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Free vibration analysis of functionally graded cylindrical nanoshells resting on Pasternak foundation based on two-dimensional analysis

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Abstract. In this paper, free vibration analysis of a functionally graded cylindrical nanoshell resting on Pasternak foundation is presented based on the nonlocal elasticity theory. A two-dimensional formulation along the axial and radial directions is presented based on the first-order shear deformation shell theory. Hamilton's principle is employed for derivation of the governing equations of motion. The solution to formulated boundary value problem is obtained based on a harmonic solution and trigonometric functions for various boundary conditions. The numerical results show influence of significant parameters such as small scale parameter, stiffness of Pasternak foundation, mode number, various boundary conditions, and selected dimensionless geometric parameters on natural frequencies of nanoshell.

Keywords: size-dependent natural vibration; functionally graded materials; cylindrical nanoshell; nonlocal parameters; various boundary conditions

1. Introduction

Nanomaterials and nanostructures have attracted many researchers for more investigations on the various aspects of them in different conditions and geometries. Experimental results on the structures in nano and micro scales indicate that the classical size-independent theories have not capability to simulate various behaviors of structures in small scales. To model the structures in micro and nano scales, some size-dependent theories have been developed by various researchers to account size-dependency. The literature review indicates that analysis of nanomaterials and nanostructures needs new theories to account small scale effects of discontinuities. Various non-classical theories have been developed by many authors for better simulation of mechanical behaviors of nanostructures in small scales. To justify originality and importance of the present paper, a deep literature survey is presented based on the previous studies.

Ansari *et al*. (2016) investigated nonlinear buckling and post-buckling analysis of cylindrical nanoshell subjected to axial loads based on a size-dependent analysis. Surface elasticity was included in governing equations based on theory of Gurtin and Murdoch. They developed the kinematic relations based on the Donnell's shell theory. The geometric nonlinearity was accounted based on the von Kármán's relations. The numerical results have been presented with and without surface stress to show that this component has significant influence on the responses.

Alibeigloo and Jafarian (2016) studied bending and free vibration analysis of cylindrical shell reinforced with carbon nanotubes based on exact theory of elasticity. Various distributions of reinforcement were defined along thickness direction. The results were calculated for simply-supported boundary condition based on Fourier series solution. The effect of other boundary conditions was investigated based on DQM. Exact solution and elastic analysis of FG cylindrical and spherical shells was studied by Tutuncu and Ozturk (2001). Jabbari *et al*. (2009) used two-dimensional thermo-elasticity relations for stress and deformation analysis of cylindrical pressure vessels with short length. FSDT was employed for size-dependent buckling analysis of FG piezoelectric cylindrical nanoshell by Mehralian *et al.* (2016). Size dependency was accounted based on modified couple stress theory. The equilibrium equations were derived based on principle of virtual work. The numerical results were presented to discuss on the influence of micro length scale parameter, length, thickness, external electric voltage and in-homogeneous index on the critical loads. Zhang *et al.* (2015) studied free vibration analysis of a functionally graded cylindrical micro shell based on four unknown shear deformation theory and strain gradient theory. After evaluation of effective material properties using Mori–Tanaka homogenization technique, the governing equations of motion have been derived based on Hamilton's principle. Size-dependent magneto-electro-elastic vibration analysis of cylindrical nano shell resting on Winkler's foundation was studied by Ke *et al.* (2014a) based on nonlocal Love's shell theory. Thermo-electro-mechanical vibration analysis of cylindrical nano shell made of piezoelectric materials was studied by Ke *et al.* (2014b) based on Love's thin shell theory for various boundary conditions. It is a well-established fact that

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computational methods are capable of dealing with a wide application range. Some advanced computer-based numerical methods for analysis of mechanical and mathematical problems were proposed by various researchers (Guo *et al.* 2019, Anitescu *et al*. 2019, Niu *et al.* 2019).

Shao (2006) investigated the thermoelastic analysis of a thick-walled cylinder under the mechanical and thermal loads. Transient thermo-elastic analysis of a FG hollow cylinder was presented by Ootao and Tanigawa (2006). Arefi *et al.* (2016) presented two-dimensional thermo-elastic analysis of functionally graded cylindrical shell subjected to fix and variable thermal and mechanical loadings, respectively. Shen and Xiang (2012) studied large amplitude vibration analyses of nanocomposite cylindrical shells reinforced by SWCNTs based on the higher-order shear deformation theory and nonlinear von Kármán-type of kinematic relations. Various patterns of reinforcements were defined for functionally graded nanocomposite materials. Sun *et al*. (2013) studied free vibration characteristics of rotating cylindrical shells with arbitrary edges based on Sanders shell theory. Shakeri et al. (2006) studied dynamic analysis of FG thick hollow cylinders under dynamic load using Galerkin finite element and Newmark methods. To solve the problem, the cylinder was divided to some sub cylinders and continuity condition was satisfied between adjacent layers for displacements and stresses.

First-order shear deformation theory (FSDT) was used to derive governing equations of motion and free vibration responses of moderately thick FG conical, cylindrical shells and annular plates by Tornabene (2009). Generalized Differential Quadrature (GDQ) method was used to convert governing equations of motion to a standard linear eigenvalue problem. Thermo-elastic free vibration and buckling analyses of FG piezoelectric cylindrical shell were studied by Sheng and Wang (2010). They used Hamilton's principle and quadratic variation of electric potential to derive governing equations of motion based on FSDT. Free vibration analysis of a two-dimensional functionally graded cylindrical shell with finite length was studied by Asgari and Akhlaghi (2011) based on three-dimensional equations of elasticity. Malekzadeh and Heydarpour (2012) investigated free vibration analysis of rotating FG cylindrical shells subjected to thermal loads based on the FSDT. They included influences of centrifugal and Coriolis forces due to rotation of the shell. The natural frequencies were presented in terms of various parameters such as angular velocity and various boundary conditions. Wave propagation analysis of carbon nanotubes was studied based on nonlocal elastic shell theory by Wang and Varadan (2007). For studying effect of various input parameters on the outputs of a mathematical problem, a statistical scrutiny may be taken into account. For example, simple MATLAB codes were provided for sensitivity analysis of computationally expensive models by Vu-Bac *et al.* (2016).

Ferreira *et al.* (2007) employed first-order theory of Donnell for natural frequencies analysis of doubly curved cross-ply composite shells based on multiquadric radial basis functions. Application of modified couple stress formulation on the vibration analysis of a sandwich nano/micro plate with various boundary conditions was developed in Reference (Arefi *et al.* 2017). Ahmadi and Najafi (2016) studied stress analysis and inter-laminar stress analysis of a rotating thin laminated cylindrical shell based on the Layerwise theory. Shokrollahi (2018) employed Kirchhoff-Love method and harmonic differential quadrature method for deformation and stress analysis of a sandwich cylindrical shell. Ahmadi and Foroutan (2019) studied resonances of FG porous cylindrical shell under two-term excitation based on classical plate theory of shells and von-Karman equation. Some important works on the functionally graded and laminated cylindrical shells can be observed in References (Santos *et al.* 2009, Loy *et al.* 1999, Pradhan *et al*. 2000).

A literature survey on the free vibration analysis, nonlocal elasticity theory and FSDT of cylindrical shell was performed. Our review indicates that free vibration analysis of functionally graded cylindrical nanoshell based on a two-dimensional analysis has not been studied by researchers and is presented for the first time in this paper. Shear strains along the axial and radial directions are accounted in our analysis based on the first-order shear deformation theory. The numerical results are presented based on analytical approach to investigate influence of significant parameters such as nonlocal parameter, two parameters of Pasternak foundation, mode number and some dimensionless geometric parameters such as length to radius, length to thickness and radius to thickness of cylinder.

2. Governing equations

We consider a functionally graded cylindrical nanoshell with thickness h and length L defined in polar coordinate system. The geometry and coordinate system of a cylindrical nanoshell are shown in Fig. 1. Note that r is the local radius and z is measured from middle surface. Relation between the local radius and z is expressed as $r = R + z$.

Hamilton's principle $\int_0^t \delta(T - U + W)dt = 0$ is used to derive governing equations of motion of a functionally graded nanoshell in which U is the strain energy, T is the kinetic energy and W is the work done by external forces in time t. The variation of strain energy is defined as

$$\delta U = \iiint_V (\sigma_x \delta\varepsilon_x + \sigma_\theta \delta\varepsilon_\theta + \sigma_z \delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\, dV, \quad (1)$$

where σ_i, ε_i are stress and strain components satisfied differential constitutive relations based on Eringen's nonlocal elasticity theory and three-dimensional Hooke's law in the following way

$$\begin{aligned}
(1 - \xi^2\nabla^2)\sigma_x &= \lambda(z)[(1-\nu)\varepsilon_x + \nu(\varepsilon_\theta + \varepsilon_z)],\\
(1 - \xi^2\nabla^2)\sigma_\theta &= \lambda(z)[(1-\nu)\varepsilon_\theta + \nu(\varepsilon_x + \varepsilon_z)],\\
(1 - \xi^2\nabla^2)\sigma_z &= \lambda(z)[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_\theta)],\\
(1 - \xi^2\nabla^2)\tau_{xz} &= k_s \frac{E(z)}{2(1+\nu)}\gamma_{xz},
\end{aligned} \quad (2)$$

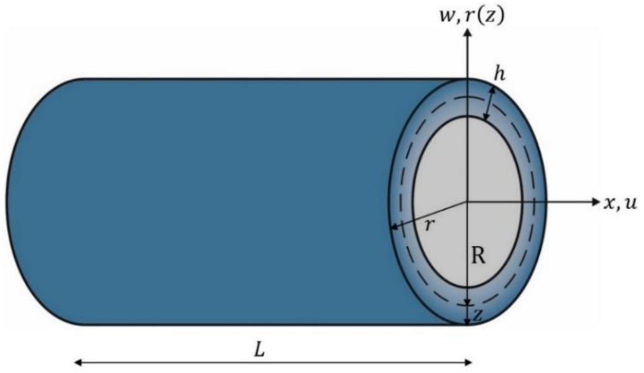

Fig. 1 The geometry and coordinate system of a cylindrical FGM nanoshell

$\lambda(z) = E(z)/(1+\nu)(1-2\nu)$ and k_s is shear stress correction factor, ξ is the nonlocal parameter, and ∇^2 is one-dimensional Laplacian defined as $\nabla^2 = \frac{\partial^2}{\partial x^2}$ due to symmetric conditions of cylindrical nanoshell. Eq.2 is derived using generized Hooke's law. It is concluded that three normal stress components and one shear stress component are assumed in our formulation. Accounting shear stress component becomes very important in short cylinders specially at both ends. We assume the constant value of Poisson's ratio ν and variation of Young's modulus $E(z)$ through the thickness of nanoshell accordance to the power-law defined as

$$E(z) = (E_t - E_b)\left(\frac{z}{h}+\frac{1}{2}\right)^n + E_b, \tag{3}$$

where E_t and E_b are the values of the Young's modulus at the top and bottom surface, respectively, n is the in-homogeneous index.

A time-dependent two-dimensional displacement field for FSDT is assumed as follows

$$\begin{aligned} u_x(x,z,t) &= u(x,t) + z\varphi_x(x,t), \\ u_z(x,z,t) &= w(x,t) + z\varphi_z(x,t), \end{aligned} \tag{4}$$

where u_x, u_z are the axial and radial displacement components. Based on first-order shear deformation theory, the both axial and radial displacments are varying linearly along the thickness direction. Based on Eqs. (4), the strain components are expressed as follows

$$\begin{aligned} \varepsilon_x &= \frac{du}{dx} + z\frac{d\varphi_x}{dx}, \quad \varepsilon_z = \varphi_z, \\ \varepsilon_\theta &= \frac{w}{R+z} + z\frac{\varphi_z}{R+z}, \quad \gamma_{xz} = \varphi_x + \frac{dw}{dx} + z\frac{d\varphi_z}{dx} \end{aligned} \tag{5}$$

The kinematic relations indicate that the present theory can be used for modeling the shear strains in cylindrical shells. Substituting variation of strain components (5) into variation of strain energy (1) and integration on thickness direction (z) simplifies variation of strain energy equation in terms of resultant forces and moments as follows

$$\delta U = \int_0^L \left[-\frac{dN_x}{dx}\delta u + \left(N_{xz} - \frac{dM_x}{dx}\right)\delta\varphi_x + \left(N_\theta - \frac{dN_{xz}}{dx}\right)\delta w + \left(M_\theta + N_z - \frac{dM_{xz}}{dx}\right)\delta\varphi_z\right]dx, \tag{6}$$

where the forces and moments are

$$\begin{aligned} \{N_x, N_z, N_{xz}\} &= \int_{-h/2}^{h/2}(R+z)\{\sigma_x, \sigma_z, \tau_{xz}\}dz, \\ \{N_\theta, M_\theta\} &= \int_{-h/2}^{h/2}\sigma_\theta\{1, z\}dz, \\ \{M_x, M_{xz}\} &= \int_{-h/2}^{h/2} z(R+z)\{\sigma_x, \tau_{xz}\}dz, \end{aligned} \tag{7}$$

The kinetic energy of the cylindrical nanoshell is defined as

$$\delta T = \int_0^L \int_{-h/2}^{h/2} 2\pi\rho(z)[\dot{u}_x\delta\dot{u}_x + \dot{u}_z\delta\dot{u}_z](R+z)dzdx, \tag{8}$$

where mass density $\rho(z)$ varies through the thickness of the nanoshell accordance to the power-law

$$\rho(z) = (\rho_t - \rho_b)\left(\frac{z}{h}+\frac{1}{2}\right)^n + \rho_b, \tag{9}$$

where ρ_t and ρ_b are the values of the density at the top and bottom surface, respectively. The superposed dot on a variable indicates time derivative.

Substituting the displacement components (4) into Eq. (8), rearranging the variables, and finally integration by parts lead to following form of variation of kinetic energy as follows

$$\delta T = -\int_0^L [(B_1\ddot{u} + B_2\ddot{\varphi}_x)\delta u + (B_2\ddot{u} + B_3\ddot{\varphi}_x)\delta\varphi_x + (B_1\ddot{w} + B_2\ddot{\varphi}_z)\delta w + (B_2\ddot{w} + B_3\ddot{\varphi}_z)\delta\varphi_z]dx, \tag{10}$$

where the mass moments of inertia B_i are calculated from

$$B_i = \int_{-h/2}^{h/2} 2\pi\rho(z)(R+z)z^{(i-1)}dz, \tag{11}$$

Finally, to complete Hamilton's principle, the virtual work done by external forces is defined as follows

$$\delta W = -\int_0^L \left[F_f\delta u_z|_{z=\frac{h}{2}}\right]dx = -\int_0^L \left[F_f\delta w + \frac{h}{2}F_f\delta\varphi_z\right]dx, \tag{12}$$

where F_f is reaction of Pasternak foundation expressed as $F_f = K_1u_z - K_2\nabla^2u_z$. Substitution of radial displacement into reaction of Pasternak foundation leads to following relation

$$F_f = K_1\left(w + \frac{h}{2}\varphi_z\right) - K_2\left(\frac{\mathrm{d}^2w}{\mathrm{d}x^2} + \frac{h}{2}\frac{\mathrm{d}^2\varphi_z}{\mathrm{d}x^2}\right), \tag{13}$$

where K_1, K_2 are two parameters of Pasternak foundation. Substitution of variations of strain energy (6), kinetic energy (10), and energy due to external works (12) into Hamilton's principle gives final governing equations of motion as

$$\begin{aligned} \delta u&: \frac{dN_x}{dx} = B_1\ddot{u} + B_2\ddot{\varphi}_x, \\ \delta w&: \frac{dN_{xz}}{dx} - N_\theta = B_1\ddot{w} + B_2\ddot{\varphi}_z + F_f, \\ \delta\varphi_x&: \frac{dM_x}{dx} - N_{xz} = B_2\ddot{u} + B_3\ddot{\varphi}_x, \\ \delta\varphi_z&: \frac{dM_{xz}}{dx} - M_\theta - N_z = B_2\ddot{w} + B_3\ddot{\varphi}_z + \frac{h}{2}F_f, \end{aligned} \tag{14}$$

where the size-dependent resultant forces and moments expressed by the displacements have the following forms

$$
\begin{aligned}
(1-\xi^2\nabla^2)N_x &= A_1\frac{du}{dx}+A_2\frac{d\varphi_x}{dx}+A_3w+(A_4+A_5)\varphi_z,\\
(1-\xi^2\nabla^2)N_z &= A_5\frac{du}{dx}+A_6\frac{d\varphi_x}{dx}+A_3w+(A_1+A_4)\varphi_z,\\
(1-\xi^2\nabla^2)M_x &= A_2\frac{du}{dx}+A_9\frac{d\varphi_x}{dx}+A_4w+(A_6+A_{10})\varphi_z,\\
(1-\xi^2\nabla^2)N_\theta &= A_3\frac{du}{dx}+A_4\frac{d\varphi_x}{dx}+A_{12}w+(A_3+A_{13})\varphi_z,\\
(1-\xi^2\nabla^2)M_\theta &= A_4\frac{du}{dx}+A_{10}\frac{d\varphi_x}{dx}+A_{13}w+(A_4+A_{14})\varphi_z,\\
(1-\xi^2\nabla^2)N_{xz} &= k_sA_7\left(\varphi_x+\frac{dw}{dx}\right)+k_sA_8\frac{d\varphi_z}{dx},\\
(1-\xi^2\nabla^2)M_{xz} &= k_sA_8\left(\varphi_x+\frac{dw}{dx}\right)+k_sA_{11}\frac{d\varphi_z}{dx},
\end{aligned}
\tag{15}
$$

Substitution of the resultant forces and moments (15) into governing equations of motion (14) gives

$$
\delta u:\ A_1\frac{d^2u}{dx^2}+A_2\frac{d^2\varphi_x}{dx^2}+A_3\frac{dw}{dx}+(A_4+A_5)\frac{d\varphi_z}{dx}=(1-\xi^2\nabla^2)(B_1\ddot{u}+B_2\ddot{\varphi}_x),
\tag{16a}
$$

$$
\delta\varphi_x:\ A_2\frac{d^2u}{dx^2}-k_sA_7\varphi_x+A_9\frac{d^2\varphi_x}{dx^2}+(A_4-k_sA_7)\frac{dw}{dx}+(A_6+A_{10}-k_sA_8)\frac{d\varphi_z}{dx}=(1-\xi^2\nabla^2)(B_2\ddot{u}+B_3\ddot{\varphi}_x),
\tag{16b}
$$

$$
\delta w:\ A_3\frac{du}{dx}+(A_4-k_sA_7)\frac{d\varphi_x}{dx}+A_{12}w-k_sA_7\frac{d^2w}{dx^2}+(A_3+A_{13})\varphi_z-k_sA_8\frac{d^2w}{dx^2}=-(1-\xi^2\nabla^2)(B_1\ddot{w}+B_2\ddot{\varphi}_z)-(1-\xi^2\nabla^2)\left[K_1\left(w+\frac{h}{2}\varphi_z\right)-K_2\left(\frac{d^2w}{dx^2}+\frac{h}{2}\frac{d^2\varphi_z}{dx^2}\right)\right],
\tag{16c}
$$

$$
\delta\varphi_z:\ (A_4+A_5)\frac{du}{dx}+(A_6+A_{10}-k_sA_8)\frac{d\varphi_x}{dx}+(A_3+A_{13})w-k_sA_8\frac{d^2w}{dx^2}-k_sA_{11}\frac{d^2\varphi_z}{dx^2}+(A_1+2A_4+A_{14})\varphi_z=-(1-\xi^2\nabla^2)(B_2\ddot{w}+B_3\ddot{\varphi}_z)-\frac{h}{2}(1-\xi^2\nabla^2)\left[K_1\left(w+\frac{h}{2}\varphi_z\right)-K_2\left(\frac{d^2w}{dx^2}+\frac{h}{2}\frac{d^2\varphi_z}{dx^2}\right)\right],
\tag{16d}
$$

Eqs. (16) are the governing equations of motion for a functionally graded cylindrical nanoshell based on first-order shear deformation theory and nonlocal elasticity. The dynamic responses of the nanoshell can be analyzed using solution of Eq. (16). In this paper, the natural frequency responses are evaluated using the analytical method. The solution is applicable for various boundary conditions. The integration constants appeared in Eq. (16) are presented in following way

$$
\begin{aligned}
\{A_1,A_2,A_9\} &= \int_{-h/2}^{h/2}\lambda(1-\nu)(R+z)\{1,z,z^2\}\mathrm{d}z,\\
\{A_3,A_4,A_{10}\} &= \int_{-h/2}^{h/2}\lambda\nu\{1,z,z^2\}\mathrm{d}z,\\
\{A_5,A_6\} &= \int_{-h/2}^{h/2}\lambda\nu(R+z)\{1,z\}\mathrm{d}z,\\
\{A_7,A_8,A_{11}\} &= \int_{-h/2}^{h/2}\frac{1-2\nu}{2}\lambda(R+z)\{1,z,z^2\}\mathrm{d}z,\\
\{A_{12},A_{13},A_{14}\} &= \int_{-h/2}^{h/2}\frac{1-\nu}{R+z}\lambda\{1,z,z^2\}\mathrm{d}z.
\end{aligned}
\tag{17}
$$

The governing equations of motion may be presented in matrix form as follows

$$
[I]\frac{d^4\{X\}}{dx^4}+[J]\frac{d^2\{X\}}{dx^2}+[S]\frac{d\{X\}}{dx}+[Q]\{X\}=[Y]\{\ddot{X}\}+[Z]\frac{d^2\{\ddot{X}\}}{dx^2},
\tag{18}
$$

where the elements of above defined matrices are expressed by

$$
\begin{aligned}
&I_{33}=\xi^2K_2,\ I_{34}=\xi^2\frac{h}{2}K_2,\ I_{44}=\left(\frac{h}{2}\right)^2\xi^2K_2,\\
&J_{11}=A_1,\ J_{12}=A_2,\ J_{22}=A_9,\\
&J_{33}=-k_sA_7-K_2-\xi^2K_1,\\
&J_{34}=-k_sA_8-\frac{h}{2}K_2-\xi^2\frac{h}{2}K_1,\\
&J_{44}=-k_sA_{11}-\left(\frac{h}{2}\right)^2(K_2+\xi^2K_1),\\
&S_{13}=A_3,\ S_{14}=A_4+A_5,\ S_{23}=A_4-k_sA_7,\\
&S_{24}=A_6+A_{10}-k_sA_8,\\
&Q_{22}=-A_7,\ Q_{33}=A_{12}+K_1,\ Q_{34}=A_3+A_{13}+\frac{h}{2}K_1,\\
&Q_{44}=A_1+2A_4+A_{14}+\left(\frac{h}{2}\right)^2K_2,\\
&Y_{11}=B_1,\ Y_{12}=B_2,\ Y_{21}=B_2,\ Y_{22}=B_3,\\
&Y_{33}=-B_1,\ Y_{34}=-B_2,\ Y_{44}=-B_3,\\
&Z_{11}=-\xi^2B_1,\ Z_{12}=-\xi^2B_2,\ Z_{21}=-\xi^2B_2,\\
&Z_{22}=-\xi^2B_3,\ Z_{33}=\xi^2B_1,\ Z_{34}=\xi^2B_2,\ Z_{44}=\xi^2B_3
\end{aligned}
\tag{19}
$$

After derivation of the governing equations of motion, the solution procedure may be presented for a functionally graded cylindrical nanoshell with various boundary conditions.

3. Analytical solution

An analytical solution for a FG cylindrical nanoshell with various boundary conditions is obtained based on properties of a trigonometric functions. The ends of cylindrical nanoshell are assumed to be simply-supported (S), clamped (C) or free (F).

The analytical solution is assumed as

$$
\begin{Bmatrix} u\\ w\\ \varphi_x\\ \varphi_z \end{Bmatrix}=\sum_{n=1}^{\infty}\begin{Bmatrix} U_m\frac{\partial X_m(x)}{\partial x}\\ W_m\frac{\partial X_m(x)}{\partial x}\\ \Theta_m^xX_m(x)\\ \Theta_m^zX_m(x) \end{Bmatrix}\mathrm{e}^{i\omega t},
\tag{20}
$$

where ω is the natural frequency, U_m, W_m, Θ_m^x, Θ_m^z are the maximum values of the displacements. The functions $X_m(x)$ are listed in Table 1 for various boundary conditions. For convenience four variables U_m, W_m, Θ_m^x, Θ_m^z can be presented as a vector

$$
\{X\}=[U_m\quad W_m\quad \Theta_m^x\quad \Theta_m^z]^T,
\tag{21}
$$

By substituting the series (20) into the governing equations expressed by displacements (16), the characteristic equation for simply-supported FGM nanoshell can be defined as

$$
([K]+\omega^2[M])\{X\}=\{0\},
\tag{22}
$$

Table 1 The admissible functions $X_m(x)$ for $\lambda_m = m\pi/a$

Boundary conditions	**Function** X_m
SS	$\sin(\lambda_m x)$
CC	$\sin^2(\lambda_m x)$
SC	$\sin(\lambda_m x)\,[\cos(\lambda_m x) - 1]$
CF	$\cos^2(\lambda_m x)\,[\sin^2(\lambda_m x) + 1]$

where the elements of the symmetric stiffness $[K]$ and mass $[M]$ matrices are expressed as follows

$$
\begin{gathered}
K_{11} = -A_1\lambda_m^2,\; K_{12} = -A_2\lambda_m^2,\; K_{13} = A_3\lambda_m,\\
K_{14} = (A_4 + A_5)\lambda_m,\\
K_{22} = -A_9\lambda_m^2 - A_7,\; K_{23} = (A_4 - k_s A_7)\lambda_m,\\
K_{24} = (\mathrm{A}_6 + \mathrm{A}_{10} - \mathrm{k_s}\mathrm{A}_8)\lambda_m,\\
K_{33} = \xi^2 K_2\lambda_m^4 + (k_s A_7 + K_2 + \xi^2 K_1)\lambda_m^2 + A_{12} + K_1,\\
K_{34} = \xi^2\frac{h}{2}K_2\lambda_m^4 + \left(k_s A_8 + \frac{h}{2}K_2 + \xi^2\frac{h}{2}K_1\right)\lambda_m^2 + A_3 + A_{13} + \frac{h}{2}K_1,\\
K_{44} = \left(\frac{h}{2}\right)^2\xi^2 K_2\lambda_m^4 + \left[k_s A_{11} + \left(\frac{h}{2}\right)^2(K_2 + \xi^2 K_1)\right]\lambda_m^2 + \mathrm{A}_1 + 2\mathrm{A}_4 + A_{14} + \left(\frac{h}{2}\right)^2 K_2,\\
M_{11} = B_1 + \xi^2 B_1\lambda_m^2,\; M_{12} = B_2 + \xi^2 B_2\lambda_m^2,\\
M_{22} = B_3 + \xi^2 B_3\lambda_m^2,\\
M_{33} = -B_1 - \xi^2 B_1\lambda_m^2,\; M_{34} = -B_2 - \xi^2 B_2\lambda_m^2\\
M_{44} = -B_3 - \xi^2 B_3\lambda_m^2,
\end{gathered}
\tag{23}
$$

The natural frequencies of cylindrical FGM nanoshell are derived using solution of characteristic equation as follows

$$\mathrm{Det}[[K] + \omega^2[M]] = 0, \tag{24}$$

It is noted that the characteristic equation for a simply-supported cylidnrical nanoshell is derived directly by substitution of solution from first row of Table 1 into Eq. (20) and then into Eq. (18). The solution procedure for other boundary conditions such as clamped-clamped and simply-clamped is obtained by integration of Eq. (18) on the length of cylinder after substitution of solution from Eq. (20). More details on the solution procedure are presented in previous paper by Arefi *et al.* (2016).

4. Results and discussion

The numerical results are presented in this section in terms of important parameters of the formulated problem. Fig. 2 shows variation of fundamental natural frequencies of simply-supported functionally graded cylindrical nanoshell in terms of nonlocal parameter ξ for various in-homogeneous index n. The obtained results indicate that with increase of nonlocal parameter, the fundamental frequencies are decreased. It is concluded that the stiffness of nanoshell is decreased with increase of nonlocal parameter. In addition, one can conclude that with increase of in-homogeneous index, the stiffness of nanoshell is decreased and consequently the fundamental natural frequencies are decreased significantly.

Fig. 3 presents distribution of fundamental natural frequencies of simply-supported cylindrical nanoshell in terms of nonlocal parameters ξ for various Winkler parameter of foundation K_1. One can conclude that with increase of Winkler parameter of foundation K_1, the stiffness of foundation is increased and consequently the fundamental natural frequencies are increased significantly. Fig. 4 shows distribution of fundamental natural frequencies of simply-supported cylindrical nanoshell in terms of nonlocal parameters ξ for various Pasternak parameter of foundation K_2. One can conclude that with increase of this parameter K_2, the stiffness of foundation is increased and then the fundamental natural frequencies are increased significantly.

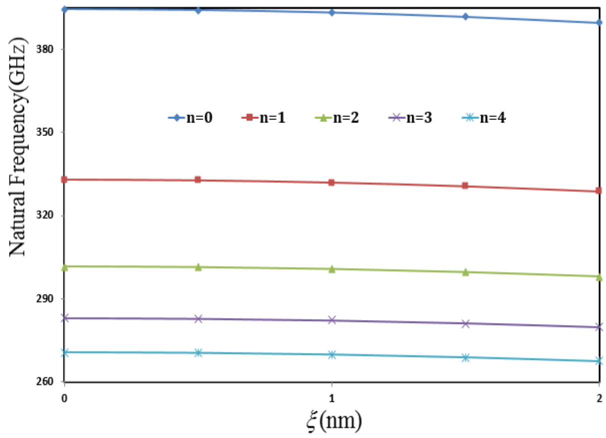

Fig. 2 The variation of fundamental natural frequencies of simply-supported functionally graded cylindrical nanoshell in terms of nonlocal parameters ξ for various in-homogeneous index n

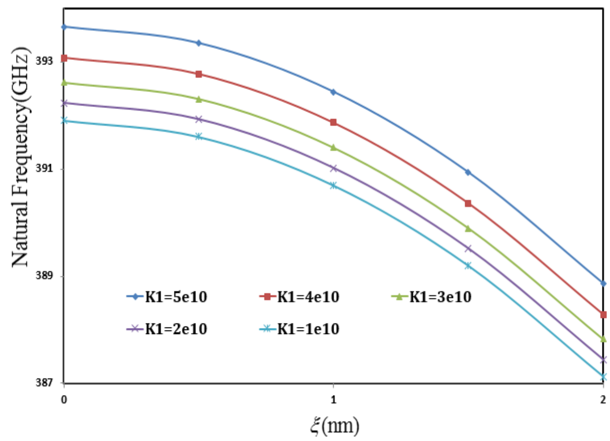

Fig. 3 The variation of fundamental natural frequencies of simply-supported functionally graded cylindrical nanoshell in terms of nonlocal parameters ξ for various Winkler's parameter K_1

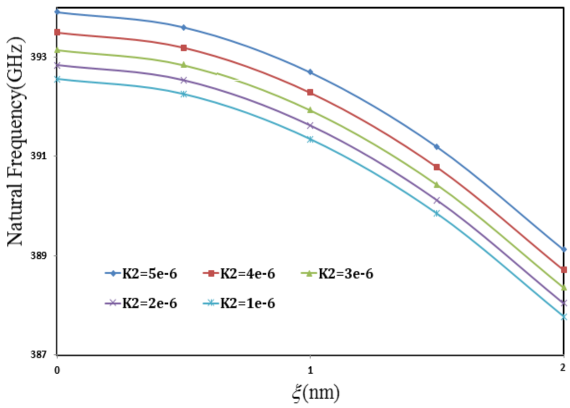

Fig. 4 The variation of fundamental natural frequencies of simply-supported functionally graded cylindrical nanoshell in terms of nonlocal parameters ξ for various Pasternak's parameter K_2

Figs. 5 and 6 present variation of natural frequencies of simply-supported functionally graded cylindrical nanoshell with various modes. Figure 5 shows variation of first three natural frequencies of functionally graded cylindrical nanoshell in terms of in-homogeneous index n. Fig. 6 shows variation of first three natural frequencies of functionally graded cylindrical nanoshell in terms of nonlocal parameter ξ. The results indicate that the frequencies of second and third mode are two and three times of fundamental frequency, respectively.

In continuation of the paper, the influence of dimensionless geometric parameters are studied on the free vibration characteristics of simply-supported functionally graded cylinderical nanoshell. Fig. 7 shows variation of fundamental natural frequencies of nanoshell in terms of small scale parameter for various dimensionless parameter of L/R. It is concluded that with increase of dimensionless parameter of L/R, the natural freqencies are decreased significantly. It can be concluded that with increase of dimensionless parameter of L/R, the stiffness of structure

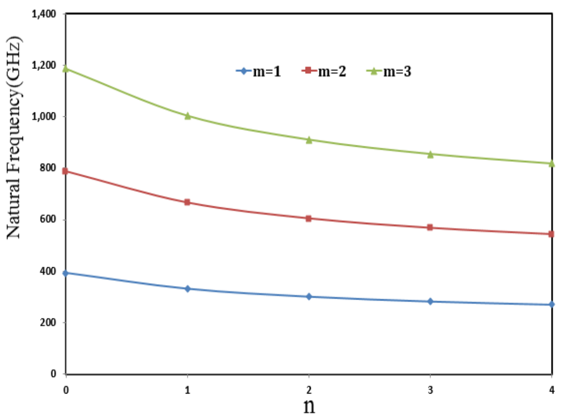

Fig. 5 The variation of natural frequencies of simply-supported functionally graded cylindrical nanoshell in terms of in-homogeneous index n for various modes m

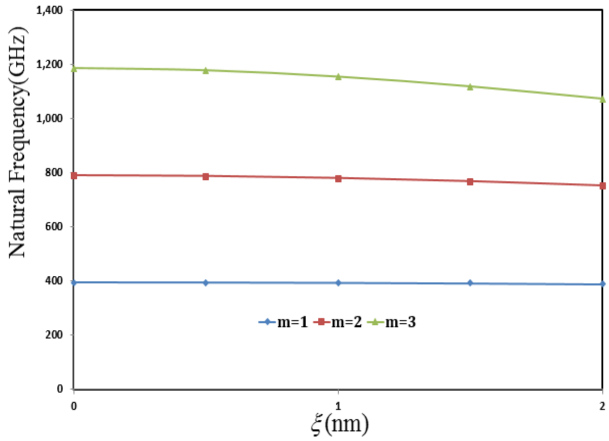

Fig. 6 The variation of natural frequencies of simply-supported functionally graded cylindrical nanoshell in terms of nonlocal parameter ξ for various modes m

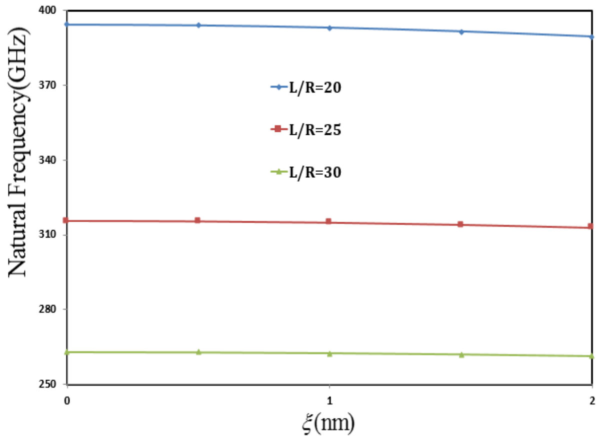

Fig. 7 Variation of fundamental natural frequencies of simply-supported nanoshell in terms of nonlocal parameter ξ for various dimensionless parameter of L/R

is decreased and consequently the natural frequencies are decreased significantly.

Figs. 8 and 9 present variation of fundamental natural frequencies in terms of small scale parameter for various dimensionless parameters of L/h and R/h, respectively. It is concluded that with increase of dimensionless parameter of L/h, the natural freqencies are decreased significantly.

One can conclude that with increase of dimensionless parameter of L/h, the stiffness of structure is decreased and consequently the natural frequencies are decreased significantly. Investigation on the influence of R/h indicates that with increase of this parameter, the stiffness of structure is increased and the natural frequencies are increased significantly.

To investigate the influence of various boundary conditions on the natural frequencies of cylindrical nanoshell, the information presented by Eq. (18) and Table 1 are used to show numerical results for simply-simply (SS), simply-clamped (CS) and clamped-clamped (CC) boundary conditions. Figs. 10 and 11 show variation of natural

frequencies of FG nanoshell for various boundary conditions in terms of nonlocal parameter and in-homogeneous index, respectively. It is concluded that the natural frequencies for CC boundary condition is 25% more than CS boundary condition and the natural frequencies for CS is 60% more than SS boundary condition. One can conclude that changing one simply-supported boundary condition leads to significant increase of natural frequency than the case the second boundary condition is changed.
Shown in Fig. 12 is variation of natural frequencies of functionally graded nanoshell in terms of various mode numbers for various boundary conditions. It is concluded that the natural frequencies are significantly increased with increase of mode number m. The numerical results indicates that natural frequencies of second and third mode shapes are approximately two and three times of fundamental fundamental natural frequency.

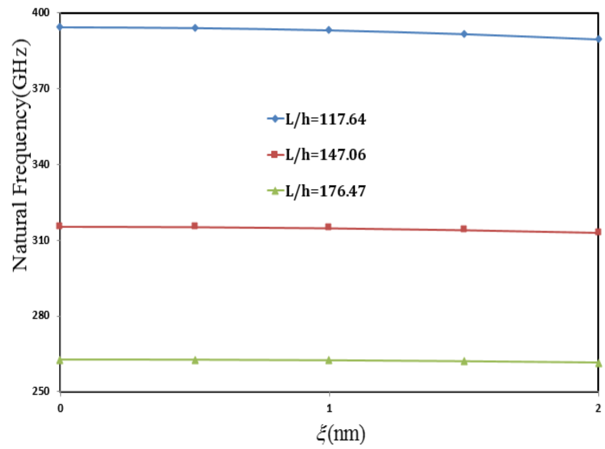

Fig. 8 Variation of fundamental natural frequencies of simply-supported nanoshell in terms of nonlocal parameter ξ for various dimensionless parameter of L/h

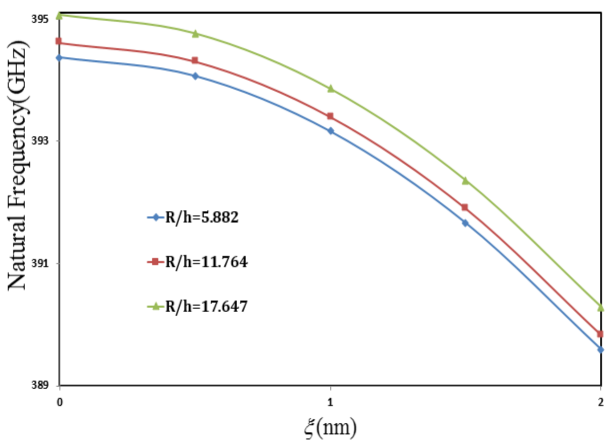

Fig. 9 Variation of fundamental natural frequencies of simply-supported nanoshell in terms of nonlocal parameter ξ for various dimensionless parameter of R/h

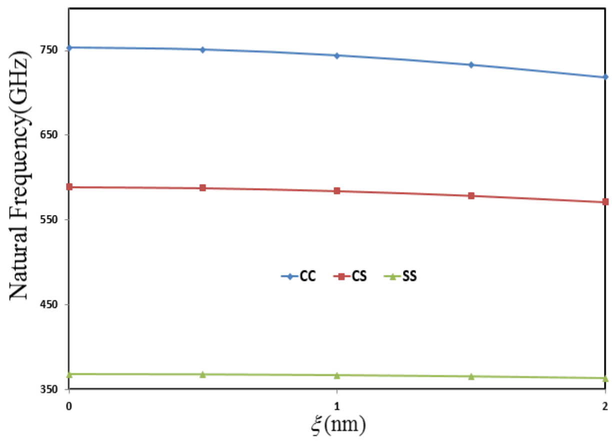

Fig. 10 Variation of fundamental natural frequencies of nanoshell in terms of nonlocal parameter ξ for various boundary conditions

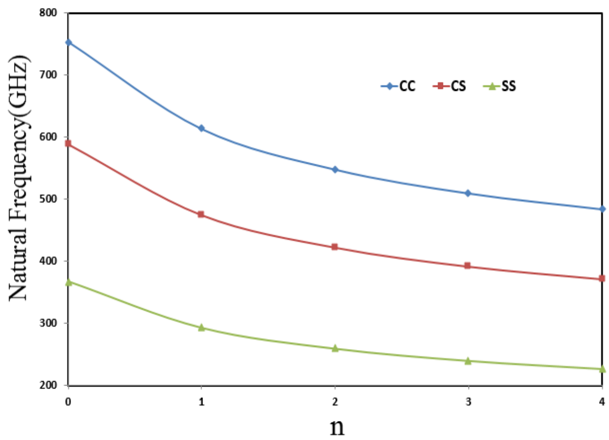

Fig. 11 Variation of fundamental natural frequencies of nanoshell in terms of in-homogeneous index n for various boundary conditions

5. Conclusions

Free vibration analysis of FG cylindrical nanoshell was studied in this work. A two-dimensional analysis was carried out based on FSDT and nonlocal elasticity theory. The cylindrical nanoshell was assumed made from functionally graded materials. Hamilton's principle was used to derive governing equations of motion in terms of four displacements and rotation components. An analytical approach was carried out to present natural frequencies of nanoshell in terms of important parameters of the problem such as nonlocal parameter, in-homogeneous index, mode number and some geometric dimensionless parameters such as ratio of length to radius and thickness on the natural frequencies of nanoshell. In addition the analytical approach used in this paper has capability to predict natural frequencies of FGM nanoshells with various boundary conditions. The main numerical results of this study are presented as follows:

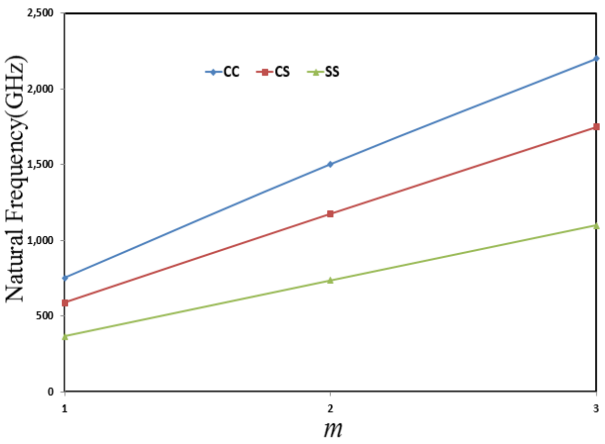

Fig. 12 Variation of fundamental natural frequencies of nanoshell in terms of mode number m for various boundary conditions

- The influence of small scale parameters of nanomaterials associated with nonlocal elasticity theory has significant influence on the natural frequencies of nanoshells. One can conclude that with increase of nonlocal parameter, the natural frequencies are decreased because of decrease of stiffness of nanoshell.
- The influence of in-homogeneous index was studied on the natural frequencies of nanoshell. Our numerical results indicate with increase of in-homogenous index, the stiffness of structure is decreased and consequently the natural frequencies are decreased significantly.
- The Pasternak's foundation can change the structural behavior of cylindrical nanoshell. With increase of two parameters of Pasternak's foundation, the stiffness of structure is increased and natural frequencies are increased importantly.
- The dimensionless geometric parameters of cylidrical nanoshell ($L/R, L/h, R/h$) have significant influences on the free vibration characteristics of nanoshell. The numerical results indicate that with increase of $(L/R, L/h)$ the stiffness of cylinder is decreased and consequently the natural frequencies are decreased significantly. In addition, with increase of R/h the stiffness of cylinder is increased and then the natural frequencies are increased.

Acknowledgments

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