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# Free vibration of AFG beams with elastic end restraints

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**Abstract.** Axially functionally graded (AFG) beams are a new class of composite structures that have continuous variations in material and/or geometrical parameters along the axial direction. In this study, the exact analytical solutions for the free vibration of AFG and uniform beams with general elastic supports are obtained by using Euler–Bernoulli beam theory. The elastic supports are modeled with linear rotational and lateral translational springs. Moreover, the material and/or geometrical properties of the AFG beams are assumed to vary continuously and together along the length of the beam according to the power-law forms. Accordingly, the accuracy, efficiency and capability of the proposed formulations are demonstrated by comparing the responses of the numerical examples with the available solutions. In the following, the effects of the elastic end restraints and AFG parameters, namely, gradient index and gradient coefficient, on the values of the first three natural frequencies of the AFG and uniform beams are investigated comprehensively. The analytical solutions are presented in tabular and graphical forms and can be used as the benchmark solutions. Furthermore, the results presented herein can be utilized for design of inhomogeneous beams with various supporting conditions.

**Keywords:** axially functionally graded beams; elastic supports; natural frequencies; free vibration; exact analysis; Euler-Bernoulli beam theory

## 1. Introduction

Axially functionally graded (AFG) beams, in which the variations of the geometrical and/or material parameters are along the length, constitute a subset of inhomogeneous structures which are made of functionally graded materials (FGM). FGM’s are advanced composites which due to thermal resistance and high stiffness are widely utilized in many aerospace, mechanical and civil engineering structures. In the two last decades, the free vibration of AFG and functionally graded (FG) beams with classical and non-classical end restraints has been investigated extensively, and is still receiving attention in literatures. Accordingly, many researchers have studied the vibration characteristics of AFG and FG beams by using the analytical or semi-analytical approaches. For instance, Elishakoff and Guede (2004) derived the analytical polynomial solutions for the vibrating axially graded (AG) beams with simply supported using the semi-inverse method. Also, the text book about the closed-form solutions in the vibration and buckling of inhomogeneous beams, columns and plates using the direct, semi-inverse and inverse methods was published by Elishakoff (2004). Furthermore, based on the semi-inverse method, Caliò and Elishakoff (2005) presented the closed-form solutions for the natural frequencies of AG beam-columns on elastic foundation. A new low-order analytical model for the free vibration analysis of non-uniform composite beams was developed by Singh *et al*. (2006). Aydogdu and Taskin (2007) investigated the free vibration of simply supported FG beam using the Navier-type solution method. Analytical solutions for the vibration of a non-uniform beam for three different types of classical boundary conditions associated with simply supported, clamped and free ends were obtained by Ece *et al*. (2007). Aydogdu (2008) analyzed the vibration and buckling of AFG simply supported beams utilizing the semi-inverse method. By using the Adomian modified decomposition method, the free vibration of non-uniform beams with general elastic end constraints was studied by Hsu et al. (2008). In the same way, Lai *et al*. (2008) investigated the free vibration of elastically end restrained non-uniform beams with tip mass and resting on an elastic foundation and subjected to an axial load. An analytical method for the free vibration analysis of FG beams was proposed by Sina *et al*. (2009). Huang and Li (2010) presented a new approach for the free vibration of AFG beams with non-uniform cross-section using the Fredholm integral equations. Analytical solutions for the free vibration of the sigmoid FGM beams with three different types of classical boundary conditions associated with simply supported, clamped and free ends were suggested by Atmane *et al*. (2011). Hein and Feklistova (2011) investigated the free vibrations of non-uniform and AFG beams with various boundary conditions using the Haar wavelets. Based on the modified wave approach, the natural frequencies and mode shapes of the arbitrary non-uniform beams were obtained by Nikkhah Bahrami *et al*. (2011). The dynamic behavior of an AFG beam with simply supported edges under action of a moving harmonic load using the Newmark method was investigated by Şimşek *et al*. (2012). Firouz-Abadi *et al*. (2013) obtained the exact solutions for the free vibrations and buckling of double tapered columns with elastic

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foundation and tip mass using the Frobenius power series. New exact vibration solutions for a class of non-uniform beams utilizing the power function solutions were presented byWang and Wang (2013a). Moreover, one of the text book about the exact solutions for the free vibration of strings, membranes, beams, and plates was published by Wang and Wang (2013b). Guo and Yang (2014) obtained an exact solution for the free and steady state forced vibrations of arbitrary non-uniform beams using the series solution. The exact frequency equations of the free vibration of exponentially non-uniform FG Timoshenko beams for various boundary conditions were derived by Tang *et al*. (2014). Sarkar and Ganguli (2014) presented the closed-form solutions for the free vibration of AFG Timoshenko beams having uniform cross-section and fixed-fixed boundary condition using the polynomial solutions. Based on the energy method, Kumar *et al*. (2015) studied the large amplitude free vibration problem of AFG slender non-uniform beams with various taper profiles and material gradation. Galeban *et al*. (2016) obtained the natural frequencies of FG thin beams made of saturated porous materials using the principle of virtual work. Hashemi *et al*. (2016) presented a general analytical solution for the free vibration analysis of a non-uniform FGM beam. By using the piecewise exponential functions and power series method, Kukla and Rychlewska (2016) investigated the free vibration of AFG beams with different boundary conditions. Based on the energy method, the natural frequencies and mode shapes of exponential tapered AFG beams on elastic foundation were obtained by Lohar *et al*. (2016a). Moreover, by the similarly method, Lohar *et al*. (2016b) studied the nonlinear free vibration of AFG non-uniform beams supported on elastic foundation. An analytical method for the free vibration of double-beam systems made up of AFG beams with elastically restrained were presented by Rezaiee-Pajand and Hozhabrossadati (2016). Yuan *et al*. (2016) derived the exact solutions for the free vibrations of axially inhomogeneous Timoshenko beams with variable cross-section using the Bessel and Hypergeometric functions. In addition, they proposed an approximate analytical method to calculate the low-order natural frequencies of Timoshenko beams accurately and efficiently. The free vibrations of AFG cantilever beams with concentrated masses attached at different points using the Ritz method were investigated by Rossit *et al*. (2017). Based on the polynomial expansion and integral technique, Huang and Rong (2017) presented a simple approach for the free vibration of axially inhomogeneous beams that are made of FGM. The free vibration analysis of FG beams using an exact transfer matrix expression was performed by Lee and Lee (2017). Zhao *et al*. (2017) analyzed the free vibration of AFG Euler-Bernoulli and Timoshenko beams with non-uniform cross-sections utilizing the Chebyshev polynomials theory. Moreover, Ghayesh (2018d) obtained the nonlinear vibration characteristics of AFG shear deformable tapered beams subjected to external harmonic excitations utilizing the third-order shear deformation beam theory. Rezaiee-Pajand and Masoodi (2018) proposed the exact second-order stiffness matrix for a FGM beam with lateral bracing. Accordingly, they calculated the exact natural frequencies and buckling load of FGM tapered beam-columns with general connections using the proposed formulations. Avcar (2019) investigated the free vibration of imperfect sigmoid and power law FG beams utilizing the first order shear deformation beam theory.

Alternatively, some researchers have used the numerical methods to analyze the free vibration of FG and AFG beams. For example, Alshorbagy *et al*. (2011) studied the free vibration characteristics of a FG beam with material graduation in axially or transversally through the thickness based on the power-law utilizing the finite element method. Shahba *et al*. (2011a) utilized the finite element method, for the free vibration and stability of AFG tapered beams. Also, the free vibration and stability analysis of AFG tapered Timoshenko beams with classical and non-classical boundary conditions were studied through the finite element approach by Shahba *et al*. (2011b). Shahba and Rajasekaran (2012) analyzed the free vibration and stability of tapered beams made of axially functionally graded materials (AFGM) based on the differential transform element method. Based on the dynamic stiffness approach and using the differential transformation method, the buckling and vibration of AFG non-uniform beams were analyzed by Rajasekaran (2013). By using the differential transform method, Ebrahimi and Dashti (2015) investigated the free vibration characteristics of a rotating double tapered FG beam. A numerical method for the free vibration of double-beam systems made up of AFG beams with elastically restrained were presented by Rezaiee-Pajand and Hozhabrossadati (2016). The free vibrations analysis of non-uniform and/or AFG beams with elastically restrained ends using the method of initial parameters in differential form were performed by Shvartsman and Majak (2016). By using the dynamic stiffness method, the free vibration of FG beams and frameworks was investigated by Banerjee and Ananthapuvirajah (2018). Cao *et al*. (2018) studied the free vibration of AFG beams with different boundary conditions using the asymptotic development method. The free vibration analysis of FG beams with non-uniform cross-section using the differential transform method was studied by Ghazaryan *et al*. (2018). Based on the symbolic-numeric method of initial parameters, the free vibration analysis of AFG tapered, stepped, and continuously segmented rods and beams with elastically restrained was investigated by Šalinić *et al*. (2018). The free vibration of tapered bidirectional FGM beams using an efficient shear deformable finite element model was performed by Nguyen and Tran (2018).

In recent years, the application of uniform and FG members in microscale and nanoscale structures has attracted the attention of researchers. For instance, the dynamic linear and nonlinear behavior of microbeams and microplates have been investigated by many researchers such as Farokhi *et al*. (2013a, b), Ghayesh *et al*. (2013a, b, c, d, 2014), Farokhi and Ghayesh (2015a, b), Ghayesh and Farokhi (2015a, b), Gholipour *et al*. (2015), Farokhi *et al*. (2016), Ghayesh *et al*. (2016), Farokhi and Ghayesh (2018a, b). Moreover, based on the modified couple stress theory and utilizing the Rayleigh–Ritz method, Akgöz and Civalek (2013) studied the free vibration analysis of AFG tapered cantilever microbeams. The vibration of AFGM nanobeams with elastic foundation and simply supported utilizing the strain gradient theory was analyzed by Zeighampour and Tadi Beni (2015). By utilizing the homotopy perturbation method in conjunction with the generalized differential quadrature method, Shafiei *et al*. (2016) analyzed the nonlinear vibration of AFG non-uniform nanobeams. The free longitudinal vibrations of FG nanorods with varying cross-section and elastic supports via a newly developed nonlocal surface energy-based integro-differential model were examined by Kiani (2016). Based on the modified couple stress theory and utilizing the Galerkin method, Farokhi *et al*. (2017) investigated the motion characteristics of bilayered extensible Timoshenko microbeams. The oscillations of FG microbeams were analyzed by Ghayesh *et al*. (2017a). By means of the backward differentiation formula, Ghayesh *et al*. (2017b) performed the vibration analysis of geometrically imperfect three-layered shear-deformable microbeams. Ghayesh (2018a) studied the dynamic behavior of FG viscoelastic microbeams. The forced nonlinear dynamics of AFG microbeams based on a shear-deformable model and the modified couple stress theory were investigated by Ghayesh (2018b). Ghayesh (2018c) analyzed the coupled nonlinear mechanical behaviour of extensible FG microbeams, when both viscoelasticity and imperfection are present. The size-dependent nonlinear oscillation characteristics of a FG microplate were investigated numerically by Ghayesh *et al*. (2018). Rahmani *et al*. (2018) studied the free vibration of deep curved FG nanobeam based on the modified couple stress theory. By utilizing the similar theory, the nonlinear free and forced vibration analysis of microbeams with different boundary conditions were analyzed by Ghorbanpour Arani and Kiani (2018). Farajpour *et al*. (2018) performed a review on the mechanics of nano-structures. Also, Ghayesh and Farajpour (2019) presented a review on the mechanics of FG nanoscale and microscale structures.

It should be noted that the tapered and/or non-prismatic beams can be considered as a special case of the AFG beams with constant material and variable geometry. So far, the free vibration of tapered and/or non-prismatic beams with classical and non-classical end restraints also has been investigated extensively by many researchers such as Conway and Dubil (1965), Mabie and Rogers (1968), Goel (1976), Downs (1977, 1978), Sato (1980), Banerjee and Williams (1985), Rao and Mirza (1989), Grossi and Bhat (1991), Lee and Kuo (1992), Lee and Lint (1992), Cortinez and Laura (1994), Naguleswaran (1994), Abrate (1995), Auciello (1995), De Rosa and Auciello (1996), Auciello and Ercolano (1997), Ho and Chen (1998), Li (2000), Auciello (2001), Kim and Kim (2001), Lee *et al*. (2002), Grossi and Albarracín (2003), Lee *et al*. (2003), Attarnejad *et al*. (2006, 2011), Taha and Essam (2013), Abdelghany *et al*. (2015), Boiangiu *et al*. (2016), and Palacio-Betancur and Aristizabal-Ochoa (2019).

Based on this brief review, it is obvious that relatively few literatures have presented an exact solution for the free vibration of the AFG Euler-Bernoulli beams with general elastic supports. Moreover, no comprehensive attempt has been made yet for the evaluation of the effects of the AFG parameters and elastic restrained supports in the free vibration analysis. Accordingly, the objective of this paper is to derive the exact expression for obtaining the exact natural frequencies of the AFG Euler-Bernoulli beams with elastic supports. In other words, for the first time, the exact solutions for the vibration characteristics of the uniform and AFG beams with elastic end restraints derived and compared. It should be noted that the material and/or geometrical properties of the AFG beam are assumed to change continuously and together along the longitudinal direction according to the power-law forms. In the following, the effects of the AFG parameters and elastic end restraints in the free vibration of the AFG and uniform beams will be investigated comprehensively. Comparing the responses of the numerical examples with the available solutions demonstrates the accuracy, efficiency and capability of the proposed formulations. Furthermore, the analytical solutions are presented in tabular and graphical forms and can be utilized as the benchmark solutions or design of inhomogeneous beams with various supporting conditions.

## 2. Free vibration formulation

In this study, the analytical solutions to obtain the exact natural frequencies of the AFG Euler-Bernoulli beam with general boundary conditions are presented.

### *2.1 AFG material and geometrical properties*

In the present work, the material and/or geometrical properties, i.e., mass per unit length and flexural rigidity of the AFG beam, shown in Fig. 1, are assumed to vary continuously and together in the axial direction according to the power-law forms and defined as

$$G(x) = \rho(x)A(x) = \rho_0 A_0 \left(1 + c\frac{x}{L}\right)^n = \rho_L A_L \left(\frac{1 + c\frac{x}{L}}{1 + c}\right)^n \tag{1a}$$

$$D(x) = E(x)I(x) = E_0 I_0 \left(1 + c\frac{x}{L}\right)^{n+2} = E_L I_L \left(\frac{1 + c\frac{x}{L}}{1 + c}\right)^{n+2} \tag{1b}$$

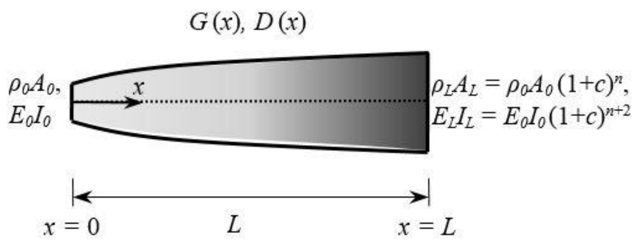
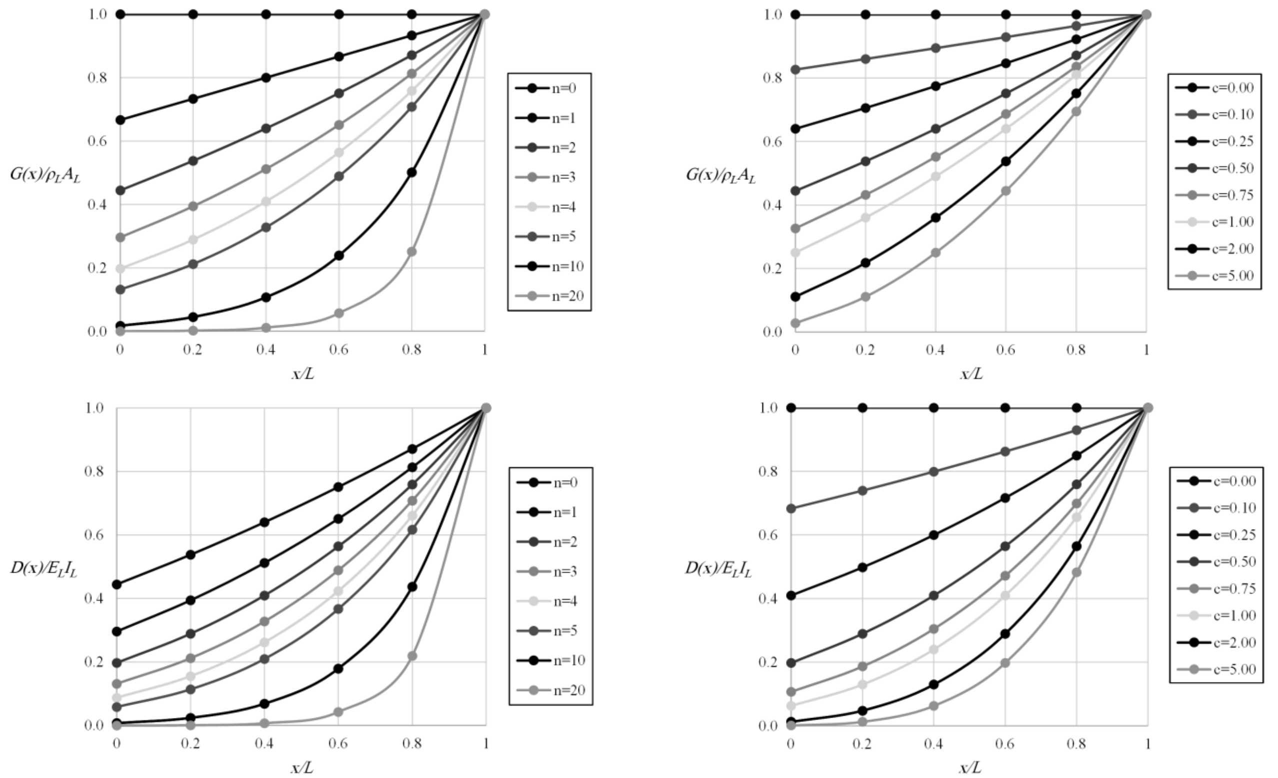



Fig. 1 Schematic of the axially functionally graded beam




(a) for various values of gradient index ($n$) in which $c$ = 0.5 (b) for various values of gradient coefficient ($c$) in which $n$ = 2

Fig. 2 Variations of mass per unit length $G(x)$ and flexural rigidity $D(x)$ of the AFG beam defined by Eq. (1)

where $x$ is the axial coordinate, $L$ is the length of the beam, $G(x) = \rho(x)A(x)$ is the mass of the beam per unit length which is depends upon the AFG variations of the mass density of the beam material $\rho(x)$, and/or cross-section area of the beam $A(x)$, and $D(x) = E(x)I(x)$ is the flexural rigidity of the beam which depends upon the AFG variations of the modulus of elasticity of the beam material $E(x)$, aznd/or moment of inertia of the beam cross-section $I(x)$. Also, $\rho_0$, $A_0$, $E_0$, and $I_o$ are the mass density, cross-section area, modulus of elasticity and moment of inertia at $x$ = 0, respectively. Similarly, $\rho_L$, $A_L$, $E_L$, and $I_L$ are the mass density, cross-section area, modulus of elasticity and moment of inertia at $x = L$, respectively. Moreover, $n$ and $c$ are the AFG parameters that $n$ is a nonnegative quantity and represents the gradient index and $c$ represents the gradient coefficient. It should be noted that the mass per unit length and flexural rigidity of the AFG beam are positive values and therefor $c > -1$. In addition, it is evident that when $c$ = 0.0, i.e., the material and geometrical properties are kept constant, the beam is uniform.

It is reminded that changing the mass per unit length $G(x)$ and flexural rigidity $D(x)$ can be expressed based on the variations of the material properties or geometrical properties or both of them. Accordingly, the tapered and/or non-prismatic beams can be considered as the special case of the AFG beams with constant material and variable geometry. For better understanding, the variations of $G(x)$ and $D(x)$ of the AFG beam, which are defined by Eq. (1), are plotted in Fig. 2 for various values of the gradient index ($n$) and gradient coefficient ($c$).

### *2.2 Governing differential equation*

The free vibration differential equation of an AFG Euler-Bernoulli beam of length $L$ with general elastic end restraints, as shown in Fig. 3, is given by (Huang and Li 2010, Singh *et al*. 2006)

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right] + G(x)\frac{\partial^2 w(x,t)}{\partial t^2} = 0, \quad 0 < x < L \tag{2}$$

where $x$ is the axial coordinate, $t$ is the time, $w(x,t)$ is the lateral displacement, $D(x)$ is the flexural rigidity of the beam and $G(x)$ is the mass per unit length.

Following the separation of variable analogy, the solution of Eq. (2) can be expressed as

$$w_i(x,t) = W_i(x)e^{j\omega_i t} \qquad (j^2 = -1) \tag{3}$$

where $\omega_i$ is the circular frequency and $W_i(x)$ is the shape function of the lateral motion of the $i$th vibration mode.

Substituting the Eq. (3) into Eq. (2), one can get (Huang and Li 2010)

$$\frac{d^2}{dx^2}\left[D(x)\frac{d^2 W_i(x)}{dx^2}\right] - G(x)\omega_i^2 W_i(x) = 0 \tag{4}$$

If Eqs. (1a) and (1b) are inserted into Eq. (4), it can be rewritten as

$$
\begin{aligned}
&\left(1+c\frac{x}{L}\right)^{n+2}\frac{d^4W_i(x)}{dx^4}\\
&+2\frac{c}{L}(n+2)\left(1+c\frac{x}{L}\right)^{n+1}\frac{d^3W_i(x)}{dx^3}\\
&+\frac{c^2}{L^2}(n+1)(n+2)\left(1+c\frac{x}{L}\right)^{n}\frac{d^2W_i(x)}{dx^2}\\
&-\frac{\rho_0A_0\omega_i^2}{E_0I_0}\left(1+c\frac{x}{L}\right)^{n}W_i(x)=0
\end{aligned}
\tag{5}
$$

Introducing the following quantity

$$X=\left(1+c\frac{x}{L}\right) \tag{6}$$

which is equal to 1 at $x$ = 0 and to 1 + $c$ at $x$ = $L$, and considering
in mind that

$$dx=\left(\frac{L}{c}\right)dX \tag{7}$$

Eq. (5) simplifies as follows (Attarnejad *et al*. 2006, Banerjee and Williams 1985)

$$
\begin{aligned}
&X^2\frac{d^4W_i(X)}{dX^4}+2(n+2)X\frac{d^3W_i(X)}{dX^3}\\
&+(n+1)(n+2)\frac{d^2W_i(X)}{dX^2}-\mu_i^4W_i(X)=0
\end{aligned}
\tag{8}
$$

where $\mu_i=L\sqrt[4]{\frac{\rho_0A_0\omega_i{}^2}{E_0I_0}}$ is the dimensionless natural frequency coefficient.

The general solution of this equation is (Attarnejad *et al*. 2006, Auciello and Ercolano 1997, Banerjee and Williams 1985)

$$
\begin{aligned}
W_i(X)=X^{-\frac{n}{2}}\Bigg[&C_1J_n\left(\frac{2\mu_i\sqrt{X}}{c}\right)+C_2Y_n\left(\frac{2\mu_i\sqrt{X}}{c}\right)\\
&+C_3I_n\left(\frac{2\mu_i\sqrt{X}}{c}\right)+C_4K_n\left(\frac{2\mu_i\sqrt{X}}{c}\right)\Bigg]
\end{aligned}
\tag{9}
$$

where $C_1$, $C_2$, $C_3$, $C_4$ are unknown constants and $J_n$, $Y_n$, $I_n$, $K_n$ are the $n$th-order Bessel functions.

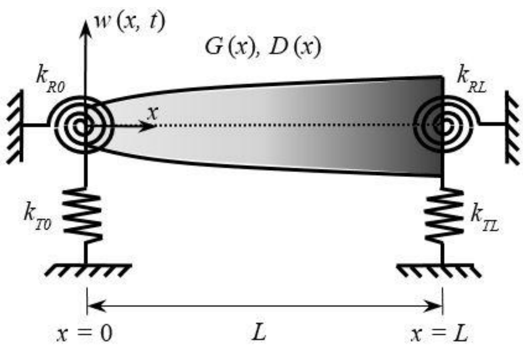



Fig. 3 Schematic of the AFG beam with elastic supports

### 2.3 Boundary conditions

The boundary conditions, in the presence of constraints with the rotational elastic stiffnesses $k_{R0}$, $k_{RL}$ and lateral translational elastic stiffnesses $k_{T0}$, $k_{TL}$ are expressed as (Hsu *et al*. 2008)

$$D(x)\frac{d^2W(x)}{dx^2}-k_{R0}\frac{dW(x)}{dx}=0 \tag{10}$$

$$\frac{d}{dx}\left[D(x)\frac{d^2W(x)}{dx^2}\right]+k_{T0}W(x)=0 \tag{11}$$

at $x$ = 0, and

$$D(x)\frac{d^2W(x)}{dx^2}+k_{RL}\frac{dW(x)}{dx}=0 \tag{12}$$

$$\frac{d}{dx}\left[D(x)\frac{d^2W(x)}{dx^2}\right]-k_{TL}W(x)=0 \tag{13}$$

at $x$ = $L$.

Substituting Eq. (1b) and utilizing Eqs. (6)-(7), the boundary conditions become

$$X^{n+2}\frac{d^2W_i(X)}{dX^2}-\frac{k_{R0}L}{E_0I_0c}\frac{dW_i(X)}{dX}=0 \tag{14}$$

$$
\begin{aligned}
&X^{n+2}\frac{d^3W_i(X)}{dX^3}+(n+2)X^{n+1}\frac{d^2W_i(X)}{dX^2}\\
&+\frac{k_{T0}L^3}{E_0I_0c^3}W_i(X)=0
\end{aligned}
\tag{15}
$$

at $X$ = 1 ($x$ = 0), and

$$X^{n+2}\frac{d^2W_i(X)}{dX^2}+\frac{k_{RL}L}{E_0I_0c}\frac{dW_i(X)}{dX}=0 \tag{16}$$

$$
\begin{aligned}
&X^{n+2}\frac{d^3W_i(X)}{dX^3}+(n+2)X^{n+1}\frac{d^2W_i(X)}{dX^2}\\
&-\frac{k_{TL}L^3}{E_0I_0c^3}W_i(X)=0
\end{aligned}
\tag{17}
$$

at $X$ = 1 + $c$ ($x$ = $L$).

It is convenient to define the following dimensionless stiffness coefficients (De Rosa and Auciello 1996)

$$
\begin{aligned}
R_0&=\frac{k_{R0}L}{E_0I_0}, & R_L&=\frac{k_{RL}L}{E_LI_L},\\
T_0&=\frac{k_{T0}L^3}{E_0I_0}, & T_L&=\frac{k_{TL}L^3}{E_LI_L}
\end{aligned}
\tag{18}
$$

the boundary conditions of Eqs. (14)-(17) can be expressed by the following non-dimensional forms

$$W_i''(1)-\frac{R_0}{c}W_i'(1)=0 \tag{19}$$

$$W_i'''(1)+(n+2)W_i''(1)+\frac{T_0}{c^3}W_i(1)=0 \tag{20}$$

$$W_i''(1+c)+\frac{R_L}{c}W_i'(1+c)=0 \tag{21}$$

$$W_i'''(1+c) + \frac{(n+2)}{1+c} W_i''(1+c) - \frac{T_L}{c^3} W_i(1+c) = 0 \tag{22}$$

where $W_i'(X) = \frac{dW_i(X)}{dX}$, $W_i''(X) = \frac{d^2W_i(X)}{d^2X}$, $W_i'''(X) = \frac{d^3W_i(X)}{d^3X}$.

### 2.4 Determination of the natural frequency

By substituting the general solution (9) into the non-dimensional boundary conditions given in Eqs. (19)-(21), a homogeneous system of four equations, for the four integration constant, can be expressed as

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{23}$$

or in compact matrix form as follows

$$\mathbf{FC} = 0 \tag{24}$$

where the constant coefficients matrix **F** for the AFG beams and/or uniform beam ($c$ = 0.0) are given explicitly in the Appendix A. In order to have a non-trivial solution, the determinant of this system must be zero

$$det\,\mathbf{F} = 0 \tag{25}$$

Table 1 First square three dimensionless natural frequency coefficients $\mu_i^2$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = var.) with different boundary conditions in Example 1

| Boundary conditions | $c$ | $\mu_i^2$ | Present | Ghazaryan *et al.* (2018) | Shafiei *et al.* (2016) | Kukla and Rychlewska (2016) | Huang and Li (2010) | Abrate (1995) | Cortinez and Laura (1994) |
|---|---|---|---|---|---|---|---|---|---|
| C-P ($T_0 = R_0 = \infty$, $T_L = \infty$, $R_L = 0$) | -0.1 | $i$ = 1 | 14.848896 | 14.848896 | 14.848890 | 14.844562 | 14.848896 | 14.848896 | 14.85 |
| | | $i$ = 2 | 47.637037 | 47.637037 | 47.636961 | 47.647237 | 47.637037 | 47.637037 | - |
| | | $i$ = 3 | 99.171635 | 99.171635 | - | 99.206918 | 99.171653 | 99.171635 | - |
| | 0.0 | $i$ = 1 | 15.418206 | 15.418206 | 15.418198 | 15.418206 | 15.418206 | 15.418206 | 15.41 |
| | | $i$ = 2 | 49.964862 | 49.964862 | 49.964773 | 49.964862 | 49.964862 | 49.964862 | - |
| | | $i$ = 3 | 104.247696 | 104.247696 | - | 104.247696 | 104.247702 | 104.24770 | - |
| | 0.1 | $i$ = 1 | 15.968710 | 15.9687099 | 15.968701 | 15.950015 | 15.9687099 | 15.9687099 | 15.96 |
| | | $i$ = 2 | 52.237227 | 52.237227 | 52.237123 | 52.198883 | 52.237227 | 52.237227 | − |
| | | $i$ = 3 | 109.202353 | 109.20235 | − | 109.134912 | 109.202354 | 109.20235 | − |
| | 0.2 | $i$ = 1 | 16.502899 | 16.502899 | 16.502889 | 16.445277 | 16.502899 | 16.502899 | 16.50 |
| | | $i$ = 2 | 54.461463 | 54.4614625 | 54.461344 | 54.360368 | 54.4614625 | 54.4614625 | − |
| | | $i$ = 3 | 114.051623 | 114.051623 | − | 113.888586 | 114.051631 | 114.051623 | − |
| | 1.0 | $i$ = 1 | 20.366601 | 20.3666 | − | − | − | − | − |
| | | $i$ = 2 | 71.047974 | 71.04797 | − | − | − | − | − |
| | | $i$ = 3 | 150.200858 | 150.20086 | − | − | − | − | − |
| | 2.0 | $i$ = 1 | 24.582600 | 24.5826 | − | − | − | − | − |
| | | $i$ = 2 | 89.983683 | 89.98368 | − | − | − | − | − |
| | | $i$ = 3 | 191.448135 | 191.44814 | − | − | − | − | − |
| C-C ($T_0 = R_0 = \infty$, $T_L = R_L = \infty$) | -0.1 | $i$ = 1 | 21.240978 | 21.240978 | 21.240968 | 21.242905 | 21.240978 | − | − |
| | | $i$ = 2 | 58.550054 | 58.550055 | 58.549953 | 58.567526 | 58.550055 | − | − |
| | | $i$ = 3 | 114.780242 | 114.780242 | − | 114.824704 | 114.780278 | − | − |
| | 0.0 | $i$ = 1 | 22.373285 | 22.373285 | 22.373274 | 22.373285 | 22.373285 | 22.3732854 | 22.375 |
| | | $i$ = 2 | 61.672823 | 61.672823 | 61.672704 | 61.672823 | 61.672823 | 61.672823 | − |
| | | $i$ = 3 | 120.903392 | 120.903392 | − | 120.903392 | 120.903400 | 120.903392 | − |
| | 0.1 | $i$ = 1 | 23.479607 | 23.479607 | 23.479594 | 23.460013 | 23.479607 | 23.479607 | 23.61 |
| | | $i$ = 2 | 64.721068 | 64.721068 | 64.720931 | 64.678046 | 64.721068 | 64.721068 | − |
| | | $i$ = 3 | 126.878017 | 126.87802 | − | 126.802905 | 126.878051 | 126.87804 | − |
| | 0.2 | $i$ = 1 | 24.563418 | 24.563418 | 24.563402 | 24.508817 | 24.563418 | 24.563418 | 25.13 |
| | | $i$ = 2 | 67.704755 | 67.704755 | 67.704599 | 67.596273 | 67.704755 | 67.704755 | − |
| | | $i$ = 3 | 132.723977 | 132.72398 | − | 132.546612 | 132.724068 | 132.72398 | − |

Table 2 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 1, $c$ = 0.5) for $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$ in Example 2

| $R_0$ | $R_L$ | $\mu_i$ | Present | Hsu *et al.* (2008) | De Rosa and Auciello (1996) | Auciello (1995) | Goel (1976) |
|---|---|---|---|---|---|---|---|
| | | $i$ = 1 | 3.488810 | 3.48881 | − | 3.4888 | 3.488 |
| 0 | 0 | $i$ = 2 | 6.997232 | 6.99720 | − | 6.9972 | 6.997 |
| | | $i$ = 3 | 10.491234 | 10.49113 | − | 10.4011 | − |
| | | $i$ = 1 | 3.491358 | 3.49136 | 3.49136 | 3.4913 | 3.491 |
| 0 | 0.01 | $i$ = 2 | 6.998324 | 6.99829 | 6.99832 | 6.9983 | 6.998 |
| | | $i$ = 3 | 10.491935 | 10.49183 | 10.49194 | 10.4918 | − |
| | | $i$ = 1 | 3.513690 | 3.51369 | 3.51369 | 3.5136 | 3.513 |
| 0 | 0.1 | $i$ = 2 | 7.008049 | 7.00802 | 7.00805 | 6.9808 | 7.008 |
| | | $i$ = 3 | 10.498203 | 10.49810 | 10.49820 | 10.4981 | − |
| | | $i$ = 1 | 3.690754 | 3.69075 | 3.69075 | 3.6907 | 3.691 |
| 0 | 1 | $i$ = 2 | 7.096086 | 7.09605 | 7.09609 | 7.0894 | 7.096 |
| | | $i$ = 3 | 10.557026 | 10.55692 | 10.55703 | 10.5569 | − |
| | | $i$ = 1 | 4.202763 | 4.20276 | 4.20276 | 4.2027 | 4.203 |
| 0 | 10 | $i$ = 2 | 7.514394 | 7.51435 | 7.51439 | 7.5143 | 7.514 |
| | | $i$ = 3 | 10.902774 | 10.90264 | 10.90277 | 10.9020 | − |
| | | $i$ = 1 | 3.591244 | 3.59124 | 3.59124 | 3.5912 | 3.633 |
| 1 | 0 | $i$ = 2 | 7.061073 | 7.06104 | 7.06107 | 7.0610 | 7.090 |
| | | $i$ = 3 | 10.537028 | 10.53692 | 10.53703 | 10.5369 | − |
| | | $i$ = 1 | 3.615164 | 3.61516 | 3.61516 | 3.5936 | 3.656 |
| 1 | 0.1 | $i$ = 2 | 7.071702 | 7.07167 | 7.07170 | 7.0621 | 7.100 |
| | | $i$ = 3 | 10.543939 | 10.54383 | 10.54394 | 10.5377 | − |
| | | $i$ = 1 | 3.786550 | 3.78655 | 3.78654 | 3.7865 | 3.826 |
| 1 | 1 | $i$ = 2 | 7.158314 | 7.15828 | 7.15831 | 7.1583 | 7.186 |
| | | $i$ = 3 | 10.602291 | 10.60218 | 10.60229 | 10.6022 | − |

Now, having the values of $n$, $c$, $T_0$, $T_L$, $R_0$ and $R_L$, positive real roots of this equation are the natural frequency coefficients $\mu_i$ of the AFG and/or uniform beams with the elastic end restraints, shown in Fig. 2. It should be added, these were calculated numerically.

## 3. Numerical results and discussion

In order to illustrate the accuracy, effectiveness and application of the presented formulations, four numerical examples are analyzed in this part. The results are compared with those obtained by other researchers. It should be noted, by using the proposed formulations, one can found the exact natural frequencies of the uniform and AFG beams with classical and non-classical boundary conditions at both ends. Accordingly, if the dimensionless stiffness coefficients are allowed to become infinity or zero, then the classical restraints can be easily recovered. For example, if $R_0 = T_0 = \infty$ and $R_L = T_L = 0$, then the beam is considered as the cantilevered beam. If $R_0 = R_L = 0$ and $T_0 = T_L = \infty$, then the frequency equation of the simply supported beam is obtained. If $R_0 = R_L = \infty$ and $T_0 = T_L = \infty$, then the beam is considered as the clamped-clamped beam.

In the following, several cases with the classical and non-classical boundary conditions will be considered and the effects of the elastic supports and AFG parameters on the first three natural frequencies of them will be studied.

### 3.1 Verification

Example 1. In this example, the first square three dimensionless natural frequency coefficients $\mu_i^2$ ($i$ = 1, 2, 3) are obtained for the clamped-pinned and clamped-clamped AFG beams ($n$ = 2, $c$ = var.). Comparison of the results with the other approaches are listed in Table 1. Based on the Table 1, it is observed that the proposed formulation for calculating the natural frequencies has a high accuracy and efficiency.

Example 2. In this case, the numerical values of $\mu_i$ ($i$ = 1, 2, 3) for the AFG beam ($n$ = 1, $c$ = 0.5) with $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$ are computed and arranged in Table 2. Table 2 shows the results of present study, as well as those of other methods. According to the findings, the predictions of the proposed technique agree well with those of other approaches.

Example 3. In this example, the first three dimensionless natural frequency coefficients of the AFG beam ($n$ = 2,

Table 3 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = 1) for $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$ in Example 3

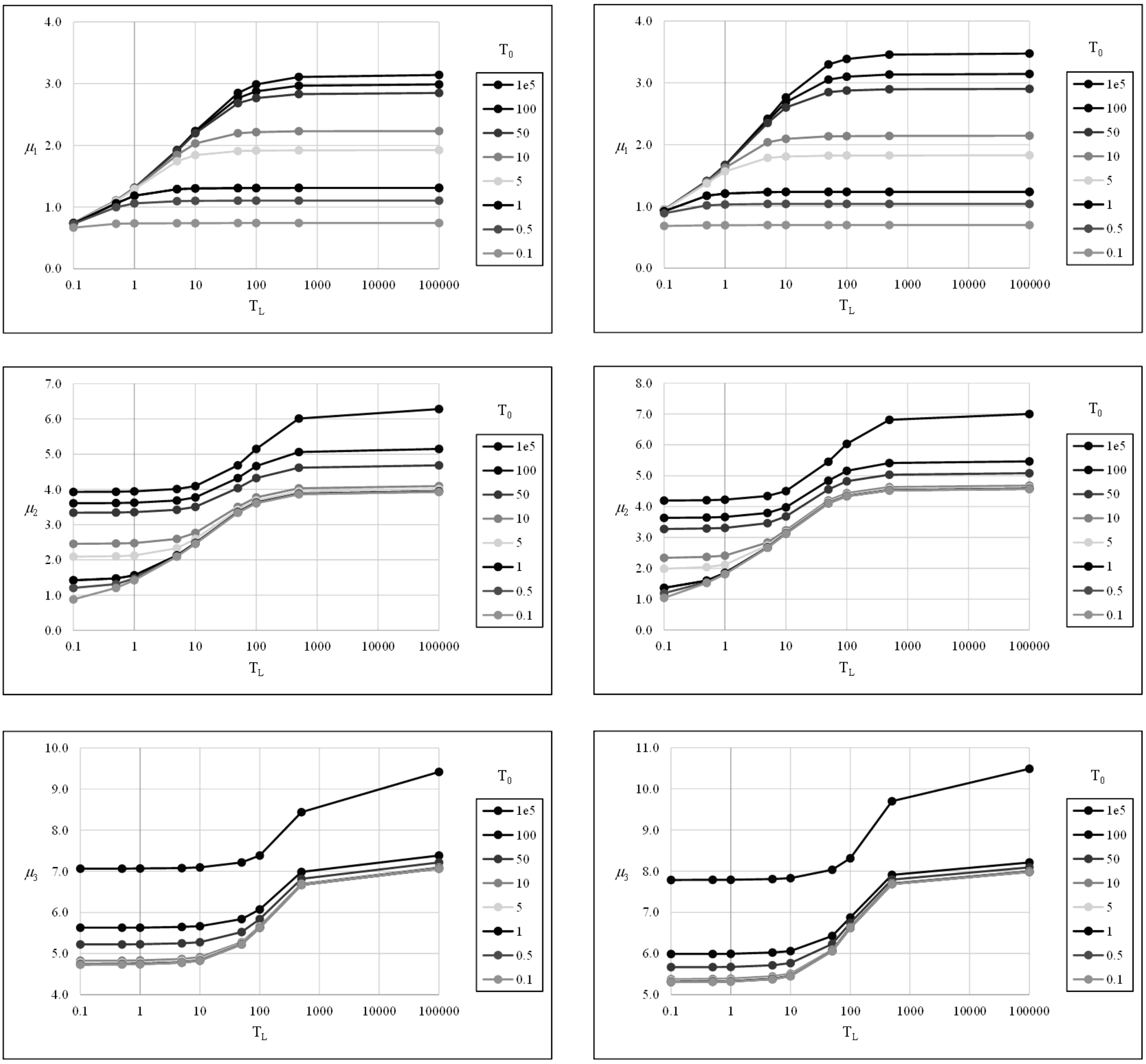
| $R_0$ | $R_L$ | $\mu_i$ | Present | Attarnejad *et al.* (2011) | Hsu *et al.* (2008) | De Rosa and Auciello (1996) | Auciello (1995) | Grossi and Bhat (1991) |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | $i$ = 1 | 3.730038 | 3.7300 | 3.73003 | 3.7300 | 3.73002 | 3.7300 |
| | | $i$ = 2 | 7.630248 | 7.6302 | 7.63020 | 7.6302 | 7.63025 | 7.4750 |
| | | $i$ = 3 | 11.421711 | 11.4217 | 11.42157 | 11.4217 | 11.42171 | 11.4201 |
| 0 | 0.01 | $i$ = 1 | 3.734549 | 3.7345 | 3.73454 | 3.7345 | − | 3.7345 |
| | | $i$ = 2 | 7.631715 | 7.6317 | 7.63167 | 7.6317 | − | 7.4696 |
| | | $i$ = 3 | 11.422607 | 11.4226 | 11.42247 | 11.4226 | − | 11.4219 |
| 0 | 0.1 | $i$ = 1 | 3.773716 | 3.7737 | 3.77371 | 3.7737 | 3.77372 | 3.8643 |
| | | $i$ = 2 | 7.644772 | 7.6448 | 7.64473 | 7.6447 | 7.64477 | 7.3921 |
| | | $i$ = 3 | 11.430612 | 11.4306 | 11.43047 | 11.4306 | 11.43061 | 11.4306 |
| 0 | 1 | $i$ = 1 | 4.063575 | 4.0636 | 4.06357 | 4.0635 | − | 4.0635 |
| | | $i$ = 2 | 7.761934 | 7.7619 | 7.76189 | 7.7619 | − | 7.7619 |
| | | $i$ = 3 | 11.505381 | 11.5054 | 11.50523 | 11.5054 | − | 11.5038 |
| 0 | 10 | $i$ = 1 | 4.754892 | 4.7549 | 4.75488 | 4.7549 | 4.75489 | 4.7625 |
| | | $i$ = 2 | 8.284662 | 8.2847 | 8.28460 | 8.2846 | 8.28466 | 8.2846 |
| | | $i$ = 3 | 11.927749 | 11.9277 | 11.92757 | 11.9277 | 11.92775 | 11.9277 |
| 1 | 0 | $i$ = 1 | 3.798407 | 3.7984 | 3.79840 | 3.7984 | − | 3.7984 |
| | | $i$ = 2 | 7.680343 | 7.6803 | 7.68029 | 7.6803 | − | 7.6803 |
| | | $i$ = 3 | 11.460457 | 11.4605 | 11.46031 | 11.4604 | − | 11.4597 |
| 1 | 0.1 | $i$ = 1 | 3.840923 | 3.8409 | 3.84091 | 3.8409 | 3.84092 | 3.8409 |
| | | $i$ = 2 | 7.694692 | 7.6947 | 7.69464 | 7.6946 | 7.69469 | 7.6946 |
| | | $i$ = 3 | 11.469301 | 11.4693 | 11.46915 | 11.4693 | 11.46930 | 11.4694 |
| 1 | 1 | $i$ = 1 | 4.124910 | 4.1249 | 4.12490 | 4.1249 | 4.12491 | 4.1249 |
| | | $i$ = 2 | 7.810550 | 7.8105 | 7.81050 | 7.8105 | 7.81055 | 7.8105 |
| | | $i$ = 3 | 11.543621 | 11.5436 | 11.54347 | 11.5436 | 11.543621 | 11.5435 |

Table 4 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = 0.4) for $R_0 = R_L = 0$ and various values of $T_0$ and $T_L$ in Example 4

| $T_0$ | $T_L$ | $\mu_i$ | Present | Hein and Feklistova (2011) | Hsu *et al.* (2008) | De Rosa and Auciello (1996) |
|---|---|---|---|---|---|---|
| 0 | 0 | $i$ = 1 | 0.000000 | − | 0.00000 | − |
| | | $i$ = 2 | 0.000000 | − | 0.00000 | − |
| | | $i$ = 3 | 5.191757 | − | 5.19176 | − |
| 0.001 | 0.001 | $i$ = 1 | 0.216565 | 0.2166 | 0.21656 | − |
| | | $i$ = 2 | 0.318023 | 0.3180 | 0.31795 | − |
| | | $i$ = 3 | 5.191778 | 5.1918 | 5.191778 | − |
| 0.01 | 0.01 | $i$ = 1 | 0.385095 | 0.3851 | 0.38510 | − |
| | | $i$ = 2 | 0.565390 | 0.5654 | 0.56539 | − |
| | | $i$ = 3 | 5.191962 | 5.1920 | 5.19196 | − |
| 0.1 | 0.1 | $i$ = 1 | 0.684617 | 0.6846 | 0.68462 | − |
| | | $i$ = 2 | 1.005276 | 1.0053 | 1.00528 | − |
| | | $i$ = 3 | 5.193807 | 5.1939 | 5.19381 | − |
| 1 | 1 | $i$ = 1 | 1.214042 | 1.2140 | 1.21404 | 1.2140 |
| | | $i$ = 2 | 1.785087 | 1.7851 | 1.78509 | 1.7851 |
| | | $i$ = 3 | 5.212235 | 5.2123 | 5.21223 | 5.2122 |

Table 4 Continued

| $T_0$ | $T_L$ | $\mu_i$ | Present | Hein and Feklistova (2011) | Hsu *et al.* (2008) | De Rosa and Auciello (1996) |
|---|---|---|---|---|---|---|
| 10 | 10 | $i$ = 1 | 2.100958 | 2.1009 | 2.10096 | 2.1010 |
| | | $i$ = 2 | 3.130230 | 3.1303 | 3.13023 | 3.1302 |
| | | $i$ = 3 | 5.393758 | 5.3938 | 5.39376 | 5.3938 |
| 100 | 100 | $i$ = 1 | 3.072413 | 3.0723 | 3.07241 | 3.0724 |
| | | $i$ = 2 | 5.066695 | 5.0668 | 5.06670 | 5.0667 |
| | | $i$ = 3 | 6.711520 | 6.7116 | 6.71152 | 6.7115 |
| 1000 | 1000 | $i$ = 1 | 3.375525 | 3.3754 | 3.37553 | 3.3755 |
| | | $i$ = 2 | 6.569634 | 6.5697 | 6.56963 | 6.5696 |
| | | $i$ = 3 | 9.288757 | 9.2888 | 9.28876 | 9.2888 |




(a) Uniform beam (b) AFG beam ($n$ = 2, $c$ = 0.5)

Fig. 4 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 for $R_0 = R_L = 0$ and various values of $T_0$ and $T_L$

Table 5 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform beam for $R_0 = R_L = 0$ and various values of $T_0$ and $T_L$

| $T_0$ | $\mu_i$ | $T_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 0.6685 | 0.7293 | 0.7348 | 0.7388 | 0.7393 | 0.7396 | 0.7397 | 0.7397 | 0.7397 |
| | $i$ = 2 | 0.8801 | 1.2050 | 1.4206 | 2.0927 | 2.4570 | 3.3401 | 3.6077 | 3.8635 | 3.9279 |
| | $i$ = 3 | 4.7319 | 4.7357 | 4.7405 | 4.7789 | 4.8280 | 5.2232 | 5.6291 | 6.6708 | 7.0671 |
| 0.5 | $i$ = 1 | 0.7293 | 0.9979 | 1.0585 | 1.0967 | 1.1005 | 1.1033 | 1.1037 | 1.1040 | 1.1040 |
| | $i$ = 2 | 1.2050 | 1.3157 | 1.4733 | 2.1062 | 2.4660 | 3.3463 | 3.6141 | 3.8701 | 3.9346 |
| | $i$ = 3 | 4.7357 | 4.7395 | 4.7442 | 4.7825 | 4.8313 | 5.2253 | 5.6305 | 6.6719 | 7.0682 |
| 1 | $i$ = 1 | 0.7348 | 1.0585 | 1.1843 | 1.2912 | 1.3010 | 1.3081 | 1.3090 | 1.3096 | 1.3098 |
| | $i$ = 2 | 1.4206 | 1.4733 | 1.5642 | 2.1250 | 2.4780 | 3.3543 | 3.6222 | 3.8784 | 3.9429 |
| | $i$ = 3 | 4.7405 | 4.7442 | 4.7489 | 4.7869 | 4.8355 | 5.2279 | 5.6323 | 6.6733 | 7.0697 |
| 5 | $i$ = 1 | 0.7388 | 1.0967 | 1.2912 | 1.7424 | 1.8418 | 1.9080 | 1.9149 | 1.9202 | 1.9215 |
| | $i$ = 2 | 2.0927 | 2.1062 | 2.1250 | 2.3334 | 2.5952 | 3.4201 | 3.6881 | 3.9456 | 4.0101 |
| | $i$ = 3 | 4.7789 | 4.7825 | 4.7869 | 4.8232 | 4.8696 | 5.2490 | 5.6468 | 6.6844 | 7.0811 |
| 10 | $i$ = 1 | 0.7393 | 1.1005 | 1.3010 | 1.8418 | 2.0323 | 2.1981 | 2.2153 | 2.2282 | 2.2313 |
| | $i$ = 2 | 2.4570 | 2.4660 | 2.4780 | 2.5952 | 2.7666 | 3.5057 | 3.7723 | 4.0308 | 4.0951 |
| | $i$ = 3 | 4.8280 | 4.8313 | 4.8355 | 4.8696 | 4.9134 | 5.2764 | 5.6656 | 6.6985 | 7.0956 |
| 50 | $i$ = 1 | 0.7396 | 1.1033 | 1.3081 | 1.9080 | 2.1981 | 2.6844 | 2.7677 | 2.8331 | 2.8488 |
| | $i$ = 2 | 3.3401 | 3.3463 | 3.3543 | 3.4201 | 3.5057 | 4.0392 | 4.3219 | 4.6160 | 4.6845 |
| | $i$ = 3 | 5.2232 | 5.2253 | 5.2279 | 5.2490 | 5.2764 | 5.5242 | 5.8375 | 6.8201 | 7.2188 |
| 100 | $i$ = 1 | 0.7397 | 1.1037 | 1.3090 | 1.9149 | 2.2153 | 2.7677 | 2.8768 | 2.9665 | 2.9885 |
| | $i$ = 2 | 3.6077 | 3.6141 | 3.6222 | 3.6881 | 3.7723 | 4.3219 | 4.6638 | 5.0596 | 5.1478 |
| | $i$ = 3 | 5.6291 | 5.6305 | 5.6323 | 5.6468 | 5.6656 | 5.8375 | 6.0762 | 6.9861 | 7.3855 |
| 500 | $i$ = 1 | 0.7397 | 1.1040 | 1.3096 | 1.9202 | 2.2282 | 2.8331 | 2.9665 | 3.0815 | 3.1104 |
| | $i$ = 2 | 3.8635 | 3.8701 | 3.8784 | 3.9456 | 4.0308 | 4.6160 | 5.0596 | 5.8053 | 6.0114 |
| | $i$ = 3 | 6.6708 | 6.6719 | 6.6733 | 6.6844 | 6.6985 | 6.8201 | 6.9861 | 7.8900 | 8.4376 |
| $10^5$ | $i$ = 1 | 0.7397 | 1.1040 | 1.3098 | 1.9215 | 2.2313 | 2.8488 | 2.9885 | 3.1104 | 3.1413 |
| | $i$ = 2 | 3.9279 | 3.9346 | 3.9429 | 4.0101 | 4.0951 | 4.6845 | 5.1478 | 6.0114 | 6.2807 |
| | $i$ = 3 | 7.0671 | 7.0682 | 7.0697 | 7.0811 | 7.0956 | 7.2188 | 7.3855 | 8.4376 | 9.4164 |

$c$ = 1) for $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$ are calculated and presented in Table 3. From Table 3, it is observed that the results of the presented method are very close to the values obtained by other techniques.

Example 4. In this case, the first three dimensionless natural frequency coefficients $\mu_i$ ($i$ = 1, 2, 3) are obtained for the AFG beam ($n$ = 2, $c$ = 0.4) with $R_0 = R_L = 0$ and various values of $T_0$ and $T_L$. Comparison of the responses with those computed by other available approaches are arranged in Table 4. According to the results, the proposed method gives a high-accuracy prediction.

### 3.2 Effects of elastic supports

In this section, the influences of the elastic supports on the natural frequencies of the uniform and AFG beams are investigated comprehensively. It is reminded that the stiffness of elastic supports are modeled with linear rotational and translational springs. In addition, for instance the behavior of the AFG beam with $n$ = 2 and $c$ = 0.5 is studied. Moreover, It should be noticed that four cases of the elastic end conditions, namely, $R_0 = R_L = 0$ and $T_0$, $T_L$ = var.; $T_0 = T_L = \infty$ and $R_0$, $R_L$ = var.; $T_0 = R_0 = \infty$ and $T_L$, $R_L$ = var.; and also $T_L = R_L = 0$ and $T_0$, $R_0$ = var. are considered. Here, var. denotes variable.

In Figs. 4 through 7, variations of the first three dimensionless natural frequency coefficients $\mu_i$ ($i$ = 1, 2, 3) of the uniform beam and AFG beam ($n$ = 2, $c$ = 0.5) with respect to various quantities of the rotational and translational stiffness coefficients (i.e., $T_0$, $R_0$, $T_L$ and $R_L$) from 0.1 (corresponding to low stiffness) up to $10^5$ (corresponding to high stiffness) in the foregoing cases are plotted. Moreover, the corresponding numerical values of the first three dimensionless natural frequency coefficients of the uniform beam are presented with different elastic supports in Tables 5, 7, 9 and 11. Similarly, Tables 6, 8, 10 and 12 indicate the corresponding numerical quantities of $\mu_i$ ($i$ = 1, 2, 3) for the AFG beam ($n$ = 2, $c$ = 0.5) with various elastic supports.

According to Fig. 4(a) and Table 5, for the free supported

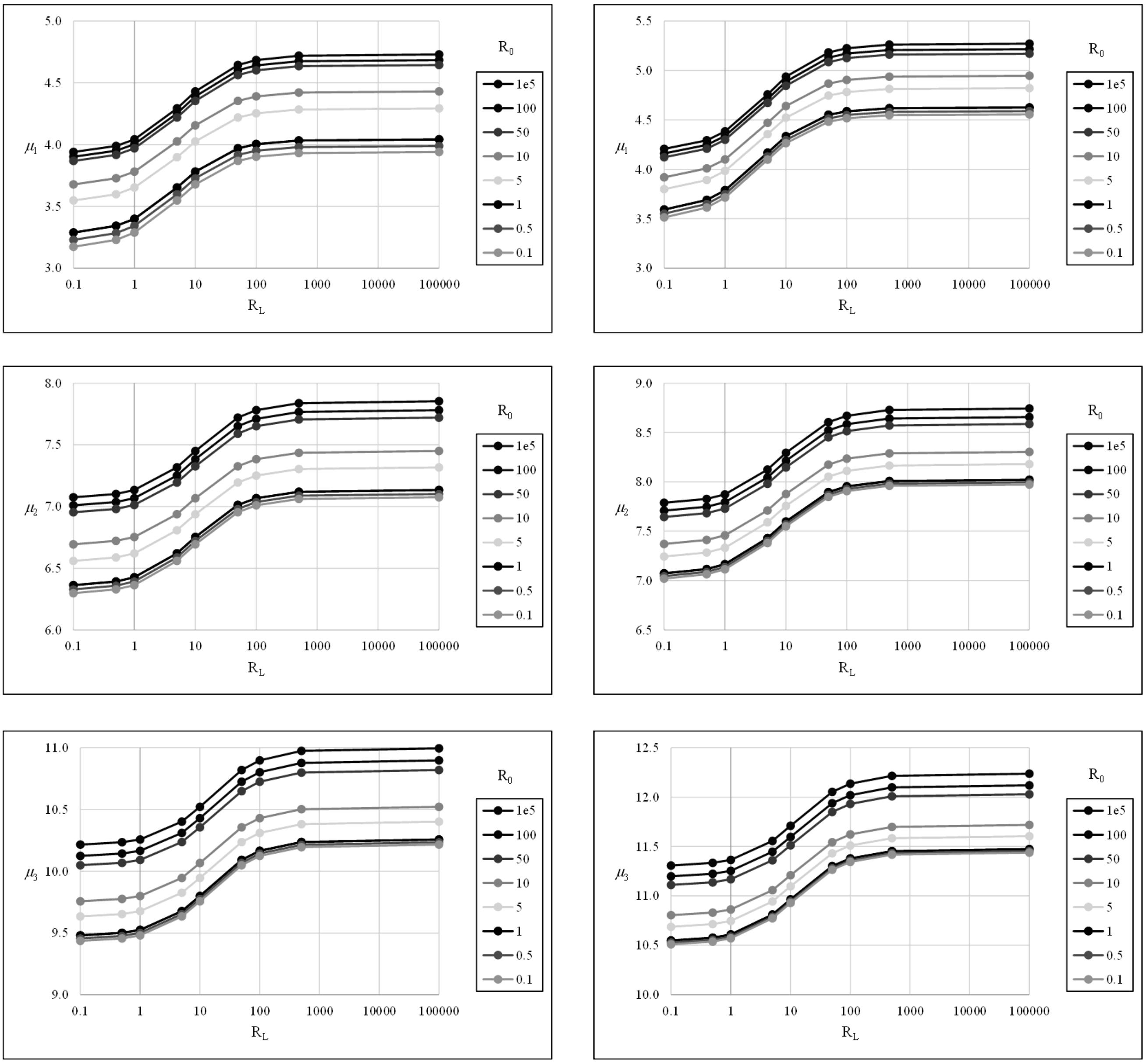
Table 6 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = 0.5) for $R_0 = R_L = 0$ and various values of $T_0$ and $T_L$

| $T_0$ | $\mu_i$ | $T_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 0.6808 | 0.6936 | 0.6949 | 0.6960 | 0.6961 | 0.6962 | 0.6963 | 0.6963 | 0.6963 |
| | $i$ = 2 | 1.0411 | 1.5262 | 1.8086 | 2.6667 | 3.1195 | 4.1032 | 4.3344 | 4.5227 | 4.5669 |
| | $i$ = 3 | 5.3058 | 5.3112 | 5.3180 | 5.3743 | 5.4482 | 6.0596 | 6.6220 | 7.6871 | 7.9824 |
| 0.5 | $i$ = 1 | 0.8871 | 1.0168 | 1.0293 | 1.0380 | 1.0390 | 1.0398 | 1.0399 | 1.0400 | 1.0400 |
| | $i$ = 2 | 1.1939 | 1.5556 | 1.8247 | 2.6721 | 3.1237 | 4.1071 | 4.3387 | 4.5271 | 4.5713 |
| | $i$ = 3 | 5.3087 | 5.3141 | 5.3209 | 5.3770 | 5.4507 | 6.0609 | 6.6229 | 7.6879 | 7.9833 |
| 1 | $i$ = 1 | 0.9199 | 1.1721 | 1.2075 | 1.2302 | 1.2327 | 1.2346 | 1.2348 | 1.2350 | 1.2351 |
| | $i$ = 2 | 1.3680 | 1.6036 | 1.8482 | 2.6792 | 3.1289 | 4.1121 | 4.3439 | 4.5326 | 4.5769 |
| | $i$ = 3 | 5.3123 | 5.3177 | 5.3245 | 5.3803 | 5.4538 | 6.0625 | 6.6240 | 7.6890 | 7.9844 |
| 5 | $i$ = 1 | 0.9405 | 1.3710 | 1.5660 | 1.7857 | 1.8075 | 1.8228 | 1.8246 | 1.8260 | 1.8264 |
| | $i$ = 2 | 1.9877 | 2.0366 | 2.1173 | 2.7433 | 3.1730 | 4.1517 | 4.3862 | 4.5766 | 4.6211 |
| | $i$ = 3 | 5.3417 | 5.3469 | 5.3535 | 5.4075 | 5.4788 | 6.0754 | 6.6329 | 7.6973 | 7.9930 |
| 10 | $i$ = 1 | 0.9427 | 1.3914 | 1.6237 | 2.0365 | 2.0948 | 2.1338 | 2.1381 | 2.1415 | 2.1423 |
| | $i$ = 2 | 2.3389 | 2.3670 | 2.4087 | 2.8387 | 3.2328 | 4.2011 | 4.4388 | 4.6314 | 4.6761 |
| | $i$ = 3 | 5.3785 | 5.3835 | 5.3898 | 5.4418 | 5.5103 | 6.0919 | 6.6442 | 7.7079 | 8.0038 |
| 50 | $i$ = 1 | 0.9444 | 1.4054 | 1.6608 | 2.3496 | 2.6020 | 2.8485 | 2.8766 | 2.8981 | 2.9032 |
| | $i$ = 2 | 3.2760 | 3.2892 | 3.3065 | 3.4649 | 3.6790 | 4.5520 | 4.8187 | 5.0328 | 5.0803 |
| | $i$ = 3 | 5.6681 | 5.6717 | 5.6763 | 5.7144 | 5.7652 | 6.2342 | 6.7410 | 7.7952 | 8.0933 |
| 100 | $i$ = 1 | 0.9446 | 1.4070 | 1.6648 | 2.3857 | 2.6870 | 3.0521 | 3.0997 | 3.1362 | 3.1450 |
| | $i$ = 2 | 3.6339 | 3.6460 | 3.6615 | 3.7957 | 3.9737 | 4.8377 | 5.1547 | 5.4123 | 5.4665 |
| | $i$ = 3 | 5.9879 | 5.9907 | 5.9941 | 6.0224 | 6.0601 | 6.4254 | 6.8731 | 7.9099 | 8.2103 |
| 500 | $i$ = 1 | 0.9448 | 1.4082 | 1.6679 | 2.4119 | 2.7501 | 3.2480 | 3.3257 | 3.3871 | 3.4021 |
| | $i$ = 2 | 4.0690 | 4.0807 | 4.0954 | 4.2191 | 4.3793 | 5.3150 | 5.8278 | 6.3924 | 6.5124 |
| | $i$ = 3 | 7.1186 | 7.1204 | 7.1225 | 7.1401 | 7.1629 | 7.3717 | 7.6641 | 8.7123 | 9.0705 |
| $10^5$ | $i$ = 1 | 0.9448 | 1.4085 | 1.6686 | 2.4180 | 2.7647 | 3.2992 | 3.3870 | 3.4571 | 3.4743 |
| | $i$ = 2 | 4.1972 | 4.2087 | 4.2233 | 4.3446 | 4.5009 | 5.4513 | 6.0326 | 6.8118 | 7.0006 |
| | $i$ = 3 | 7.7888 | 7.7905 | 7.7927 | 7.8104 | 7.8330 | 8.0334 | 8.3124 | 9.7003 | 10.4877 |

beam with two translational springs (corresponding to $R_0 = R_L = 0$ and $T_0$, $T_L$ = var.), increase of $T_0$ and $T_L$ from the low translational stiffnesses ($T_0 = T_L = 0.1$) to the high translational stiffnesses ($T_0 = T_L = 10^5$), the first three dimensionless natural frequency coefficients of the uniform beam increase from 0.6685, 0.8801 and 4.7319, and approach 3.1413, 6.2807 and 9.4164 (close to the behavior of pinned-pinned beam), respectively. Correspon-dingly, Fig. 4(b) and Table 6 show that when the elastic translational stiffnesses rise from 0.1 to $10^5$, $\mu_i$ ($i$ = 1, 2, 3) of the AFG beam ($n$ = 2, $c$ = 0.5) increase from 0.6808, 1.0411 and 5.3058, and tend to 3.4743, 7.0006 and 10.4877, respectively.

From Fig. 4 and Tables 5 and 6, it is observed that increasing $T_0$ and $T_L$ form 0.1 to $10^5$, can increase the first dimensionless natural frequency coefficient $\mu_1$ of the uniform and AFG beams to a maximum of 4.70 and 5.10 times, respectively. Moreover, in this case and with the same situations, the natural frequency coefficients of the AFG beam ($n$ = 2, $c$ = 0.5) are always greater than $\mu_i$ ($i$ = 1, 2, 3) of the uniform beam. Also, irrespective of type of the beam, by increase of the stiffness of the translational springs, the first three dimensionless natural frequency coefficients always increase.

Based on the Fig. 5(a) and Table 7, for the simply supported beam with two rotational springs (corresponding to $T_0 = T_L = \infty$ and $R_0$, $R_L$ = var.), as $R_0$ and $R_L$ increasing from low rotational stiffnesses ($R_0 = R_L = 0.1$) to high rotational stiffnesses ($R_0 = R_L = 10^5$), $\mu_i$ ($i$ = 1, 2, 3) of the uniform beam increase from 3.1727, 6.2989 and 9.4353, and approach limit value 4.7300, 7.8532 and 10.9956 (close to the behavior of clamped-clamped beam), respectively. Similarly, according to Fig. 5(b) and Table 8, when the elastic rotational stiffnesses $R_0$ and $R_L$ increase from 0.1 up to $10^5$, the first three dimensionless natural frequency coefficients of the AFG beam ($n$ = 2, $c$ = 0.5) rise from 3.5136, 7.0216 and 10.5100, to 5.2714, 8.7436 and 12.2373, respectively.




(a) Uniform beam (b) AFG beam ($n$ = 2, $c$ = 0.5)

Fig. 5 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 for $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$

Table 7 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform beam for $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$

| $R_0$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 3.1727 | 3.2287 | 3.2881 | 3.5478 | 3.6781 | 3.8684 | 3.9024 | 3.9321 | 3.9398 |
| | $i$ = 2 | 6.2989 | 6.3290 | 6.3637 | 6.5600 | 6.6946 | 6.9542 | 7.0102 | 7.0617 | 7.0756 |
| | $i$ = 3 | 9.4353 | 9.4558 | 9.4801 | 9.6331 | 9.7566 | 10.0493 | 10.1234 | 10.1951 | 10.2150 |
| 0.5 | $i$ = 1 | 3.2287 | 3.2836 | 3.3417 | 3.5980 | 3.7274 | 3.9172 | 3.9512 | 3.9809 | 3.9887 |
| | $i$ = 2 | 6.3290 | 6.3588 | 6.3932 | 6.5881 | 6.7221 | 6.9812 | 7.0371 | 7.0886 | 7.1025 |
| | $i$ = 3 | 9.4558 | 9.4762 | 9.5004 | 9.6529 | 9.7760 | 10.0683 | 10.1424 | 10.2140 | 10.2340 |
| 1 | $i$ = 1 | 3.2881 | 3.3417 | 3.3988 | 3.6520 | 3.7806 | 3.9702 | 4.0043 | 4.0340 | 4.0418 |
| | $i$ = 2 | 6.3637 | 6.3932 | 6.4273 | 6.6207 | 6.7541 | 7.0125 | 7.0685 | 7.1199 | 7.1338 |
| | $i$ = 3 | 9.4801 | 9.5004 | 9.5245 | 9.6764 | 9.7991 | 10.0910 | 10.1650 | 10.2367 | 10.2566 |

Table 7 Continued

| $R_0$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 5 | $i$ = 1 | 3.5478 | 3.5980 | 3.6520 | 3.8974 | 4.0257 | 4.2188 | 4.2540 | 4.2848 | 4.2929 |
| | $i$ = 2 | 6.5600 | 6.5881 | 6.6207 | 6.8077 | 6.9384 | 7.1953 | 7.2516 | 7.3034 | 7.3175 |
| | $i$ = 3 | 9.6331 | 9.6529 | 9.6764 | 9.8250 | 9.9459 | 10.2357 | 10.3097 | 10.3815 | 10.4016 |
| 10 | $i$ = 1 | 3.6781 | 3.7274 | 3.7806 | 4.0257 | 4.1557 | 4.3537 | 4.3900 | 4.4219 | 4.4303 |
| | $i$ = 2 | 6.6946 | 6.7221 | 6.7541 | 6.9384 | 7.0682 | 7.3262 | 7.3831 | 7.4356 | 7.4499 |
| | $i$ = 3 | 9.7566 | 9.7760 | 9.7991 | 9.9459 | 10.0657 | 10.3551 | 10.4294 | 10.5016 | 10.5218 |
| 50 | $i$ = 1 | 3.8684 | 3.9172 | 3.9702 | 4.2188 | 4.3537 | 4.5629 | 4.6018 | 4.6361 | 4.6451 |
| | $i$ = 2 | 6.9542 | 6.9812 | 7.0125 | 7.1953 | 7.3262 | 7.5911 | 7.6503 | 7.7053 | 7.7203 |
| | $i$ = 3 | 10.0493 | 10.0683 | 10.0910 | 10.2357 | 10.3551 | 10.6480 | 10.7242 | 10.7987 | 10.8196 |
| 100 | $i$ = 1 | 3.9024 | 3.9512 | 4.0043 | 4.2540 | 4.3900 | 4.6018 | 4.6413 | 4.6761 | 4.6852 |
| | $i$ = 2 | 7.0102 | 7.0371 | 7.0685 | 7.2516 | 7.3831 | 7.6503 | 7.7103 | 7.7660 | 7.7811 |
| | $i$ = 3 | 10.1234 | 10.1424 | 10.1650 | 10.3097 | 10.4294 | 10.7242 | 10.8013 | 10.8765 | 10.8975 |
| 500 | $i$ = 1 | 3.9321 | 3.9809 | 4.0340 | 4.2848 | 4.4219 | 4.6361 | 4.6761 | 4.7114 | 4.7206 |
| | $i$ = 2 | 7.0617 | 7.0886 | 7.1199 | 7.3034 | 7.4356 | 7.7053 | 7.7660 | 7.8224 | 7.8377 |
| | $i$ = 3 | 10.1951 | 10.2140 | 10.2367 | 10.3815 | 10.5016 | 10.7987 | 10.8765 | 10.9527 | 10.9740 |
| $10^5$ | $i$ = 1 | 3.9398 | 3.9887 | 4.0418 | 4.2929 | 4.4303 | 4.6451 | 4.6852 | 4.7206 | 4.7300 |
| | $i$ = 2 | 7.0756 | 7.1025 | 7.1338 | 7.3175 | 7.4499 | 7.7203 | 7.7811 | 7.8377 | 7.8532 |
| | $i$ = 3 | 10.2150 | 10.2340 | 10.2566 | 10.4016 | 10.5218 | 10.8196 | 10.8975 | 10.9740 | 10.9956 |

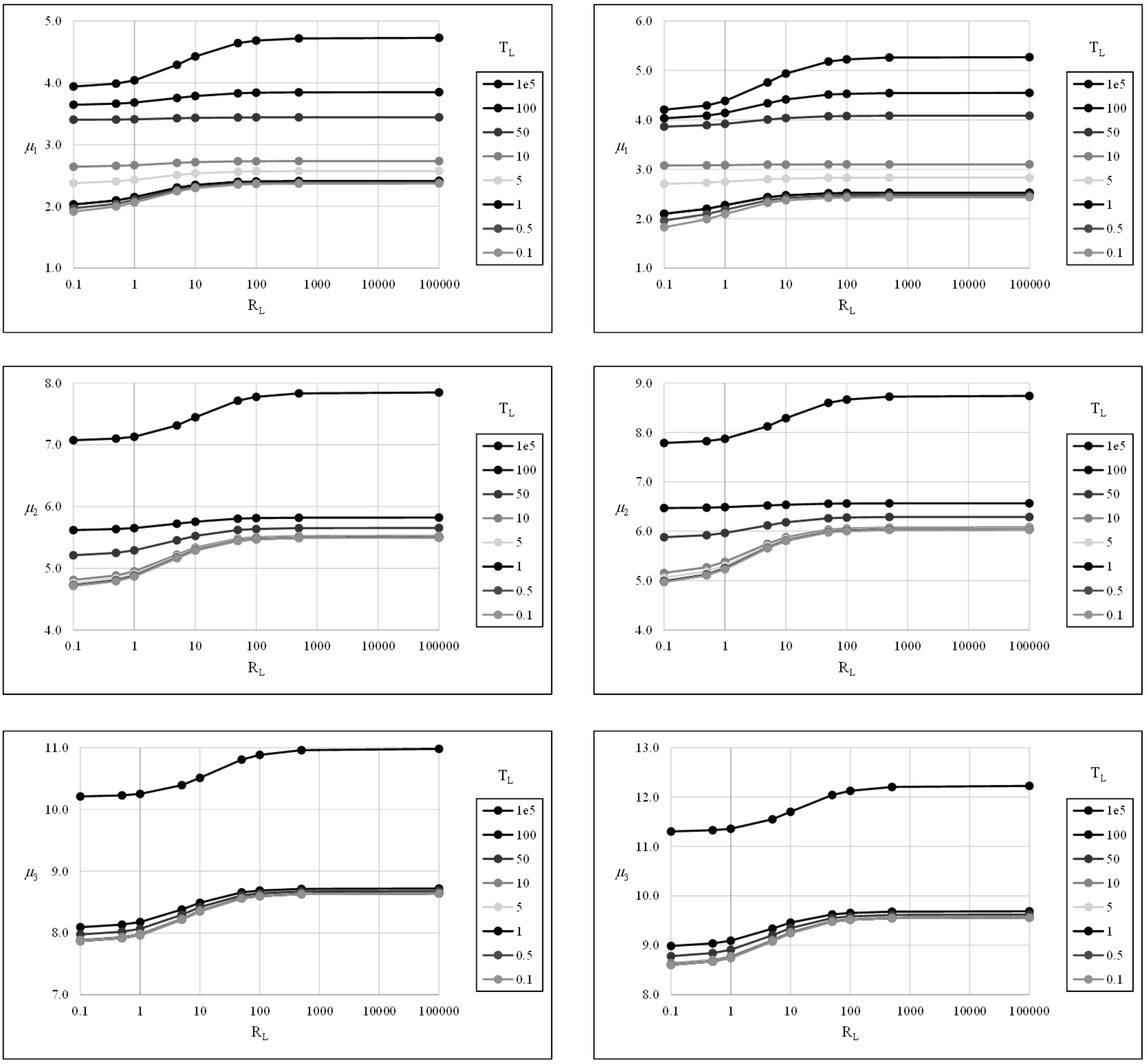
Table 8 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = 0.5) for $T_0 = T_L = \infty$ and various values of $R_0$ and $R_L$

| $R_0$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 3.5136 | 3.6134 | 3.7135 | 4.0984 | 4.2648 | 4.4818 | 4.5177 | 4.5482 | 4.5561 |
| | $i$ = 2 | 7.0216 | 7.0650 | 7.1145 | 7.3814 | 7.5511 | 7.8479 | 7.9071 | 7.9601 | 7.9741 |
| | $i$ = 3 | 10.5100 | 10.5380 | 10.5710 | 10.7730 | 10.9278 | 11.2647 | 11.3435 | 11.4175 | 11.4377 |
| 0.5 | $i$ = 1 | 3.5512 | 3.6497 | 3.7487 | 4.1312 | 4.2973 | 4.5145 | 4.5505 | 4.5811 | 4.5890 |
| | $i$ = 2 | 7.0461 | 7.0891 | 7.1383 | 7.4040 | 7.5732 | 7.8698 | 7.9290 | 7.9820 | 7.9960 |
| | $i$ = 3 | 10.5277 | 10.5556 | 10.5885 | 10.7901 | 10.9446 | 11.2812 | 11.3600 | 11.4340 | 11.4541 |
| 1 | $i$ = 1 | 3.5927 | 3.6899 | 3.7877 | 4.1678 | 4.3337 | 4.5514 | 4.5875 | 4.6183 | 4.6262 |
| | $i$ = 2 | 7.0746 | 7.1174 | 7.1662 | 7.4305 | 7.5992 | 7.8955 | 7.9547 | 8.0078 | 8.0218 |
| | $i$ = 3 | 10.5490 | 10.5768 | 10.6096 | 10.8105 | 10.9647 | 11.3009 | 11.3797 | 11.4538 | 11.4739 |
| 5 | $i$ = 1 | 3.7988 | 3.8904 | 3.9838 | 4.3561 | 4.5228 | 4.7451 | 4.7823 | 4.8141 | 4.8223 |
| | $i$ = 2 | 7.2445 | 7.2857 | 7.3328 | 7.5905 | 7.7569 | 8.0528 | 8.1124 | 8.1659 | 8.1801 |
| | $i$ = 3 | 10.6870 | 10.7142 | 10.7463 | 10.9438 | 11.0962 | 11.4310 | 11.5099 | 11.5841 | 11.6044 |
| 10 | $i$ = 1 | 3.9192 | 4.0086 | 4.1002 | 4.4711 | 4.6398 | 4.8669 | 4.9051 | 4.9379 | 4.9464 |
| | $i$ = 2 | 7.3710 | 7.4114 | 7.4576 | 7.7119 | 7.8776 | 8.1747 | 8.2350 | 8.2891 | 8.3035 |
| | $i$ = 3 | 10.8044 | 10.8312 | 10.8629 | 11.0582 | 11.2095 | 11.5440 | 11.6233 | 11.6979 | 11.7183 |
| 50 | $i$ = 1 | 4.1214 | 4.2084 | 4.2985 | 4.6718 | 4.8462 | 5.0856 | 5.1264 | 5.1615 | 5.1705 |
| | $i$ = 2 | 7.6446 | 7.6839 | 7.7291 | 7.9806 | 8.1473 | 8.4521 | 8.5147 | 8.5712 | 8.5862 |
| | $i$ = 3 | 11.1119 | 11.1381 | 11.1691 | 11.3614 | 11.5121 | 11.8505 | 11.9317 | 12.0086 | 12.0296 |
| 100 | $i$ = 1 | 4.1614 | 4.2481 | 4.3381 | 4.7126 | 4.8885 | 5.1310 | 5.1724 | 5.2081 | 5.2173 |
| | $i$ = 2 | 7.7098 | 7.7490 | 7.7941 | 8.0456 | 8.2132 | 8.5207 | 8.5842 | 8.6414 | 8.6567 |
| | $i$ = 3 | 11.1975 | 11.2236 | 11.2545 | 11.4466 | 11.5977 | 11.9384 | 12.0204 | 12.0981 | 12.1194 |

Table 8 Continued

| $R_0$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 500 | $i$ = 1 | 4.1974 | 4.2839 | 4.3737 | 4.7495 | 4.9269 | 5.1723 | 5.2144 | 5.2506 | 5.2599 |
| | $i$ = 2 | 7.7717 | 7.8109 | 7.8559 | 8.1079 | 8.2763 | 8.5869 | 8.6512 | 8.7093 | 8.7248 |
| | $i$ = 3 | 11.2835 | 11.3095 | 11.3404 | 11.5327 | 11.6843 | 12.0277 | 12.1108 | 12.1895 | 12.2111 |
| $10^5$ | $i$ = 1 | 4.2070 | 4.2934 | 4.3832 | 4.7593 | 4.9371 | 5.1834 | 5.2256 | 5.2620 | 5.2714 |
| | $i$ = 2 | 7.7888 | 7.8279 | 7.8729 | 8.1250 | 8.2937 | 8.6052 | 8.6697 | 8.7281 | 8.7436 |
| | $i$ = 3 | 11.3079 | 11.3340 | 11.3648 | 11.5572 | 11.7090 | 12.0533 | 12.1366 | 12.2157 | 12.2373 |

From Fig. 5 and Tables 7 and 8, it is concluded that increase of $R_0$ and $R_L$ form 0.1 to $10^5$, can increase $\mu_1$ of the uniform and AFG beams up to 1.49 and 1.50 times, respectively. Furthermore, in this case and with the same conditions, the natural frequencies of the AFG beam ($n$ = 2, $c$ = 0.5) are always larger than the uniform ones. Moreover,




(a) Uniform beam (b) AFG beam ($n$ = 2, $c$ = 0.5)

Fig. 6 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 for $T_0 = R_0 = \infty$ and various values of $T_L$ and $R_L$

Table 9 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform beam for $T_0 = R_0 = \infty$ and various values of $T_L$ and $R_L$

| $T_L$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 1.9163 | 1.9970 | 2.0641 | 2.2437 | 2.2970 | 2.3531 | 2.3613 | 2.3681 | 2.3698 |
| | $i$ = 2 | 4.7166 | 4.7925 | 4.8694 | 5.1649 | 5.2892 | 5.4459 | 5.4712 | 5.4926 | 5.4981 |
| | $i$ = 3 | 7.8675 | 7.9142 | 7.9659 | 8.2161 | 8.3532 | 8.5608 | 8.5983 | 8.6309 | 8.6394 |
| 0.5 | $i$ = 1 | 1.9697 | 2.0422 | 2.1033 | 2.2696 | 2.3197 | 2.3727 | 2.3804 | 2.3868 | 2.3885 |
| | $i$ = 2 | 4.7204 | 4.7961 | 4.8727 | 5.1671 | 5.2910 | 5.4473 | 5.4725 | 5.4938 | 5.4993 |
| | $i$ = 3 | 7.8683 | 7.9150 | 7.9666 | 8.2168 | 8.3537 | 8.5612 | 8.5986 | 8.6312 | 8.6397 |
| 1 | $i$ = 1 | 2.0304 | 2.0943 | 2.1491 | 2.3007 | 2.3470 | 2.3965 | 2.4037 | 2.4097 | 2.4112 |
| | $i$ = 2 | 4.7251 | 4.8005 | 4.8768 | 5.1700 | 5.2933 | 5.4489 | 5.4740 | 5.4953 | 5.5008 |
| | $i$ = 3 | 7.8693 | 7.9160 | 7.9676 | 8.2175 | 8.3544 | 8.5616 | 8.5991 | 8.6316 | 8.6401 |
| 5 | $i$ = 1 | 2.3756 | 2.4043 | 2.4301 | 2.5084 | 2.5342 | 2.5627 | 2.5670 | 2.5705 | 2.5714 |
| | $i$ = 2 | 4.7638 | 4.8365 | 4.9099 | 5.1925 | 5.3117 | 5.4624 | 5.4868 | 5.5074 | 5.5128 |
| | $i$ = 3 | 7.8776 | 7.9241 | 7.9754 | 8.2237 | 8.3596 | 8.5653 | 8.6025 | 8.6348 | 8.6432 |
| 10 | $i$ = 1 | 2.6426 | 2.6549 | 2.6662 | 2.7022 | 2.7147 | 2.7288 | 2.7310 | 2.7327 | 2.7332 |
| | $i$ = 2 | 4.8133 | 4.8822 | 4.9520 | 5.2210 | 5.3349 | 5.4795 | 5.5029 | 5.5228 | 5.5279 |
| | $i$ = 3 | 7.8880 | 7.9342 | 7.9852 | 8.2315 | 8.3661 | 8.5699 | 8.6067 | 8.6388 | 8.6471 |
| 50 | $i$ = 1 | 3.4021 | 3.4064 | 3.4106 | 3.4259 | 3.4321 | 3.4397 | 3.4409 | 3.4419 | 3.4422 |
| | $i$ = 2 | 5.2120 | 5.2506 | 5.2905 | 5.4524 | 5.5252 | 5.6218 | 5.6379 | 5.6517 | 5.6553 |
| | $i$ = 3 | 7.9758 | 8.0192 | 8.0669 | 8.2954 | 8.4195 | 8.6076 | 8.6416 | 8.6713 | 8.6791 |
| 100 | $i$ = 1 | 3.6456 | 3.6635 | 3.6818 | 3.7557 | 3.7888 | 3.8329 | 3.8403 | 3.8467 | 3.8483 |
| | $i$ = 2 | 5.6202 | 5.6355 | 5.6517 | 5.7219 | 5.7562 | 5.8046 | 5.8130 | 5.8203 | 5.8222 |
| | $i$ = 3 | 8.0946 | 8.1334 | 8.1760 | 8.3788 | 8.4889 | 8.6565 | 8.6871 | 8.7138 | 8.7207 |
| 500 | $i$ = 1 | 3.8801 | 3.9216 | 3.9664 | 4.1725 | 4.2819 | 4.4502 | 4.4815 | 4.5091 | 4.5164 |
| | $i$ = 2 | 6.6700 | 6.6764 | 6.6836 | 6.7218 | 6.7459 | 6.7893 | 6.7983 | 6.8065 | 6.8088 |
| | $i$ = 3 | 8.9982 | 9.0033 | 9.0090 | 9.0381 | 9.0559 | 9.0868 | 9.0931 | 9.0988 | 9.1003 |
| $10^5$ | $i$ = 1 | 3.9395 | 3.9884 | 4.0415 | 4.2923 | 4.4295 | 4.6442 | 4.6843 | 4.7197 | 4.7290 |
| | $i$ = 2 | 7.0738 | 7.1006 | 7.1318 | 7.3146 | 7.4465 | 7.7158 | 7.7765 | 7.8329 | 7.8483 |
| | $i$ = 3 | 10.2096 | 10.2284 | 10.2507 | 10.3939 | 10.5127 | 10.8073 | 10.8848 | 10.9608 | 10.9821 |

Table 10 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = 0.5) for $T_0 = R_0 = \infty$ and various values of $T_L$ and $R_L$

| $T_L$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 1.8268 | 1.9883 | 2.0974 | 2.3198 | 2.3721 | 2.4222 | 2.4290 | 2.4347 | 2.4361 |
| | $i$ = 2 | 4.9720 | 5.1064 | 5.2340 | 5.6570 | 5.8083 | 5.9809 | 6.0070 | 6.0288 | 6.0343 |
| | $i$ = 3 | 8.6022 | 8.6712 | 8.7458 | 9.0824 | 9.2488 | 9.4787 | 9.5176 | 9.5509 | 9.5595 |
| 0.5 | $i$ = 1 | 1.9640 | 2.0909 | 2.1810 | 2.3728 | 2.4192 | 2.4640 | 2.4702 | 2.4753 | 2.4766 |
| | $i$ = 2 | 4.9790 | 5.1127 | 5.2397 | 5.6605 | 5.8111 | 5.9829 | 6.0090 | 6.0307 | 6.0362 |
| | $i$ = 3 | 8.6035 | 8.6725 | 8.7470 | 9.0833 | 9.2496 | 9.4793 | 9.5181 | 9.5514 | 9.5600 |
| 1 | $i$ = 1 | 2.1011 | 2.1997 | 2.2726 | 2.4340 | 2.4742 | 2.5134 | 2.5188 | 2.5233 | 2.5244 |
| | $i$ = 2 | 4.9878 | 5.1207 | 5.2468 | 5.6650 | 5.8146 | 5.9855 | 6.0114 | 6.0330 | 6.0385 |
| | $i$ = 3 | 8.6052 | 8.6740 | 8.7486 | 9.0845 | 9.2506 | 9.4799 | 9.5188 | 9.5520 | 9.5606 |
| 5 | $i$ = 1 | 2.7060 | 2.7298 | 2.7492 | 2.7983 | 2.8119 | 2.8257 | 2.8277 | 2.8293 | 2.8297 |
| | $i$ = 2 | 5.0608 | 5.1862 | 5.3052 | 5.7008 | 5.8432 | 6.0065 | 6.0313 | 6.0520 | 6.0573 |
| | $i$ = 3 | 8.6184 | 8.6868 | 8.7609 | 9.0940 | 9.2584 | 9.4854 | 9.5239 | 9.5568 | 9.5653 |

Table 10 Continued

| $T_L$ | $\mu_i$ | $R_L$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 10 | $i$ = 1 | 3.0784 | 3.0823 | 3.0857 | 3.0946 | 3.0972 | 3.0999 | 3.1003 | 3.1007 | 3.1007 |
| | $i$ = 2 | 5.1562 | 5.2710 | 5.3803 | 5.7465 | 5.8796 | 6.0333 | 6.0568 | 6.0764 | 6.0814 |
| | $i$ = 3 | 8.6352 | 8.7031 | 8.7765 | 9.1059 | 9.2682 | 9.4924 | 9.5303 | 9.5628 | 9.5712 |
| 50 | $i$ = 1 | 3.8642 | 3.8926 | 3.9194 | 4.0072 | 4.0385 | 4.0745 | 4.0800 | 4.0846 | 4.0857 |
| | $i$ = 2 | 5.8807 | 5.9217 | 5.9630 | 6.1193 | 6.1836 | 6.2639 | 6.2767 | 6.2876 | 6.2904 |
| | $i$ = 3 | 8.7811 | 8.8433 | 8.9101 | 9.2053 | 9.3497 | 9.5496 | 9.5837 | 9.6129 | 9.6204 |
| 100 | $i$ = 1 | 4.0340 | 4.0873 | 4.1402 | 4.3351 | 4.4146 | 4.5142 | 4.5303 | 4.5439 | 4.5474 |
| | $i$ = 2 | 6.4712 | 6.4792 | 6.4875 | 6.5222 | 6.5382 | 6.5600 | 6.5637 | 6.5669 | 6.5677 |
| | $i$ = 3 | 8.9856 | 9.0375 | 9.0929 | 9.3373 | 9.4574 | 9.6256 | 9.6546 | 9.6795 | 9.6860 |
| 500 | $i$ = 1 | 4.1732 | 4.2527 | 4.3348 | 4.6723 | 4.8292 | 5.0459 | 5.0832 | 5.1154 | 5.1237 |
| | $i$ = 2 | 7.5052 | 7.5232 | 7.5434 | 7.6487 | 7.7135 | 7.8262 | 7.8490 | 7.8696 | 7.8751 |
| | $i$ = 3 | 10.2865 | 10.2872 | 10.2880 | 10.2921 | 10.2946 | 10.2988 | 10.2997 | 10.3004 | 10.3006 |
| $10^5$ | $i$ = 1 | 4.2069 | 4.2932 | 4.3830 | 4.7589 | 4.9367 | 5.1828 | 5.2250 | 5.2613 | 5.2707 |
| | $i$ = 2 | 7.7876 | 7.8266 | 7.8715 | 8.1229 | 8.2912 | 8.6019 | 8.6663 | 8.7246 | 8.7401 |
| | $i$ = 3 | 11.3041 | 11.3299 | 11.3605 | 11.5513 | 11.7018 | 12.0439 | 12.1268 | 12.2056 | 12.2272 |

regardless of type of the beam, $\mu_i$ ($i$ = 1, 2, 3) always increase by increasing the stiffness of the rotational springs.

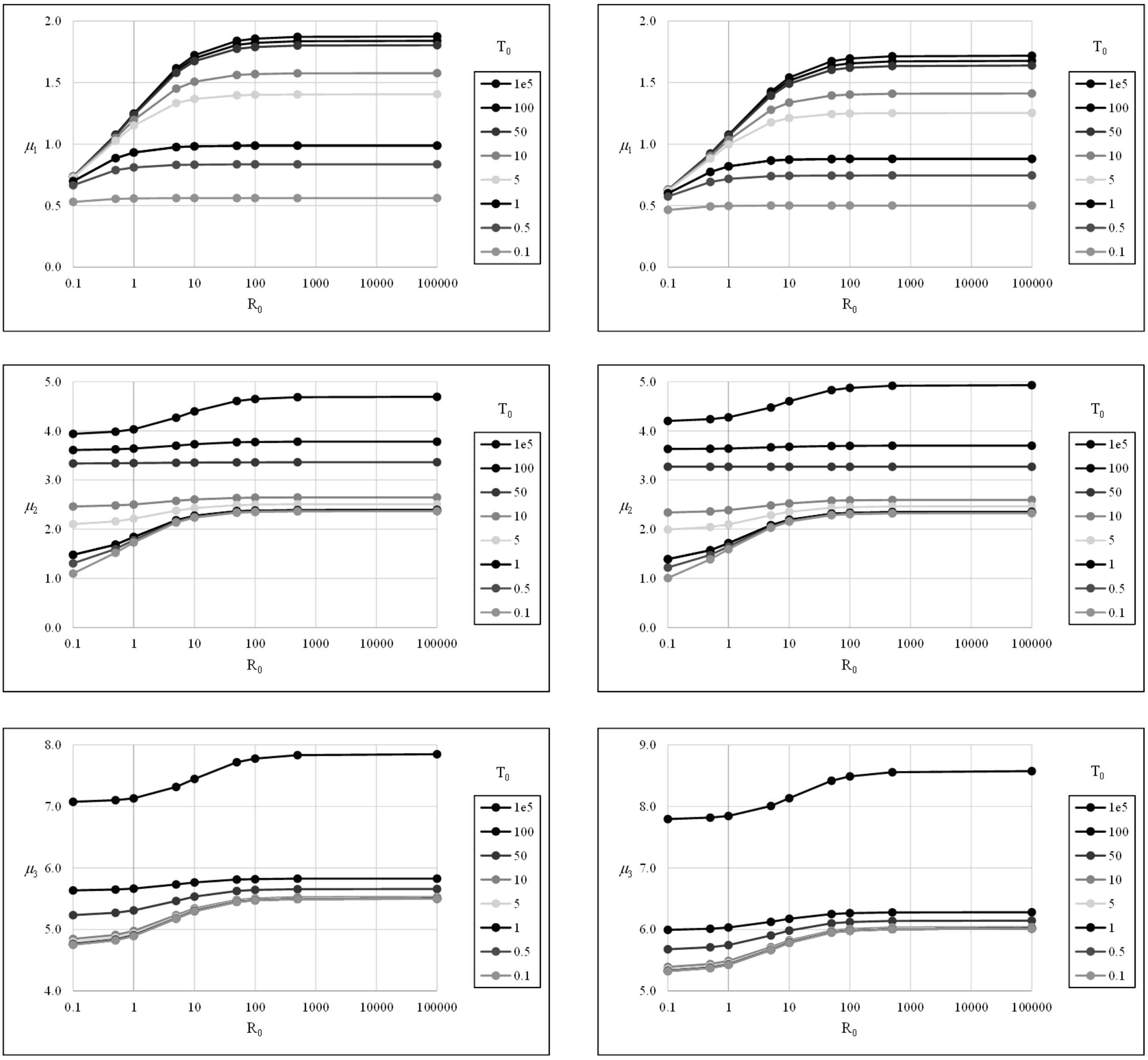
As seen in Fig. 6(a) and Table 9, for the clamped-free supported beam with the translational and rotational springs at $x = L$ (corresponding to $T_0 = R_0 = \infty$ and $T_L$, $R_L$ = var.), increasing $T_L$ and $R_L$ from 0.1 up to $10^5$, the first three dimensionless natural frequency coefficients of uniform beam increase from 1.9163, 4.7166 and 7.8675, and tend to 4.7290, 7.8483 and 10.9821 (close to the behavior of clamped-clamped beam), respectively. Similarly, Fig. 6(b) and Table 10 show that whenever the translational and rotational elastic stiffnesses (i.e., $T_L$ and $R_L$) rise from 0.1 to $10^5$, the first three dimensionless natural frequency coefficients of the AFG beam ($n$ = 2, $c$ = 0.5) increase from 1.8268, 4.9720 and 8.6022, and approach 5.2715, 8.7401 and 12.2272, respectively.

From Fig. 6 and Tables 9 and 10, as the stiffnesses of $T_L$ and $R_L$ increase form 0.1 up to $10^5$, the first dimensionless natural frequency coefficient $\mu_1$ of the uniform and AFG beams can increase up to 2.47 and 2.89 times, respectively. Moreover, in this case and for the low values of $T_L$ and $R_L$, the natural frequency coefficients of the AFG beam ($n$ = 2, $c$ = 0.5) are smaller than the corresponding $\mu_i$ ($i$ = 1, 2, 3) of the uniform beam. Also, irrespective of type of the beam, by increase of the stiffness of the elastic supports, the first three dimensionless natural frequency coefficients always increase.

Based on the Fig. 7(a) and Table 11, for the free supported beam with the translational and rotational springs at $x$ = 0 (corresponding to $T_L = R_L = 0$ and $T_0$, $R_0$ = var.), by increase of $T_0$ and $R_0$ from 0.1 to $10^5$, the first three dimensionless natural frequency coefficients of the uniform beam rise from 0.5294, 1.1015 and 4.7509, to 1.8751, 4.6930 and 7.8498 (close to the behavior of clamped-free beam), respectively. Correspondingly, Fig. 7(b) and Table 12 indicate that as the translational and rotational elastic stiffnesses (i.e., $T_0$ and $R_0$) increase from 0.1 up to $10^5$, $\mu_i$ ($i$ = 1, 2, 3) of the AFG beam ($n$ = 2, $c$ = 0.5) increase from 0.4654, 1.0089 and 5.3185, and tend to 1.7182, 4.9285 and 8.5748, respectively.

In Fig. 7 and Tables 11 and 12, it is founded that increasing $T_0$ and $R_0$ form 0.1 to $10^5$, can raise the first dimensionless natural frequency coefficient $\mu_1$ of the uniform and AFG beams up to 3.54 and 3.69 times, respectively. Furthermore, in this case and with the same situations, the natural frequency coefficients of the AFG beam ($n$ = 2, $c$ = 0.5) are not always greater than $\mu_i$ ($i$ = 1, 2, 3) of the uniform beam. In other words, for the low values of $T_0$ and $R_0$, the first natural frequency coefficient of the uniform beam is higher than $\mu_1$ of the AFG beam. Moreover, regardless of type of the beam, $\mu_i$ ($i$ = 1, 2, 3) always increase by increasing the stiffness of the elastic supports.

According to the Figs. 4-7 and Tables 5-12, regardless of type of the beam, the effects of the elastic supports should be considered in the problem of free vibration. Moreover, it is evident that by increase of the stiffness of the elastic supports, the natural frequency always increases. It should be added that depending on the values of stiffnesses of end restraints, the increase of $\mu_i$ ($i$ = 1, 2, 3) of the uniform and/or AFG beams can be insignificant or more considerable. Nevertheless, in the most cases, the rise of the first natural frequency coefficient of the beam is more significant when the stiffnesses of end restraints increase. Furthermore, irrespective of type of the beam, the influence of the translational stiffness on the values of $\mu_i$ ($i$ = 1, 2, 3) is more considerable than the rotational stiffness. Accordingly and based on the studied cases, increasing stiffness of translational springs from 0.1 to $10^5$, can raise the first dimensionless natural frequency coefficient $\mu_1$ of the uniform beam ($c$ = 0.0) and AFG beam ($n$ = 2, $c$ = 0.5)




(a) Uniform beam (b) AFG beam ($n$ = 2, $c$ = 0.5)

Fig. 7 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 for $T_L = R_L = 0$ and various values of $T_0$ and $R_0$

Table 11 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform beam for $T_L = R_L = 0$ and various values of $T_0$ and $R_0$

| $T_0$ | $\mu_i$ | $R_0$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 0.5294 | 0.5547 | 0.5582 | 0.5609 | 0.5613 | 0.5616 | 0.5616 | 0.5616 | 0.5616 |
| | $i$ = 2 | 1.1015 | 1.5198 | 1.7314 | 2.1384 | 2.2389 | 2.3391 | 2.3532 | 2.3649 | 2.3678 |
| | $i$ = 3 | 4.7509 | 4.8215 | 4.8936 | 5.1749 | 5.2948 | 5.4472 | 5.4718 | 5.4927 | 5.4981 |
| 0.5 | $i$ = 1 | 0.6664 | 0.7879 | 0.8107 | 0.8305 | 0.8331 | 0.8351 | 0.8354 | 0.8356 | 0.8356 |
| | $i$ = 2 | 1.3073 | 1.5986 | 1.7811 | 2.1587 | 2.2547 | 2.3513 | 2.3650 | 2.3762 | 2.3790 |
| | $i$ = 3 | 4.7546 | 4.8250 | 4.8968 | 5.1772 | 5.2967 | 5.4485 | 5.4731 | 5.4939 | 5.4993 |
| 1 | $i$ = 1 | 0.6982 | 0.8857 | 0.9316 | 0.9755 | 0.9815 | 0.9863 | 0.9869 | 0.9874 | 0.9875 |
| | $i$ = 2 | 1.4819 | 1.6894 | 1.8414 | 2.1840 | 2.2746 | 2.3666 | 2.3797 | 2.3905 | 2.3932 |
| | $i$ = 3 | 4.7593 | 4.8294 | 4.9009 | 5.1800 | 5.2990 | 5.4502 | 5.4747 | 5.4955 | 5.5008 |

Table 11 Continued

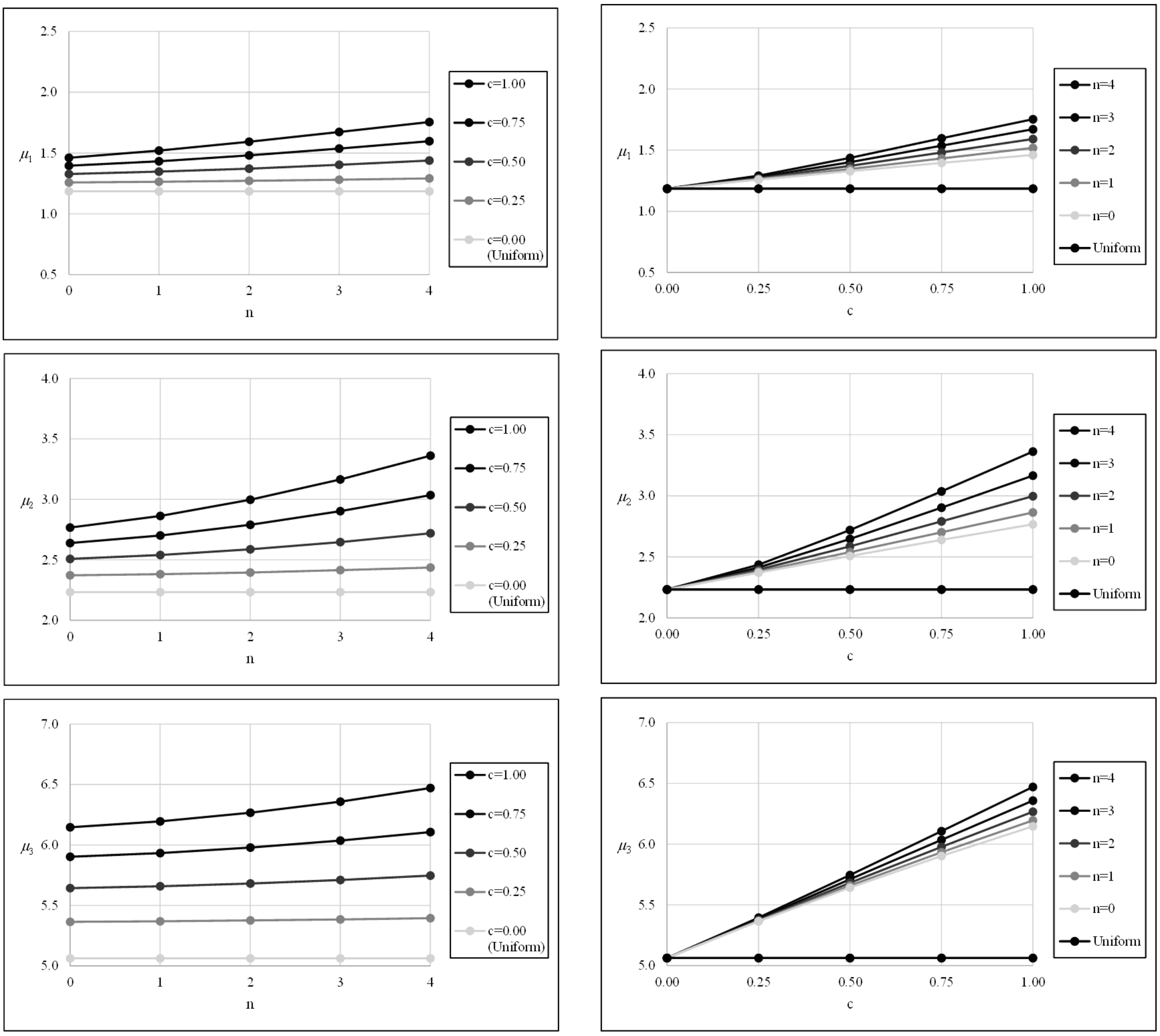
| $T_0$ | $\mu_i$ | $R_0$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 5 | $i$ = 1 | 0.7278 | 1.0261 | 1.1494 | 1.3327 | 1.3669 | 1.3969 | 1.4009 | 1.4041 | 1.4049 |
| | $i$ = 2 | 2.1059 | 2.1615 | 2.2138 | 2.3773 | 2.4314 | 2.4907 | 2.4995 | 2.5068 | 2.5086 |
| | $i$ = 3 | 4.7971 | 4.8647 | 4.9336 | 5.2026 | 5.3176 | 5.4640 | 5.4878 | 5.5079 | 5.5131 |
| 10 | $i$ = 1 | 0.7318 | 1.0503 | 1.1957 | 1.4508 | 1.5078 | 1.5621 | 1.5695 | 1.5755 | 1.5771 |
| | $i$ = 2 | 2.4612 | 2.4835 | 2.5051 | 2.5790 | 2.6064 | 2.6382 | 2.6431 | 2.6472 | 2.6482 |
| | $i$ = 3 | 4.8453 | 4.9096 | 4.9751 | 5.2312 | 5.3411 | 5.4814 | 5.5042 | 5.5236 | 5.5286 |
| 50 | $i$ = 1 | 0.7350 | 1.0709 | 1.2370 | 1.5796 | 1.6738 | 1.7744 | 1.7892 | 1.8016 | 1.8047 |
| | $i$ = 2 | 3.3391 | 3.3414 | 3.3436 | 3.3523 | 3.3559 | 3.3607 | 3.3615 | 3.3622 | 3.3624 |
| | $i$ = 3 | 5.2330 | 5.2696 | 5.3075 | 5.4624 | 5.5324 | 5.6257 | 5.6413 | 5.6547 | 5.6581 |
| 100 | $i$ = 1 | 0.7354 | 1.0735 | 1.2424 | 1.5978 | 1.6980 | 1.8066 | 1.8227 | 1.8362 | 1.8396 |
| | $i$ = 2 | 3.6100 | 3.6240 | 3.6386 | 3.6998 | 3.7285 | 3.7678 | 3.7745 | 3.7803 | 3.7818 |
| | $i$ = 3 | 5.6328 | 5.6475 | 5.6630 | 5.7304 | 5.7632 | 5.8097 | 5.8178 | 5.8248 | 5.8266 |
| 500 | $i$ = 1 | 0.7357 | 1.0757 | 1.2468 | 1.6127 | 1.7178 | 1.8327 | 1.8500 | 1.8643 | 1.8680 |
| | $i$ = 2 | 3.8718 | 3.9086 | 3.9488 | 4.1393 | 4.2434 | 4.4062 | 4.4367 | 4.4637 | 4.4709 |
| | $i$ = 3 | 6.6723 | 6.6788 | 6.6861 | 6.7248 | 6.7492 | 6.7930 | 6.8022 | 6.8105 | 6.8127 |
| $10^5$ | $i$ = 1 | 0.7358 | 1.0762 | 1.2479 | 1.6164 | 1.7227 | 1.8393 | 1.8568 | 1.8713 | 1.8750 |
| | $i$ = 2 | 3.9381 | 3.9822 | 4.0307 | 4.2664 | 4.3987 | 4.6090 | 4.6487 | 4.6837 | 4.6930 |
| | $i$ = 3 | 7.0738 | 7.1007 | 7.1321 | 7.3155 | 7.4477 | 7.7173 | 7.7780 | 7.8345 | 7.8498 |

Table 12 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam ($n$ = 2, $c$ = 0.5) for $T_L = R_L = 0$ and various values of $T_0$ and $R_0$

| $T_0$ | $\mu_i$ | $R_o$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 0.1 | $i$ = 1 | 0.4654 | 0.4929 | 0.4967 | 0.4999 | 0.5003 | 0.5006 | 0.5006 | 0.5007 | 0.5007 |
| | $i$ = 2 | 1.0089 | 1.3898 | 1.5948 | 2.0321 | 2.1550 | 2.2873 | 2.3069 | 2.3233 | 2.3274 |
| | $i$ = 3 | 5.3185 | 5.3698 | 5.4244 | 5.6636 | 5.7815 | 5.9490 | 5.9782 | 6.0035 | 6.0101 |
| 0.5 | $i$ = 1 | 0.5747 | 0.6929 | 0.7171 | 0.7390 | 0.7419 | 0.7443 | 0.7445 | 0.7448 | 0.7448 |
| | $i$ = 2 | 1.2209 | 1.4774 | 1.6508 | 2.0545 | 2.1721 | 2.2999 | 2.3190 | 2.3348 | 2.3389 |
| | $i$ = 3 | 5.3214 | 5.3726 | 5.4270 | 5.6655 | 5.7831 | 5.9502 | 5.9794 | 6.0046 | 6.0112 |
| 1 | $i$ = 1 | 0.5990 | 0.7729 | 0.8195 | 0.8667 | 0.8733 | 0.8788 | 0.8795 | 0.8800 | 0.8801 |
| | $i$ = 2 | 1.3918 | 1.5738 | 1.7167 | 2.0822 | 2.1934 | 2.3157 | 2.3340 | 2.3493 | 2.3532 |
| | $i$ = 3 | 5.3250 | 5.3760 | 5.4302 | 5.6679 | 5.7851 | 5.9517 | 5.9808 | 6.0059 | 6.0125 |
| 5 | $i$ = 1 | 0.6215 | 0.8835 | 0.9966 | 1.1764 | 1.2123 | 1.2446 | 1.2489 | 1.2524 | 1.2533 |
| | $i$ = 2 | 1.9930 | 2.0462 | 2.0988 | 2.2841 | 2.3546 | 2.4388 | 2.4520 | 2.4630 | 2.4659 |
| | $i$ = 3 | 5.3540 | 5.4035 | 5.4561 | 5.6871 | 5.8012 | 5.9636 | 5.9920 | 6.0166 | 6.0230 |
| 10 | $i$ = 1 | 0.6245 | 0.9024 | 1.0336 | 1.2784 | 1.3372 | 1.3949 | 1.4030 | 1.4096 | 1.4113 |
| | $i$ = 2 | 2.3395 | 2.3642 | 2.3893 | 2.4864 | 2.5274 | 2.5795 | 2.5880 | 2.5952 | 2.5971 |
| | $i$ = 3 | 5.3904 | 5.4380 | 5.4886 | 5.7111 | 5.8213 | 5.9786 | 6.0062 | 6.0301 | 6.0363 |
| 50 | $i$ = 1 | 0.6270 | 0.9185 | 1.0669 | 1.3934 | 1.4917 | 1.6032 | 1.6204 | 1.6348 | 1.6385 |
| | $i$ = 2 | 3.2727 | 3.2727 | 3.2727 | 3.2728 | 3.2728 | 3.2728 | 3.2728 | 3.2728 | 3.2728 |
| | $i$ = 3 | 5.6761 | 5.7088 | 5.7438 | 5.9014 | 5.9817 | 6.0995 | 6.1206 | 6.1390 | 6.1438 |
| 100 | $i$ = 1 | 0.6273 | 0.9206 | 1.0713 | 1.4104 | 1.5157 | 1.6379 | 1.6570 | 1.6731 | 1.6773 |
| | $i$ = 2 | 3.6322 | 3.6370 | 3.6420 | 3.6647 | 3.6764 | 3.6939 | 3.6970 | 3.6998 | 3.7005 |
| | $i$ = 3 | 5.9923 | 6.0108 | 6.0308 | 6.1242 | 6.1740 | 6.2499 | 6.2638 | 6.2761 | 6.2793 |

Table 12 Continued

| $T_0$ | $\mu_i$ | $R_o$ | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0.1 | 0.5 | 1 | 5 | 10 | 50 | 100 | 500 | $10^5$ |
| 500 | $i$ = 1 | 0.6275 | 0.9222 | 1.0749 | 1.4244 | 1.5357 | 1.6670 | 1.6878 | 1.7054 | 1.7099 |
| | $i$ = 2 | 4.0727 | 4.0976 | 4.1253 | 4.2659 | 4.3502 | 4.4952 | 4.5244 | 4.5508 | 4.5579 |
| | $i$ = 3 | 7.1186 | 7.1200 | 7.1216 | 7.1300 | 7.1355 | 7.1458 | 7.1480 | 7.1500 | 7.1506 |
| $10^5$ | $i$ = 1 | 0.6276 | 0.9227 | 1.0758 | 1.4280 | 1.5407 | 1.6744 | 1.6956 | 1.7136 | 1.7182 |
| | $i$ = 2 | 4.2033 | 4.2372 | 4.2753 | 4.4760 | 4.6020 | 4.8277 | 4.8745 | 4.9171 | 4.9285 |
| | $i$ = 3 | 7.7942 | 7.8170 | 7.8437 | 8.0068 | 8.1325 | 8.4176 | 8.4881 | 8.5560 | 8.5748 |



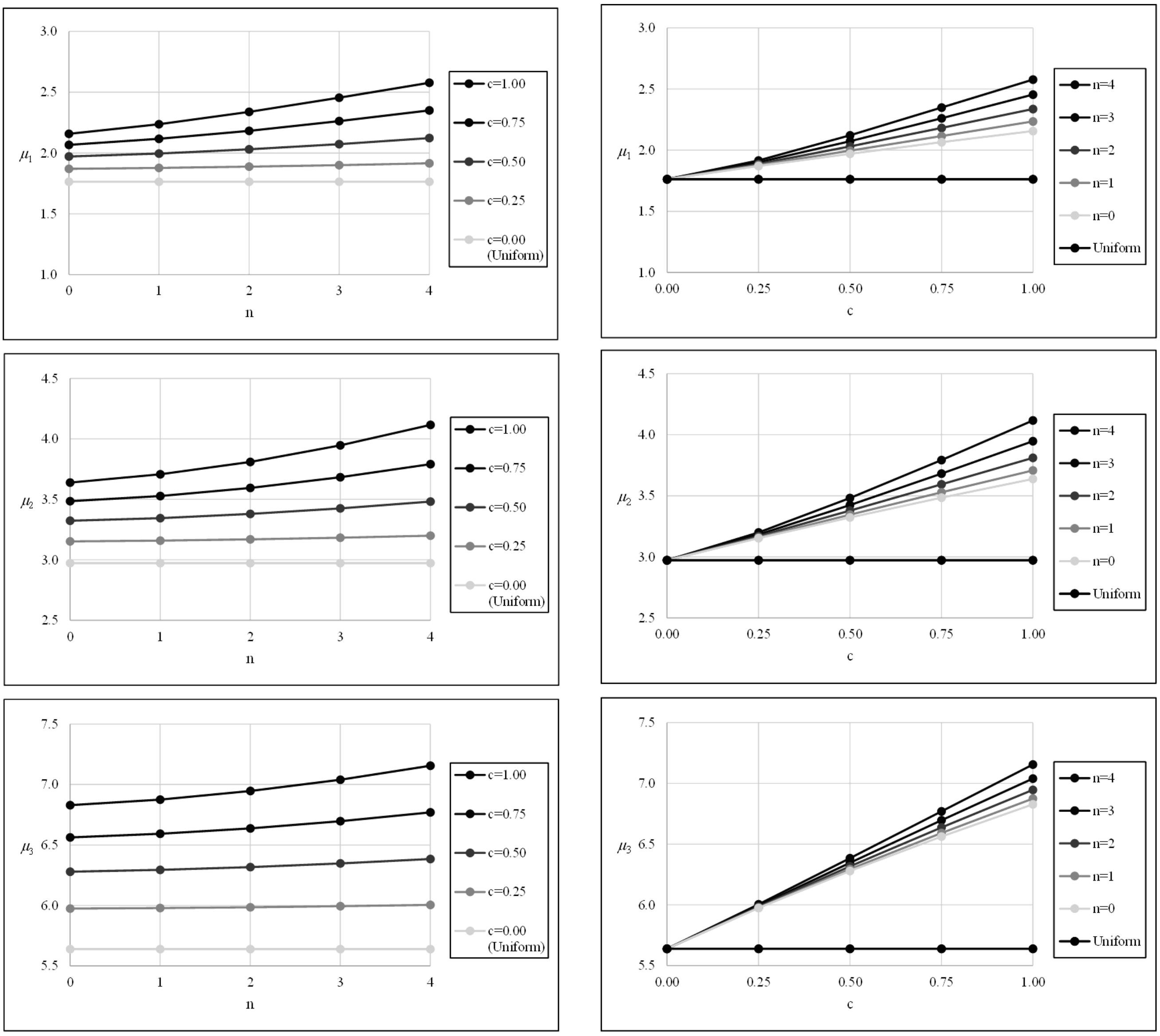

(a) with respect to various values of gradient index ($n$) (b) with respect to various values of gradient coefficient ($c$)

Fig. 8 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam with symmetric elastic boundary conditions ($R_0 = T_0 = R_L = T_L = 1$)

to a maximum of 4.70 and 5.10 times, respectively. In addition, with the same conditions and in the most cases,the natural frequencies of the AFG beam are greater than those of the uniform beam.

### 3.3 Effects of the AFG parameters

In this part, the influences of the AFG parameters, namely, the gradient index $n$ and gradient coefficient $c$ on




(a) with respect to various values of gradient index ($n$) (b) with respect to various values of gradient coefficient ($c$)

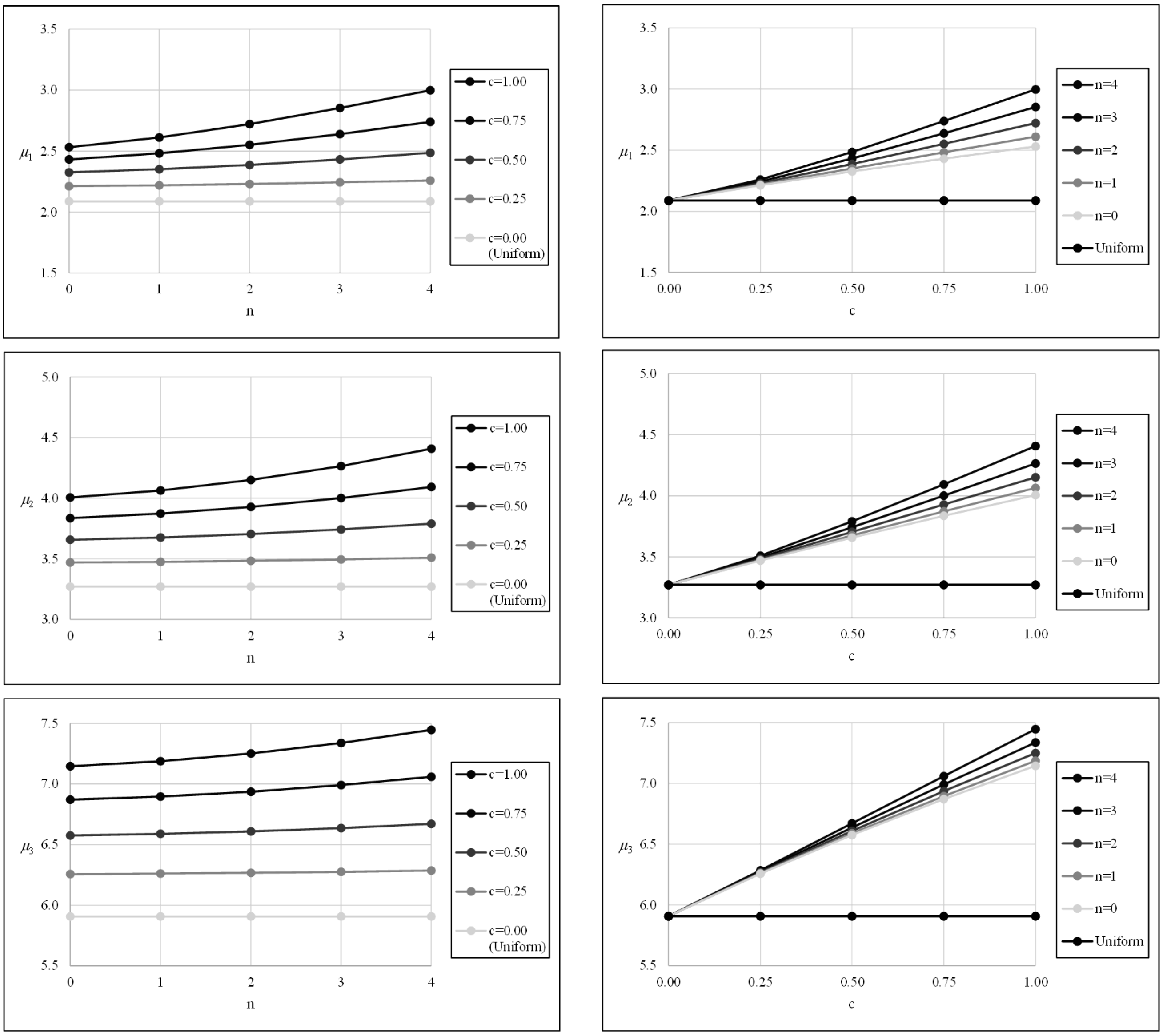
Fig. 9 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam with symmetric elastic boundary conditions ($R_0 = T_0 = R_L = T_L = 5$)

the natural frequencies of the AFG beams with the non-classical and classical boundary conditions are studied comprehensively. It is reminded that the foregoing parameters were previously introduced by Eq. (1) at section 2.1. Moreover, it is clear that when $c$ = 0.0, the beam is uniform. Accordingly, three cases of the symmetric non-classical end conditions, i.e., $R_0 = T_0 = R_L = T_L = 1, 5, 10$ (corresponding to moderate stiffnesses) and six types of the classical boundary conditions, namely, C-C, P-C, C-P, P-P, F-C and C-F are considered. Here, C means clamped, P denotes pinned, and F means free.

In Figs. 8, 9 and 10, changing of the first three dimensionless natural frequency coefficients $\mu_i$ ($i$ = 1, 2, 3) of the AFG beam with respect to increase of the values of $n$ and $c$, in three cases of the non-classical symmetric elastic supports, i.e., $R_0 = T_0 = R_L = T_L = 1, 5, 10$ are drawn, respectively. Furthermore, Tables 13, 14 and 15 show the corresponding numerical quantities of $\mu_i$ ($i$ = 1, 2, 3) for different values of the gradient index $n$ and gradient coefficient $c$.

From Fig. 8 and Table 13, it is observed that for the AFG beam with $R_0 = T_0 = R_L = T_L = 1$, as the AFG parameters $n$ and $c$ increase to 4 and 1.00, respectively, the first three dimensionless natural frequency coefficients of the AFG beam can raise by about 48%, 51%, and 28% versus the corresponding $\mu_i$ ($i$ = 1, 2, 3) of the uniform beam, respectively. Also, by increase of the AFG parameters the first three dimensionless natural frequency coefficients always increase.

According to Fig. 9 and Table 14, it is concluded that whereas $R_0 = T_0 = R_L = T_L = 5$, the rise of the gradient index $n$ and gradient coefficient $c$ up to 4 and 1.00, respectively, can increase the values of $\mu_i$ ($i$ = 1, 2, 3) of the AFG beam by about 47%, 38%, and 27% with respect to those of the




(a) with respect to various values of gradient index ($n$) (b) with respect to various values of gradient coefficient ($c$)

Fig. 10 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam with symmetric elastic boundary conditions ($R_0 = T_0 = R_L = T_L = 10$)

Table 13 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform and AFG beam ($n$ = var., $c$ = var.) with symmetric elastic boundary conditions ($R_0 = T_0 = R_L = T_L = 1$)

| $\mu_i$ | Uniform beam | AFG beam | | | | | |
|---|---|---|---|---|---|---|---|
| | | $n$ | $c$ | | | | |
| | | | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| $i$ = 1 | 1.1856 | | 1.1856 | 1.2582 | 1.3280 | 1.3954 | 1.4606 |
| $i$ = 2 | 2.2333 | 0 | 2.2333 | 2.3719 | 2.5074 | 2.6391 | 2.7667 |
| $i$ = 3 | 5.0631 | | 5.0631 | 5.3651 | 5.6431 | 5.9023 | 6.1463 |
| $i$ = 1 | 1.1856 | | 1.1856 | 1.2637 | 1.3468 | 1.4325 | 1.5190 |
| $i$ = 2 | 2.2333 | 1 | 2.2333 | 2.3816 | 2.5401 | 2.7019 | 2.8632 |
| $i$ = 3 | 5.0631 | | 5.0631 | 5.3697 | 5.6589 | 5.9336 | 6.1960 |
| $i$ = 1 | 1.1856 | | 1.1856 | 1.2710 | 1.3719 | 1.4807 | 1.5920 |
| $i$ = 2 | 2.2333 | 2 | 2.2333 | 2.3957 | 2.5870 | 2.7905 | 2.9974 |
| $i$ = 3 | 5.0631 | | 5.0631 | 5.3762 | 5.6814 | 5.9782 | 6.2668 |

Table 13 Continued

| $\mu_i$ | Uniform beam | AFG beam | | | | | |
|---|---|---|---|---|---|---|---|
| | | $n$ | $c$ | | | | |
| | | | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| $i$ = 1 | 1.1856 | | 1.1856 | 1.2803 | 1.4026 | 1.5368 | 1.6725 |
| $i$ = 2 | 2.2333 | 3 | 2.2333 | 2.4142 | 2.6473 | 2.9025 | 3.1650 |
| $i$ = 3 | 5.0631 | | 5.0631 | 5.3847 | 5.7107 | 6.0362 | 6.3586 |
| $i$ = 1 | 1.1856 | | 1.1856 | 1.2913 | 1.4377 | 1.5971 | 1.7542 |
| $i$ = 2 | 2.2333 | 4 | 2.2333 | 2.4369 | 2.7200 | 3.0358 | 3.3617 |
| $i$ = 3 | 5.0631 | | 5.0631 | 5.3951 | 5.7467 | 6.1072 | 6.4710 |

Table 14 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform and AFG beam ($n$ = var., $c$ = var.) with symmetric elastic boundary conditions ($R_0 = T_0 = R_L = T_L = 5$)

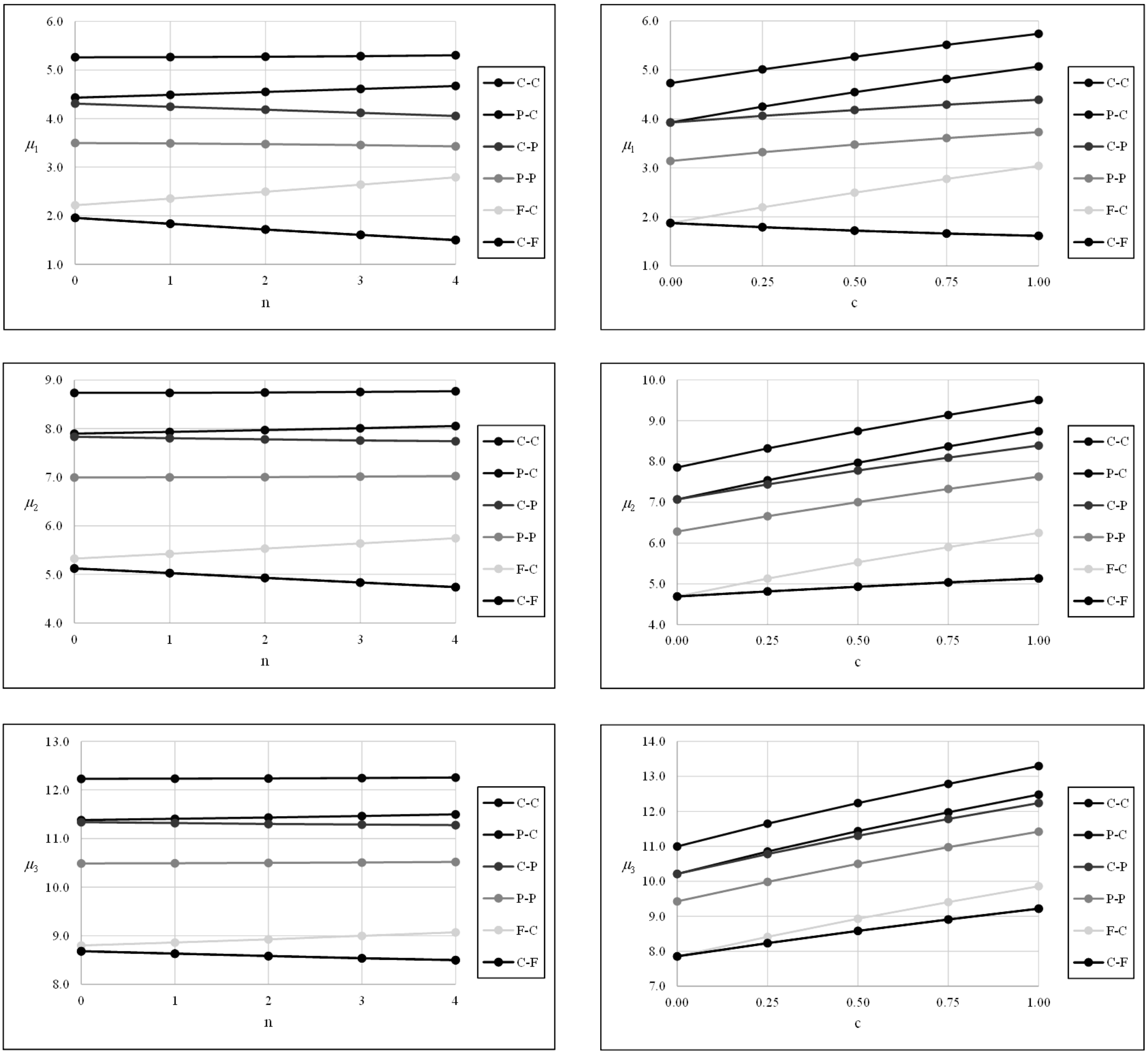
| $\mu_i$ | Uniform beam | AFG beam | | | | | |
|---|---|---|---|---|---|---|---|
| | | $n$ | $c$ | | | | |
| | | | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| $i$ = 1 | 1.7635 | | 1.7635 | 1.8701 | 1.9706 | 2.0660 | 2.1571 |
| $i$ = 2 | 2.9729 | 0 | 2.9729 | 3.1529 | 3.3229 | 3.4843 | 3.6382 |
| $i$ = 3 | 5.6393 | | 5.6393 | 5.9740 | 6.2797 | 6.5631 | 6.8287 |
| $i$ = 1 | 1.7635 | | 1.7635 | 1.8775 | 1.9960 | 2.1159 | 2.2356 |
| $i$ = 2 | 2.9729 | 1 | 2.9729 | 3.1595 | 3.3454 | 3.5282 | 3.7071 |
| $i$ = 3 | 5.6393 | | 5.6393 | 5.9783 | 6.2946 | 6.5927 | 6.8757 |
| $i$ = 1 | 1.7635 | | 1.7635 | 1.8876 | 2.0304 | 2.1823 | 2.3374 |
| $i$ = 2 | 2.9729 | 2 | 2.9729 | 3.1695 | 3.3794 | 3.5944 | 3.8102 |
| $i$ = 3 | 5.6393 | | 5.6393 | 5.9849 | 6.3171 | 6.6372 | 6.9464 |
| $i$ = 1 | 1.7635 | | 1.7635 | 1.9005 | 2.0730 | 2.2616 | 2.4543 |
| $i$ = 2 | 2.9729 | 3 | 2.9729 | 3.1830 | 3.4250 | 3.6824 | 3.9467 |
| $i$ = 3 | 5.6393 | | 5.6393 | 5.9936 | 6.3472 | 6.6965 | 7.0399 |
| $i$ = 1 | 1.7635 | | 1.7635 | 1.9159 | 2.1224 | 2.3495 | 2.5777 |
| $i$ = 2 | 2.9729 | 4 | 2.9729 | 3.1999 | 3.4818 | 3.7919 | 4.1158 |
| $i$ = 3 | 5.6393 | | 5.6393 | 6.0045 | 6.3847 | 6.7700 | 7.1555 |

Table 15 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform and AFG beam ($n$ = var., $c$ = var.) with symmetric elastic boundary conditions ($R_0 = T_0 = R_L = T_L = 10$)

| $\mu_i$ | Uniform beam | AFG beam | | | | | |
|---|---|---|---|---|---|---|---|
| | | $n$ | $c$ | | | | |
| | | | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| $i$ = 1 | 2.0883 | | 2.0883 | 2.2123 | 2.3259 | 2.4316 | 2.5313 |
| $i$ = 2 | 3.2709 | 0 | 3.2709 | 3.4694 | 3.6572 | 3.8358 | 4.0062 |
| $i$ = 3 | 5.9069 | | 5.9069 | 6.2569 | 6.5756 | 6.8702 | 7.1458 |
| $i$ = 1 | 2.0883 | | 2.0883 | 2.2197 | 2.3514 | 2.4822 | 2.6116 |
| $i$ = 2 | 3.2709 | 1 | 3.2709 | 3.4750 | 3.6763 | 3.8731 | 4.0646 |
| $i$ = 3 | 5.9069 | | 5.9069 | 6.2607 | 6.5885 | 6.8960 | 7.1869 |
| $i$ = 1 | 2.0883 | | 2.0883 | 2.2300 | 2.3870 | 2.5522 | 2.7210 |
| $i$ = 2 | 3.2709 | 2 | 3.2709 | 3.4834 | 3.7049 | 3.9287 | 4.1511 |
| $i$ = 3 | 5.9069 | | 5.9069 | 6.2665 | 6.6089 | 6.9364 | 7.2512 |

Table 15 Continued

| $\mu_i$ | Uniform beam | $n$ | AFG beam | | | | |
|---|---|---|---|---|---|---|---|
| | | | $c$ | | | | |
| | | | 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| $i$ = 1 | 2.0883 | | 2.0883 | 2.2433 | 2.4322 | 2.6389 | 2.8528 |
| $i$ = 2 | 3.2709 | 3 | 3.2709 | 3.4946 | 3.7429 | 4.0022 | 4.2654 |
| $i$ = 3 | 5.9069 | | 5.9069 | 6.2746 | 6.6366 | 6.9912 | 7.3380 |
| $i$ = 1 | 2.0883 | | 2.0883 | 2.2595 | 2.4861 | 2.7387 | 2.9980 |
| $i$ = 2 | 3.2709 | 4 | 3.2709 | 3.5087 | 3.7903 | 4.0938 | 4.4080 |
| $i$ = 3 | 5.9069 | | 5.9069 | 6.2847 | 6.6715 | 7.0601 | 7.4468 |




(a) with respect to various values of gradient index ($n$) in which $c$ = 0.5

(b) with respect to various values of gradient coefficient ($c$) in which $n$ = 2

Fig. 11 Plot first three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the AFG beam with classical boundary conditions uniform beam, respectively.

As seen in Fig. 10 and Table 15, It is founded that for the AFG beam with $R_0$ = $T_0$ = $R_L$ = $T_L$ = 10, when the parameters *n* and *c* increase to 4 and 1.00, respectively, the first three dimensionless natural frequency coefficients of the AFG beam can raise by about 44%, 35%, and 26% versus the uniform ones, respectively. Moreover, the values of $\mu_i$ (*i* = 1, 2, 3) always increase by increasing the AFG parameters.

Based on the Figs. 8-10 and Tables 13-15, it is concluded that with the symmetric elastic boundary conditions, as the gradient index and gradient coefficient increase, $\mu_i$ (*i* = 1, 2, 3) of the AFG beam always increase linearly. This effect is more pronounced when the gradient coefficient *c* increases. In other words, for the symmetric elastic supports, the influence of the coefficient *c* on the natural frequency of the AFG beam is more significant than the gradient index *n*. Moreover, regardless of the values of gradient index and gradient coefficient, by increasing the stiffness of the symmetric elastic boundary conditions, $\mu_i$ (*i* = 1, 2, 3) always increase.

In the following, variations of the first three dimensionless natural frequency coefficients $\mu_i$ (*i* = 1, 2, 3) of the AFG beam with the classical boundary conditions versus different values of the gradient index *n* in which *c* = 0.5 and various quantities of the gradient coefficient *c* in which *n* = 2 are depicted in Figs. 11(a) and (b), respectively. Moreover, the corresponding numerical values of the first three dimensionless natural frequency coefficients $\mu_i$ (*i* = 1, 2, 3) of the AFG beam in the aforementioned cases, namely, *n* = var. whereas *c* = 0.5 and *c* = var. whereas *n* = 2, with the classical boundary conditions are reported in Tables 16 and 17, respectively. It should be added, for the sake comparing the results of Xing and Wang (2013) for the uniform beams (*c* = 0.0) and with the classical boundary conditions are inserted in the mentioned tables, too.

According to the Fig. 11 and Tables 16 and 17, it is observed that as the gradient index and gradient coefficient increase, in the most cases, $\mu_i$ (*I* = 1, 2, 3) of the AFG beam increase linearly. Nevertheless, by increase of *n*, the first three dimensionless natural frequency coefficients decrease in the C-P and C-F beams. Also, $\mu_1$ of the C-F beam reduces when the parameter *c* increases. In fact, in the latter case, the first natural frequency coefficient of the uniform beam (*c* = 0.0) is greater than the corresponding $\mu_1$ of the AFG beam. In addition, irrespective of type of the classical boundary conditions, the effect of the gradient coefficient *c* on the natural frequency of the AFG beam is more considerable than parameter *n*. Accordingly and based on the investigated cases, as the parameters *n* and *c* increase, the first dimensionless natural frequency coefficient $\mu_1$ of the F-C beam can increase by about 49% (corresponding to *n* = 4 and *c* = 0.5) and 62% (corresponding to *n* = 2 and *c* = 1.0) versus the uniform ones, respectively. In other words, the F-C case is more sensitive than the other classical boundary conditions. Furthermore, it is concluded that for the specified quantities of the AFG parameters *n* and *c*, the C-C and C-F boundary conditions have the maximum and minimum values of $\mu_i$ (*i* = 1, 2, 3), respectively.

Table 16 First three dimensionless natural frequency coefficients $\mu_i$, *i* = 1, 2, 3 of the uniform and AFG beam (*n* = var., *c* = 0.5) with classical boundary conditions

| Boundary condistions | $\mu_i$ | Uniform beam | | AFG beam (*c* = 0.5) | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Present | Xing and Wang (2013) | *n* | | | | |
| | | | | 0 | 1 | 2 | 3 | 4 |
| C-C ($T_0 = R_0 = \infty$, $T_L = R_L = \infty$) | *i* = 1 | 4.7300 | 4.730039 | 5.2609 | 5.2635 | 5.2715 | 5.2848 | 5.3037 |
| | *i* = 2 | 7.8532 | 7.853195 | 8.7351 | 8.7373 | 8.7438 | 8.7547 | 8.7699 |
| | *i* = 3 | 10.9956 | 10.995581 | 12.2308 | 12.2325 | 12.2376 | 12.2461 | 12.2580 |
| P-C ($T_0 = \infty$, $R_0 = 0$, $T_L = R_L = \infty$) | *i* = 1 | 3.9266 | − | 4.4293 | 4.4879 | 4.5474 | 4.6085 | 4.6716 |
| | *i* = 2 | 7.0686 | − | 7.8962 | 7.9307 | 7.9686 | 8.0097 | 8.0541 |
| | *i* = 3 | 10.2102 | − | 11.3809 | 11.4057 | 11.4335 | 11.4643 | 11.4980 |
| C-P ($T_0 = R_0 = \infty$, $T_L = \infty$, $R_L = 0$) | *i* = 1 | 3.9266 | 3.926601 | 4.3082 | 4.2457 | 4.1829 | 4.1195 | 4.0550 |
| | *i* = 2 | 7.0686 | 7.068577 | 7.8325 | 7.8038 | 7.7786 | 7.7572 | 7.7396 |
| | *i* = 3 | 10.2102 | 10.210160 | 11.3366 | 11.3174 | 11.3014 | 11.2885 | 11.2789 |
| P-P ($T_0 = \infty$, $R_0 = 0$, $T_L = \infty$, $R_L = 0$) | *i* = 1 | 3.1416 | 3.141593 | 3.4972 | 3.4888 | 3.4748 | 3.4552 | 3.4301 |
| | *i* = 2 | 6.2832 | 6.283185 | 6.9933 | 6.9972 | 7.0039 | 7.0132 | 7.0254 |
| | *i* = 3 | 9.4248 | 9.424778 | 10.4871 | 10.4912 | 10.4982 | 10.5079 | 10.5205 |
| F-C ($T_0 = R_0 = 0$, $T_L = R_L = \infty$) | *i* = 1 | 1.8751 | − | 2.2155 | 2.3522 | 2.4937 | 2.6398 | 2.7903 |
| | *i* = 2 | 4.6941 | − | 5.3241 | 5.4261 | 5.5300 | 5.6361 | 5.7445 |
| | *i* = 3 | 7.8548 | − | 8.7982 | 8.8610 | 8.9272 | 8.9967 | 9.0697 |
| C-F ($T_0 = R_0 = \infty$, $T_L = R_L = 0$) | *i* = 1 | 1.8751 | 1.875104 | 1.9567 | 1.8349 | 1.7183 | 1.6069 | 1.5007 |
| | *i* = 2 | 4.6941 | 4.694090 | 5.1249 | 5.0272 | 4.9305 | 4.8343 | 4.7385 |
| | *i* = 3 | 7.8548 | 7.854753 | 8.6834 | 8.6314 | 8.5832 | 8.5388 | 8.4984 |

Table 17 First three dimensionless natural frequency coefficients $\mu_i$, $i$ = 1, 2, 3 of the uniform and AFG beam ($n$ = 2, $c$ = var.) with classical boundary conditions

| Boundary condistions | $\mu_i$ | Uniform beam | | AFG beam ($n$ = 2) | | | | |
|---|---|---|---|---|---|---|---|---|
| | | Present | Xing and Wang (2013) | $c$ = 0.00 | 0.25 | 0.50 | 0.75 | 1.00 |
| C-C ($T_0 = R_0 = \infty$, $T_L = R_L = \infty$) | $i$ = 1 | 4.7300 | 4.730039 | 4.7300 | 5.0121 | 5.2715 | 5.5133 | 5.7409 |
| | $i$ = 2 | 7.8532 | 7.853195 | 7.8532 | 8.3190 | 8.7438 | 9.1372 | 9.5054 |
| | $i$ = 3 | 10.9956 | 10.995581 | 10.9956 | 11.6464 | 12.2376 | 12.7834 | 13.2931 |
| P-C ($T_0 = \infty$, $R_0 = 0$, $T_L = R_L = \infty$) | $i$ = 1 | 3.9266 | − | 3.9266 | 4.2530 | 4.5474 | 4.8179 | 5.0696 |
| | $i$ = 2 | 7.0686 | − | 7.0686 | 7.5388 | 7.9686 | 8.3671 | 8.7406 |
| | $i$ = 3 | 10.2102 | − | 10.2102 | 10.8503 | 11.4335 | 11.9732 | 12.4781 |
| C-P ($T_0 = R_0 = \infty$, $T_L = \infty$, $R_L = 0$) | $i$ = 1 | 3.9266 | 3.926601 | 3.9266 | 4.0621 | 4.1829 | 4.2925 | 4.3931 |
| | $i$ = 2 | 7.0686 | 7.068577 | 7.0686 | 7.4389 | 7.7786 | 8.0948 | 8.3919 |
| | $i$ = 3 | 10.2102 | 10.210160 | 10.2102 | 10.7809 | 11.3014 | 11.7832 | 12.2341 |
| P-P ($T_0 = \infty$, $R_0 = 0$, $T_L = \infty$, $R_L = 0$) | $i$ = 1 | 3.1416 | 3.141593 | 3.1416 | 3.3213 | 3.4748 | 3.6095 | 3.7300 |
| | $i$ = 2 | 6.2832 | 6.283185 | 6.2832 | 6.6582 | 7.0039 | 7.3264 | 7.6302 |
| | $i$ = 3 | 9.4248 | 9.424778 | 9.4248 | 9.9851 | 10.4982 | 10.9746 | 11.4217 |
| F-C ($T_0 = R_0 = 0$, $T_L = R_L = \infty$) | $i$ = 1 | 1.8751 | − | 1.8751 | 2.1952 | 2.4937 | 2.7749 | 3.0414 |
| | $i$ = 2 | 4.6941 | − | 4.6941 | 5.1304 | 5.5300 | 5.9021 | 6.2526 |
| | $i$ = 3 | 7.8548 | − | 7.8548 | 8.4136 | 8.9272 | 9.4060 | 9.8569 |
| C-F ($T_0 = R_0 = \infty$, $T_L = R_L = 0$) | $i$ = 1 | 1.8751 | 1.875104 | 1.8751 | 1.7881 | 1.7183 | 1.6605 | 1.6113 |
| | $i$ = 2 | 4.6941 | 4.694090 | 4.6941 | 4.8168 | 4.9305 | 5.0362 | 5.1351 |
| | $i$ = 3 | 7.8548 | 7.854753 | 7.8548 | 8.2333 | 8.5832 | 8.9104 | 9.2189 |

## 4. Conclusions

In this paper, the analytical solutions for obtaining the exact natural frequencies of the AFG and uniform beams restrained with two rotational and two translational elastic springs were presented. In this way, based on the Euler-Bernoulli beam theory and general boundary conditions, the governing differential equation of motion was solved accurately by using the Bessel functions. Accordingly, the constant coefficients matrix of the AFG beams and/or uniform beam ($c$ = 0.0) was derived with the general elastic supports. Then, by taking the constant coefficients matrix determinant equal to zero and finding the positive real roots, the natural frequencies were obtained. The mass per unit length and the flexural rigidity of the AFG beams were assumed to vary continuously and together along the length direction according to the power-law forms. In the following, after the proposed formulation was verified, the effects of the AFG parameters and flexible ends on the first three natural frequencies of the AFG and uniform beams were investigated comprehensively. The analytical solutions were presented in tabular and graphical forms and could be utilized as either the benchmark problems or design of composite beams with various supporting conditions.

According to the results of this study, the following important points are concluded:

- Depending on the values of stiffnesses of end restraints, the effects of the flexibility of supports would be more significant or less considerable in the free vibration problem of the beam structures.
- As the stiffness of end restraints increases, the natural frequencies of the beam always increase.
- The effect of the translational stiffness on the natural frequencies of the AFG and/or uniform beam with the general boundary conditions is more considerable than the rotational stiffness. For example, in the free supported beam with two translational springs, increasing stiffness of the translational springs from 0.1 to $10^5$, can raise the first dimensionless natural frequency coefficient $\mu_1$ of the uniform beam ($c$ = 0.0) and AFG beam ($n$ = 2, $c$ = 0.5) to a maximum of 4.70 and 5.10 times, respectively.
- In most cases, the natural frequencies of the AFG beam are greater than those of the uniform beam with the same supports.
- For the composite beam with the symmetric elastic boundary conditions, as the AFG parameters $n$ and $c$ increase, the natural frequencies of the beam always increase.
- The effect of the gradient coefficient $c$ on the natural frequencies of the AFG beam with the classical or non-classical boundary conditions is more significant than the gradient index $n$. For example, as the parameters $n$ and $c$ rise, the first dimensionless natural frequency coefficient of the F-C beam can increase by about 49% (corresponding to $n$ = 4 and $c$ = 0.5) and 62% (corresponding to $n$ = 2 and $c$ = 1.0) versus the uniform ones, respectively.

- For the specified quantities of the AFG parameters *n* and *c*, the C-C and C-F beams have the maximum and minimum values of the first three dimensionless natural frequency coefficients, respectively.

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*CC*

## Appendix A

For the AFG beams and/or uniform beam ($c \to 0.0$) the terms of the constant coefficients matrix, **F** are as follows

$$F_{11} = -\mu J_n\left(\frac{2\mu}{c}\right) + [R_0 + c(n+1)]J_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A1)}$$

$$F_{12} = -\mu Y_n\left(\frac{2\mu}{c}\right) + [R_0 + c(n+1)]Y_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A2)}$$

$$F_{13} = \mu I_n\left(\frac{2\mu}{c}\right) - [R_0 + c(n+1)]I_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A3)}$$

$$F_{14} = \mu K_n\left(\frac{2\mu}{c}\right) + [R_0 + c(n+1)]K_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A4)}$$

$$F_{21} = T_0 J_n\left(\frac{2\mu}{c}\right) + \mu^3 J_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A5)}$$

$$F_{22} = T_0 Y_n\left(\frac{2\mu}{c}\right) + \mu^3 Y_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A6)}$$

$$F_{23} = T_0 I_n\left(\frac{2\mu}{c}\right) + \mu^3 I_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A7)}$$

$$F_{24} = T_0 K_n\left(\frac{2\mu}{c}\right) - \mu^3 K_{n+1}\left(\frac{2\mu}{c}\right) \quad \text{(A8)}$$

$$F_{31} = -\sqrt{1+c}\,\mu J_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) - [R_L(1+c) - c(n+1)]J_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A9)}$$

$$F_{32} = -\sqrt{1+c}\,\mu Y_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) - [R_L(1+c) - c(n+1)]Y_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A10)}$$

$$F_{33} = \sqrt{1+c}\,\mu I_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) + [R_L(1+c) - c(n+1)]I_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A11)}$$

$$F_{34} = \sqrt{1+c}\,\mu K_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) - [R_L(1+c) - c(n+1)]K_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A12)}$$

$$F_{41} = -T_L(1+c)^2 J_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) + \mu^3\sqrt{1+c}\,J_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A13)}$$

$$F_{42} = -T_L(1+c)^2 Y_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) + \mu^3\sqrt{1+c}\,Y_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A14)}$$

$$F_{43} = -T_L(1+c)^2 I_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) + \mu^3\sqrt{1+c}\,I_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A15)}$$

$$F_{44} = -T_L(1+c)^2 K_n\left(\frac{2\mu\sqrt{1+c}}{c}\right) - \mu^3\sqrt{1+c}\,K_{n+1}\left(\frac{2\mu\sqrt{1+c}}{c}\right) \quad \text{(A16)}$$