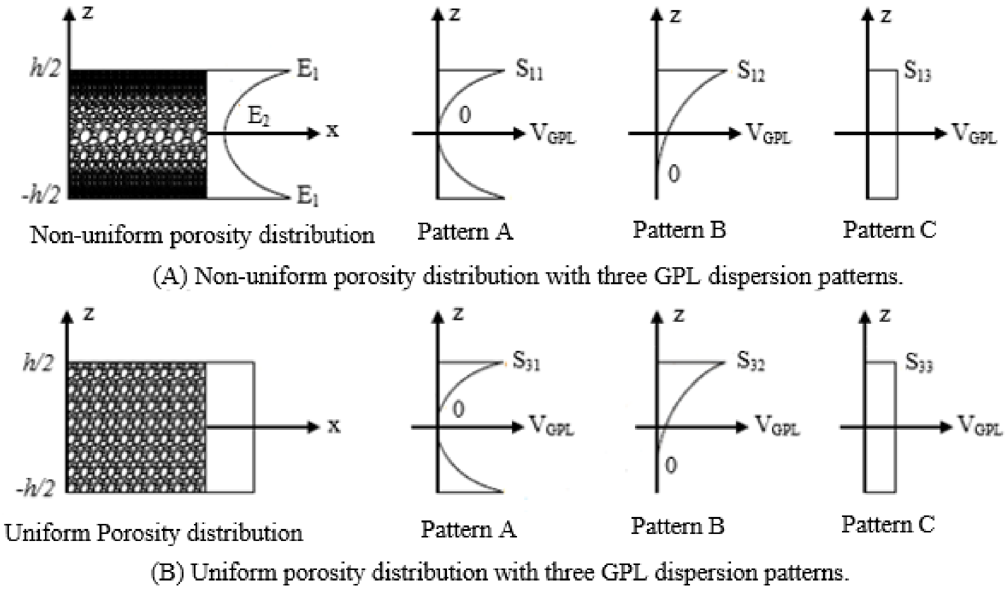

Fig. 2 Porosity distribution and GPL dispersion patterns

$$E_1 = \frac{3}{8}\left(\frac{1+\xi_L^{GPL}\eta_L^{GPL}V_{GPL}}{1-\eta_L^{GPL}V_{GPL}}\right)E_M + \frac{5}{8}\left(\frac{1+\xi_W^{GPL}\eta_W^{GPL}V_{GPL}}{1-\eta_W^{GPL}V_{GPL}}\right)E_M \tag{13}$$

in which E_m is Young's modulus of the metal and

$$\xi_L^{GPL} = 2l_{GPL}/t_{GPL}, \eta_L^{GPL} = \frac{(E_{GPL}/E_M)-1}{(E_{GPL}/E_M)+\xi_L^{GPL}}, \quad \xi_W^{GPL} = 2w_{GPL}/t_{GPL}, \eta_W^{GPL} = \frac{(E_{GPL}/E_M)-1}{(E_{GPL}/E_M)+\xi_W^{GPL}} \tag{14}$$

w_{GPL}, l_{GPL} and t_{GPL} denote GPLs' average width, length, and thickness, respectively. Finally, Poisson's ratio of GPL-reinforced metal matrix implementing rule of mixture can be expressed by

$$\upsilon_1 = \upsilon_{GPL}V_{GPL} + \upsilon_M V_M \tag{15}$$

where V_M is the volume fraction of metal matrix ($V_M = 1-V_{GPL}$).

3. Formulation of the problem

Consider a cantilevered column resting on elastic foundation K subjected to a follower force p, the column is assumed to be graded in the thickness direction, as shown in Fig.1. The governing partial differential equation of the column resting on an elastic foundation subjected to a follower force is given by the following equation of motion

$$\frac{\partial^2}{\partial x^2}(EI_x\frac{\partial^2 w}{\partial x^2}) + p\frac{\partial^2 w}{\partial x^2} + K.w + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \quad 0 \le x \le L \tag{16}$$

Where E is the modulus of elasticity, I is the column inertia, *p* is the follower force, K is the modulus of elastic foundation, v is the lateral displacement of the column, ρ is the mass density per unit length of the column and A is the cross section of the column. For a non-uniform column resting on an elastic foundation, the governing equation can be expressed as (Mahmoud *et al*. 2011)

$$D_{fgm}I_x\frac{\partial^4}{\partial x^4} + 2\frac{\partial D_{fgm}I_x}{\partial x}\frac{\partial^3 w}{\partial x^3} + \frac{\partial^2 D_{fgm}I_x}{\partial x^2} + p\frac{\partial^2 w}{\partial x^2} + K.w + \rho A\frac{\partial^2 w}{\partial t^2} = 0, \quad 0 \le x \le L \tag{17}$$

For analysis of the natural frequency, Eq. (17) is formulated as an eigenvalue problem by assuming the following periodic function

$$w(x,t) = W(x)e^{-i\omega t} \tag{18}$$

where $W(x)$ is the mode shape of the transverse motion of the beam, therefore

$$D_{fgm}I_x\frac{d^4W}{dx^4} + 2\frac{dD_{fgm}I_x}{dx}\frac{d^3W}{dx^3} + 2\frac{d^2D_{fgm}I_x}{dx^2}\frac{dW}{dx} + p\frac{d^2W}{dx^2} + KW + \rho A\omega^2 W = 0 \tag{19}$$

4. Implementation of boundary conditions

Eq. (19) is a fourth-order ordinary differential equation. Thus, it requires four boundary conditions, which are as follows:

For Clamped–Free supports (C-F) with a follower force *P*, where η is the tangency coefficient of a follower force *P*.

$$W = \frac{dW}{dx} = 0 \quad at \quad x = 0$$
$$\frac{d^2W}{dx^2} = \frac{d^3W}{dx^3} + (1-\eta)P\frac{dW}{dx} = 0 \quad at \quad x = L \tag{20}$$

5. Solution procedure

In this stage, the GDQ approach is used to solve the governing equations of columns [A brief review of GDQ method is given in Appendix]. Quantities have been implemented in Eq. (19) by changing variables in the following form

$$W = \frac{w}{L}, X = \frac{x}{L}, S = \frac{D_{fgm}I}{E_m I_0}, P = p\frac{D_{fgm}L^2}{E_m I_0},$$
$$K = \frac{D_{fgm}L^4}{E_m I_0}, \Omega^2 = \frac{\rho_{GPL}}{\rho_m}L^4 A\omega^2 \tag{21}$$

Also, according to the GDQ method, the governing Eq. (19) should be rewritten in discretized form. Therefore Eq. (19) at a sample grid point (x_i) can be written as

$$S_{xi}\sum_{j=1}^{N} c_{ij}^{(4)}W_j + 2S_{xi}^{(1)}\sum_{j=1}^{N} c_{i,j}^{(3)}W_j + S_{xi}^{(2)}\sum_{j=1}^{N} c_{ij}^{(2)}W_j +$$
$$P\sum_{j=1}^{N} c_{ij}^{(2)}W_j + KW_i + \Omega W_i = 0, \tag{22}$$
$$i = 1, 2, ..., N$$

In Eq. (23) $c_{i,j}^{(4)}$, $c_{i,j}^{(3)}$ and $c_{i,j}^{(2)}$ are the weighting coefficients of the fourth-, third- and second-order derivatives. Where W_i i=1, 2, . . . , N, is the functional value at the grid point x_i and $S_{xi}^{(1)}$, $S_{xi}^{(2)}$ are the first- and second-order derivatives of S_x at x_i respectively. The boundary conditions Eq. (20) should also be rewritten in discretized form. For Clamped-Free supports (C-F) with a follower force P

$$W_1 = \sum_{j=1}^{N} c_{1j}^{(1)}W_j = 0,$$
$$\sum_{j=1}^{N} c_{Nj}^{(2)}W_j = \sum_{j=1}^{N} c_{Nj}^{(3)}W_j + (1-\eta)p\sum_{j=1}^{N} c_{Nj}^{(1)}W_j = 0 \tag{23}$$

Applying the GDQ procedure, the whole system of differential equations has been discretized and the global assembling leads to the linear algebraic equations where the natural frequencies for FGM beam are obtained. A Matlab program has been used to solve the non-dimensional governing differential equation of non-uniform column and get the normalized frequencies (Ω) and the corresponding mode shapes. In Table 1 convergence and validation study of the normalized natural frequency is considered for an isotropic column without elastic foundation. As can be seen, a fast rate of convergence of the method is evident for different boundary conditions and it is found that only 17 DQ grid in the thickness direction can yield accurate results.

Table 1 Convergence behavior and accuracy of the first two normalized frequencies of non-uniform column with P=K=0

Natural Frequency	Present	Mahmoud *et al.* (2011)
Ω_1		
α=-0.5	3.3274	3.327
α=0	3.5159	3.516
α=0.5	3.6732	3.673
Ω_2		
α=-0.5	19.5039	19.503
α=0	22.0344	22.034
α=0.5	24.0793	24.079

Also, the comparison shows that the present results agree very well with similar ones obtained by Mahmoud *et al.* (2011).

6. Benchmark results

It is assumed that the thickness of the substrate is an arbitrary continuous and smooth function of z, that is

$$S_x = \mu x + 1, \mu = \alpha(\frac{D_{fgm}}{E_m}) \tag{24}$$

where S_x is the non-dimensional bending stiffness.

Fig. 3 shows the effect of GPL weight fractions on the first and second non-dimensional frequencies. As observed, the critical load (P_{cr}) of the FG porous column decreases with increasing GPL weight fraction. The effect of elastic foundation on the first and second natural frequency of a non-uniform FG porous column for different amount of follower load (P) is shown in Fig. 4. It can be seen that the first and second natural frequency of structure increases with increasing the amount of elastic coefficient. It should be noted that the amount of elastic coefficient does not have any effect on the critical load (P_{cr}) of structure.

Influences of porosity coefficient and non-dimensional bending stiffness S(x) on vibration frequency of GPL reinforced non-uniform column are shown in Figs. 5 and 6. It is clear that a porous non-uniform column has lower natural frequencies than a perfect column (e_0=0). In other words, increasing porosity coefficient results in smaller natural frequencies due to the reduction in the bending rigidity of the nanocomposite structure. Therefore, for better understanding of mechanical behavior of nanocomposite columns, it is crucial to consider porosities inside the material structure. Results indicate that the natural frequency increases with increasing non-dimensional bending stiffness S(x).

The combined effects of porosity distribution and GPL distribution pattern on the fundamental frequency are investigated in Fig. 7 in which the fundamental natural frequency at various GPL weight fractions is presented. Symmetric GPL pattern A is proved to be the best dispersion method, followed by the uniform pattern C which is slightly better than the asymmetric pattern B.

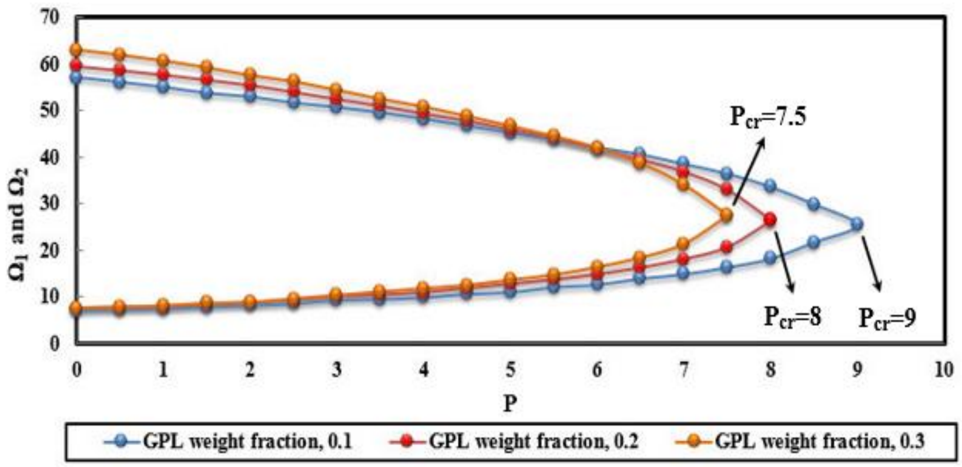

Fig. 3 First and second frequency of Clamped-Free GPLs-reinforcement column (μ=0, e_0=0.2, K=0)

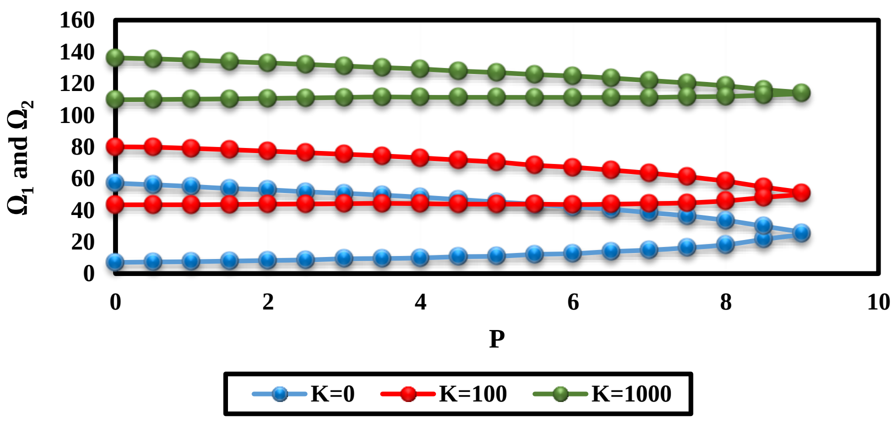

Fig. 4 First and second frequency of Clamped-Free GPLs-reinforcement column with various P and elastic coefficient (μ=0, e_0=0.2, wt. 0.1%)

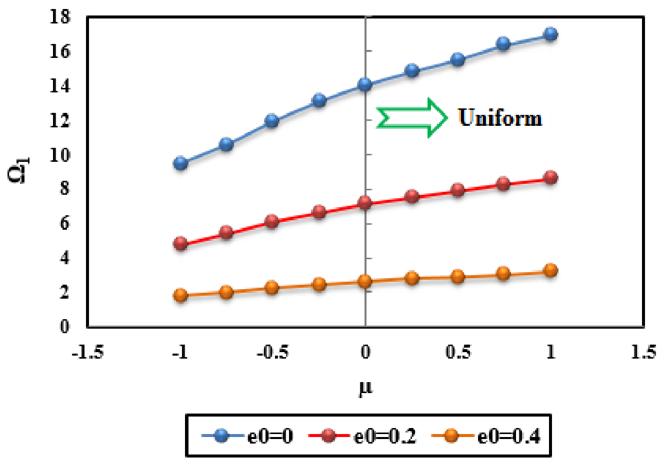

Fig. 5 First frequency of Clamped-Free GPLs-reinforcement column with various μ and porosity coefficient (wt. 0.1%, K=0)

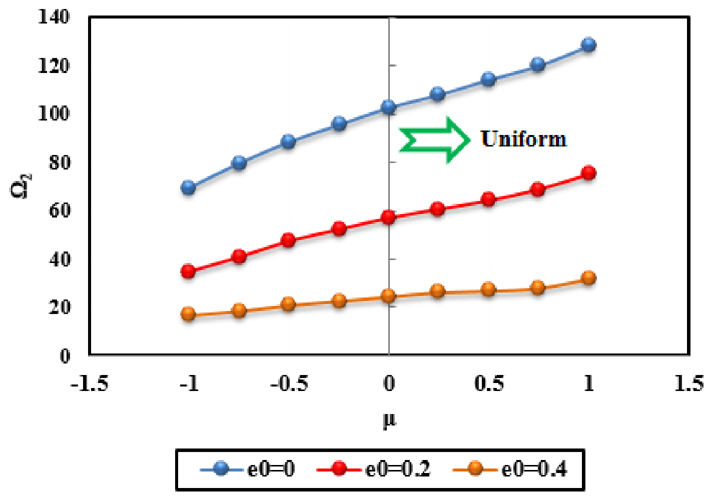

Fig. 6 Second frequency of Clamped-Free GPLs-reinforcement column with various μ and porosity coefficient (wt. 0.1%, K=0)

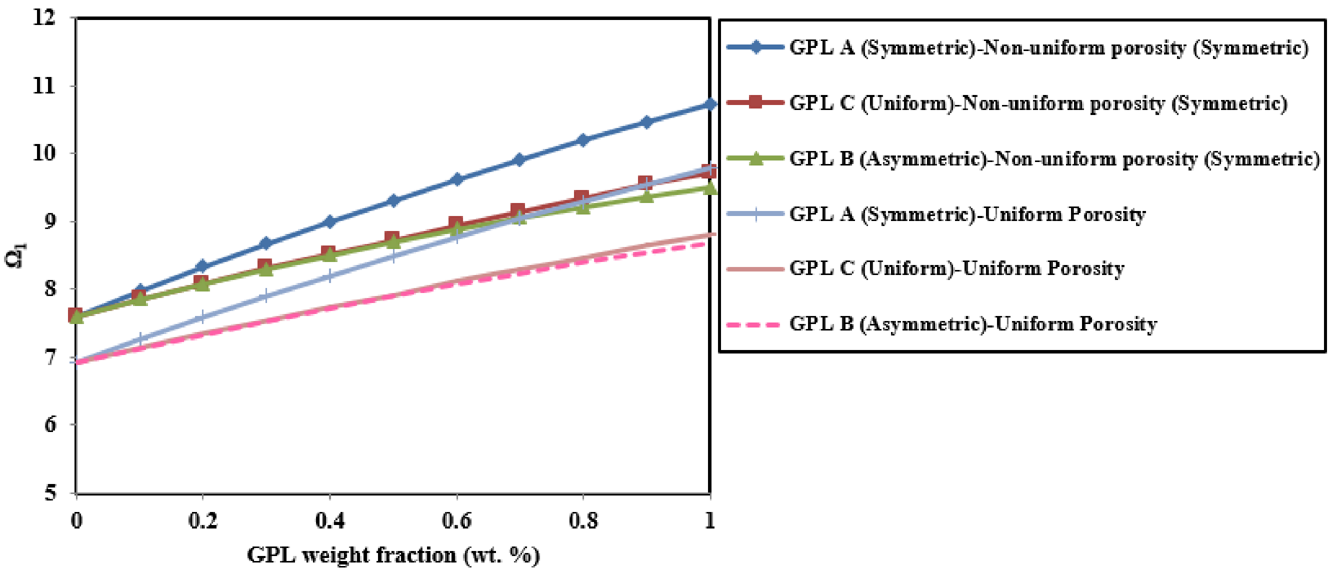

Fig. 7 The effect of GPL on the fundamental frequency of nanocomposite F-C columns (K=0, e_0=0.2).

Results indicate that columns with non-uniform symmetric porosity distribution and symmetric GPL pattern A have the largest fundamental frequencies, i.e., the highest effective stiffness under the same GPL weight fraction, suggesting that a nanocomposite column in which both internal pores and nanofillers are symmetrically distributed can offer the best structural performance.

7. Conclusions

The present work is concerned with free vibration analysis of non-uniform FG porous column resting on an elastic foundation and subjected to follower force where the internal pores and graphene platelets (GPLs) are distributed in the matrix uniformly or non-uniformly according to three different patterns. The elastic modulus of the nanocomposite is obtained by using Halpin-Tsai micromechanics model. The differential quadrature method as an efficient and robust numerical approach is used to discretize the governing equations and to implement the boundary F-C boundary condition. It can be concluded that the non-uniformity parameters, elastic and porosity coefficients have a significant effect on the dynamic response of non-uniform nanocomposite columns. From this study some conclusions can be made as following:

- It is observed that the maximum vibration frequency obtained in the case of symmetric porosity and GPL distribution, while the minimum vibration frequency is obtained using uniform porosity distribution.
- Based on the results the frequency parameter is sensitive to the value of coefficient of column porosity significantly. The results imply that the

frequency parameter decreases as the coefficient of column porosity increases.

- It is observed that higher values of Winkler foundation constant leads to increase in bending rigidity and natural frequency of the column.
- Results indicate that the critical load (P_{cr}) of the FG porous column decreases with increasing GPL weight fractions.

Results show that for better understanding of mechanical behavior of nanocomposite columns, it is crucial to consider porosities inside the material structure.

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Appendix

In Generalized Differential Quadrature Method (GDQM), the *n*th order partial derivative of a continuous function $f(x,z)$ with respect to x at a given point x_i can be approximated as a linear summation of weighted function values at all the discrete points in the domain of *x*, that is

$$\frac{\partial^n f(x_i,z)}{\partial x^n} = \sum_{k=1}^{N} c_{ik}^n f(x_{ik},z) \quad (i=1,2,...,N, n=1,2,...,N-1) \qquad (1)$$

Where *N* is the number of sampling points and c_{ij}^n is the x^i dependent weight coefficient. To determine the weighting coefficients c_{ij}^n, the Lagrange interpolation basic functions are used as the test functions, and explicit formulas for computing these weighting coefficients can be obtained as (Bert and Malik 1996)

$$c_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_j)}, i, j = 1,2,...,N, i \neq j \qquad (2)$$

where

$$M^{(1)}(x_i) = \prod_{j=1, i\neq j}^{N} (x_i - x_j) \qquad (3)$$

and for higher order derivatives, one can use the following relations iteratively

$$c_{ij}^{(n)} = n(c_{ii}^{(n-1)} c_{ij}^{1} - \frac{c_{ij}^{(n-1)}}{(x_i - x_j)}), \quad i, j = 1,2,...,N, \qquad (4)$$

$$i \neq j, n = 2,3,...,N-1$$

$$c_{ii}^{(n)} = -\sum_{j=1, i\neq j}^{N} c_{ij}^{(n)} \quad i = 1,2,...,N, \quad n = 1,2,...,N-1 \qquad (5)$$

A simple and natural choice of the grid distribution is the uniform grid-spacing rule. However, it was found that nonuniform grid-spacing yields result with better accuracy. Hence, in this work, the Chebyshev-Gauss-Lobatto quadrature points are used

$$x_i = \frac{1}{2}(1 - \cos(\frac{i-1}{N-1}\pi)) \quad i = 1,2,...,N \qquad (6)$$