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Nonlinear dynamic analysis of spiral stiffened cylindrical shells rested on elastic foundation

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Abstract. In this paper, an analytical approach for the free vibration analysis of spiral stiffened functionally graded (SSFG) cylindrical shells is investigated. The SSFG shell is resting on linear and non-linear elastic foundation with damping force. The elastic foundation for the linear model is according to Winkler and Pasternak parameters and for the non-linear model, one cubic term is added. The material constitutive of the stiffeners is continuously changed through the thickness. Using the Galerkin method based on the von Kármán equations and the smeared stiffeners technique, the non-linear vibration problem has been solved. The effects of different geometrical and material parameters on the free vibration response of SSFG cylindrical shells are adopted. The results show that the angles of stiffeners and elastic foundation parameters strongly effect on the natural frequencies of the SSFG cylindrical shell.

Keywords: FG cylindrical shells; non-linear free vibration; spiral stiffeners; damping; elastic foundation

1. Introduction

The stiffened FG cylindrical shells have more application in a wide range of engineering structures, including submarines, bridges, aircraft, satellites, ships and offshore structures. Thus, research in the field of these structures vibration analysis has been interested in scientists from many years ago, and a great number of researches have been done on vibration analysis of stiffened shell structures.

Free vibration behavior of FG porous cylindrical shells by using a sinusoidal shear deformation theory (SDT) and the Rayleigh-Ritz method was investigated by Wang and Wu (2017). Lee and Kwak (2015) presented the vibration analysis of a cylindrical shell using the Rayleigh-Ritz method. They constructed the model of dynamic based on the Donnell–Mushtari theory. Sofiyev *et al*. (2017) analyzed the non-linear dynamic analysis of composite cylindrical shells with the elastic mediums and using the SDT. Non-linear dynamic analysis of composite cylindrical shells under the periodic radial and axial loading was invoked by Dey and Ramachandra (2017). Free vibration analysis of FG cylindrical shells with different shell theories using the semi-analytical method was investigated by Khayat *et al*. (2018). Mochida *et al*. (2012) analyzed the vibration of shallow-shells with doubly curved by using the Galerkin-method for several boundary conditions. Darabi *et al*. (2008) used the large deflection theory for analyzing the dynamic response of FG shells with axial compression. The stability behavior of FG cylindrical shells under periodic axial loading was studied by Sofiyev (2005). In this work, the Love's shell theory and Lagrange–Hamilton type principle was used. Sheng and Wang (2008) proposed the vibration analysis of fluid-conveying FG cylindrical shell and resting on an elastic foundation under thermo-mechanical loading. Choe *et al*. (2018b) investigated the vibration behavior of the composite shell by considering axis-symmetric geometry with doubly-curved using the unified Jacobi-Ritz method. Also, Choe *et al*. (2018a) proposed the dynamic behavior of coupled FG doubly-curved revolution shells. The effect of thermal gradient load on thermo-elastic vibrational behavior of sandwich plates reinforced by Carbon Nanotube agglomerations was analyzed by Safaei *et al*. (2018). Malikan *et al*. (2018) addressed the effect of sinusoidal corrugated geometries on the vibrational response of viscoelastic nanoplates. Safaei *et al*. (2019) investigated the Frequency-dependent forced vibration analysis of nanocomposite sandwich plate under thermo-mechanical loads. Qin *et al*. (2018) reported the free vibration analysis of rotating cylindrical shells coupled with moderately thick annular plate. The vibration analysis of FG composite shell reinforced by carbon nanotube was presented by Zghal *et al*. (2018). Shen *et al*. (2018) presented the vibration response of FG reinforced by graphene for composite cylindrical panels with elastic medium under thermal loading. Three-dimensional free vibration analysis of cylindrical shells with continuous grading reinforcement was presented by Yas and Garmsiri (2010). Kiani *et al*. (2018b) analyzed the free vibration behavior of FG-CNT reinforced composite skew cylindrical shells. They used the first shear deformation theory (FSDT) and Chebyshev-Ritz formulation. Javed *et al*. (2016) addressed the free vibration behavior of composite

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cylindrical shells with non-uniform thickness walls. Kiani *et al*. (2018a) presented the free vibration study of composite conical panels reinforced with FG-CNTs. The higher order thermo-elastic analysis of FG-CNTRC cylindrical vessels surrounded by a Pasternak foundation was investigated by Mohammadi *et al*. (2019).

In the studies mentioned above the effects of the stiffeners on dynamic analysis of cylindrical shells have not been considered. Some studies have been done on the dynamic behavior of cylindrical shells reinforced by stiffeners.

Dung and Nam (2014) presented the dynamic stability of stiffened FG cylindrical shells with elastic foundations subjected to external loading using the Galerkin method. The non-linear dynamic behaviors of imperfect stiffened FG cylindrical shells with elastic foundations under damping and mechanical loads by using the FSDT, and Runge-Kutta method were studied by Duc and Thang (2015). Chen *et al*. (2015) investigated the vibration behavior of stiffened conical shells by using the Flügge theory. Duc *et al*. (2017) presented the vibration analysis of the orthogonal stiffened FG elliptical shells with elastic foundation under thermal loading. The free vibration demeanor of orthogonally stiffened cylindrical shells was investigated by Torkamani *et al*. (2009). They used the similitude theory for developing the scaling laws. Nejati *et al*. (2017) studied the thermal buckling of nanocomposite stiffened cylindrical shells reinforced by functionally graded wavy carbon nanotubes with temperature-dependent properties.

A review of studies shows that few researches have been presented on the free vibration of stiffened FG cylindrical shells utilizing analytical approaches. Soong (1969) emphasizes the importance of the spiral stiffener, as follows: "In the stability analysis of stiffened shells, the field has been completely devoted to the conventional-type stiffened shells, i.e., ring frames in the circumferential direction and/or stringers in the axial direction. The advantages of their geometric simplicity are obvious from an engineering point of view, but whether the conventional stiffening is most efficient as a least-weight design is questionable. Since the conventional stringers and rings are designed to resist deformations in the axial direction and the circumferential direction, respectively, their effectiveness cannot be fully utilized if the critical mode for a particular type of load is such that the average deformation occurs in a direction inclined to these two principal axes. An obvious example is a stringer stiffened cylinder under external pressure and, to a less extent, a ring- stiffened cylinder subjected to axial compression or a ring-and-stringer-stiffened cylinder under torsion. With the rapid advance of space technology and the extreme importance of weight saving in aeronautical and space structures, other feasible stiffener arrangements should be explored for possible gains". The main innovation of the present work is to study the influence of spiral stiffener on the behavior of non-linear free vibration of FG cylindrical shells, which yields new and interesting results. The modeling of spiral stiffener in non-linear dynamic analysis of FG cylindrical shells was first proposed by the authors of this paper.

In the present study, the Galerkin method is adopted for investigation of the non-linear free vibration behavior of stiffened FG cylindrical shells reinforced by spiral stiffeners, embedded in elastic media with damping force. The elastic foundation is formulated based on two linear parameters (Winkler and Pasternak) and a cubic nonlinearity. The material constitutive of the shell and stiffeners is continuously changed along the thickness direction based on a simple power law distribution. For modeling of shells, the classical plate theory of shells and smeared stiffeners technique are used. The presented results are compared with those available in the literature. The effects of the stiffener's angle, geometrical and material parameters, elastic foundation, and damping coefficient are investigated on the vibration response of SSFG cylindrical shells.

2. The basic formulation

2.1 FG material properties

The SSFG cylindrical shell with elastic foundation and linear damping that the coordinate system (x, y, z) is illustrated in Fig. 1, which the axes of x, y and z are the axial, circumferential, and radial coordinate variables of the SSFG cylindrical shell, respectively. The cylindrical shell has radius R, thickness h and length L and the stiffeners have thickness h_s, width d, spacing s and angle of stiffeners θ, β. The cylindrical shell and the stiffeners are considered to consist of metals and ceramics mixture in two situations. The first case with external stiffeners, the inner surface of the cylinder shell ($z = h/2$) is rich metal, and the outer surface ($z = -h/2$) is rich ceramic and in order to keep material continuity, the lower surface of the stiffener is made of ceramic and the upper surface is made of metal. In the second case, a reverse order is used for internal stiffeners.

The Young's modulus (E) and density (ρ) of the FG shell and stiffeners can be written as follows (Foroutan *et al*. 2012, He *et al*. 2012, Dung and Nam 2014)

Shell

$$E(z) = E_o + (E_i - E_o)\left(\frac{2z+h}{2h}\right)^k ; \quad -\frac{h}{2} \le z \le \frac{h}{2} \quad (1a)$$
$$\rho(z) = \rho_o + (\rho_i - \rho_o)\left(\frac{2z+h}{2h}\right)^k$$

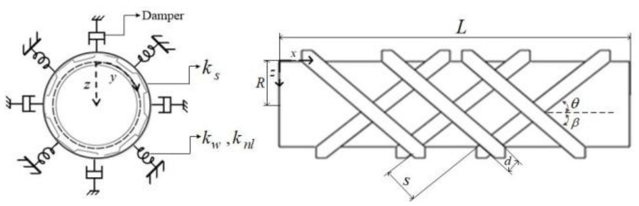

Fig. 1 Presentation SSFG cylindrical shell resting on damping and elastic foundation

Stiffeners

$$E_s(z) = E_i + (E_o - E_i)\left(\frac{2z+h}{2h_s}\right)^K$$

$$\rho_s(z) = \rho_i + (\rho_o - \rho_i)\left(\frac{2z+h}{2h_s}\right)^K \tag{1b}$$

$$\text{External:} -\left(\frac{h}{2}+h_s\right) \le z \le -\frac{h}{2}$$

$$\text{Internal:} \frac{h}{2} \le z \le \left(\frac{h}{2}+h_s\right)$$

where ρ, ρ_s, E, E_s and k, K are the density, Young's modulus and material index of the FG shell and stiffeners, respectively.

2.2 The theoretical formulation

2.2.1 Governing equations

With regard to the relations of non-linear von Kármán strain- displacement Brush and Almroth (1975), the components of strain on the middle plane of cylindrical shells can be given by

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} u_{,x} + \frac{1}{2}w_{,x}^2 \\ v_{,y} - \frac{w}{R} + \frac{1}{2}w_{,y}^2 \\ u_{,y} + v_{,x} + w_{,x}w_{,y} \end{Bmatrix} \ \& \ \begin{Bmatrix} \chi_x \\ \chi_y \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ w_{,xy} \end{Bmatrix} \tag{2}$$

where $u = u(x,y,t), v = v(x,y,t), w = w(x,y,t)$ are the components of displacement through the axes of x,y,z, respectively. $\varepsilon_x^0, \varepsilon_y^0$, and γ_{xy}^0 are the normal and shear strains at the middle plane, respectively and $\chi_x, \chi_y, \chi_{xy}$ are the curvatures and twist change of the shell.

The components of strain through the thickness at a distance z from the middle plane of the shell are depicted by

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} - z \begin{Bmatrix} \chi_x \\ \chi_y \\ 2\chi_{xy} \end{Bmatrix} \tag{3}$$

According to Eq. (2), the compatibility equation can be presented as follows

$$\varepsilon_{x,yy}^0 + \varepsilon_{y,xx}^0 - \gamma_{xy,xy}^0 = -w_{,xx}/R + w_{,xy}^2 - w_{,xx}w_{,yy} \tag{4}$$

The relations of stress-strain for FG cylindrical shells can be written as follows

$$\begin{Bmatrix} \sigma_x^{sh} \\ \sigma_y^{sh} \\ \tau_{xy}^{sh} \end{Bmatrix} = \begin{bmatrix} \frac{E(z)}{1-\nu^2} & \frac{\nu E(z)}{1-\nu^2} & 0 \\ \frac{\nu E(z)}{1-\nu^2} & \frac{E(z)}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E(z)}{2(1+\nu)} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{5}$$

where ν is the Poisson's ratio and considered to be constant. σ_x^{sh}, σ_y^{sh}, and τ_{xy}^{sh} are normal stress in axial (x), circumferential (y) directions and shearing stress of cylindrical shell, respectively.

The stress-strain relations of the spiral stiffeners are as follow (Shaterzadeh and Foroutan 2016)

$$\begin{Bmatrix} \sigma_x^s \\ \sigma_y^s \\ \tau_{xy}^s \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} Z_1\varepsilon_x \\ Z_2\varepsilon_y \\ Z_3\gamma_{xy} \end{Bmatrix} \tag{6}$$

where

$$\begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} = \begin{Bmatrix} \left(\cos^3\theta + \cos^3\beta\right) \\ \left(\sin\theta\cos^2\theta + \sin\beta\cos^2\beta\right) \\ \left(\cos^2\theta - \cos^2\beta\right) \end{Bmatrix}$$

$$\begin{Bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{Bmatrix} = \begin{Bmatrix} \left(\sin^2\theta\cos\theta + \sin^2\beta\cos\beta\right) \\ \left(\sin^3\theta + \sin^3\beta\right) \\ \left(\sin\theta\cos\theta + \sin\beta\cos\beta\right) \end{Bmatrix}$$

$$\begin{Bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{Bmatrix} = \begin{Bmatrix} 2\left(\sin\theta\cos^2\theta - \sin\beta\cos^2\beta\right) \\ 2\left(\sin^2\theta\cos\theta - \sin^2\beta\cos\beta\right) \\ \left(\sin^2\theta - \sin^2\beta\right) \end{Bmatrix} \tag{7}$$

$$\begin{Bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{Bmatrix} = \frac{h_s dE_s}{sh_s}\sin(\gamma) \begin{Bmatrix} 1/\left(\sin\theta + \sin\beta\right) \\ 1/\left(\cos\theta + \cos\beta\right) \\ 1/2 \end{Bmatrix}$$

where the γ is the sum of the two angles of stiffeners ($\gamma = \theta + \beta$) and σ_x^s, σ_y^s and τ_{xy}^s are the normal and shearing stress components of the stiffeners, respectively. To consider the effect of the stiffeners on the shell, the smeared stiffeners technique is used. By integrating the stress-strain relations, the resultant forces, and moments for SSFG cylindrical shells can be obtained (Shaterzadeh and Foroutan 2016).

The equilibrium equations of cylindrical shells with regard to the classical shell theory that in this, transverse shear deformation is ignored and assumed $u \ll w$ and $v \ll w$, $\rho_1 u_{,tt} \to 0$, $\rho_1 v_{,tt} \to 0$ (Volmir 1972, Bich *et al*. 2013, Ghiasian *et al*. 2013, Vasiliev and Morozov 2018)

$$\begin{aligned} &M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy}w_{,xy} \\ &+N_y\left(w_{,yy} + \frac{1}{R}\right) - k_w + k_s\left(w_{,xx} + w_{,yy}\right) \\ &+k_{nl}w^3 = \rho_1 w_{,tt} + 2\rho_1 c w_{,t} \end{aligned} \tag{8}$$

where k_s, k_w, and k_{nl} are the Pasternak, Winkler, and non-linear cubic parameters of the elastic foundation, respectively. Also, c is damping coefficient and the mass density ρ_1 can be calculated as

$$\rho_1 = \left(\rho_o + \frac{\rho_i - \rho_o}{k+1}\right)h + 2\left(\rho_i + \frac{\rho_o - \rho_i}{K+1}\right)\frac{dh_s}{S} \tag{9}$$

Substituting the force-moment relations versus to strain relations into Eq. (4) (compatibility equation) and the third part of Eq. (8) and using the Eq. (2) and a stress function (φ), the following equations of the system can be derived as (Shaterzadeh and Foroutan 2016)

$$\begin{aligned}&A^*_{11}\varphi_{,xxxx}+(A^*_{33}-A^*_{12}+A^*_{21})\varphi_{,xxyy}+A^*_{22}\varphi_{,yyyy}\\&+A^{**}_{21}w_{,xxxx}+(A^{**}_{11}+A^{**}_{22}-2A^{**}_{36})w_{,xxyy}\\&+A^{**}_{12}w_{,yyyy}+\frac{1}{R}w_{,xx}+\left[w^2_{,xy}-w_{,xx}w_{,yy}\right]\\&-2w_{,xy}+w_{,xx}+w_{,yy}=0\end{aligned} \tag{10}$$

$$\begin{aligned}&\rho_1 w_{,tt}+2\rho_1 c w_{,t}+B^{**}_{11}w_{,xxxx}\\&+(B^{**}_{12}+B^{**}_{21}+4B^{**}_{36})w_{,xxyy}\\&+B^{**}_{22}w_{,yyyy}-B^*_{21}\varphi_{,xxxx}\\&-(B^*_{11}+B^*_{22}-2B^*_{36})\varphi_{,xxyy}\\&-B^*_{12}\varphi_{,yyyy}-\frac{1}{R}\varphi_{,xx}-\varphi_{,yy}w_{,xx}+2\varphi_{,xy}w_{,xy}\\&-\varphi_{,xx}w_{,yy}+k_w w-k_s\left(w_{,xx}w_{,yy}\right)-k_{nl}w^3=0\end{aligned} \tag{11}$$

The coefficients $A^*_{ij}, A^{**}_{ij}, B^*_{ij}$ *and* B^{**}_{ij} are defined in Appendix.

2.2.2 Boundary conditions

Suppose the SSFG cylindrical shell is simply supported surrounded by the linear and non-linear elastic foundation. So, boundary conditions of the cylindrical shell are considered as

$$w=0,\ M_x=0,\ \ at\ \ x=0;\ \ L \tag{12}$$

The deflection of the cylindrical shells is considered as (Volmir 1972, Bich *et al*. 2012)

$$w=f\left(t\right)\sin\frac{m\pi x}{L}\sin\frac{ny}{R} \tag{13}$$

where $f(t)$ is the amplitude and n, m denote the number of full wave and half wave in the circumferential and axial directions, respectively.

To obtain the stress function φ, the Eq. (13) is substituted into Eq. (10) and then the resulting partial differential equation is solved. If Eq. (11) is denoted by $\Gamma=0$, with substituting φ and w according to f in Γ, and applying the Galerkin's method in the ranges $0\le y\le 2\pi R$ and $0\le x\le L$, we have

$$w=\int_0^L\int_0^{2\pi R}\sin\frac{m\pi x}{L}\sin\frac{ny}{R}\Gamma dydx \tag{14}$$

After carrying out Galerkin's orthogonality integration appeared in Eq. (14), the discretized equation of motion is obtained as

$$\begin{aligned}&\ddot{f}(t)+2c\dot{f}(t)+(a_1+a_2k_w+a_3k_s)f(t)\\&+(a_4+a_5k_{nl})f^3(t)=0\end{aligned} \tag{15}$$

where

$$\begin{aligned}&a_1=\frac{1}{L^4\rho_1}\left(D+\frac{BB^*}{A}\right),\ \ a_2=\frac{1}{\rho_1}\\&a_3=\frac{(\lambda n)^2+(m\pi)^2}{L^2\rho_1},\ \ a_4=\frac{G}{L^4\rho_1},\ \ a_5=\frac{9}{16\rho_1}\end{aligned} \tag{16}$$

The constant coefficients A, B, B^*, D and G are defined in Appendix.

3. Vibration analysis

3.1 Linear analysis

The linear form of Eq. (15) for investigation of free vibration of SSFG cylindrical shells without damping, becomes

$$\ddot{f}(t)+(a_1+a_2k_w+a_3k_s)f(t)=0 \tag{17}$$

The natural frequency of SSFG cylindrical shells is

$$\omega_{mn}=\sqrt{\left(a_1+a_2k_w+a_3k_s\right)} \tag{18}$$

3.2 Non-linear analysis

Consider the SSFG simply supported cylindrical shell with an elastic foundation and linear damping.

If Eq. (15) is denoted by $\Lambda=0$, by considering $f(t)=\eta sin(\Omega t)$and substituting in Λ, based on procedure like Galerkin's method in the ranges $0\le t\le\frac{\pi}{2\omega}$ we have

$$\int_0^{\pi/2\omega}\sin\Omega t\Lambda dt \tag{19}$$

After carrying out integration appeared in Eq. (19), the relation of frequency versus amplitude of SSFG cylindrical shell with an elastic foundation is obtained

$$\Omega^2-\frac{4}{\pi}c\Omega=\left(a_1+a_2k_w+a_3k_s\right)+\frac{3}{4}\left(a_4+a_5k_{nl}\right)\eta^2 \tag{20}$$

where η is the non-linear vibration amplitude of $f(t)$.

By defining non-dimensional frequency ratio ($\xi=\Omega/\omega_{mn}$) in Eq. (20), the relation of frequency-amplitude for non-linear free vibration is

$$\xi^2-\frac{4c}{\pi\omega_{mn}}\xi=1+\frac{3}{4}\frac{\left(a_4+a_5k_{nl}\right)}{\omega_{mn}{}^2}\eta^2 \tag{21}$$

4. Numerical results

4.1 Validation of this study

To validation of the present study, in Tables 1 and 2, the obtained isotropic cylindrical shells natural frequencies with and without elastic foundation are validated with those

Table 1 Comparison of the natural frequency of cylindrical shells with Winkler foundation ($m = 1, L = 1$ m, $R = 0.5$ m, $E = 7 \times 10^{10}$ N/m^2, $\nu = 0.3, d = 0.0025$ m, $h_s = 0.01$ m)

n	Present	Paliwal *et al*. (1996)		Sofiyev (2009)	
			Erorrs (%)		Erorrs (%)
1	0.67480	0.67882	0.60	0.67921	0.65
2	0.36223	0.36394	0.47	0.36463	0.66
3	0.20670	0.20526	0.70	0.20804	0.65
4	0.13747	0.12745	7.29	0.13824	0.56

Table 2 Comparison of the natural frequencies of cylindrical shell ($L = 0.2$ m, $R = 0.1$ m, $h = 0.247 \times 10^{-3}$ m, $m = 1, E = 7.12 \times 10^{10}$ N/m^2, $\rho = 2796$ kg/m^3, $\nu = 0.31$)

m	n	Present	Qin *et al*. (2017)		Pellicano (2007)	
				Errors (%)		Errors (%)
1	7	486.0	484.6	0.2	484.6	0.2
1	8	490.3	489.6	0.1	489.6	0.1
1	9	545.8	546.2	0.07	546.2	0.07
1	6	555.8	553.3	0.4	553.3	0.4
1	10	634.8	636.8	0.3	636.8	0.3
2	10	962.3	968.1	0.5	968.1	0.5
2	11	976.6	983.4	0.6	983.4	0.6

Table 3 Comparison of the natural frequencies of stiffened FG cylindrical shell resting on a linear elastic foundation ($L = 0.75$ m, $R = 0.5$ m, $R/h = 250$, $m = 1$, $E_m = 7 \times 10^{10}$ N/m^2, $\rho_m = 2702$ kg/m^3, $E_c = 38 \times 10^{10}$ N/m^2, $\rho_c = 3800$ kg/m^3, $\nu = 0.3, d_s = d_r = 0.0025$ m, $h_s = h_r = 0.01$ m, $k = 1, K = 0$)

		Present	Dung and Nam (2014)	Errors (%)
Un-stiffened				
$k_w = 10^5$	$k_s = 10^4$	1781.01	1781.01	0.00
$k_w = 10^5$	$k_s = 5 \times 10^4$	2121.24	2121.24	0.00
$k_w = 10^6$	$k_s = 10^4$	1819.45	1819.45	0.00
$k_w = 10^6$	$k_s = 5 \times 10^4$	2153.62	2153.62	0.00
External stiffeners				
$k_w = 10^5$	$k_s = 10^4$	2555.62	2555.62	0.00
$k_w = 10^5$	$k_s = 5 \times 10^4$	2689.42	2689.42	0.00
$k_w = 10^6$	$k_s = 10^4$	2574.67	2574.67	0.00
$k_w = 10^6$	$k_s = 5 \times 10^4$	2707.53	2707.53	0.00
External stiffeners				
$k_w = 10^5$	$k_s = 10^4$	2568.27	2568.27	0.00
$k_w = 10^5$	$k_s = 5 \times 10^4$	2665.83	2665.83	0.00
$k_w = 10^6$	$k_s = 10^4$	2587.23	2587.23	0.00
$k_w = 10^6$	$k_s = 5 \times 10^4$	2684.10	2684.10	0.00

present by Paliwal *et al*. (1996), Sofiyev (2009), Pellicano (2007) and Qin *et al*. (2017), respectively. Also, in Tables 3, the obtained natural frequencies of stiffened FG cylindrical shells with a linear elastic medium are compared with those of Dung and Nam (2014). They analyzed the stiffened circular shells consist of metal orthogonal stiffeners (Ring and stringer stiffeners). These comparisons show that good agreements are obtained.

It should be noted that the error between the present result and that literature by Paliwal *et al*. (1996) for $n = 4$ is due to the fact that in this study for $n = 1$, results have been computed by computer and for $n > 1$, the natural frequencies are obtained by linear approximation.

In Figs. 2-4, the natural frequencies of the cylindrical shells for the various number of full waves without stiffeners and with external and internal stiffeners are compared with those of Sewall and Naumann (1968) and Sewall *et al*. (1964). They experimentally analyzed the vibration response of the cylindrical shells. These comparisons also show that good agreements are obtained.

4.1 Free vibration results of SSFG cylindrical shells

In this sub-section, linear and non-linear dynamic analysis of SSFG cylindrical shells resting on a linear and

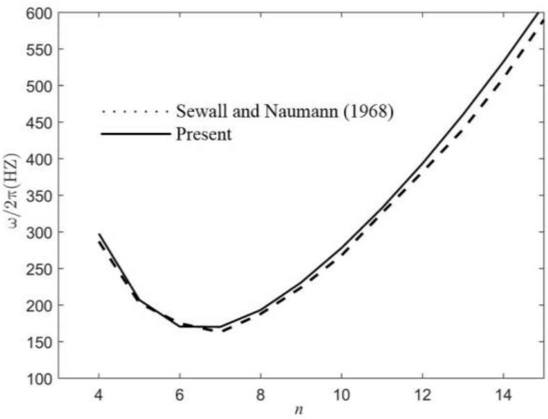

Fig. 2 Comparison of the natural frequencies of isotropic cylindrical shells without stiffeners ($m = 1$)

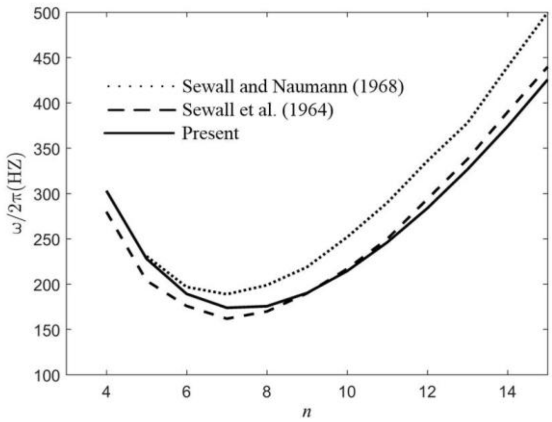

Fig. 3 Comparison of the natural frequencies of isotropic cylindrical shell with external stiffeners ($m = 1$)

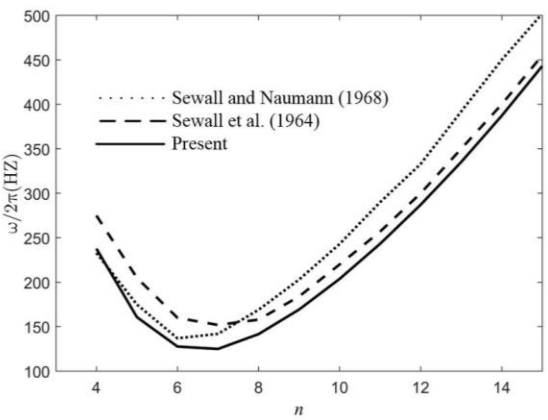

Fig. 4 Comparison of the natural frequencies of isotropic cylindrical shells with internal stiffeners ($m = 1$)

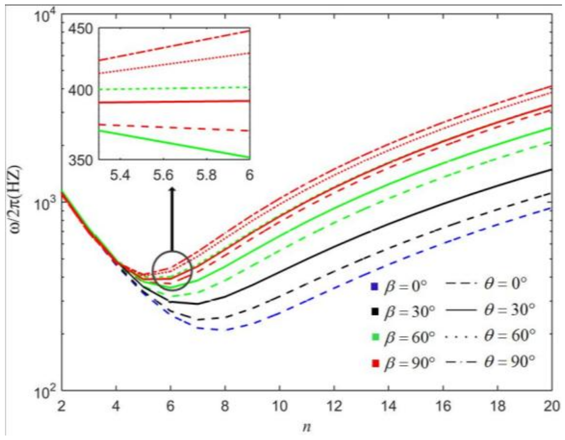

Fig. 5 The natural frequency responses of internal SSFG cylindrical shells with various angle of stiffeners without elastic foundation ($K = k = 1$)

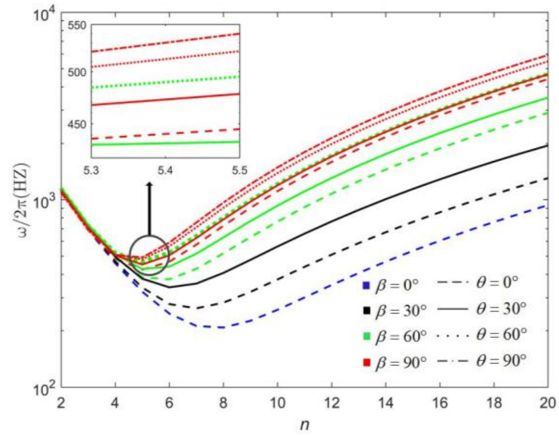

Fig. 6 The natural frequency responses of external SSFG cylindrical shells with various angle of stiffeners without elastic foundation ($K = k = 1$)

non-linear elastic medium along with linear damping are illustrated. The effect of various geometrical and material specifications such as the angle of stiffeners, radius, and thickness of the FG shells, the volume fraction of FG material, damping coefficient and also elastic foundation parameters on vibration responses of SSFG cylindrical shells are presented. Two different types of stiffened FG cylindrical shell are examined. In the first model, external stiffeners exist on the outer layer of the FG cylindrical shell, and the exterior layer of the cylindrical shell is rich metal while the interior layer is rich ceramic. The FG material direction is in reverse order in the second model, and the stiffeners are founded on the interior layer of the FG cylindrical shell. In the present study, unless defined, the number of stiffeners is assumed to be thirty ($n_s = 30$) which are distributed uniformly along the length of the FG cylindrical shell. The SSFG cylindrical shell with foundation parameters $k_s = 2.5 \times 10^4 \mathrm{N/m}$, $k_w = 5 \times 10^5\ \mathrm{N/m^3}$, $k_{nl} = 3 \times 10^{13}\ \mathrm{N/m^5}$ is assumed to be made of aluminum (Al) $E_m = 70$ GPa, $\rho_m = 2702\ \mathrm{kg/m^3}$, $\nu_m = 0.3$ and alumina (Al_2O_3) $E_c = 380$ GPa, $\rho_c = 3800\ \mathrm{kg/m^3}$, $\nu_c = 0.3$ (by assumption the similar Poisson's ratio for metal and ceramic). Also, the half waves number (m) are assumed to be equal to 1. The geometrical specifications of the FG shell and stiffeners are assumed with $R = 0.5$ m, $L = 0.75$ m, $h = 0.002$ m, $h_s = 0.01$ m, $d = 0.0025$ m.

The effect of angle of stiffeners on the natural frequency response of SSFG cylindrical shells with internal and external stiffeners is demonstrated in Figs. 5 and 6, respectively. In this research, the effects of different values of stiffener's angle are examined. According to Figs. 5 and 6, the effect of angle of stiffeners in the higher mode number is considerable. Also, for cylindrical shells with internal and external stiffeners, minimum of natural frequency response happen when the angle between both series of stiffeners is $0°$ ($\theta = \beta = 0°$).

As the angle of the stiffener increases to $90°$ the natural frequency increases, and these changes from angle $60°$ to the next, are not so noticeable. Therefore, it can be concluded that in order to achieve the maximum natural frequency for stiffened cylindrical shells, it is preferable

Table 4 Effect of R/h ratio and volume-fraction index k on the fundamental frequency of natural vibration (rad/s) of SSFG cylindrical shells

R/h	k	Un-stiffened	Internal stiffeners	External stiffeners
100	0.2	2114.5 (6[a])	3264.4 (6)	2642.9 (5)
	1	2597.4 (6)	2756.4 (5)	2987.6 (5)
	5	2981.9 (6)	2254.2 (5)	3183.8 (5)
150	0.2	1750.9 (7)	2720.6 (6)	2488.8 (5)
	1	21.38.3 (7)	2468.6 (6)	2797.7 (5)
	5	2455.6 (7)	2143.9 (5)	2829.4 (6)
200	0.2	1490.9 (7)	2495.5 (6)	2420.5 (5)
	1	1841.1 (7)	2365.6 (6)	2715.3 (5)
	5	2112.9 (7)	2109.7 (5)	2716.1 (6)
250	0.2	1353.6 (8)	2375.5 (6)	2376.5 (5)
	1	1654.1 (8)	2323.0 (6)	2665.4 (5)
	5	1899.5 (8)	2094.0 (5)	2675.4 (6)

*The numbers in the parenthesis denote the number of full wave (n)

that the angle of the stiffener be at least 60°. Also, it can be seen that increasing the γ (the sum of the two angles of stiffeners) leads to increasing the natural frequency.The effect of the volume-fraction index and radius-to-thickness ratio on the natural frequency of the FG cylindrical shells is demonstrated in Table 4. According to this table, when R/h increases, the value of natural frequency decreases. When the metal proportion of the shells decreases, the natural frequency for SSFG cylindrical shells with external and internal stiffeners increases and decreases, respectively.

The influence of the angle of stiffeners on non-linear free vibration of SSFG cylindrical shells with internal and external stiffeners is demonstrated in Figs. 7 and 8, respectively. According to Figs. 7 and 8, by increasing the amplitude of non-linear vibration, the values of the frequency increase. With notice to these figures, when the angle between both series of stiffeners is bigger than 60° the obtained vibration response is maximum. Also, minimum vibration response of SSFG cylindrical shells with internal and external stiffeners happen when the angle between both series of stiffeners is 0°.

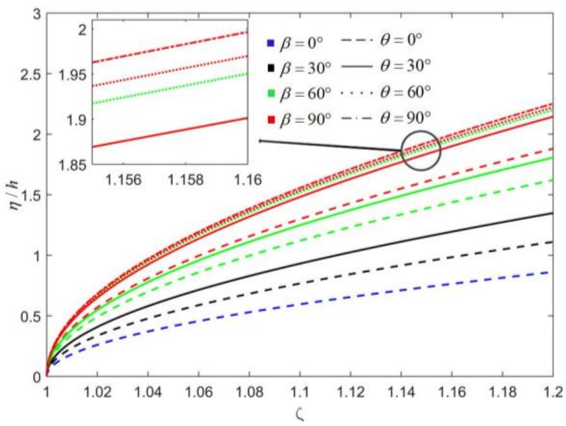

Fig. 7 The frequency ratio according to amplitude for non-linear free vibration of SSFG cylindrical shells with various angle of internal stiffeners without elastic foundation ($K = k = 1$)

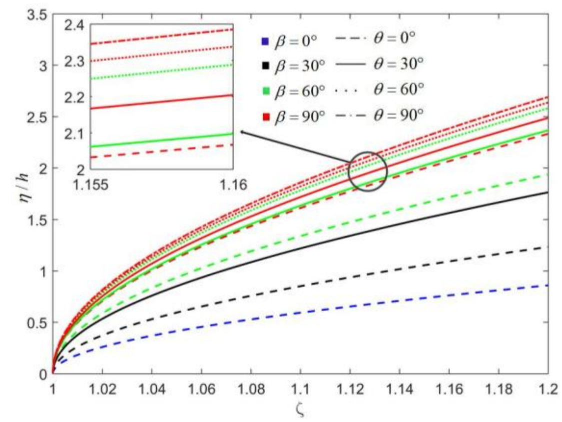

Fig. 8 The frequency ratio according to amplitude for non-linear free vibration of SSFG cylindrical shells with various angle of external stiffeners without elastic foundation ($K = k = 1$)

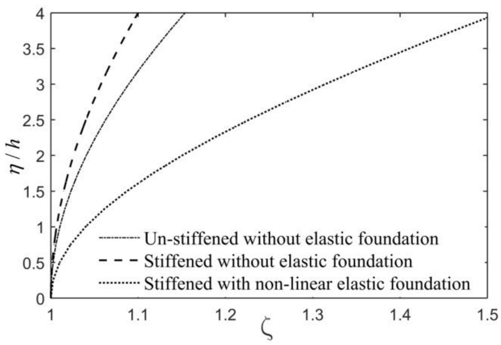

Fig. 9 The frequency ratio according to amplitude for non-linear free vibration of SSFG cylindrical shells ($K = k = 1$)

The effect of stiffeners and non-linear elastic medium on the vibration analysis of SSFG cylindrical shells is shown in Fig. 9. One can show that stiffeners increase the non-linear free vibration amplitude, but the non-linear elastic medium decreases the non-linear free vibration amplitude.

In Fig. 10, the effects of linear elastic medium parameters on the non-linear vibration response of SSFG cylindrical shells is demonstrated. According to this figure, the parameter of Pasternak elastic medium has a more considerable effect on non-linear vibration results with respect to the Winkler elastic medium coefficient.

The influence of material constitutive of the SSFG shells on the natural frequency is shown in Fig. 11. According to this figure, full ceramic stiffened shells and full metallic stiffened shells have the highest and the lowest natural frequency, respectively.

In Fig. 12 the result of damping coefficient on the non-linear free vibration behavior is shown. According to this figure, increasing the damping coefficient leads to reduce the amplitude of vibration.

The effect of damping on the frequency and amplitude of the non-linear free vibration responses of SSFG cylindrical shells are investigated in Fig. 13. Considering this figure, damping leads to decreasing the amplitude exponentially with time, but the effect of damping on the frequency is very small. Also, Nayfeh and Mook (2008) presented that for the first approximation, the frequency

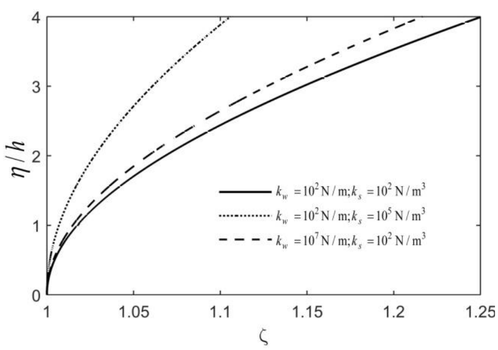

Fig. 10 The influence of linear elastic foundation coefficients on the non-linear free vibration response of internal SSFG cylindrical shell ($\theta = 0°, \beta = 90°, K = k = 1$)

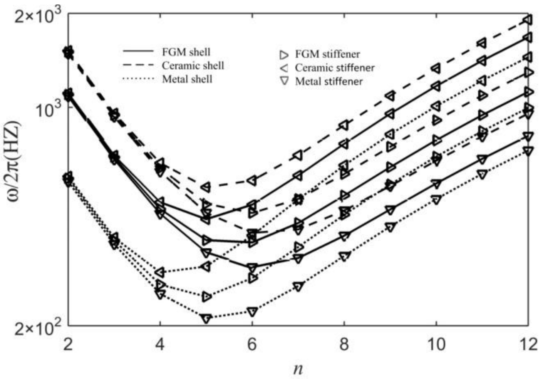

Fig. 11 The effect of material properties on the natural frequency of internal SSFG cylindrical shells without elastic foundation ($\theta = 0°, \beta = 90°$)

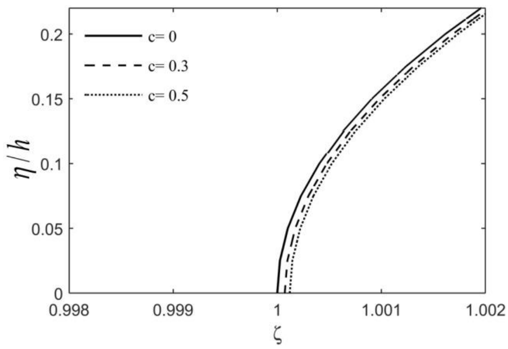

Fig. 12 The influence of damping coefficient on non-linear free vibration responses ($\theta = 0°, \beta = 90°, K = k = 1$)

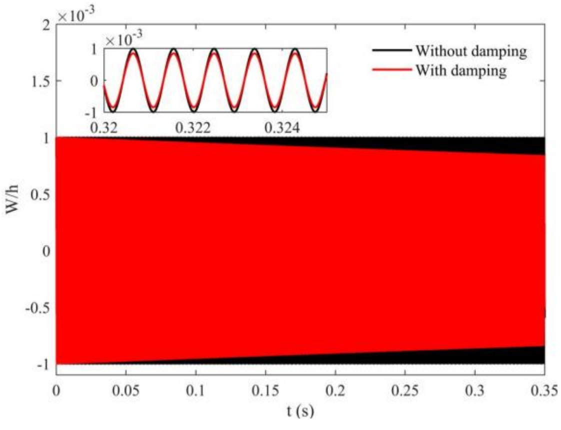

Fig. 13 The nonlinear free vibration responses of SSFG cylindrical shell ($\theta = 0°, \beta = 90°, K = k = 1$)

is not affected by the damping. In more investigations, it was observed that in the high initial condition and high damping coefficient, the linear damping has a low effect on the frequency, and weakly leads to decreasing the frequency.

5. Conclusions

An analytical method was used to study the non-linear free vibration behavior of SSFG cylindrical shells. The SSFG shell is resting on linear and non-linear elastic foundation with damping force. The elastic foundation for the linear model is according to Winkler and Pasternak parameters, and for the non-linear model, one cubic term is added. Meanwhile, it is assumed the cylindrical shell is surrounded by linear damping, too. The material constitutive of the shell and stiffeners was considered to be continuously varied along the thickness direction. According to the location of the stiffeners, two different models of stiffened FG cylindrical shells with internal and external stiffeners were formulated. The effect of various parameters including material properties, geometrical dimensions, angle of stiffeners, damping coefficient and elastic foundation parameters on the non-linear free vibration response of the SSFG cylindrical shells was examined and the following conclusions were obtained

- The maximum natural frequency and amplitude of SSFG cylindrical shells are occurred when the angle between both series of stiffeners is 90°. But when the angle between both series of stiffeners is 60° to 90°, the significant difference is not observed in the answer.
- The minimum natural frequency and the frequency-amplitude response of SSFG cylindrical shell happen when the angle between both series of stiffeners is 0°.
- Increasing γ leads to increase the natural frequency of SSFG cylindrical shells.
- Linear elastic foundation and stiffeners increase the value of frequency-amplitude response of FG cylindrical shells while the non-linear elastic foundation has the reverse effect.
- When the metal proportion of shells decreases, the natural frequency of shell with external stiffeners increases and with internal stiffeners decreases, respectively.
- The Pasternak elastic foundation coefficient has a more considerable effect on frequency-amplitude response rather than the Winkler elastic foundation coefficient.
- The natural frequency decreases by increasing the radius-to-thickness ratio.
- Increasing the damping coefficient leads to reduce the amplitude of vibration.

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Appendix

$$
\begin{aligned}
A &= J_{11}^{*}m^4\pi^4 + \left(J_{33}^{*} - J_{12}^{*} - J_{21}^{*}\right)m^2n^2\pi^2\lambda^2 + J_{22}^{*}n^4\lambda^4 \\
B &= J_{21}^{**}m^4\pi^4 + \left(J_{11}^{*} + J_{22}^{*} - 2J_{36}^{**}\right)m^2n^2\pi^2\lambda^2 \\
&\quad + J_{12}^{**}n^4\lambda^4 - \frac{L^2}{R}m^2n^2,\ G = \left(\frac{n^4\lambda^4}{16J_{11}^{*}} + \frac{m^4\pi^4}{16J_{22}^{*}}\right) \\
B^{*} &= A_{21}^{*}m^4\pi^4 + \left(A_{11}^{*} + A_{22}^{*} - 2J_{36}^{**}\right)m^2n^2\pi^2\lambda^2 \\
&\quad + J_{12}^{**}n^4\lambda^4 - \frac{L^2}{R}m^2n^2,\ \lambda = \frac{L}{R} \\
D &= A_{11}^{**}m^4\pi^4 + \left(A_{12}^{**} + A_{21}^{**} + 4A_{36}^{**}\right)m^2n^2\pi^2\lambda^2 \\
&\quad + A_{22}^{**}n^4\lambda^4
\end{aligned} \tag{A1}
$$

where

$$
\begin{aligned}
&\Delta = J_{11}J_{22} - J_{12}J_{21},\ J_{22}^{*} = \frac{J_{22}}{\Delta},\ J_{12}^{*} = \frac{J_{12}}{\Delta} \\
&J_{11}^{*} = \frac{J_{11}}{\Delta},\ J_{21}^{*} = \frac{J_{21}}{\Delta},\ J_{33}^{*} = \frac{1}{J},\ J_{36}^{*} = \frac{J_{36}}{J_{33}} \\
&J_{11}^{**} = J_{22}^{*}J_{14} - J_{12}^{*}J_{24},\ J_{12}^{**} = J_{22}^{*}J_{15} - J_{12}^{*}J_{25} \\
&J_{21}^{**} = J_{11}^{*}J_{24} - J_{21}^{*}J_{14},\ J_{22}^{**} = J_{11}^{*}J_{25} - J_{21}^{*}J_{15} \\
&A_{11}^{*} = J_{22}^{*}J_{14} - J_{21}^{*}J_{15},\ A_{21}^{*} = J_{11}^{*}J_{15} - J_{12}^{*}J_{14} \\
&A_{12}^{*} = J_{22}^{*}J_{24} - J_{21}^{*}J_{25},\ A_{22}^{*} = J_{11}^{*}J_{25} - J_{12}^{*}J_{24} \\
&A_{11}^{**} = J_{11}^{**}J_{14} - J_{21}^{**}J_{15} - J_{41},\ A_{36}^{**} = J_{36}^{**}J_{36} - J_{63} \\
&A_{12}^{**} = J_{12}^{**}J_{14} - J_{22}^{**}J_{15} - J_{42} \\
&A_{21}^{**} = J_{11}^{**}J_{24} - J_{21}^{**}J_{25} - J_{51} \\
&A_{22}^{**} = J_{12}^{**}J_{24} - J_{22}^{**}J_{25} - J_{52}
\end{aligned} \tag{A2}
$$

where

$$
\begin{aligned}
J_{11} &= \frac{E_1}{1-\nu^2} + Z_1E_{1s}(\cos^3\theta + \cos^3\beta) \\
J_{12} &= \frac{E_1\nu}{1-\nu^2} + Z_1E_{1s}(\sin^2\theta\cos\theta + \sin^2\beta\cos\beta) \\
J_{14} &= \frac{E_2}{1-\nu^2} + Z_1E_{2s}(\cos^3\theta + \cos^3\beta) \\
J_{15} &= \frac{E_2\nu}{1-\nu^2} + Z_1E_{2s}(\sin^2\theta\cos\theta + \sin^2\beta\cos\beta) \\
J_{21} &= \frac{E_1\nu}{1-\nu^2} + Z_2E_{1s}(\sin\theta\cos^2\theta + \sin\beta\cos^2\beta) \\
J_{22} &= \frac{E_1}{1-\nu^2} + Z_2E_{1s}(\sin^3\theta + \sin^3\beta) \\
J_{24} &= \frac{E_2\nu}{1-\nu^2} + Z_2E_{2s}(\sin\theta\cos^2\theta + \sin\beta\cos^2\beta) \\
J_{25} &= \frac{E_2}{1-\nu^2} + Z_2E_{2s}(\sin^3\theta + \sin^3\beta)
\end{aligned} \tag{A3}
$$

$$
\begin{aligned}
J_{33} &= \frac{E_1}{2(1+\nu)} + 2Z_3E_{1s}(\sin\theta\cos\theta + \sin\beta\cos\beta) \\
J_{36} &= \frac{E_2}{2(1+\nu)} + 2Z_3E_{2s}(\sin\theta\cos\theta + \sin\beta\cos\beta) \\
J_{51} &= \frac{E_3\nu}{1-\nu^2} + Z_2E_{3s}(\sin\theta\cos^2\theta + \sin\beta\cos^2\beta) \\
J_{41} &= \frac{E_3}{1-\nu^2} + Z_1E_{3s}(\cos^3\theta + \cos^3\beta) \\
J_{42} &= \frac{E_3\nu}{1-\nu^2} + Z_1E_{3s}(\sin^2\theta\cos\theta + \sin^2\beta\cos\beta) \\
J_{55} &= \frac{E_3}{1-\nu^2} + Z_2E_{3s}(\sin^3\theta + \sin^3\beta) \\
J_{63} &= \frac{E_3}{2(1+\nu)} + 2Z_3E_{3s}(\sin\theta\cos\theta + \sin\beta\cos\beta)
\end{aligned} \tag{A3}
$$

where

$$
\begin{aligned}
E_1 &= \int_{-h/2}^{h/2} E_{sh}(z)dz = \left(E_o + \frac{E_i - E_o}{k+1}\right)h \\
E_2 &= \int_{-h/2}^{h/2} zE_{sh}(z)dz = \frac{(E_i - E_o)kh^2}{2(k+1)(k+2)} \\
E_3 &= \int_{-\frac{h}{2}}^{\frac{h}{2}} z^2E_{sh}(z)dz \\
&= \left[\frac{E_o}{12} + (E_i - E_o)\left(\frac{1}{k+3} + \frac{1}{k+2} + \frac{1}{4k+4}\right)\right]h^3 \\
E_{1s} &= \int_{-\left(\frac{h}{2}+h_s\right)}^{-\frac{h}{2}} E_s(z)dz \\
&= \int_{h/2}^{h/2+h_s} E_s(z)dz = \left(E_i + \frac{E_o - E_i}{k_2+1}\right)h_s \\
E_{2s} &= \int_{-\left(\frac{h}{2}+h_s\right)}^{-\frac{h}{2}} zE_s(z)dz = \int_{\frac{h}{2}}^{\frac{h}{2}+h_s} zE_s(z)dz \\
&= \frac{E_i}{2}\, hh_s\left(\frac{h_s}{h} + 1\right) + (E_o - E_i)\, hh_s\left(\frac{1}{k_2+2}\frac{h_s}{h} + \frac{1}{2k_2+2}\right) \\
E_{3s} &= \int_{-\left(\frac{h}{2}+h_s\right)}^{-\frac{h}{2}} z^2E_s(z)dz = \int_{\frac{h}{2}}^{\frac{h}{2}+h_s} z^2E_s(z)dz \\
&= \frac{E_i}{3}h_s^3\left(\frac{3}{4}\frac{h^2}{h_s^2} + \frac{3}{2}\frac{h}{h_s} + 1\right) \\
&\quad + (E_o - E_i)h_s^3\left[\frac{1}{k_2+3} + \frac{1}{k_2+2}\frac{h}{h_s} + \frac{1}{4(k_2+1)}\frac{h^2}{h_s^2}\right]
\end{aligned} \tag{A4}
$$