Steel and Composite Structures, *Vol. 33, No. 1* (*2019*) *133-142*
DOI: https://doi.org/10.12989/scs.2019.33.1.133

Static stability analysis of axially functionally graded tapered micro columns with different boundary conditions

Bekir Akgöz*

Department of Civil Engineering, Faculty of Engineering, Akdeniz University, 07058, Antalya, Turkey

(*Received June 22, 2019, Revised September 3, 2019, Accepted September 5, 2019*)

Abstract. In the present study, microstructure-dependent static stability analysis of inhomogeneous tapered micro-columns is performed. It is considered that the micro column is made of functionally graded materials and has a variable cross-section. The material and geometrical properties of micro column vary continuously throughout the axial direction. Euler-Bernoulli beam and modified couple stress theories are used to model the nonhomogeneous micro column with variable cross section. Rayleigh-Ritz solution method is implemented to obtain the critical buckling loads for various parameters. A detailed parametric study is performed to examine the influences of taper ratio, material gradation, length scale parameter, and boundary conditions. The validity of the present results is demonstrated by comparing them with some related results available in the literature. It can be emphasized that the size-dependency on the critical buckling loads is more prominent for bigger length scale parameter-to-thickness ratio and changes in the material gradation and taper ratio affect significantly the values of critical buckling loads.

Keywords: critical buckling load; size dependency; variable cross section; axially functionally graded materials; Rayleigh-Ritz method

1. Introduction

In many engineering applications such as civil, mechanical, structural, aerospace, and automotive, it is expected that the structural components are to be safety, functional, long lasting, aesthetic, and economical. The use of nonuniform, laminated, sandwiched, nonhomogeneous and reinforced elements with composite and alloy materials may be efficacious to maintain the desired characteristics as lighter, durable, cheaper, and more efficient.

A most recent example on the use of composite materials in aviation is Boeing 787 Dreamliner. Boeing 787 Dreamliner represents a new generation of carbon composite airliners. The chief breakthrough material technology on this airplane is the increased use of composites. The 787 is 50 percent composite by weight. A majority of the primary structure is made of composite materials, most notably the fuselage. Composite materials in aircrafts have many advantages. They allow a lighter, simpler structure, and more economical which increases airplane efficiency, reduces fuel consumption and reduces weight-based maintenance and fees. They do not fatigue or corrode, which reduces scheduled maintenance and increases productive time. Composites resist impacts better and are designed for easy visual inspection (Boeing 2019).

There are various types of composites. As one of them, the conventional layered composites consist of individual layers with different material properties. As a disadvantage of them, delamination may occur due to undesired high stresses between two adjacent layers. As a relatively new kind of composites, called as functionally graded materials (FGMs), which consist of at least two materials with different properties that the properties are changed gradually and continuously throughout one or more directions. The concept of FGMs was firstly introduced by Japanese scientists during a spacecraft project as a thermal barrier material for propulsion and airframe structural systems of the spacecraft in 1984.

As stated before, the material properties of FGMs vary gradually along the length and thickness or both of them. Consequently, structural elements made of FGMs are widely used in space transportation, nuclear reactors, defense industries, biomedical, enhanced sports equipment, and chemical plants. For this reason, it is crucial to determine the mechanical behaviors of structures made of FGMs. Consequently, many theoretical studies have been performed on this topic (Li *et al*. 2006, Aydogdu 2008, Sahraee and Saidi 2009, Anandrao *et al*. 2010, Kiani and Eslami 2010, Atmane *et al*. 2011, Hamzehkolaei *et al*. 2011, Wattanasakulpong *et al*. 2011, Kocaturk and Akbas 2013, Saidi *et al*. 2013, Atmane *et al*. 2015, Bennai *et al*. 2015, Hamidi *et al*. 2015, Nguyen *et al*. 2015, Tagrara *et al*. 2015, Mahmoud and Tounsi 2017, Avcar 2019).

Due to rapid developments in technology, the small-sized structures, in which the characteristics dimensions of them are on the order of microns and sub-microns, have been extensively used in nano-and micro-electro mechanical systems (NEMS and MEMS) (Younis *et al*. 2003, Li *et al*. 2004, He *et al*. 2009). Microbeam or microcolumn is one of the essential structures frequently used in MEMS/NEMS such as atomic force microscopes

*Corresponding author, Ph.D.,
E-mail: bekirakgoz@akdeniz.edu.tr

http://www.techno-press.org/?journal=scs&subpage=6
ISSN: 1229-9367 (Print), 1598-6233 (Online)

(Lee and Chang 2009, Payam and Fathipour 2009), micro-resonators (He *et al*. 2009), micro-switches (Gusso *et al*. 2019), micro-actuators (Ak *et al*. 2017). Size influence on the mechanical deformation behaviors of such structures has been experimentally proved (Lam and Chong 2001, Lam *et al*. 2003, Lei *et al*. 2016, Li *et al*. 2018).

It is considerable that continuum mechanics approaches may be an efficient way to predict the mechanical responses of micro-and nano-sized structures instead of expensive and laborious experiments and time consuming and limited atomistic modeling as molecular dynamic simulation. However, the continuum models elaborated by classical (macro) elasticity theory fail to take into account this size effect on the mechanical characteristics of such structures due to the lack of any additional material length scale parameters. After that, various non-classical continuum theories have been developed like couple stress theory (Mindlin and Tiersten 1962, Koiter 1964, Toupin 1964), micro polar theory (Eringen 1967), nonlocal elasticity theory (Eringen 1972, 1983), strain gradient theories (Fleck and Hutchinson 1993, 2001, Vardoulakis and Sulem 1995, Aifantis 1999), and nonlocal strain gradient theory (Lim *et al*. 2015).

Modified couple stress theory (MCST) is a popular higher-order continuum theory which is evolved by Yang *et al*. (2002). This theory can be defined as a special case of modified strain gradient theory (MSGT). This simpler and useful theory contains only one length scale parameter. MCST has been widely used to investigate static and dynamic responses of microbeams (Park and Gao 2006, Ma *et al*. 2008, 2010, Akgöz and Civalek 2011, Nateghi *et al*. 2012, Salamat-Talab *et al*. 2012, Simsek and Reddy 2013, Al-Basyouni *et al*. 2015, Jahangiri *et al*. 2015, Khorshidi *et al*. 2016, Park *et al*. 2016, Ehyaei and Akbarizadeh 2017, Amar *et al*. 2018, Hadi *et al*. 2018, Jia *et al*. 2018, Khaniki and Rajasekaran 2018, Rahmani *et al*. 2018b, Ebrahimi and Mahmoodi 2019, Thanh *et al*. 2019). On the other hand, nonlocal and strain gradient theories have also been utilized to model the small-sized structures (Simsek 2011, Nguyen *et al*. 2014, Belkorissat *et al*. 2015, She *et al*. 2017, 2018, Arefi 2018, Houari *et al*. 2018, Karami *et al*. 2018, Nazemnezhad and Kamali 2018, Rahmani *et al*. 2018a, Shafiei and She 2018).

As stated before, size-dependent continuum modeling of such structures has attracted much interest from researchers over the last two decades. However, most of these researches are related to the investigation mechanical characteristics of micro-/nano-sized homogeneous and prismatic structures with constant cross-section. Researches on the static and dynamic responses of axially functionally graded (AFG) tapered small-sized structures have relatively been limited (Simsek 2012, Akgöz and Civalek 2013a, b, Marques *et al*. 2014, Shafiei *et al*. 2016a, b, c, 2019, Ghayesh *et al*. 2017, Shafiei and Kazemi 2017, Shafiei *et al*. 2017, Ghayesh 2018a, b, c, d, 2019, Ghayesh and Farokhi 2018a, b, Nguyen and Tran 2018, Rezaiee-Pajand *et al*. 2018).

The aim of the present study is to perform the size-dependent stability analysis of AFG tapered micro columns based on modified couple stress and Euler-Bernoulli beam theories. It is assumed that the micro column is made of axially functionally graded materials and the material properties and the cross section of it change continuously and smoothly throughout the longitudinal direction. Rayleigh-Ritz method is utilized to obtain the critical buckling loads for different boundary conditions. A detailed parametric study is performed to peruse the influences of various parameters like taper ratio, variation type, length scale parameter-to-thickness ratio, boundary conditions, material property variation on the critical buckling loads of AFG tapered micro columns.

2. Formulation

The strain energy U in a linear elastic isotropic material based on the modified couple stress theory can be written by (Yang *et al*. 2002, Park and Gao 2006)

$$U = \frac{1}{2}\int_0^L \int_A \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}^s \chi_{ij}^s\right) dA\, dx \tag{1}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

$$\chi_{ij}^s = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) \tag{3}$$

$$\theta_i = \frac{1}{2} e_{ijk} u_{k,j} \tag{4}$$

where u_i, θ_i, ε_{ij}, and χ_{ij}^s denote the components of the displacement vector $\mathbf{u}$, the rotation vector $\boldsymbol{\theta}$, the strain tensor $\boldsymbol{\varepsilon}$, and the symmetric rotation gradient tensor $\boldsymbol{\chi}^s$, respectively. Also, e_{ijk} is the permutation symbol.

On the other hand, the components of the classical stress tensor $\boldsymbol{\sigma}$ (conjugated with the strain tensor) and deviatoric part of couple stress tensor $\mathbf{m}^s$ (conjugated with the rotation gradient tensor) can be expressed by (Yang *et al*. 2002, Ma *et al*. 2008)

$$\sigma_{ij} = \lambda\varepsilon_{mm}\delta_{ij} + 2G\varepsilon_{ij} \tag{5}$$

$$m_{ij}^s = 2Gl^2\chi_{ij}^s \tag{6}$$

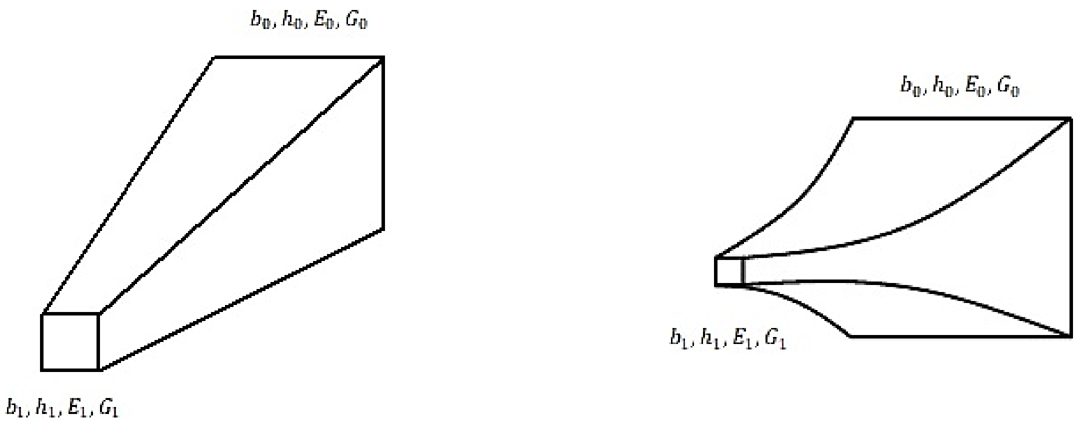
where δ_{ij} is Kronecker delta and l is material length scale parameter related to rotation gradients. Furthermore, λ and G are the Lamé constants defined as follows

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad G = \frac{E}{2(1+\nu)} \tag{7}$$

where E and ν are the Young's modulus and Poisson's ratio, respectively.

The axial and transverse displacements of any point of an initially straight column (see Fig. 1) based on Euler-Bernoulli beam theory can be respectively described as

$$u_1(x,z) = -z\frac{\mathrm{d}w(x)}{\mathrm{d}x}, \quad u_3(x,z) = w(x) \tag{8}$$



(a) Linearly tapered (b) Parabolic or exponential tapered

Fig. 1 Schematic representations of tapered one-dimensional structures

Substituting Eq. (8) into Eq. (2), we obtain the non-zero strain component as

$$\varepsilon_{11} = -z\frac{d^2w}{dx^2} \tag{9}$$

and the non-zero components of rotation gradient are determined by substituting Eq. (9) into Eqs. (3)-(4)

$$\chi_{12}^s = \chi_{21}^s = -\frac{1}{2}\frac{d^2w}{dx^2} \tag{10}$$

The non-zero components of classical and deviatoric part of couple stress tensors can be achieved by implementing Eqs. (9)-(10) in Eqs. (5)-(6) as (by ignoring Poisson effect)

$$\sigma_{11} = -Ez\frac{d^2w}{dx^2} \tag{11}$$

$$m_{12}^s = m_{21}^s = -Gl_2^2\frac{d^2w}{dx^2} \tag{12}$$

Substituting above equations into Eq. (1) with some mathematical manipulations yields an expression for the strain energy U (Kong *et al*. 2009, Akgöz and Civalek 2011)

$$U = \frac{1}{2}\int_0^L (EI + GAl^2)\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{13}$$

where I and A are the moment of inertia and cross section area of the microcolumn, respectively.

3. Buckling problem of a nonhomogeneous and nonuniform micro column

In the present study, it is assumed that the material and geometrical properties of micro column change continuously and smoothly along the longitudinal direction. Consequently, the strain energy U in Eq. (13) can be rewritten for AFG tapered micro-columns shown in Fig. 1 as following

$$U = \frac{1}{2}\int_0^L \left(E_{(x)}I_{(x)} + G_{(x)}A_{(x)}l^2\right)\left(\frac{d^2w}{dx^2}\right)^2 dx \tag{14}$$

in which $E_{(x)}$, $I_{(x)}$, $G_{(x)}$, and $A_{(x)}$ are the variable elasticity modulus, second moment of inertia, shear modulus, and cross-section area, respectively. They can be expressed by related to x as follows

$$E_{(x)} = E_0\left(1 - f_e\frac{x}{L}\right) \tag{15}$$

$$G_{(x)} = G_0\left(1 - g_e\frac{x}{L}\right) \tag{16}$$

$$I_{(x)} = I_0\left(1 - b\frac{x}{L}\right)^a \tag{17}$$

$$A_{(x)} = A_0\left(1 - b\frac{x}{L}\right)^a \tag{18}$$

4. Implementation of Rayleigh-Ritz Method

The total potential energy of the axially functionally graded micro columns with variable cross section subjected to axial load can be expressed as

$$\Pi = U + V \tag{19}$$

where U is the strain energy of micro column given Eq. (14) and V is the energy of axial pressure load can be defined as

$$V = -\frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{20}$$

in which P is the axial pressure load. Substituting Eqs. (14) and (20) in Eq. (19) yields the following relation as

$$\Pi = \frac{1}{2}\int_0^L \left(E_{(x)}I_{(x)} + G_{(x)}A_{(x)}l^2\right)\left(\frac{d^2w}{dx^2}\right)^2 dx - \frac{P}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx \tag{21}$$

Table 1 Values of p and q for four different boundary conditions

Boundary conditions	p	q
Simply Supported-Simply Supported (S-S)	1	1
Clamped-Free (C-F)	2	0
Clamped-Clamped (C-C)	2	2
Clamped-Simply Supported (C-S)	2	1

According to Rayleigh-Ritz method, $w(x)$ can be described in the polynomial form as

$$w(x) = \sum_{i=1}^{N} c_i \phi_i(x) \tag{22}$$

where c_i are unknown constant coefficients and $\phi_i(x)$ is an admissible function that must be satisfy only the geometric boundary conditions and N is the number of polynomials. In the present study, $\phi_i(x)$ is chosen as following (Pradhan and Chakraverty 2013)

$$\phi_i(x) = x^p (L-x)^q x^{i-1}, p, q = 0,1,2 \text{ and } i = 1,2,\ldots,N \tag{23}$$

Superscripts p and q are related with boundary conditions and the values of them for different boundary conditions are presented in Table 1 (Pradhan and Chakraverty 2013).

The stationary points of total potential energy can be expressed by taking partial derivative of total potential energy in Eq. (21) with respect to the unknown constant coefficients as

$$\frac{\partial \Pi}{\partial c_i} = 0, \quad i = 1,2,\ldots,N \tag{24}$$

5. Numerical results and discussion

In order to demonstrate the validity and sensitivity of the present analysis, the non-dimensional critical buckling loads of homogeneous tapered columns obtained based on classical theory are compared with those of the previous

Table 2 Comparison of dimensionless critical buckling loads for homogeneous columns with linear variation of flexural rigidity for $a = 1$

b	C–F		S–S		C–S		C–C	
	Exact	Present	Exact	Present	Exact	Present	Exact	Present
0.1	2.393	2.3928	9.372	9.3716	19.17	19.1686	37.48	37.4765
0.3	2.235	2.2351	8.343	8.3434	17.03	17.0353	33.27	33.2733
0.5	2.062	2.0621	7.256	7.2556	14.74	14.7394	28.70	28.6970
0.7	1.865	1.8653	6.069	6.0693	12.18	12.1772	23.48	23.4828
0.9	1.621	1.6211	4.667	4.6667	9.029	9.0294	16.70	16.7001

Table 3 Comparison of dimensionless critical buckling loads for homogeneous columns with quadratic variation of flexural rigidity for $a = 2$

b	C–F		S–S		C–S		C–C	
	Exact	Present	Exact	Present	Exact	Present	Exact	Present
0.1	2.319	2.3191	8.893	8.8934	18.19	18.1893	35.56	35.5610
0.3	2.012	2.0115	7.005	7.0048	14.29	14.2915	27.91	27.9067
0.5	1.683	1.6830	5.198	5.1981	10.53	10.5273	20.48	20.4808
0.7	1.318	1.3180	3.459	3.4588	6.869	6.8682	13.23	13.2287
0.9	0.862	0.8616	1.710	1.7106	3.164	3.1647	5.864	5.8640

Table 4 Comparison of dimensionless critical buckling loads for homogeneous columns with cubic variation of flexural rigidity for $a = 3$

b	C–F		S–S		C–S		C–C	
	Exact	Present	Exact	Present	Exact	Present	Exact	Present
0.1	2.246	2.2464	8.436	8.4344	17.25	17.2517	33.73	33.7289
0.3	1.798	1.7977	5.840	5.8404	11.92	11.9231	23.29	23.2913
0.5	1.336	1.3364	3.628	3.6278	7.362	7.3622	14.35	14.3485
0.7	0.853	0.8533	1.821	1.8208	3.634	3.6344	7.045	7.0449
0.9	0.321	0.3215	0.467	0.4686	0.875	0.8790	1.670	1.6711

Table 5 Dimensionless critical buckling loads for axially functionally graded micro columns with linear variation of flexural rigidity for $a = 1,\ f_e = g_e = 0.5,\ l = 0.5h_0$

b	C–F		S–S		C–S		C–C	
	CT	MCST	CT	MCST	CT	MCST	CT	MCST
0.1	1.9924	4.2913	6.8625	14.7809	13.9478	30.0413	27.1659	58.5113
0.3	1.8449	3.9737	6.0525	13.0362	12.2982	26.4885	23.9588	51.6036
0.5	1.6830	3.6249	5.1981	11.1958	10.5273	22.6742	20.4808	44.1126
0.7	1.4980	3.2265	4.2692	9.1952	8.5567	18.4299	16.5405	35.6258
0.9	1.2658	2.7263	3.1735	6.8353	6.1413	13.2274	11.4677	24.6996

Table 6 Dimensionless critical buckling loads for axially functionally graded micro columns with quadratic variation of flexural rigidity for $a = 2,\ f_e = g_e = 0.5,\ l = 0.5h_0$

b	C–F		S–S		C–S		C–C	
	CT	MCST	CT	MCST	CT	MCST	CT	MCST
0.1	1.9238	4.1435	6.4868	13.9716	13.1924	28.4145	25.7056	55.3658
0.3	1.6384	3.5289	5.0139	10.7993	10.2055	21.9810	19.9048	42.8718
0.5	1.3364	2.8785	3.6278	7.8137	7.3622	15.8571	14.3485	30.9044
0.7	1.0057	2.1661	2.3242	5.0060	4.6549	10.0259	9.0209	19.4296
0.9	0.6034	1.2996	1.0664	2.2969	2.0127	4.3350	3.7846	8.1514

Table 7 Dimensionless critical buckling loads for axially functionally graded micro columns with cubic variation of flexural rigidity for $a = 3,\ f_e = g_e = 0.5,\ l = 0.5h_0$

b	C–F		S–S		C–S		C–C	
	CT	MCST	CT	MCST	CT	MCST	CT	MCST
0.1	1.8563	3.9982	6.1280	13.1987	12.4723	26.8633	24.3133	52.3670
0.3	1.4435	3.1091	4.1251	8.8848	8.4242	18.1445	16.4561	35.4440
0.5	1.0290	2.2162	2.4674	5.3144	5.0477	10.8719	9.8696	21.2576
0.7	0.6127	1.3197	1.1713	2.5228	2.3859	5.1388	4.6565	10.0293
0.9	0.1987	0.4280	0.2723	0.5864	0.5343	1.1507	1.0225	2.2023

work for different values of a as 1, 2, and 3 in Tables 2-4, respectively. It is clearly seen from the tables that there is an excellent agreement between the exact (Wang *et al*. 2005) and present results for all boundary conditions, taper ratios, and variation types of cross section. It is notable that the number of polynomials is chosen as ten ($N = 10$) in the present analysis.

Tables 5-7 show the variation of non-dimensional classical and non-classical critical buckling loads for axially functionally graded micro columns with linear, quadratic, and cubic variation of flexural rigidity with respect to various taper ratios, respectively. It is found from the tables that an increase in taper ratio gives rise to a decrement in the critical buckling loads. It is also observed that the critical buckling loads obtained based on modified couple stress theory (MCST) are always greater than those predicted by classical theory (CT). The reason of differences between the dimensionless critical buckling loads based on CT and MCST can be explained as an increment in the bending rigidity of the structure due to size effect for non-classical model. In addition, it is seen from these tables that the critical buckling loads for $a = 1$ are bigger than those for $a = 2$ and $a = 3$. This situation can be explained as the decrease in cross section as well as second moment of inertia for linear variation type is less than those of quadratic and cubic variation types. It should be indicated that the results of CT are calculated by letting $l = 0$ in the related formulations.

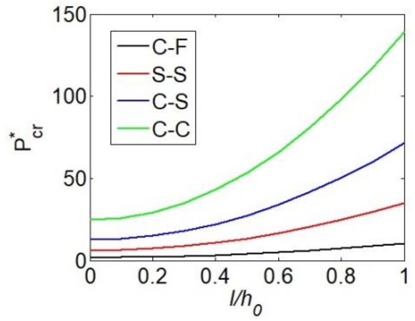


Fig. 2 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling loads of the micro column for $f_e = g_e = 0.25, a = 1, b = 0.5$

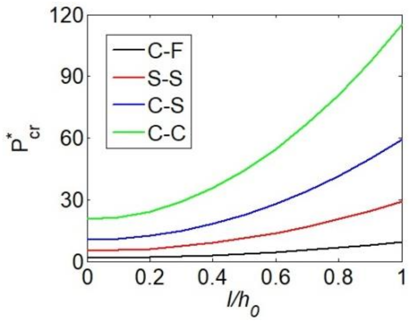


Fig. 3 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.5, a = 1, b = 0.5$

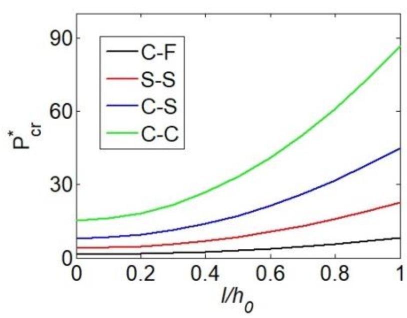


Fig. 4 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling loads of the micro column for $f_e = g_e = 0.75, a = 1, b = 0.5$

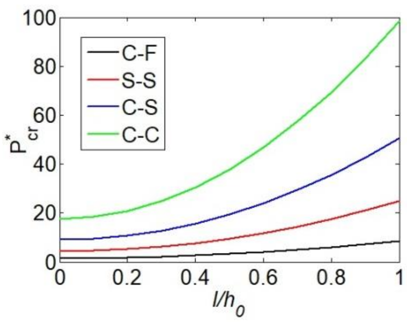


Fig. 5 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.25, a = 2, b = 0.5$

Effects of length scale parameter-to-thickness ratio (l/h_0) on the dimensionless critical buckling loads of the axially functionally graded tapered micro column are respectively illustrated in Figs. 2-4 for $f_e = g_e = 0.25, 0.5$ and 0.75 and $a = 1, b = 0.5$. It is notable that the results for $l/h_0 = 0$ represent the classical buckling loads and an increase in l/h_0 leads an increment in the critical buckling loads. Also, it is clearly seen from the figures that the dimensionless critical buckling loads of C-C are the biggest while those of C-F are the smallest. Moreover, it can be observed that the dimensionless critical buckling loads decrease by increasing f_e and g_e.

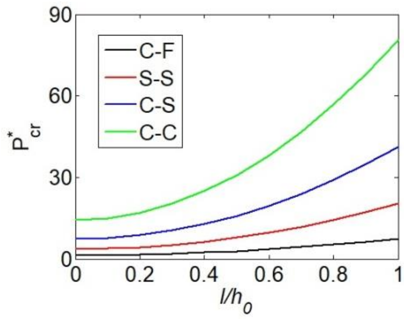


Fig. 6 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.5, a = 2, b = 0.5$

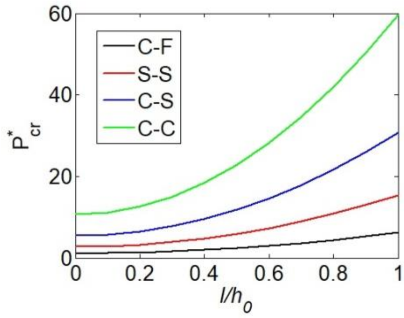


Fig. 7 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.75, a = 2, b = 0.5$

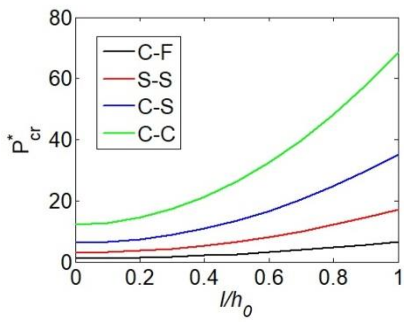


Fig. 8 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.25, a = 3, b = 0.5$

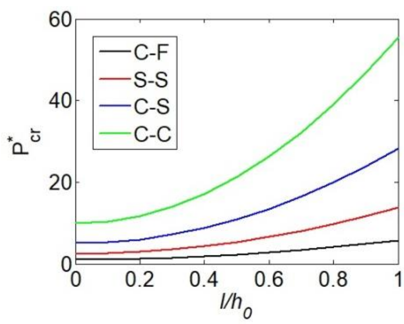


Fig. 9 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.5, a = 3, b = 0.5$

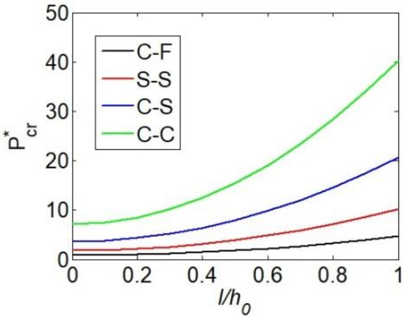


Fig. 10 Influence of length scale parameter-to-thickness ratio on the dimensionless critical buckling load of the micro column for $f_e = g_e = 0.75, a = 3, b = 0.5$

Figs. 5-7 and Figs. 8-10 depict the variation of dimensionless critical buckling loads of the axially functionally graded tapered micro column with respect to length scale parameter-to-thickness ratio with various values of f_e and g_e for quadratic ($a = 2$) and cubic ($a = 3$) variations. Similar interpretations with Figs. 2-4 can be made for Figs. 5-7 and Figs. 8-10. Additionally, it is observed from the figures that the non-dimensional critical buckling loads tend to decrease by increasing the value of a.

6. Conclusions

In this paper, microstructure-dependent buckling behavior of nonuniform nonhomogeneous micro columns is examined. It is assumed that the micro column is made of axially functionally graded materials and the material properties and the cross section of it change continuously and smoothly throughout the longitudinal direction. Critical buckling loads are obtained by Rayleigh-Ritz method for four different boundary conditions. Effects of several parameters such as taper ratio, variation type, length scale parameter-to-thickness ratio, boundary conditions, material property variation on the critical buckling loads of axially functionally graded tapered micro columns are investigated in detail. Main observations from the numerical results can be outlined as following:

- The present critical buckling loads agree very well with the previously published exact results in the literature.
- An increase in the taper ratio gives rise to a decrease in the critical buckling loads.
- The size-dependent critical buckling loads are always larger than the classical ones.
- C–C and C–F boundary conditions have the biggest and lowest critical buckling loads, respectively.
- The critical buckling loads decrease as the values of a, f_e, and g_e increase.
- The benchmark results will be a useful reference for the related works in the future.

Acknowledgments

The financial support of the Scientific Research Projects Unit of Akdeniz University is gratefully acknowledged.

References

Aifantis, E.C. (1999), "Gradient deformation models at nano, micro, and macroscales", *J. Eng. Mater. Technol.*, **121**, 189-202. https://doi.org/10.1115/1.2812366

Ak, C., Yildiz, A., Akdagli, A. and Bicer, M.B. (2017), "Computing the pull-in voltage of fixed-fixed micro-actuators by artificial neural network", *Microsyst. Technol.*, **23**(8), 3537-3546. https://doi.org/10.1007/s00542-016-3128-4

Akgöz, B. and Civalek, O. (2011), "Buckling analysis of cantilever carbon nanotubes using the strain gradient elasticity and modified couple stress theories", *J. Comput. Theor. Nanostruct.*, **8**(9), 1821-1827. https://doi.org/10.1166/jctn.2011.1888

Akgöz, B. and Civalek, O. (2013a), "Buckling analysis of linearly tapered micro-columns based on strain gradient elasticity", *Struct. Eng. Mech., Int. J.*, **48**(2), 195-205. https://doi.org/10.12989/sem.2013.48.2.195

Akgöz, B. and Civalek, O. (2013b), "Free vibration analysis of axially functionally graded tapered Bernoulli-Euler microbeams based on the modified couple stress theory", *Compos. Struct.*, **98**, 314-322. https://doi.org/10.1016/j.compstruct.2012.11.020

Al-Basyouni, K.S., Tounsi, A. and Mahmoud, S.R. (2015), "Size dependent bending and vibration analysis of functionally graded micro beams based on modified couple stress theory and neutral surface position", *Compos. Struct.*, **125**, 621-630. https://doi.org/10.1016/j.compstruct.2014.12.070

Amar, L.H.H., Kaci, A., Yeghnem, R. and Tounsi, A. (2018), "A new four-unknown refined theory based on modified couple stress theory for size-dependent bending and vibration analysis of functionally graded micro-plate", *Steel Compos. Struct., Int. J.*, **26**(1), 89-102. https://doi.org/10.12989/scs.2018.26.1.089

Anandrao, K.S., Gupta, R.K., Ramchandran, P. and Rao, G.V. (2010), "Thermal post-buckling analysis of uniform slender functionally graded material beams", *Struct. Eng. Mech., Int. J.*, **36**(5), 545-560. https://doi.org/10.12989/sem.2010.36.5.545

Arefi, M. (2018), "Nonlocal free vibration analysis of a doubly curved piezoelectric nano shell", *Steel Compos. Struct., Int. J.*, **27**(4), 479-493. https://doi.org/10.12989/scs.2018.27.4.479

Atmane, H.A., Tounsi, A., Ziane, N. and Mechab, I. (2011), "Mathematical solution for free vibration of sigmoid functionally graded beams with varying cross-section", *Steel Compos. Struct., Int. J.*, **11**(6), 489-504. https://doi.org/10.12989/scs.2011.11.6.489

Atmane, H.A., Tounsi, A., Bernard, F. and Mahmoud, S.R. (2015), "A computational shear displacement model for vibrational analysis of functionally graded beams with porosities", *Steel Compos. Struct., Int. J.*, **19**(2), 369-384. https://doi.org/10.12989/scs.2015.19.2.369

Avcar, M. (2019), "Free vibration of imperfect sigmoid and power law functionally graded beams", *Steel Compos. Struct., Int. J.*, **30**(6), 603-615. https://doi.org/10.12989/scs.2019.30.6.603

Aydogdu, M. (2008), "Semi-inverse method for vibration and buckling of axially functionally graded beams", *J. Reinforced Plast. Comp.*, **27**(7), 683-691. https://doi.org/10.1177/0731684407081369

Belkorissat, I., Houari, M.S.A., Tounsi, A., Bedia, E.A.A. and Mahmoud, S.R. (2015), "On vibration properties of functionally graded nano-plate using a new nonlocal refined four variable model", *Steel Compos. Struct., Int. J.*, **18**(4), 1063-1081. https://doi.org/10.12989/scs.2015.18.4.1063

Bennai, R., Atmane, H.A. and Tounsi, A. (2015), "A new higher-order shear and normal deformation theory for functionally graded sandwich beams", *Steel Compos. Struct., Int. J.*, **19**(3), 521-546. https://doi.org/10.12989/scs.2015.19.3.521

Boeing (2019), https://www.boeing.com/commercial/787/by-design/#/advanced-composite-use

Ebrahimi, F. and Mahmoodi, F. (2019), "A modified couple stress theory for buckling analysis of higher order inhomogeneous microbeams with porosities", *Proc. Inst. Mech. Eng. C-J. Mech. Eng. Sci.*, **233**(8), 2855-2866. https://doi.org/10.1177/0954406218791642

Ehyaei, J. and Akbarizadeh, M.R. (2017), "Vibration analysis of micro composite thin beam based on modified couple stress", *Struct. Eng. Mech., Int. J.*, **64**(4), 403-411.

Eringen, A.C. (1967), "Theory of micropolarplates", *Z. Angew. Math. Phys.*, **18**, 12-30. https://doi.org/10.1007/BF01593891

Eringen, A.C. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**, 1-16. https://doi.org/10.1016/0020-7225(72)90070-5

Eringen, A.C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys.*, **54**, 4703-4710. https://doi.org/10.1063/1.332803

Fleck, N.A. and Hutchinson, J.W. (1993), "A phenomenological theory for strain gradient effects in plasticity", *J. Mech. Phys. Solids*, **41**, 1825-1857. https://doi.org/10.1016/0022-5096(93)90072-N

Fleck, N.A. and Hutchinson, J.W. (2001), "A reformulation of strain gradient plasticity", *J. Mech. Phys. Solids*, **49**, 2245-2271. https://doi.org/10.1016/S0022-5096(01)00049-7

Ghayesh, M.H. (2018a), "Mechanics of tapered AFG shear-deformable microbeams", *Microsyst. Technol.*, **24**(4), 1743-1754. https://doi.org/10.1007/s00542-018-3764-y

Ghayesh, M.H. (2018b), "Nonlinear vibration analysis of axially functionally graded shear-deformable tapered beams", *Appl. Math. Model.*, **59**, 583-596. https://doi.org/10.1016/j.apm.2018.02.017

Ghayesh, M.H. (2018c), "Nonlinear vibrations of axially functionally graded Timoshenko tapered beams", *J. Comput. Nonlinear Dyn.*, **13**(4), 041002. https://doi.org/10.1115/1.4039191

Ghayesh, M.H. (2018d), "Vibration analysis of shear-deformable AFG imperfect beams", *Compos. Struct.*, **200**, 910-920. https://doi.org/10.1016/j.compstruct.2018.03.091

Ghayesh, M.H. (2019), "Resonant dynamics of axially functionally graded imperfect tapered Timoshenko beams", *J. Vib. Control*, **25**(2), 336-350. https://doi.org/10.1177/1077546318777591

Ghayesh, M.H. and Farokhi, H. (2018a), "Bending and vibration analyses of coupled axially functionally graded tapered beams", *Nonlinear Dyn.*, **91**(1), 17-28. https://doi.org/10.1007/s11071-017-3783-8

Ghayesh, M.H. and Farokhi, H. (2018b), "Mechanics of tapered axially functionally graded shallow arches", *Compos. Struct.*, **188**, 233-241. https://doi.org/10.1016/j.compstruct.2017.11.017

Ghayesh, M.H., Farokhi, H., Gholipour, A. and Tavallaeinejad, M. (2017), "Nonlinear bending and forced vibrations of axially functionally graded tapered microbeams", *Int. J. Eng. Sci.*, **120**, 51-62. https://doi.org/10.1016/j.ijengsci.2017.03.010

Gusso, A., Viana, R.L., Mathias, A.C. and Caldas, I.L. (2019), "Nonlinear dynamics and chaos in micro/ nanoelectromechanical beam resonators actuated by two-sided electrodes", *Chaos Soliton Fract.*, **122**, 6-16. https://doi.org/10.1016/j.chaos.2019.03.004

Hadi, A., Nejad, M.Z., Rastgoo, A. and Hosseini, M. (2018), "Buckling analysis of FGM Euler-Bernoulli nano-beams with 3D-varying properties based on consistent couple-stress theory", *Steel Compos. Struct., Int. J.*, **26**(6), 663-672. https://doi.org/10.12989/scs.2018.26.6.663

Hamidi, A., Houari, M.S.A., Mahmoud, S.R. and Tounsi, A. (2015), "A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates", *Steel Compos. Struct., Int. J.*, **18**(1), 235-253. https://doi.org/10.12989/scs.2015.18.1.235

Hamzehkolaei, N.S., Malekzadeh, P. and Vaseghi, J. (2011), "Thermal effect on axisymmetric bending of functionally graded circular and annular plates using DQM", *Steel Compos. Struct., Int. J.*, **11**(4), 341-358. https://doi.org/10.12989/scs.2011.11.4.341

He, X.J., Wu, Q., Wang, Y., Song, M.X. and Yin, J.H. (2009), "Numerical simulation and analysis of electrically actuated microbeam-based MEMS capacitive switch", *Microsyst. Technol.*, **15**(2), 301-307. https://doi.org/10.1007/s00542-008-0702-4

Houari, M.S.A., Bessaim, A., Bernard, F., Tounsi, A. and Mahmoud, S.R. (2018), "Buckling analysis of new quasi-3D FG nanobeams based on nonlocal strain gradient elasticity theory and variable length scale parameter", *Steel Compos. Struct., Int. J.*, **28**(1), 13-24. https://doi.org/10.12989/scs.2018.28.1.013

Jahangiri, R., Jahangiri, H. and Khezerloo, H. (2015), "FGM micro-gripper under electrostatic and intermolecular Van-der Waals forces using modified couple stress theory", *Steel Compos. Struct., Int. J.*, **18**(6), 1541-1555. https://doi.org/10.12989/scs.2015.18.6.1541

Jia, X.L., Ke, L.L., Zhong, X.L., Sun, Y., Yang, J. and Kitipornchai, S. (2018), "Thermal-mechanical-electrical buckling behavior of functionally graded micro-beams based on modified couple stress theory", *Compos. Struct.*, **202**, 625-634. https://doi.org/10.1016/j.compstruct.2018.03.025

Karami, B., Janghorban, M. and Tounsi, A. (2018), "Nonlocal strain gradient 3D elasticity theory for anisotropic spherical nanoparticles", *Steel Compos. Struct., Int. J.*, **27**(2), 201-216. https://doi.org/10.12989/scs.2018.27.2.201

Khaniki, H.B. and Rajasekaran, S. (2018), "Mechanical analysis of non-uniform bi-directional functionally graded intelligent micro-beams using modified couple stress theory", *Mater. Res. Express.*, **5**(5), 055703. https://doi.org/10.1088/2053-1591/aabe62

Khorshidi, M.A., Shariati, M. and Emam, S.A. (2016), "Postbuckling of functionally graded nanobeams based on modified couple stress theory under general beam theory", *Int. J. Mech. Sci.*, **110**, 160-169. https://doi.org/10.1016/j.ijmecsci.2016.03.006

Kiani, Y. and Eslami, M.R. (2010), "Thermal buckling analysis of functionally graded material beams", *Int. J. Mech. Mater. Des.*, **6**(3), 229-238. https://doi.org/10.1007/s10999-010-9132-4

Kocaturk, T. and Akbas, S.D. (2013), "Thermal post-buckling analysis of functionally graded beams with temperature-dependent physical properties", *Steel Compos. Struct., Int. J.*, **15**(5), 481-505. https://doi.org/10.12989/scs.2013.15.5.481

Koiter, W.T. (1964), "Couple stresses in the theory of elasticity", *I and II. Proc. K. Ned. Akad. Wet. B*, **67**, 17-44.

Kong, S., Zhou, S., Nie, Z. and Wang, K. (2009), "Static and dynamic analysis of micro beams based on strain gradient elasticity theory", *Int. J. Eng. Sci.*, **47**, 487-498. https://doi.org/10.1016/j.ijengsci.2008.08.008

Lam, D.C.C. and Chong, A.C.M. (2001), "Model and experiments on strain gradient hardening in metallic glass", *Mat. Sci. Eng. A-Struct.*, **318**(1-2), 313-319. https://doi.org/10.1016/S0921-5093(01)01329-6

Lam, D.C.C., Yang, F., Chong, A.C.M., Wang, J. and Tong, P. (2003), "Experiments and theory in strain gradient elasticity", *J. Mech. Phys. Solids.*, **51**(8), 1477-1508. https://doi.org/10.1016/S0022-5096(03)00053-X

Lee, H.L. and Chang, W.J. (2009), "Effects of damping on the vibration frequency of atomic force microscope cantilevers using the Timoshenko beam model", *Jpn. J. Appl. Phys.*, **48**(6), 065005. https://doi.org/10.1143/JJAP.48.065005

Lei, J., He, Y.M., Guo, S., Li, Z.K. and Liu, D.B. (2016), "Size-dependent vibration of nickel cantilever microbeams: Experiment and gradient elasticity", *Aip. Adv.*, **6**(10), 105202. https://doi.org/10.1063/1.4964660

Li, B., Tang, X.S., Xie, H.M. and Xin, Z. (2004), "Strain analysis in MEMS/NEMS structures and devices by using focused ion beam system", *Sensor Actuator A-Phys.*, **111**(1), 57-62. https://doi.org/10.1016/j.sna.2003.07.014

Li, S.R., Zhang, J.H. and Zhao, Y.G. (2006), "Thermal post-buckling of functionally graded material Timoshenko beams", *Appl. Math. Mech.*, **27**(6), 803-810. https://doi.org/10.1007/s10483-006-0611-y

Li, Z.K., He, Y.M., Lei, J., Guo, S., Liu, D.B. and Wang, L. (2018), "A standard experimental method for determining the material length scale based on modified couple stress theory", *Int. J. Mech. Sci.*, **141**, 198-205. https://doi.org/10.1016/j.ijmecsci.2018.03.035

Lim, C.W., Zhang, G. and Reddy, J.N. (2015), "A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation", *J. Mech. Phys. Solids.*, **78**, 298-313. https://doi.org/10.1016/j.jmps.2015.02.001

Ma, H.M., Gao, X.L. and Reddy, J.N. (2008), "A microstructure-dependent Timoshenko beam model based on a modified couple stress theory", *J. Mech. Phys. Solids.*, **56**(12), 3379-3391. https://doi.org/10.1016/j.jmps.2008.09.007

Ma, H.M., Gao, X.L. and Reddy, J.N. (2010), "A nonclassical Reddy-Levinson beam model based on a modified couple stress theory", *Int. J. Multiscale Comput. Eng.*, **8**(2), 167-180. https://doi.org/10.1615/IntJMultCompEng.v8.i2.30

Mahmoud, S.R. and Tounsi, A. (2017), "A new shear deformation plate theory with stretching effect for buckling analysis of functionally graded sandwich plates", *Steel Compos. Struct.*, *Int. J.*, **24**(5), 569-578. https://doi.org/10.12989/scs.2017.24.5.569

Marques, L., da Silva, L.S. and Rebelo, C. (2014), "Rayleigh-Ritz procedure for determination of the critical load of tapered columns", *Steel Compos. Struct.*, *Int. J.*, **16**(1), 47-60. https://doi.org/10.12989/scs.2014.16.1.047

Mindlin, R.D. and Tiersten, H.F. (1962), "Effects of couple-stresses in linear elasticity", *Arch. Ration. Mech. Anal.*, **11**, 415-448. https://doi.org/10.1007/BF00253946

Nateghi, A., Salamat-talab, M., Rezapour, J. and Daneshian, B. (2012), "Size dependent buckling analysis of functionally graded micro beams based on modified couple stress theory", *Appl. Math. Model.*, **36**(10), 4971-4987. https://doi.org/10.1016/j.apm.2011.12.035

Nazemnezhad, R. and Kamali, K. (2018), "Free axial vibration analysis of axially functionally graded thick nanorods using nonlocal Bishop's theory", *Steel Compos. Struct.*, *Int. J.*, **28**(6), 749-758. https://doi.org/10.12989/scs.2018.28.6.749

Nguyen, D.K. and Tran, T.T. (2018), "Free vibration of tapered BFGM beams using an efficient shear deformable finite element model", *Steel Compos. Struct.*, *Int. J.*, **29**(3), 363-377. https://doi.org/10.12989/scs.2018.29.3.363

Nguyen, N.T., Kim, N.I. and Lee, J. (2014), "Analytical solutions for bending of transversely or axially FG nonlocal beams", *Steel Compos. Struct.*, *Int. J.*, **17**(5), 639-663. https://doi.org/10.12989/scs.2014.17.5.641

Nguyen, K.T., Thai, T.H. and Vo, T.P. (2015), "A refined higher-order shear deformation theory for bending, vibration and buckling analysis of functionally graded sandwich plates", *Steel Compos. Struct.*, *Int. J.*, **18**(1), 91-120. https://doi.org/10.12989/scs.2015.18.1.091

Park, S.K. and Gao, X.L. (2006), "Bernoulli-Euler beam model based on a modified couple stress theory", *J. Micromech. Microeng.*, **16**(11), 2355-2359. https://doi.org/10.1088/0960-1317/16/11/015

Park, W.T., Han, S.C., Jung, W.Y. and Lee, W.H. (2016), "Dynamic instability analysis for S-FGM plates embedded in Pasternak elastic medium using the modified couple stress theory", *Steel Compos. Struct.*, *Int. J.*, **22**(6), 1239-1259. https://doi.org/10.12989/scs.2016.22.6.1239

Payam, A.F. and Fathipour, M. (2009), "Modeling and dynamic analysis of atomic force microscope based on Euler-Bernoulli beam theory", *Dig. J. Nanomater. Bios.*, **4**(3), 565-578.

Pradhan, K.K. and Chakraverty, S. (2013), "Free vibration of Euler and Timoshenko functionally graded beams by Rayleigh–Ritz method", *Compos. Part B-Eng.*, **51**, 175-184. https://doi.org/10.1016/j.compositesb.2013.02.027

Rahmani, O., Deyhim, S. and Hosseini, S.A.H. (2018a), "Size dependent bending analysis of micro/nano sandwich structures based on a nonlocal high order theory", *Steel Compos. Struct.*, *Int. J.*, **27**(3), 371-388. https://doi.org/10.12989/scs.2018.27.3.371

Rahmani, O., Hosseini, S.A.H., Ghoytasi, I. and Golmohammadi, H. (2018b), "Free vibration of deep curved FG nano-beam based on modified couple stress theory", *Steel Compos. Struct.*, *Int. J.*, **26**(5), 607-620. https://doi.org/10.12989/scs.2018.26.5.607

Rezaiee-Pajand, M., Masoodi, A.R. and Alepaighambar, A. (2018), "Lateral-torsional buckling of functionally graded tapered I-beams considering lateral bracing", *Steel Compos. Struct.*, *Int. J.*, **28**(4), 403-414. https://doi.org/10.12989/scs.2018.28.4.403

Sahraee, S. and Saidi, A.R. (2009), "Free vibration and buckling analysis of functionally graded deep beam-columns on two-parameter elastic foundations using the differential quadrature method", *Proc. Inst. Mech. Eng. C-J. Mech. Eng. Sci.*, **223**(6), 1273-1284. https://doi.org/10.1243/09544062JMES1349

Saidi, H., Houari, M.S.A., Tounsi, A. and Bedia, E.A. (2013), "Thermo-mechanical bending response with stretching effect of functionally graded sandwich plates using a novel shear deformation theory", *Steel Compos. Struct.*, *Int. J.*, **15**(2), 221-245. https://doi.org/10.12989/scs.2013.15.2.221

Salamat-Talab, M., Nateghi, A. and Torabi, J. (2012), "Static and dynamic analysis of third-order shear deformation FG micro beam based on modified couple stress theory", *Int. J. Mech. Sci.*, **57**(1), 63-73. https://doi.org/10.1016/j.ijmecsci.2012.02.004

Shafiei, N. and Kazemi, M. (2017), "Buckling analysis on the bi-dimensional functionally graded porous tapered nano-/micro-scale beams", *Aerosp. Sci. Technol.*, **66**, 1-11. https://doi.org/10.1016/j.ast.2017.02.019

Shafiei, N. and She, G.L. (2018), "On vibration of functionally graded nano-tubes in the thermal environment", *Int. J. Eng. Sci.*, **133**, 84-98. https://doi.org/10.1016/j.ijengsci.2018.08.004

Shafiei, N., Kazemi, M. and Ghadiri, M. (2016a), "Comparison of modeling of the rotating tapered axially functionally graded Timoshenko and Euler-Bernoulli microbeams", *Physica E*, **83**, 74-87. https://doi.org/10.1016/j.physe.2016.04.011

Shafiei, N., Kazemi, M. and Ghadiri, M. (2016b), "Nonlinear vibration of axially functionally graded tapered microbeams", *Int. J. Eng. Sci.*, **102**, 12-26. https://doi.org/10.1016/j.ijengsci.2016.02.007

Shafiei, N., Mousavi, A. and Ghadiri, M. (2016c), "On size-dependent nonlinear vibration of porous and imperfect functionally graded tapered microbeams", *Int. J. Eng. Sci.*, **106**, 42-56. https://doi.org/10.1016/j.ijengsci.2016.05.007

Shafiei, N., Kazemi, M. and Fatahi, L. (2017), "Transverse vibration of rotary tapered microbeam based on modified couple stress theory and generalized differential quadrature element method", *Mech. Adv. Mater. Struct.*, **24**(3), 240-252.

https://doi.org/10.1080/15376494.2015.1128025

Shafiei, N., Ghadiri, M. and Mahinzare, M. (2019), "Flapwise bending vibration analysis of rotary tapered functionally graded nanobeam in thermal environment", *Mech*. *Adv*. *Mater*. *Struct*., **26**(2), 139-155. https://doi.org/10.1080/15376494.2017.1365982

She, G.L., Yuan, F.G. and Ren, Y.R. (2017), "Thermal buckling and post-buckling analysis of functionally graded beams based on a general higher-order shear deformation theory", *Appl*. *Math*. *Model*., **47**, 340-357. https://doi.org/10.1016/j.apm.2017.03.014

She, G.L., Yan, K.M., Zhang, Y.L., Liu, H.B. and Ren, Y.R. (2018), "Wave propagation of functionally graded porous nanobeams based on non-local strain gradient theory", *Eur*. *Phys*. *J*. *Plus*, **133**, 368. https://doi.org/10.1140/epjp/i2018-12196-5

Simsek, M. (2011), "Forced vibration of an embedded single-walled carbon nanotube traversed by a moving load using nonlocal Timoshenko beam theory", *Steel Compos*. *Struct*., *Int*. *J*., **11**(1), 59-76. https://doi.org/10.12989/scs.2011.11.1.059

Simsek, M. (2012), "Nonlocal effects in the free longitudinal vibration of axially functionally graded tapered nanorods", *Comp*. *Mater*. *Sci*., **61**, 257-265. https://doi.org/10.1016/j.commatsci.2012.04.001

Simsek, M. and Reddy, J.N. (2013), "A unified higher order beam theory for buckling of a functionally graded microbeam embedded in elastic medium using modified couple stress theory", *Compos*. *Struct*., **101**, 47-58. https://doi.org/10.1016/j.compstruct.2013.01.017

Tagrara, S.H., Benachour, A., Bouiadjra, M.B. and Tounsi, A. (2015), "On bending, buckling and vibration responses of functionally graded carbon nanotube-reinforced composite beams", *Steel Compos*. *Struct*., *Int*. *J*., **19**(5), 1259-1277. https://doi.org/10.12989/scs.2015.19.5.1259

Thanh, C.L., Tran, L.V., Vu-Huu, T. and Abdel-Wahab, M. (2019), "The size-dependent thermal bending and buckling analyses of composite laminate microplate based on new modified couple stress theory and isogeometric analysis", *Comput*. *Method*. *Appl*. *M*ech., **350**, 337-361. https://doi.org/10.1016/j.cma.2019.02.028

Toupin, R.A. (1964), "Theories of elasticity with couple-stresses", *Arch*. *Ration*. *Mech*. *Anal*., **17**, 85-112. https://doi.org/10.1007/BF00253050

Vardoulakis, I. and Sulem, J. (1995), *Bifurcation Analysis in Geomechanics*, CRC Press.

Wang, C.M., Wang, C.Y. and Reddy, J.N. (2005), *Exact Solutions for Buckling of Structural Members*, CRC Press.

Wattanasakulpong, N., Prusty, B.G. and Kelly, D.W. (2011), "Thermal buckling and elastic vibration of third-order shear deformable functionally graded beams", *Int*. *J*. *Mech*. *Sci*., **53**(9), 734-743. https://doi.org/10.1016/j.ijmecsci.2011.06.005

Yang, F., Chong, A.C.M., Lam, D.C.C. and Tong, P. (2002), "Couple stress based strain gradient theory for elasticity", *Int*. *J*. *Solids Struct*., **39**, 2731-2743. https://doi.org/10.1016/S0020-7683(02)00152-X

Younis, M.I., Abdel-Rahman, E.M. and Nayfeh, A. (2003), "A reduced-order model for electrically actuated microbeam-based MEMS", *J*. *Microelectromech*. *Sci*, **12**(5), 672-680. https://doi.org/10.1109/JMEMS.2003.818069

CC