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# The application of nonlocal elasticity to determine vibrational behavior of FG nanoplates

A.M. Fattahi [*1], Babak Safaei [2a] and E. Moaddab [3b]

[1] *Department of Mechanical Engineering, Tabriz Branch, Islamic Azad University, Tabriz, Iran*
[2] *Department of Mechanical Engineering, Tsinghua University, Beijing 100084, China*
[3] *Seraj Institute of Higher Education, Tabriz, Iran*

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**Abstract.** Nonlocal elasticity and Reddy plant theory are used to study the vibration response of functionally graded (FG) nanoplates resting on two parameters elastic medium called Pasternak foundation. Nonlocal higher order theory accounts for the effects of both scale and the effect of transverse shear deformation, which becomes significant where stocky and short nanoplates are concerned. It is assumed that the properties of FG nanoplate follow a power law through the thickness. In addition, Poisson's ratio is assumed to be constant in this model. Both Winkler-type and Pasternak-type foundation models are employed to simulate the interaction of nanoplate with surrounding elastic medium. Using Hamilton's principle, size-dependent governing differential equations of motion and corresponding boundary conditions are derived. A differential quadrature approach is being utilized to discretize the model and obtain numerical solutions for various boundary conditions. The model is validated by comparing the results with other published results.

**Keywords:** vibration; functionally graded nanoplate; nonlocal theory; elastic foundation

## 1. Introduction

Many research has been conducted on the behaviors of nanostructures from different mechanical, chemical and electrical viewpoints. Most of these methods have not found widespread use because they are very complex, costly and time-consuming. Therefore, Investigations are generally carried out by non-classical continuous environment modelling by Lian *et al*. (2010). Fleck and Hutchinson (1993) extracted high-order planar equations using strain gradient elasticity theory. In this work, high-order elements of stress related to coupled stress theory were obtained. Lam (2003) introduced extended strain gradient method in which three parameters of strain gradient, strain deviation and stretch gradient tensors were used to take into account the effect of size in explaining planar behavior. Fattahi and Mondali (2013) investigated elastic transition in short-fiber composites for plane strain case, in the other work they studied stress transfer in platelet reinforced composites (Fattahi and Mondali 2014). Kong *et al*. (2009) obtained static and dynamic responses of Euler-Bernoulli beam using extended strain gradient method and investigated the effects of thickness and dimensional parameters on static deformation and vibrational behavior of plate. Safaei and Fattahi (2017) investigated free vibration response of embedded single-layered graphene sheets. Wang *et al*. (2010) rearranged Timoshenko's beam equations using strain gradient model. The beam was composed of functionally graded materials. The vibrational behavior of the beam was also investigated.

The classical continuum mechanics theories are not capable of predicting and explaining size-dependent behaviors, which occur in micron-and submicron-scale structures. However, non-classical continuum theories such as non-local theory can acceptably interpret size-dependencies. Hence, some of studies have investigated vibration and dynamic analysis of a functionally graded (FG) nonobeam using non-local theory and Molecular Dynamic simulation (Moheimani and Ahmadian 2012, Damadam *et al*. 2018, Alizadeh and Fattahi 2019, Pourasghar *et al*. 2015, Sahmani and Fattahi 2017a, b). Peddieson *et al*. (2003) were the first to introduce a model for the investigation of Euler-Bernoulli nano-beam based on non-local elasticity theory. Several works have also been published since then on the analysis of one-dimensional systems based on non-local theory (Aydogdu 2009, Reddy and Pang 2008, Reddy 2010, Roque *et al*. 2011). Investigation of the stability of nano-sheets under in-sheet forces was a great breakthrough in the design of such systems enabling buckling and vibrations of these sheets to be investigated (Xia and Yang 2003, Pradhan and Murmu 2009).

Pasharavesh *et al*. (2011) considered nonlocal effects in inertial and forcing terms to investigate the transverse vibrations of nonlinear clamped-clamped and cantilever beam using approximate mode shapes. While they found that the resonance frequency of cantilever increases with increase of size, that of doubly clamped beam reduces with

∗Corresponding author, Ph.D., Professor,
E-mail: a.fattahi@iaut.ac.ir
[a] Ph.D.
[b] Ph.D.


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size. Mohammadsalehi *et al*. (2017) studied the vibration behavior of viscoelastic nano plates with DQM method. They considered the effects of structural damping coefficient, boundary conditions, aspect ratio, and nonlocal parameters on nano plates vibration behavior. Qin *et al*. (2017) employed Rayleigh-Ritz method for vibration analysis of shells and plates with different boundary conditions. In the other work Qin *et al*. (2018) investigated free vibration analysis of cylindrical shells coupled with moderately thick annular plates. Jalali *et al*. (2018) investigated the free vibration analysis of annular FG disk considering variable thickness with GDQM method. They performed the effect of thickness on critical speed and natural frequency. In another work Jalali *et al*. (2018) applied vibration analysis of FG micro beams in the thermal environment by modified couple stress theory.

In some papers, analytical solutions and nan-local ekasticity theory have been presented for the investigation of uniform nano-sheet vibrational behavior (Fattahi and Sahmani 2017a, b). In an attempt to investigate the effect of non-local modulus on frequency and bending of Nano-sheets, Aghababaei and Reddy (2009) studied bending and vibration of inotrope nano-sheets on simple support and provided an analytical solution. The sheet was analyzed based on third-order shear theory. Wang and Wang (2011) studied the vibration of nano-functional graded sheet by changing the characteristics. Sheets obeyed first-order shear theory. They used an analytical solution for this problem and investigated the effect of non-local modulus on sheet frequency. It was observed that increase of non-local modulus decreased the frequency of nano-sheet.

There are different theories for modeling phenomena at nano scale such as coupled stress theory, strain gradient and non-local theory. Among them, more attention has been paid to non-local theory as evidenced by the higher number of papers published using this theory possibly because it considers a wide range of inter-atomic interactions. Jung and Han (2013), showed the application of non-local elasticity model for the simulation of nano-scale phenomena is suitable and acceptable. Azizi *et al*. (2015a, b) by using carbon nanotubes as reinforcing fibers used this theory to address the exceptional mechanical and electrical properties of such nanotube-based composites. More recently, many researchers (Fattahi and Sahmani 2017a, b, c, Fattahi and Safaei 2017, Sahmani and Fattahi 2016, Safaei *et al*. 2017, 2018, 2019, Moradi-Dastjerdi *et al*. 2017, 2018, Moradi-Dastjerdi and Payganeh 2017a, b, Ghanati and Safaei 2018, Qin *et al*. 2019, Pourasghar and Chen 2019, Pourasghar *et al*. 2018, Pourasghar and Kamarian 2013) have shown that the use of non-local theory and mesh-free method promising in studying, FG and nano beam, plate, cylinder, panels vibration.

In this study, we have extended the application of non-local theory to the vibrations of nano-sheets fabricated from FG materials on elastic substrates by developing a generalized differential squares numerical method. To achieve minimum grid points needed for the calculations, we also performed convergence tests and obtained minimum point numbers in generalized differential squares method. The development of common plate theories for functionally graded plates and solving the equations using discretization method for different boundary conditions are among the novelties of this work.

## 2. Non-local elasticity theory

In this section dominant equations along with different boundary conditions are obtained using calculation, changes and Hamilton's principle. To do so it is first necessary to investigate the principles of non-local elasticity theory. In the classic method, when large-scale beams and sheets are investigated, it is assumed that the distance between atoms is negligible compared to the length of the object. The effect of characteristic length is not considered in the equations; while in the problems of nano-beams and nano-sheets, due to their small sizes, this effect cannot be neglected and this parameter enters the static and dynamic analyses as an effective factor. In the following, a summary is presented on the principles of non-local elasticity theory.

As mentioned above, in classic elasticity, stress tensor $\sigma$ at physical point $x$ is a function of strain tensor $\varepsilon$ at that point. In non-local elasticity theory presented by Eringen, stress tensor $\sigma$ at point $x$ in physical environment $\Omega$ depends on strain tensor $\varepsilon$ by an integral equation. In other words, structural equation of non-local elasticity theory is presented as an integral as

$$\sigma(x) = \iiint \alpha(|x' - x|, \tau) C\varepsilon(x') dv \tag{1}$$

where $\alpha(|x' - x|, \tau)$ is a function which is known as non-local modulus and is in fact a weight function for integral equation. $|x' - x|$ is the distance between local point $x$ and non-local point $x'$. $C$ is the fourth-order elasticity tensor which exists in classic theory. $\tau$ in equation is a parameter which is determined by the ratio of internal specific length of nanostructure $\bar{a}$ and external specific length $l$ and shows the significance of small scale in the structural integral equation of non-local elasticity theory. In fact $\tau$ is defined as

$$\tau = \frac{e_0 \bar{a}}{l} = \sqrt{\frac{\mu}{l^2}} \tag{2}$$

where $e_0$ is a physical parameter and is determined by matching non-local elasticity theory and results obtained from experiments or simulations. $\mu = (e_0 a)^2$ is known as small scale parameter. In structural integral equation, when $\tau$ tends to zero the effect of integral and nano-localness of stress and strain function disappears and the equation tends to classical structural equation $\sigma = C : \varepsilon$. Therefore, non-local modulus $\alpha(|x' - x|, \tau)$ should be such that when $\tau$ tends to zero, it tends to Dirac delta function; i.e.

$$\lim_{\tau \to 0} \alpha(|x' - x|, \tau) = \delta(|x' - x|) \tag{3}$$

Also function $\alpha$ should have its maximum value at local point. By defining a suitable non-local modulus capable of satisfying all required conditions, the differential form of structural equation of non-local elasticity theory can be obtained from its integral form. By defining non-local modulus $\alpha$ as

$$\alpha(|x,\tau|) = K_0\left(\frac{|x|/\sqrt{\mu}}{2\pi\mu}\right) \tag{4}$$

where $K_0$ is generalized Bessel function and by replacing modulus $\alpha$ in the above integral equation, differential form of structural equation of non-local elasticity theory is obtained as

$$(1-\mu\nabla^2)\sigma = C:\varepsilon \tag{5}$$

Elemental form of the above equation is expressed as

$$(1-\mu\nabla^2)\sigma_{ij} = C_{ijkl}:\varepsilon_{kl} \tag{6}$$

If small scale can be neglected, $\mu$ and therefore $\tau$ tend to zero and above equations transform to classic structural equation $\sigma = C:\varepsilon$.

Non-local modulus has the following interesting characteristics: Its integral value should be 1 at all points of the material volume.

$$\iiint \alpha(|x|)dv = 1 \tag{7}$$

Its maximum value is at $x = x'$ and it is decreased by increasing $|x' - x|$.

When non-local parameter tends to zero, non-local modulus has to tend to Dirac delta function so that the non-local theory including the limit of classical elasticity becomes non-local by setting the parameter at zero. For small internal specific lengths, i.e. when non-local modulus tends to one, non-local theory must be an approximation of crystal dynamics theory. By matching curves of planar wave propagation with curves of crystal dynamics or experiments, $\alpha$ can be determined for a certain material. In the following examples of some functions have been presented that have found applications:

(a) One-dimensional modules

$$\alpha(|x|,\tau) = \begin{cases} \frac{1}{l\tau}\left(1-\frac{|x|}{l\tau}\right), & |x| < l\tau \\ 0, & |x| \geq l\tau \end{cases} \tag{8}$$

(1)

$$\alpha(|x|,\tau) = \frac{1}{2l\tau}e^{-\frac{|x|}{l\tau}} \tag{9}$$

$$\alpha(|x|,\tau) = \frac{1}{l\sqrt{\pi\tau}}e^{(-x^2/l^2\tau)} \tag{10}$$

(b) Two-dimensional modules

$$\alpha(|x|,\tau) = (2\pi l^2\tau)^{-1}K_0(\sqrt{x.x}/l\tau) \tag{11}$$

where $K_0$ is modified Bessel function.

$$\alpha(|x|,\tau) = (\pi l^2\tau)^{-1}\exp(-x.x/l^2\tau) \tag{12}$$

(c) Three-dimensional modules

$$\alpha(|x|,t) = \frac{1}{8(\pi t)^{3/2}}\exp(-x.x/4t)\ , \qquad t = l^2\tau/4 \tag{13}$$

$$\alpha(|x|,\tau) = (4\pi l^2\tau^2)^{-1}(x.x)^{-1/2}\exp\left(-\sqrt{x.x}/l\tau\right) \tag{14}$$

For the modulus expressed in Eq. (12) it can be shown that the integral equation can be similarly shown in integral form

$$(1-\tau^2 l^2\nabla^2)\sigma = C:\varepsilon \tag{15}$$

This equation can be used for one-dimensional problem as

$$\left(1-\tau^2 l^2\frac{\partial^2}{\partial x^2}\right)\sigma = C:\varepsilon \tag{16}$$

## 3. Extracting motion equations

A uniform FG sheet with side length $L_1$, width $L_2$ and thickness h is assumed as shown in Fig. 1. Coordinate system ($x$, y, z) is introduced at one corner of FG sheet midplane such that x, y and z axes are assumed to be along the length , width and depth (thickness) directions of the nanoplate, respectively. $k_w$ is Winkler modulus parameter corresponding to normal pressure and $k_G$ is Pasternak modulus parameter relevant to transverse shear stress.

As mentioned above displacement field is considered as

$$u = u_0 + z\phi_x - c_1 z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \tag{17}$$

$$v = v_0 + z\phi_y - c_1 z^3\left(\phi_y + \frac{\partial w_0}{\partial x}\right) \tag{18}$$

$$w = w_0 \tag{19}$$

Therefore strain field is obtained as

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} + z\frac{\partial\phi_x}{\partial x} - c_1 z^3\left(\frac{\partial\phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \tag{20}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} + z\frac{\partial\phi_y}{\partial y} - c_1 z^3\left(\frac{\partial\phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \tag{21}$$

$$2\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = (1-3c_1 z^2)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \tag{22}$$

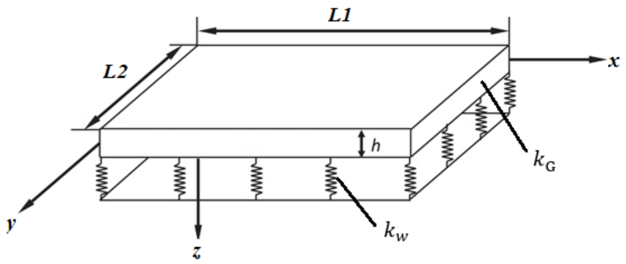


Fig. 1 Schematic diagram of a FG sheet

$$2\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = (1 - 3c_1 z^2)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \tag{23}$$

$$\begin{aligned} 2\varepsilon_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + z\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y}\right) \\ &- c_1 z^3\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial w_0}{\partial y\,\partial x}\right) \end{aligned} \tag{24}$$

The relation of strain stress for planar stress is as follows

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} & 0 \\ 0 & 0 & \mathcal{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \tag{25}$$

$$\begin{Bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \mathcal{Q}_{66} \begin{Bmatrix} \varepsilon_{xz} \\ \varepsilon_{yz} \end{Bmatrix} \tag{26}$$

where $\mathcal{Q}_{ij}$ are coefficients of hardness matrices and is defined as

$$\begin{aligned} \mathcal{Q}_{11} &= \mathcal{Q}_{22} = \frac{E}{1-\nu^2} \\ \mathcal{Q}_{12} &= \mathcal{Q}_{21} = \frac{E\upsilon}{1-\upsilon^2} \\ \mathcal{Q}_{22} &= \frac{E}{2(1+\upsilon)} \end{aligned} \tag{27}$$

Hamilton's principle is

$$\int_0^t \delta U - \delta V - \delta K \; dt = 0 \tag{28}$$

where $U$ is the total energy of the object. $V$ is the work of external forces and K is the total kinetic energy of the object.

Strain energy is expressed as

$$\begin{aligned} \delta U &= \int_0^t \iiint \sigma_{ij}\,\delta\varepsilon_{ij}\; dVdt \\ &= \int_0^t \iiint \sigma_{xx}\,\delta\varepsilon_{xx} + \sigma_{yy}\,\delta\varepsilon_{yy} \\ &+ 2\sigma_{xy}\,\delta\varepsilon_{xy} + 2\sigma_{xz}\,\delta\varepsilon_{xz} \\ &+ \sigma_{yz}\,\delta\varepsilon_{yz}\; dVdt \end{aligned} \tag{29}$$

In the following terms within the integral equation are integrated to the variable z

$$\begin{aligned} \iiint \sigma_{xx}\,\delta\varepsilon_{xx}\; dV = \iint & N_{xx}\frac{\delta\,\partial u_0}{\partial x} + M_{xx}\frac{\delta\,\partial \phi_x}{\partial x} \\ &- c_1 P_{xx}\left(\frac{\delta\,\partial \phi_x}{\partial x} + \frac{\delta\,\partial^2 w_0}{\partial x^2}\right) dxdy \end{aligned} \tag{30}$$

$$\begin{aligned} \iiint \sigma_{yy}\,\delta\varepsilon_{yy}\; dV = \iint & N_{yy}\frac{\delta\,\partial v_0}{\partial y} + M_{yy}\frac{\delta\,\partial \phi_y}{\partial y} \\ &- c_1 P_{yy}\left(\frac{\delta\,\partial \phi_y}{\partial y} + \frac{\delta\,\partial^2 w_0}{\partial y^2}\right) dxdy \end{aligned} \tag{31}$$

$$\begin{aligned} \iiint 2\sigma_{xy}\,\delta\varepsilon_{xy}\; dV = \iint & N_{xy}\left(\frac{\delta\,\partial v_0}{\partial x} + \frac{\delta\,\partial u_0}{\partial x}\right) \\ &+ M_{xy}\left(\frac{\delta\,\partial \phi_x}{\partial y} + \frac{\delta\,\partial \phi_y}{\partial x}\right) \\ &- c_1 P_{xy}\left(\frac{\delta\,\partial \phi_x}{\partial y} + \frac{\delta\,\partial \phi_y}{\partial x}\right. \\ &\left. + 2\frac{\delta\,\partial^2 w_0}{\partial y^2}\right) dxdy \end{aligned} \tag{32}$$

$$\begin{aligned} \iiint 2\sigma_{xz}\,\delta\varepsilon_{xz}\; dV = \iint & Q_x\left(\delta\phi_x + \frac{\delta\,\partial w_0}{\partial x}\right) \\ &- c_2 R_x\left(\delta\phi_x + \frac{\delta\,\partial w_0}{\partial x}\right) dxdy \end{aligned} \tag{33}$$

$$\begin{aligned} \iiint 2\sigma_{yz}\,\delta\varepsilon_{yz}\; dV = \iint & Q_y\left(\delta\phi_y + \frac{\delta\,\partial w_0}{\partial y}\right) \\ &- c_2 R_y\left(\delta\phi_y + \frac{\delta\,\partial w_0}{\partial y}\right) dxdy \end{aligned} \tag{34}$$

In which resultant forces are defined as

$$\begin{Bmatrix} N_{\alpha\beta} \\ M_{\alpha\beta} \\ P_{\alpha\beta} \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} dz \qquad \alpha = 1,2 \quad \beta = 1,2 \tag{35}$$

$$\begin{Bmatrix} Q_\alpha \\ R_\alpha \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_{\alpha z} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dz \qquad , \; \alpha = 1,2 \tag{36}$$

$$\overline{M}_{\alpha\beta} = M_{\alpha\beta} - c_1 P_{\alpha\beta} \qquad , \; \alpha = 1,2 \; , \; \beta = 1,2 \tag{37}$$

$$\overline{Q}_\alpha = Q_\alpha - c_2 R_\alpha \qquad \alpha = 1,2 \tag{38}$$

where N is the resultant tension force and is perpendicular to cross-section, and M and P are stress torque resultant and higher-order stress torque resultant respectively. $Q_\alpha$ is shear force and $R_\alpha$ is higher order stress shear force. Using Eqs. (35)-(36) resultant forces are obtained as

$$\begin{aligned} \mathcal{L}(N_{xx}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{Q}_{11}\varepsilon_{xx} + \mathcal{Q}_{12}\varepsilon_{yy}\right) dz \\ &= a_0\frac{\partial u_0}{\partial x} + a_1\frac{\partial \phi_x}{\partial x} - c_1 a_3\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \\ &\quad b_0\frac{\partial v_0}{\partial y} + b_1\frac{\partial \phi_y}{\partial y} - c_1 b_3\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \end{aligned} \tag{39}$$

$$\begin{aligned} \mathcal{L}(N_{yy}) &= \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{Q}_{12}\varepsilon_{xx} + \mathcal{Q}_{11}\varepsilon_{yy}\right) dz \\ &= b_0\frac{\partial u_0}{\partial x} + b_1\frac{\partial \phi_x}{\partial x} - c_1 b_3\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \\ &\quad a_0\frac{\partial v_0}{\partial y} + a_1\frac{\partial \phi_y}{\partial y} - c_1 a_3\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \end{aligned} \tag{40}$$

$$\mathcal{L}(\mathrm{M_{xx}}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{Q}_{11}\varepsilon_{xx} + \mathcal{Q}_{12}\varepsilon_{yy}\right) zdz = a_1\frac{\partial u_0}{\partial x} + a_2\frac{\partial \phi_x}{\partial x} - c_1 a_4\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) b_1\frac{\partial v_0}{\partial y} + b_2\frac{\partial \phi_y}{\partial y} - c_1 b_4\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \quad (41)$$

$$\mathcal{L}(\mathrm{M_{yy}}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{Q}_{12}\varepsilon_{xx} + \mathcal{Q}_{11}\varepsilon_{yy}\right) zdz = b_1\frac{\partial u_0}{\partial x} + b_2\frac{\partial \phi_x}{\partial x} - c_1 b_4\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) a_1\frac{\partial v_0}{\partial y} + a_2\frac{\partial \phi_y}{\partial y} - c_1 a_4\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \quad (42)$$

$$\mathcal{L}(\overline{\mathrm{M}}_{xx}) = (a_1 - c_1 a_3)\frac{\partial u_0}{\partial x} + (a_2 - c_1 a_4)\frac{\partial \phi_x}{\partial x} + (-c_1 a_4 + {c_1}^2 a_6)\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) (b_1 - c_1 b_3)\frac{\partial u_0}{\partial x} + (b_2 - c_1 b_4)\frac{\partial \phi_x}{\partial x} + (-c_1 b_4 + {c_1}^2 b_6)\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \quad (43)$$

$$\mathcal{L}(\overline{\mathrm{M}}_{xx}) = (b_1 - c_1 b_3)\frac{\partial u_0}{\partial x} + (b_2 - c_1 b_4)\frac{\partial \phi_x}{\partial x} + (-c_1 b_4 + {c_1}^2 b_6)\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) (a_1 - c_1 a_3)\frac{\partial u_0}{\partial x} + (a_2 - c_1 a_4)\frac{\partial \phi_x}{\partial x} + (-c_1 51_4 + {c_1}^2 a_6)\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \quad (44)$$

$$\mathcal{L}(\mathrm{P_{xx}}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{Q}_{11}\varepsilon_{xx} + \mathcal{Q}_{12}\varepsilon_{yy}\right) z^3 dz = a_3\frac{\partial u_0}{\partial x} + a_4\frac{\partial \phi_x}{\partial x} - c_1 a_6\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) + b_3\frac{\partial v_0}{\partial y} + b_4\frac{\partial \phi_y}{\partial y} - c_1 b_6\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \quad (45)$$

$$\mathcal{L}(\mathrm{P_{yy}}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\mathcal{Q}_{12}\varepsilon_{xx} + \mathcal{Q}_{11}\varepsilon_{yy}\right) z^3 dz = b_3\frac{\partial u_0}{\partial x} + b_4\frac{\partial \phi_x}{\partial x} - c_1 b_6\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) a_3\frac{\partial v_0}{\partial y} + a_4\frac{\partial \phi_y}{\partial y} - c_1 a_6\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w_0}{\partial y^2}\right) \quad (46)$$

$$\mathcal{L}(\mathrm{N_{xy}}) = d_0\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) + d_1\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - c_1 d_3\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x\,\partial y}\right) \quad (47)$$

$$\mathcal{L}(\mathrm{M_{xy}}) = d_1\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) + d_2\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - c_1 d_4\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x\,\partial y}\right) \quad (48)$$

$$\mathcal{L}(\mathrm{M_{xy}}) = d_3\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) + d_4\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - c_1 d_6\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x\,\partial y}\right) \quad (49)$$

$$\mathcal{L}(\mathrm{M_{xy}}) = d_3\left(\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y}\right) + d_4\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) - c_1 d_6\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} + 2\frac{\partial^2 w_0}{\partial x\,\partial y}\right) \quad (50)$$

$$\mathcal{L}(\mathrm{Q_x}) = (d_0 - 3c_1 d_2)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \quad (51)$$

$$\mathcal{L}(\mathrm{Q_y}) = (d_0 - 3c_1 d_2)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \quad (52)$$

$$\mathcal{L}(\mathrm{Q_x}) = (d_0 - 3c_1 d_2)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \quad (53)$$

$$\mathcal{L}(\mathrm{Q_y}) = (d_0 - 3c_1 d_2)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \quad (54)$$

$$\mathcal{L}(\mathrm{R_x}) = (d_2 - 3c_1 d_4)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \quad (55)$$

$$\mathcal{L}(\mathrm{R_x}) = (d_2 - 3c_1 d_4)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \quad (56)$$

$$\mathcal{L}(\mathrm{R_y}) = (d_2 - 3c_1 d_4)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \quad (57)$$

$$\mathcal{L}(\mathrm{R_y}) = (d_2 - 3c_1 d_4)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \quad (58)$$

$$\mathcal{L}(\overline{\mathrm{R}}_x) = (d_0 - 2c_2 d_2 + {c_2}^2 d_4)\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \quad (59)$$

$$\mathcal{L}(\overline{\mathrm{R}}_y) = (d_0 - 2c_2 d_2 + {c_2}^2 d_4)\left(\phi_y + \frac{\partial w_0}{\partial y}\right) \quad (60)$$

$$\mathcal{L} = 1 - \mu\nabla^2 = 1 - \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad (61)$$

where $\mu$ is $\mu = (e_0 a)^2$.

Now the integral of relations (7) are obtained using integration by part. After all simplifications and separating coefficients related to changes of each part of these equations, which are connected to equilibrium equations, are obtained as

$$\delta u_0: \quad -\frac{\partial N_{xx}}{\partial x}-\frac{\partial N_{xy}}{\partial y} \tag{62}$$

$$\delta v_0: \quad -\frac{\partial N_{yy}}{\partial y}-\frac{\partial N_{xy}}{\partial x} \tag{63}$$

$$\delta\phi_x: \quad -\frac{\partial \bar{M}_{xx}}{\partial x}-\frac{\partial \bar{M}_{xy}}{\partial y}+\bar{Q}_x \tag{64}$$

$$\delta\phi_y: \quad -\frac{\partial \bar{M}_{yy}}{\partial y}-\frac{\partial \bar{M}_{xy}}{\partial x}+\bar{Q}_y \tag{65}$$

$$\begin{aligned}\delta w_0: \quad & -c_1\frac{\partial^2 P_{xx}}{\partial x^2}-c_1\frac{\partial^2 P_{yy}}{\partial y^2}\\ & -2c_1\frac{\partial^2 P_{xy}}{\partial y\,\partial x}-\frac{\partial \bar{Q}_x}{\partial x}-\frac{\partial \bar{Q}_y}{\partial y}\end{aligned} \tag{66}$$

Now terms for kinetic energy are obtained. Since no rotational speed is present speeds related to particles in different directions are defined as

$$\dot{u}=\dot{u_0}+z\dot{\phi}_x-c_1z^3\left(\dot{\phi}_x+\frac{\partial \dot{w}_0}{\partial x}\right) \tag{67}$$

$$\dot{v}=\dot{v_0}+z\dot{\phi}_y-c_1z^3\left(\dot{\phi}_y+\frac{\partial \dot{w}_0}{\partial y}\right) \tag{68}$$

$$\dot{w}=\dot{w}_0 \tag{69}$$

Kinetic energy changes are defined as

$$\delta K=\int_0^t \iiint \rho(\dot{u}\delta\dot{u}+\dot{v}\delta\dot{v}+\dot{w}\delta\dot{w})\,dxdydt \tag{70}$$

By integrating to z we have

$$\begin{aligned}&\int_0^t \iiint \rho u\delta\dot{u}\ dVdt\\ &=\int_0^t \iint m_0\frac{\partial u_0}{\partial t}\frac{\partial \delta u_0}{\partial t}+m_1\frac{\partial u_0}{\partial t}\frac{\partial \delta\phi_x}{\partial t}\\ &-c_1m_3\frac{\partial u_0}{\partial t}\left(\frac{\partial \delta\phi_x}{\partial t}+\frac{\partial^2 \delta w_0}{\partial x\,\partial t}\right)+m_1\frac{\partial \phi_x}{\partial t}\frac{\partial \delta u_0}{\partial t}\\ &+m_2\frac{\partial \phi_x}{\partial t}\frac{\partial \delta\phi_x}{\partial t}-c_1m_4\frac{\partial \phi_x}{\partial t}\left(\frac{\partial \delta\phi_x}{\partial t}+\frac{\partial^2 \delta w_0}{\partial x\,\partial t}\right)\\ &-c_1m_3\left(\frac{\partial^2 w_0}{\partial x\,\partial t}+\frac{\partial \phi_x}{\partial t}\right)\frac{\partial \delta u_0}{\partial t}\\ &-c_1m_4\left(\frac{\partial^2 w_0}{\partial x\,\partial t}+\frac{\partial \phi_x}{\partial t}\right)\frac{\partial \delta\phi_x}{\partial t}\ dxdydt\end{aligned} \tag{71}$$

$$\begin{aligned}&\int_0^t \iiint \rho v\delta\dot{v}\ dVdt\\ &=\int_0^t \iint m_0\frac{\partial v_0}{\partial t}\frac{\partial \delta v_0}{\partial t}+m_1\frac{\partial v_0}{\partial t}\frac{\partial \delta\phi_y}{\partial t}\\ &-c_1m_3\frac{\partial v_0}{\partial t}\left(\frac{\partial \delta\phi_y}{\partial t}+\frac{\partial^2 \delta w_0}{\partial y\,\partial t}\right)+m_1\frac{\partial \phi_y}{\partial t}\frac{\partial \delta v_0}{\partial t}\\ &+m_2\frac{\partial \phi_y}{\partial t}\frac{\partial \delta\phi_y}{\partial t}-c_1m_4\frac{\partial \phi_y}{\partial t}\left(\frac{\partial \delta\phi_y}{\partial t}+\frac{\partial^2 \delta w_0}{\partial y\,\partial t}\right)\\ &-c_1m_3\left(\frac{\partial^2 w_0}{\partial y\,\partial t}+\frac{\partial \phi_y}{\partial t}\right)\frac{\partial \delta v_0}{\partial t}\\ &-c_1m_4\left(\frac{\partial^2 w_0}{\partial y\,\partial t}+\frac{\partial \phi_y}{\partial t}\right)\frac{\partial \delta\phi_y}{\partial t}\ dxdydt\end{aligned} \tag{72}$$

$$\int_0^t \iiint m_0 w\delta\dot{w}\ \ dVdt \tag{73}$$

Now the integral of relations (27) are obtained using integration by part. After all simplifications and separating coefficients related to changes of each variable of these equations which are connected to equilibrium equations are obtained as

$$\begin{aligned}\delta u_0: &-m_0\frac{\partial^2 u_0}{\partial t^2}-m_1\frac{\partial^2 \phi_x}{\partial t^2}\\ &+c_1m_3\frac{\partial^2 \phi_x}{\partial t^2}+c_1m_3\frac{\partial^3 w}{\partial t^2\,\partial x}\end{aligned} \tag{74}$$

$$\begin{aligned}\delta v_0: &-m_0\frac{\partial^2 v_0}{\partial t^2}-m_1\frac{\partial^2 \phi_y}{\partial t^2}\\ &+c_1m_3\frac{\partial^2 \phi_y}{\partial t^2}+c_1m_3\frac{\partial^3 w}{\partial t^2\,\partial y}\end{aligned} \tag{75}$$

$$\begin{aligned}\delta\phi_x: &-m_1\frac{\partial^2 u_0}{\partial t^2}-m_2\frac{\partial^2 \phi_x}{\partial t^2}+c_1m_4\frac{\partial^2 \phi_x}{\partial t^2}\\ &+c_1m_4\frac{\partial^3 w}{\partial t^2\,\partial x}+c_1m_3\frac{\partial^2 u_0}{\partial t^2}+c_1m_4\frac{\partial^2 \phi_x}{\partial t^2}\\ &-c_1{}^2m_6\frac{\partial^2 \phi_x}{\partial t^2}\delta\phi_x-c_1{}^2m_6\frac{\partial^3 w}{\partial t^2\,\partial x}\end{aligned} \tag{76}$$

$$\begin{aligned}\delta\phi_y: &-m_1\frac{\partial^2 v_0}{\partial t^2}-m_2\frac{\partial^2 \phi_y}{\partial t^2}+c_1m_4\frac{\partial^2 \phi_y}{\partial t^2}\\ &+c_1m_4\frac{\partial^3 w}{\partial t^2\,\partial y}+c_1m_3\frac{\partial^2 v_0}{\partial t^2}+c_1m_4\frac{\partial^2 \phi_y}{\partial t^2}\\ &-c_1{}^2m_6\frac{\partial^2 \phi_y}{\partial t^2}\delta\phi_y-c_1{}^2m_6\frac{\partial^3 w}{\partial t^2\,\partial y}\end{aligned} \tag{77}$$

$$\begin{aligned}\delta w: &\ c_1{}^2m_6\frac{\partial^4 w_0}{\partial t^2\,\partial x^2}-c_1m_4\frac{\partial^3 \phi_x}{\partial t^2\,\partial x}+c_1{}^2m_6\frac{\partial^3 \phi_x}{\partial t^2\,\partial x}\\ &-c_1m_3\frac{\partial^3 u_0}{\partial t^2\,\partial x}+c_1{}^2m_6\frac{\partial^4 w_0}{\partial t^2\,\partial y^2}-c_1m_3\frac{\partial^3 v_0}{\partial t^2\,\partial y}\\ &-c_1m_4\frac{\partial^3 \phi_y}{\partial t^2\,\partial y}+c_1{}^2m_6\frac{\partial^3 \phi_y}{\partial t^2\,\partial y}-m_0\frac{\partial^2 w_0}{\partial t^2}\end{aligned} \tag{78}$$

where coefficients of m are defined as

$$\begin{Bmatrix} m_0 \\ m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_6 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} z^0 \\ z^1 \\ z^2 \\ z^3 \\ z^4 \\ z^6 \end{Bmatrix} \rho(z)dz \tag{79}$$

where q is external transverse force which is exerted on the upper surfaces. $f_u$ and $f_v$ are extended longitudinal forces along the direction of x and y respectively. M, V and P are torque resultant, and stress force resultant on the edges of the sheet. $M_R$ is the torque resultant of higher order stress on the edges of the sheet which are obtained by shear third order theory. $k_w$ and $k_G$ are elastic substrate coefficients. Finally, by replacing Eqs. (17)-(18) and (28) in Hamilton's equation and setting the coefficients related to changes of each variables of equilibrium equation at zero in the form of obtained force terms we have

$$\begin{aligned} \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= f_u + m_0 \frac{\partial^2 u_0}{\partial t^2} + m_1 \frac{\partial^2 \phi_x}{\partial t^2} \\ &-c_1 m_3 \frac{\partial^2 \phi_x}{\partial t^2} - c_1 m_3 \frac{\partial^3 w}{\partial t^2 \partial x} \end{aligned} \tag{80}$$

$$\begin{aligned} \frac{\partial N_{yy}}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= f_v + m_0 \frac{\partial^2 v_0}{\partial t^2} + m_1 \frac{\partial^2 \phi_y}{\partial t^2} \\ &-c_1 m_3 \frac{\partial^2 \phi_y}{\partial t^2} - c_1 m_3 \frac{\partial^3 w}{\partial t^2 \partial y} \end{aligned} \tag{81}$$

$$\begin{aligned} \frac{\partial \overline{M}_{xx}}{\partial x} + \frac{\partial \overline{M}_{xy}}{\partial y} + \overline{Q}_x &= (m_1 - c_1 m_3) \frac{\partial^2 u_0}{\partial t^2} \\ &+(m_2 - 2c_1 m_4 + c_1{}^2 m_6) \frac{\partial^2 \phi_x}{\partial t^2} \\ &+(c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 w}{\partial t^2 \partial x} \end{aligned} \tag{82}$$

$$\begin{aligned} \frac{\partial \overline{M}_{yy}}{\partial y} + \frac{\partial \overline{M}_{xy}}{\partial x} + \overline{Q}_y &= (m_1 - c_1 m_3) \frac{\partial^2 v_0}{\partial t^2} \\ &+(m_2 - 2c_1 m_4 + c_1{}^2 m_6) \frac{\partial^2 \phi_y}{\partial t^2} \\ &+(c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 w}{\partial t^2 \partial y} \end{aligned} \tag{83}$$

$$\begin{aligned} & c_1 \frac{\partial^2 P_{xx}}{\partial x^2} + c_1 \frac{\partial^2 P_{yy}}{\partial y^2} + 2c_1 \frac{\partial^2 P_{xy}}{\partial y \partial x} \\ & + \frac{\partial \overline{Q}_x}{\partial x} + \frac{\partial \overline{Q}_y}{\partial y} - (k_w w - k_G \nabla^2 w) \\ = {} & m_0 \frac{\partial^2 w_0}{\partial t^2} + (c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 \phi_x}{\partial t^2 \partial x} \\ & +q + (c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 \phi_x}{\partial t^2 \partial x} \\ & -c_1{}^2 m_6 \left( \frac{\partial^4 w_0}{\partial t^2 \partial y^2} + \frac{\partial^4 w_0}{\partial t^2 \partial x^2} \right) \\ & +c_1 m_3 \left( \frac{\partial^3 u_0}{\partial t^2 \partial x} + \frac{\partial^3 v_0}{\partial t^2 \partial y} \right) \end{aligned} \tag{84}$$

To obtain equilibrium equations in the form of displacement terms, terms related to forces or their equivalents must be replaced but forces are not explicitly in the form of displacement terms. The operator $\mathcal{L}$ is used; and since this operator is linear, by applying it in equilibrium equation and replacement of Eqs. (1)-(3)-(15) the following equations can be obtained in the form of replacement terms

$$\begin{aligned} & a_0 \frac{\partial^2 u_0}{\partial x^2} + d_0 \frac{\partial^2 u_0}{\partial y^2} + (b_0 + d_0) \frac{\partial^2 v_0}{\partial x \partial y} \\ & +(a_1 - c_1 a_3) \frac{\partial^2 \phi_x}{\partial x^2} + (d_1 - c_1 d_3) \frac{\partial^2 \phi_x}{\partial y^2} \\ & +(b_1 - c_1 b_3 + d_1 - c_1 d_3) \frac{\partial^2 \phi_y}{\partial x \partial y} - c_1 a_3 \frac{\partial^3 w_0}{\partial x^3} \\ & +(-c_1 b_3 - 2c_1 d_3) \frac{\partial^3 w_0}{\partial x \partial y^2} = \mathcal{L}\, \dot{u} \end{aligned} \tag{85}$$

$$\begin{aligned} & a_0 \frac{\partial^2 v_0}{\partial y^2} + d_0 \frac{\partial^2 v_0}{\partial x^2} + (b_0 + d_0) \frac{\partial^2 u_0}{\partial x \partial y} \\ & +(a_1 - c_1 a_3) \frac{\partial^2 \phi_y}{\partial y^2} + (d_1 - c_1 d_3) \frac{\partial^2 \phi_y}{\partial x^2} \\ & +(b_1 - c_1 b_3 + d_1 - c_1 d_3) \frac{\partial^2 \phi_x}{\partial x \partial y} - c_1 a_3 \frac{\partial^3 w_0}{\partial y^3} \\ & +(-c_1 b_3 - 2c_1 d_3) \frac{\partial^3 w_0}{\partial y \partial x^2} = \mathcal{L}\, \dot{v} \end{aligned} \tag{86}$$

$$\begin{aligned} & (a_1 - c_1 a_3) \frac{\partial^2 u_0}{\partial x^2} \\ & +(d_1 - c_1 d_3) \frac{\partial^2 u_0}{\partial y^2} (b_1 - c_1 b_3 + d_1 - c_1 d_3) \frac{\partial^2 v_0}{\partial x \partial y} \\ & +(a_2 - 2c_1 a_4 + c_1{}^2 a_6) \frac{\partial^2 \phi_x}{\partial x^2} \\ & +(d_2 - 2c_1 d_4 + c_1{}^2 d_6) \frac{\partial^2 \phi_x}{\partial y^2} + (-c_1 a_4 + c_1{}^2 a_6) \frac{\partial^3 w_0}{\partial x^3} \\ & +(b_2 - 2c_1 b_4 + c_1{}^2 b_6 + d_2 - 2c_1 d_4 + c_1{}^2 d_6) \frac{\partial^2 \phi_y}{\partial x \partial y} \\ & +(-c_1 b_4 + c_1{}^2 b_6 - 2c_1 d_4 + 2c_1{}^2 d_6) \frac{\partial^3 w_0}{\partial x \partial y^2}) \\ & +(-d_0 - 2c_2 d_2 - c_2{}^2 d_4) \left( \phi_x + \frac{\partial w_0}{\partial x} \right) = \mathcal{L}\, \dot{\Phi}_x \end{aligned} \tag{87}$$

$$\begin{aligned} & (a_1 - c_1 a_3) \frac{\partial^2 v_0}{\partial y^2} + (d_1 - c_1 d_3) \frac{\partial^2 v_0}{\partial x^2} \\ & +(b_1 - c_1 b_3 + d_1 - c_1 d_3) \frac{\partial^2 u_0}{\partial x \partial y} \\ & +(a_2 - 2c_1 a_4 + c_1{}^2 a_6) \frac{\partial^2 \phi_y}{\partial y^2} \\ & +(d_2 - 2c_1 d_4 + c_1{}^2 d_6) \frac{\partial^2 \phi_y}{\partial x^2} \\ & +(-c_1 a_4 + c_1{}^2 a_6) \frac{\partial^3 w_0}{\partial y^3} \\ & +(b_2 - 2c_1 b_4 + c_1{}^2 b_6 + d_2 - 2c_1 d_4 + c_1{}^2 d_6) \frac{\partial^2 \phi_x}{\partial x \partial y} \end{aligned} \tag{88}$$

$$+(-c_1b_4+c_1{}^2b_6-2c_1d_4+2c_1{}^2d_6)\frac{\partial^3 w_0}{\partial y\,\partial x^2})$$
$$(-d_0-2c_2d_2-c_2{}^2d_4)\left(\phi_y+\frac{\partial w_0}{\partial y}\right)=\mathcal{L}\,\dot{\Phi}_y \tag{88}$$

$$c_1a_3\frac{\partial^3 u_0}{\partial x^3}+c_1a_3\frac{\partial^3 v_0}{\partial y^3}+(c_1a_4-c_1{}^2a_6)\frac{\partial^3\phi_x}{\partial x^3}$$
$$+(c_1a_4-c_1{}^2a_6)\frac{\partial^3\phi_y}{\partial x^3}+(2c_1d_3+c_1b_3)\frac{\partial^3 u_0}{\partial y^2\,\partial x}$$
$$+(2c_1d_3+c_1b_3)\frac{\partial^3 v_0}{\partial x^2\,\partial y}$$
$$+(c_1b_4-c_1{}^2b_6-2c_1{}^2d_6+2c_1d_4)\frac{\partial^3\phi_x}{\partial y^2\,\partial x} \tag{89}$$
$$+(c_1b_4-c_1{}^2b_6-2c_1{}^2d_6+2c_1d_4)$$
$$-c_1{}^2a_6\left(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^4 w}{\partial y^4}\right)+(-2c_1{}^2b_6-4c_1{}^2d_6)\frac{\partial^4 w}{\partial y^2\,\partial x^2}$$
$$+(d_0-2c_1d_2+c_2{}^2d_4)\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)$$
$$-\mathcal{L}(k_w w-k_G\nabla^2 w)=\mathcal{L}\,\dot{w}$$

where coefficients a, b and c are defined as

$$\begin{Bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_6\end{Bmatrix}=\int_{-h/2}^{h/2}Q_{11}\begin{Bmatrix}1\\z\\z^2\\z^3\\z^4\\z^6\end{Bmatrix}dz$$
$$\begin{Bmatrix}b_0\\b_1\\b_2\\b_3\\b_4\\b_6\end{Bmatrix}=\int_{-h/2}^{h/2}Q_{22}\begin{Bmatrix}1\\z\\z^2\\z^3\\z^4\\z^6\end{Bmatrix}dz \tag{90}$$
$$\begin{Bmatrix}d_0\\d_1\\d_2\\d_3\\d_4\\d_6\end{Bmatrix}=\int_{-h/2}^{h/2}Q_{66}\begin{Bmatrix}1\\z\\z^2\\z^3\\z^4\\z^6\end{Bmatrix}dz$$

And also $\dot{u}$ , $\dot{v}$ , $\dot{\Phi}_x$ , $\dot{\Phi}_y$ and $\dot{w}$ are defined as

$$\dot{u}=f_u+m_0\frac{\partial^2 u_0}{\partial t^2}+m_1\frac{\partial^2\phi_x}{\partial t^2}$$
$$-c_1m_3\frac{\partial^2\phi_x}{\partial t^2}-c_1m_3\frac{\partial^3 w}{\partial t^2\,\partial x} \tag{91}$$

$$\dot{v}=f_v+m_0\frac{\partial^2 v_0}{\partial t^2}+m_1\frac{\partial^2\phi_y}{\partial t^2}$$
$$-c_1m_3\frac{\partial^2\phi_y}{\partial t^2}-c_1m_3\frac{\partial^3 w}{\partial t^2\,\partial y} \tag{92}$$

$$\dot{\Phi}_x=(m_1-c_1m_3)\frac{\partial^2 u_0}{\partial t^2}$$
$$+(m_2-2c_1m_4+c_1{}^2m_6)\frac{\partial^2\phi_x}{\partial t^2} \tag{93}$$
$$+(c_1m_4-c_1{}^2m_6)\frac{\partial^3 w}{\partial t^2\,\partial x} \tag{93}$$

$$\dot{\Phi}_y=(m_1-c_1m_3)\frac{\partial^2 v_0}{\partial t^2}$$
$$+(m_2-2c_1m_4+c_1{}^2m_6)\frac{\partial^2\phi_y}{\partial t^2} \tag{94}$$
$$+(c_1m_4-c_1{}^2m_6)\frac{\partial^3 w}{\partial t^2\,\partial y}$$

$$\dot{w}=+m_0\frac{\partial^2 w_0}{\partial t^2}+(c_1m_4-\ c_1{}^2m_6)\frac{\partial^3\phi_x}{\partial t^2\,\partial x}$$
$$+(c_1m_4-\ c_1{}^2m_6)\frac{\partial^3\phi_x}{\partial t^2\,\partial x}$$
$$-c_1{}^2m_6\left(\frac{\partial^4 w_0}{\partial t^2\,\partial y^2}+\frac{\partial^4 w_0}{\partial t^2\,\partial x^2}\right) \tag{95}$$
$$+c_1m_3\left(\frac{\partial^3 u_0}{\partial t^2\,\partial x}+\frac{\partial^3 v_0}{\partial t^2\,\partial y}\right)$$

The obtained boundary conditions would be

$$\begin{aligned}
&N_{xx}=P_1 \text{ or } u_0=\bar{u}_0 \text{ in } x=0 \text{ and } x=L_1\\
&N_{xy}=0 \text{ or } u_0=\bar{u}_0 \text{ in } y=0 \text{ and } y=L_2\\
&N_{yy}=P_2 \text{ or } v_0=\bar{v}_0 \text{ in } y=0 \text{ and } y=L_2\\
&N_{xy}=0 \text{ or } v_0=\bar{v}_0 \text{ in } x=0 \text{ and } x=L_1\\
&M_{xx}=M_1 \text{or } \phi_x=\bar{\phi}_x \text{ in } x=0 \text{ and } x=L_1\\
&M_{xy}=0 \text{ or } \phi_x=\bar{\phi}_x \text{ in } y=0 \text{ and } y=L_2\\
&M_{yy}=M_2 \text{ or } \phi_y=\bar{\phi}_y \text{ in} y=0 \text{ and } y=L_2\\
&M_{yx}=0 \text{ or } \phi_y=\bar{\phi}_y \text{ in } x=0 \text{ and } x=L_1\\
&c_1\frac{\partial P_{xx}}{\partial x}+c_1\frac{\partial P_{xy}}{\partial y}+\bar{Q}_x-\bar{N}_{xx}\frac{\partial w_0}{\partial x}=V_1\\
&\text{or } w_0=\bar{w}_0 \text{ in } x=0 \text{ and } x=L_1
\end{aligned} \tag{96}$$

## 4. Making equations dimensionless

To simplify the equations and increase precision, the equations are made dimensionless. To do so first we can eliminate the time dependency of displacement fields by defining them as

$$\begin{aligned}
u_0(x,y,t)&=\tilde{u}_0(x,y)e^{i\omega t}\\
v_0(x,y,t)&=\tilde{v}_0(x,y)e^{i\omega t}\\
w_0(x,y,t)&=\widetilde{w}_0(x,y)e^{i\omega t}\\
\phi_x(x,y,t)&=\widetilde{\phi}_x(x,y)e^{i\omega t}\\
\phi_y(x,y,t)&=\widetilde{\phi}_y(x,y)e^{i\omega t}
\end{aligned} \tag{97}$$

This is done because free vibrations of displacement field changes are harmonic with time. Then all parameters are made dimensionless as

$$\begin{aligned}
&X=x/L_1 && 0<X<1\\
&Y=y/L_2 && 0<Y<1\\
&Z=z/h && 0<Z<1\\
&\bar{h}=\frac{h}{L_1},\quad \frac{L_1}{L_2}=\alpha, && \tau=\mu/L_1{}^2
\end{aligned} \tag{98}$$

$$\mathcal{U}_0=\frac{\tilde{u}_0}{L_1},\quad \mathcal{V}_0=\frac{\tilde{v}_0}{L_2},\quad \mathcal{W}_0=\tilde{w}_0/L_1$$
$$\bar{E}_c=\frac{E_c}{E_c},\quad \bar{E}_m=\frac{E_m}{E_c},$$
$$\bar{\rho}_c=\frac{\rho_c}{\rho_c},\quad \bar{\rho}_m=\rho_m/\rho_c \tag{98}$$
$$\bar{\mathcal{N}}_{xx}=\frac{\bar{N}_{xx}}{\bar{h}LE_c},\quad \bar{\mathcal{N}}_{yy}=\frac{\bar{N}_{yy}}{\bar{h}LE_c},\quad \bar{\mathcal{N}}_{xy}=\frac{\bar{N}_{xy}}{\bar{h}L_1E_c}$$
$$\varpi=\omega h\sqrt{\frac{\rho_c}{E_c}},\quad K_g=\frac{k_g}{E_cL\bar{h}},\quad K_w=\frac{k_wL}{E_c\bar{h}}$$

$$\begin{aligned}&a_0\frac{\partial^2\mathcal{U}_0}{\partial X^2}+\alpha^2 d_0\frac{\partial^2\mathcal{U}_0}{\partial Y^2}+\alpha(b_0+d_0)\frac{\partial^2\mathcal{U}_0}{\partial X\,\partial Y}\\&+(a_1-c_1a_3)\frac{\partial^2\phi_x}{\partial X^2}+(d_1-c_1d_3)\alpha^2\frac{\partial^2\phi_x}{\partial Y^2}\\&+(b_1-c_1b_3+d_1-c_1d_3)\alpha\frac{\partial^2\phi_y}{\partial X\,\partial Y}\\&-c_1a_3\frac{\partial^3\mathcal{W}_0}{\partial X^3}+(-c_1b_3-2c_1d_3)\alpha^2\frac{\partial^3\mathcal{W}_0}{\partial X\,\partial Y^2}=-\omega^2\bar{\mathcal{L}}\,\dot{u}\end{aligned} \tag{99}$$

$$\begin{aligned}&\alpha a_0\frac{\partial^2\mathcal{V}_0}{\partial Y^2}+\frac{d_0}{\alpha\frac{\partial^2\mathcal{V}_0}{\partial X^2}}+(b_0+d_0)\alpha\frac{\partial^2\mathcal{U}_0}{\partial X\,\partial Y}\\&+(a_1-c_1a_3)\alpha^2\frac{\partial^2\phi_y}{\partial Y^2}+(d_1-c_1d_3)\frac{\partial^2\phi_y}{\partial X^2}\\&+(b_1-c_1b_3+d_1-c_1d_3)\alpha\frac{\partial^2\phi_x}{\partial X\,\partial Y}\\&-c_1a_3\alpha^3\frac{\partial^3\mathcal{W}_0}{\partial Y^3}+(-c_1b_3-2c_1d_3)\alpha\frac{\partial^3\mathcal{W}_0}{\partial Y\,\partial X^2}\\&=-\omega^2\bar{\mathcal{L}}\,\dot{v}\end{aligned} \tag{100}$$

$$\begin{aligned}&(a_1-c_1a_3)\frac{\partial^2\mathcal{U}_0}{\partial X^2}+(d_1-c_1d_3)\alpha^2\frac{\partial^2\mathcal{U}_0}{\partial Y^2}\\&+(b_1-c_1b_3+d_1-c_1d_3)\frac{\partial^2\mathcal{V}_0}{\partial X\,\partial Y}\\&+(a_2-2c_1a_4+c_1{}^2a_6)\frac{\partial^2\phi_x}{\partial X^2}\\&+(d_2-2c_1d_4+c_1{}^2d_6)\alpha^2\frac{\partial^2\phi_x}{\partial Y^2}\\&+(-c_1a_4+c_1{}^2a_6)\frac{\partial^3\mathcal{W}_0}{\partial X^3}\\&+(b_2-2c_1b_4+c_1{}^2b_6+d_2\\&-2c_1d_4+c_1{}^2d_6)\alpha\frac{\partial^2\phi_y}{\partial X\,\partial Y}+(-c_1b_4+c_1{}^2b_6\\&+(-d_0-2c_2d_2-c_2{}^2d_4)\left(\phi_x+\frac{\partial\mathcal{W}_0}{\partial X}\right)=-\omega^2\bar{\mathcal{L}}\,\dot{\Phi}_x\end{aligned} \tag{101}$$

$$\begin{aligned}&(a_1-c_1a_3)\alpha\frac{\partial^2\mathcal{V}_0}{\partial Y^2}+(d_1-c_1d_3)/\alpha\frac{\partial^2\mathcal{V}_0}{\partial X^2}\\&+(b_1-c_1b_3+d_1-c_1d_3)\alpha\frac{\partial^2\mathcal{U}_0}{\partial X\,\partial Y}\\&+(a_2-2c_1a_4+c_1{}^2a_6)\alpha^2\frac{\partial^2\phi_y}{\partial Y^2}\\&+(d_2-2c_1d_4+c_1{}^2d_6)\frac{\partial^2\phi_y}{\partial X^2}\\&+(-c_1a_4+(b_2-2c_1b_4+c_1{}^2b_6+d_2\\&-2c_1d_4+c_1{}^2d_6)\alpha\frac{\partial^2\phi_x}{\partial X\,\partial Y}\\&+(-c_1b_4+c_1{}^2b_6-2c_1d_4+2c_1{}^2d_6)\alpha\frac{\partial^3\mathcal{W}_0}{\partial Y\,\partial X^2})\\&+(-d_0-2c_2d_2-c_2{}^2d_4)\left(\phi_y+\alpha\frac{\partial\mathcal{W}_0}{\partial Y}\right)\\&=-\omega^2\bar{\mathcal{L}}\,\dot{\Phi}_y\end{aligned} \tag{102}$$

$$\begin{aligned}&c_1a_3\frac{\partial^3\mathcal{U}_0}{\partial X^3}+c_1a_3\alpha^2\frac{\partial^3\mathcal{V}_0}{\partial Y^3}+(c_1a_4-c_1{}^2a_6)\frac{\partial^3\phi_x}{\partial X^3}\\&+(c_1a_4-c_1{}^2a_6)\frac{\partial^3\phi_y}{\partial X^3}\\&+(2c_1d_3+c_1b_3)\alpha^2\frac{\partial^3\mathcal{U}_0}{\partial Y^2\,\partial X}(2c_1d_3+c_1b_3)\frac{\partial^3\mathcal{V}_0}{\partial X^2\,\partial Y}\\&+(c_1b_4-c_1{}^2b_6-2c_1{}^2d_6+2c_1d_4)\alpha^2\frac{\partial^3\phi_x}{\partial Y^2\,\partial X}\\&+(c_1b_4-c_1{}^2b_6-2c_1{}^2d_6+2c_1d_4)\alpha\frac{\partial^3\phi_y}{\partial X^2\,\partial Y}\\&-c_1{}^2a_6\left(\frac{\partial^4\mathcal{W}}{\partial X^4}+\alpha^4\frac{\partial^4\mathcal{W}}{\partial Y^4}\right)\\&+(-2c_1{}^2b_6-4c_1{}^2d_6)\alpha^2\frac{\partial^4\mathcal{W}}{\partial Y^2\,\partial X^2}\\&+(d_0-2c_1d_2+c_2{}^2d_4)\\&\left(\frac{\partial^2 w}{\partial X^2}+\frac{\partial\phi_x}{\partial X}+\alpha^2\frac{\partial^2\mathcal{W}}{\partial Y^2}+\alpha\frac{\partial\phi_y}{\partial Y}\right)\\&-\bar{\mathcal{L}}(K_w w-K_G\nabla^2 w)=-\omega^2\bar{\mathcal{L}}\,\dot{w}\end{aligned} \tag{103}$$

where coefficients a, b and d are defined as

$$\begin{Bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_6\end{Bmatrix}=\int_{-1/2}^{-1/2}\mathcal{Q}_{11}/E_c\begin{Bmatrix}1*\bar{h}\\z*\bar{h}^2\\z^2*\bar{h}^3\\z^3*\bar{h}^4\\z^4*\bar{h}^5\\z^6*\bar{h}^7\end{Bmatrix}dz$$
$$\begin{Bmatrix}b_0\\b_1\\b_2\\b_3\\b_4\\b_6\end{Bmatrix}=\int_{-1/2}^{1/2}\mathcal{Q}_{22}/E_c\begin{Bmatrix}1*\bar{h}\\z*\bar{h}^2\\z^2*\bar{h}^3\\z^3*\bar{h}^4\\z^4*\bar{h}^5\\z^6*\bar{h}^7\end{Bmatrix}dz \tag{104}$$
$$\begin{Bmatrix}d_0\\d_1\\d_2\\d_3\\d_4\\d_6\end{Bmatrix}=\int_{-1/2}^{1/2}\mathcal{Q}_{66}/E_c\begin{Bmatrix}1*\bar{h}\\z*\bar{h}^2\\z^2*\bar{h}^3\\z^3*\bar{h}^4\\z^4*\bar{h}^5\\z^6*\bar{h}^7\end{Bmatrix}dz$$

Operator $\bar{\mathcal{L}}$ is transformed to

$$\bar{\mathcal{L}}=1-\tau\nabla^2=1-\tau\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \tag{105}$$

In above equations $\acute{\mathcal{u}}$, $\acute{\mathcal{v}}$, $\dot{\Phi}_x$, $\dot{\Phi}_y$ and $\acute{\mathcal{w}}$ are defined as

$$\acute{\mathcal{u}} = f_u + m_0 \frac{\partial^2 \mathcal{U}_0}{\partial t^2} + m_1 \frac{\partial^2 \phi_x}{\partial t^2} - c_1 m_3 \frac{\partial^2 \phi_x}{\partial t^2} - c_1 m_3 \frac{\partial^3 \mathcal{W}}{\partial t^2\, \partial X} \tag{106}$$

$$\acute{\mathcal{v}} = f_v + m_0 \frac{\partial^2 \mathcal{V}_0}{\partial t^2} + m_1 \frac{\partial^2 \phi_y}{\partial t^2} - c_1 m_3 \frac{\partial^2 \phi_y}{\partial t^2} - \alpha c_1 m_3 \frac{\partial^3 \mathcal{W}}{\partial t^2\, \partial Y} \tag{107}$$

$$\dot{\Phi}_x = (m_1 - c_1 m_3) \frac{\partial^2 \mathcal{U}_0}{\partial t^2} + (m_2 - 2c_1 m_4 + c_1{}^2 m_6) \frac{\partial^2 \phi_x}{\partial t^2} + (c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 \mathcal{W}}{\partial t^2\, \partial X} \tag{108}$$

$$\dot{\Phi}_y = (m_1 - c_1 m_3)/\alpha \frac{\partial^2 \mathcal{V}_0}{\partial t^2} + (m_2 - 2c_1 m_4 + c_1{}^2 m_6) \frac{\partial^2 \phi_y}{\partial t^2} + (c_1 m_4 - c_1{}^2 m_6)\alpha \frac{\partial^3 \mathcal{W}}{\partial t^2\, \partial Y} \tag{109}$$

$$\acute{\mathcal{w}} = +m_0 \frac{\partial^2 \mathcal{W}_0}{\partial t^2} + (c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 \phi_x}{\partial t^2\, \partial X} + (c_1 m_4 - c_1{}^2 m_6) \frac{\partial^3 \phi_x}{\partial t^2\, \partial X} - c_1{}^2 m_6 \left( \alpha^2 \frac{\partial^4 \mathcal{W}_0}{\partial t^2\, \partial Y^2} + \frac{\partial^4 \mathcal{W}_0}{\partial t^2\, \partial X^2} \right) + c_1 m_3 \left( \frac{\partial^3 \mathcal{U}_0}{\partial t^2\, \partial X} + \frac{\partial^3 \mathcal{V}_0}{\partial t^2\, \partial Y} \right) \tag{110}$$

## 5. Conclusions

In this section equilibrium equations obtained in the previous sections are discretized and solved using differential squares method. These equations were solved using a code in software MATLAB. In order to investigate the precision and stability of the solutions obtained from differential squares method; they were compared and validated with results published in other references (Jung and Han 2013). In this work the upper part of the sheet consisted of ceramic phase and its lower part was pure metal. FG material which was considered here was composed of silicon nitride and stainless steel with the following characteristics. The boundary conditions considered here were a combination of simple and clamped boundary conditions. The characteristics of materials and properties of different materials used in this research are presented in Tables 1 and 2.

Dimensionless frequencies for functional graded sheet are presented in Table 3 based on sheet thickness, local parameter and aspect ratio for simple support condition. It is obvious that numerical results presented in this research had very small differences of about 0.01% from the results of reported in other references. The results showed that the natural frequencies predicted by numerical method of extended differential squares matched greatly with those obtained from analytical method.

In this work, the vibrations of nano-sheets prepared from FG materials placed on elastic substrates have been investigated. This analysis was performed based on non-local elasticity theory and the application of numerical method of generalized differential squares. In order to obtain minimum grid points required for the calculations, convergence test was performed and minimum point numbers in generalized differential squares method was obtained. As was seen, the number of points required for the gridding of the sheet for conducting the analysis, common numerical methods such as finite elements were smaller and this method had higher convergence speed and power. Therefore, this method can be applied as a dynamic and strong method for solving complicated problems in mechanics including nano-mechanics.

Table 1 Characteristics of metal and ceramic phases

| Properties material | $\mathbf{E}$ (GPa) | $\boldsymbol{\rho}$ ($\mathbf{Kg/m^3}$) | $\boldsymbol{\nu}$ |
|---|---|---|---|
| $Si_3N_4$ | 348.46 | 2370 | 0.24 |
| SUS304 | 201.049 | 8166 | 0.32 |

Table 2 Parameters used in the problem

| $\boldsymbol{\mu}$ ($\mathbf{nm^2}$) | 0,1,2 |
|---|---|
| $L_1/h$ | 10, 20 |
| $K_G$ (GPa/nm) | 0.005 |
| $K_w$ (GPa/nm) | 0.5 |
| $L_1/L_2$ | 1,2 |
| Boundary condition | SSSS |
| Coefficient of shear stress | 5/6 |

Table 3 Comparison of dimensionless frequencies for a sheet with a simple support according to reference (Jung and Han 2013), analytical and numerical solutions

| $\mathbf{L_1/h}$ | $\mathbf{L_1/L_2}$ | $\boldsymbol{\mu}$ | Reference | Present solution |
|---|---|---|---|---|
| 1 | 10 | 0 | 0.0441 | 0.0442 |
| | | 1 | 0.0403 | 0.0403 |
| | | 2 | 0.0374 | 0.0373 |
| | 20 | 0 | 0.0113 | 0.0115 |
| | | 1 | 0.0103 | 0.0102 |
| | | 2 | 0.0096 | 0.0097 |
| 2 | 10 | 0 | 0.01055 | 0.01054 |
| | | 1 | 0.0863 | 0.0864 |
| | | 2 | 0.0748 | 0.0748 |
| | 20 | 0 | 0.0279 | 0.0279 |
| | | 1 | 0.0229 | 0.0229 |
| | | 2 | 0.0198 | 0.0198 |

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