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Free vibration of imperfect sigmoid and power law functionally graded beams

Mehmet Avcar*

Department of Civil Engineering, Faculty of Engineering, Suleyman Demirel University, Cunur, Isparta, Turkey

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Abstract. In the present work, free vibration of beams made of imperfect functionally graded materials (FGMs) including porosities is investigated. Because of faults during process of manufacture, micro voids or porosities may arise in the FGMs, and this situation causes imperfection in the structure. Therefore, material properties of the beams are assumed to vary continuously through the thickness direction according to the volume fraction of constituents described with the modified rule of mixture including porosity volume fraction which covers two types of porosity distribution over the cross section, i.e., even and uneven distributions. The governing equations of power law FGM (P-FGM) and sigmoid law FGM (S-FGM) beams are derived within the frame works of classical beam theory (CBT) and first order shear deformation beam theory (FSDBT). The resulting equations are solved using separation of variables technique and assuming FG beams are simply supported at both ends. To validate the results numerous comparisons are carried out with available results of open literature. The effects of types of volume fraction function, beam theory and porosity volume fraction, as well as the variations of volume fraction index, span to depth ratio and porosity volume fraction, on the first three non-dimensional frequencies are examined in detail.

Keywords: vibration; power-law FGM; sigmoid FGM; porosity volume fraction; classical beam theory; first order shear deformation beam theory

1. Introduction

Traditionally layered composites that made up of a variety of materials are commonly used in structural elements to ensure high performance. However, the sharp discontinuity between the material properties at the interface of two different types of material may pose severe material imperfections regarding stress concentrations. Japanese material scientists suggested FGMs in 1984 to eliminate such defects (Koizumi 1993, 1997). In FGMs, the volume fraction of dissimilar materials changes as a function of position throughout specific dimension of the structure to achieve the desired function. The graded structure of the material protects metals against corrosion, oxidation, and wear while minimizing imperfections such as interface and surface cracks and failure of the ceramic coating. FGMs are used in a wide variety of applications, among these are aviation, space crafts, automobiles, defense industry, electronics, biomedical sectors and other engineering practices (Delale and Erdogan 1983, Bao and Wang 1995, Lee and Erdogan 1995, Kieback *et al*. 2003, Attia *et al*. 2015, Bellifa *et al*. 2016, Chakraverty and Pradhan 2016, Abdelaziz *et al*. 2017, Civalek 2017, Abualnour *et al*. 2018, Fourn *et al*. 2018, Younsi *et al*. 2018, Civalek and Baltacioglu 2019). Beams form integral parts of the complex body; prior knowledge of their vibration behavior is essential for an engineer before finalizing the design of a given structure. Therefore, the vibration analysis of beams composed of FGMs has been one of the interests of the literature for many years (Aydogdu and Taskin 2007, Li 2008, Sina *et al*. 2009, Li *et al*. 2010, Şimşek 2010a, b, Thai and Vo 2012, Wattanasakulpong and Ungbhakorn 2012, Nguyen *et al*. 2013, 2017, Pradhan and Chakraverty 2013, 2014, Al-Basyouni *et al*. 2015, Bourada *et al*. 2015, Ebrahimi and Dashti 2015, Wattanasakulpong and Mao 2015, Jing *et al*. 2016, Avcar and Alwan 2017, Chen and Chang 2017, Kahya and Turan 2017, Avcar and Mohammed 2018, Chen and Chang 2018, Nguyen and Tran 2018, Rahmani *et al*. 2018).

Microvoids or porosities may arise in the FGMs because of some technical issues met through the process of manufacturing, i.e., due to the major difference in solidification temperatures between material constituents during sintering (Zhu *et al*. 2001) and production of FGM samples using multi-step sequential infiltration technique (Wattanasakulpong *et al*. 2012). The existence of porosities inside FGMs may cause reduction of the density and strength of materials, and so it is vital to consider the porosity effect on the mechanical behavior of FGM structures. However, the number of studies on the vibration of structures composed of FGMs including porosities is still limited. Wattanasakulpong and Ungbhakorn (2014) examined the linear and nonlinear vibration problems of beams made of FGMs with porosities. Wattanasakulpong and Chaikittiratana (2015) presented the flexural vibration analysis of imperfect FGM beams carrying porosities. Atmane *et al*. (2015, 2017) investigated the bending; free vibration and buckling analysis of functionally graded (FG) beams including porosities with and without elastic foundation. Yahia *et al*. (2015) examined the effects of the

*Corresponding author, Ph.D.,
E-mail: mehmetavcar@sdu.edu.tr; mehmetavcar@yahoo.com

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volume fraction distributions and porosity volume fraction on wave propagation of functionally graded plate adopting different higher order plate theories. Al Rjoub and Hamad (2017) studied the free vibration of FG beams with porosities adopting different beam theories. Galeban *et al*. (2016) presented the free vibration analysis of FG porous beams whose pores are saturated with fluid. Ebrahimi and Habibi (2016) reported the vibration characteristics of rectangular plates made of saturated porous FGMs. Akbaş (2017) investigated the thermal effects on the free vibration of FG porous deep beams. Ebrahimi and Hashemi (2017) studied the thermo-mechanical vibration behavior of non-uniform beams made of FG porous material under different thermal loadings. Fouda *et al*. (2017) presented a modified porosity model to study the static bending, the buckling and free vibrations of porous FG beams. Mirjavadi *et al*. (2017) examined the thermo-mechanical vibration of two-dimensional FG porous nanobeam. Akbaş (2018) investigated the forced vibration of an FG deep beam with porosity effect under a harmonic external distributed load. Heshmati and Daneshmand (2018) examined the effect of different profile variations on vibrational properties of non-uniform beams made of graded porous materials. Wattanasakulpong *et al*. (2018) investigated the vibration of FG porous beams with different patterns of porosity distributions.

The variations of material properties of FGMs are commonly expressed with power-law function (P-FGM) by most researchers. However, in P-FGMs the stress concentrations appear in one of the interfaces in which the material is continuous but quickly changes. For overcoming this problem, Chung and Chi (2001) proposed the concept of the sigmoid functionally graded materials (S-FGM) which is composed of the volume fraction using two power-law functions, and so on the variation of material properties becomes smooth at both ends. The number of studies dealing with the mechanical behavior of structures composed of S-FGMs is exceedingly rare in comparison with P-FGMs. Chi and Chung (2002, 2003) examined the crack propagating through coating–substrate composites composed of S-FGMs. Chi and Chung (2006a, b) presented theoretical solutions for the mechanical behavior of rectangular S-FGM plates under transverse loading. Ben-Oumrane *et al*. (2009) investigated the bending response of S-FGM thick beam subjected to uniformly distributed transverse loading. Mahi *et al*. (2010) examined the free vibration of a beam made of symmetric S-FGMs. Fereidoon *et al*. (2011) studied the bending of FG and FG coated thin plates composed of S-FGMs under transverse loading. Han *et al*. (2009) gave solutions for the geometrical non-linear analysis of S-FGM plates and shells. Atmane *et al*. (2011) presented a theoretical investigation for the free vibration of S-FGM beams with variable cross-section. Beldjelili *et al*. (2016) investigated the hygro-thermo-mechanical bending behavior of S-FGM plate resting on variable two-parameter elastic foundations adopting a refined plate theory. Hamed *et al*. (2016) studied the vibration characteristics of S-FGM nonlocal nanobeams. Park *et al*. (2016) performed the dynamic instability analysis for S-FGM plates embedded in an elastic medium. Aldousari (2017) considered the static bending analysis of an S-FGM beam subjected to a uniform distributed load. Wang and Zu (2017) investigated the large-amplitude vibration of S- FGM thin plates with porosities. Meradjah *et al*. (2018) analyzed the bending and free vibration of S-FGM beams. Wang and Zu (2018) examined the natural frequencies and nonlinear forced responses of moving porous S-FGM plates. Zhou and Zhang (2019) analyzed the uncertain natural frequency of S-FGM beams with axially varying stochastic properties.

In the light of foregoing literature review, it is seen that the free vibration of imperfect functionally graded beams including even, and uneven porosity distributions has not been analyzed yet considering sigmoid and power law volume fraction functions in the frameworks of classical and first order shear deformation beam theories. Besides, the interaction between adopted volume fraction functions, porosity distributions, and beam theories also have not been examined. For this purpose, an attempt is made to address these issues in the present paper. The material properties of the beams are assumed to vary continuously through the thickness direction according to the volume fraction of constituents described with the modified rule of the mixture including porosity volume fraction which covers two types of porosity distributions over the cross-section, i.e., even, and uneven distributions. The governing equations of P-FGM and S-FGM beams are derived within the frame works of CBT and FSDBT. The resulting equations are solved using separation of variables technique and assuming FG beams are simply supported at both ends. To validate the results numerous comparisons are carried out with available results of open literature. The effects of types of volume fraction function, beam theory and porosity volume fraction, as well as the variations of volume fraction index, span to depth ratio and porosity volume fraction, on the first three non-dimensional frequencies are examined in detail.

2. Material gradient of imperfect FG beams

In the present investigation, a straight FG imperfect beam of length, L, width b, thickness, h, having rectangular cross-section and simply supported at both ends is considered. The Cartesian coordinate system O(x,y,z) is located on the left edge of central axis of beam, where x-, y-, and z-axes are taken along the length, width and depth of the beam, as given in Fig. 1.

FG beam is composed of ceramic and metal constituents, where material composition at the upper surface is assumed to be ceramic-rich and varies continuously to the metal-rich one located in the lower

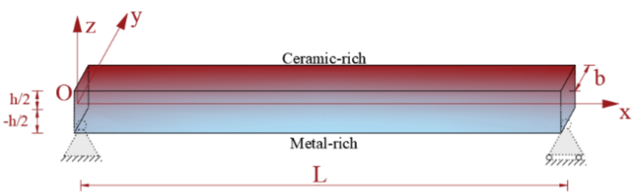


Fig. 1 Geometry of imperfect FG beam

(a) Even (IP-I) porosity distribution (b) Uneven (IP-II) porosity distribution

Fig. 2 The cross sections of imperfect FG beam

surface. Therefore, Young's modulus, E, and mass density ρ, of FG beam vary continuously through the thickness direction according to the function of the volume fractions of the constituents while Poisson's ratio v assume to be constant since its effect was found negligible (Delale and Erdogan 1983). Besides, imperfect FG beam is supposed to include porosities spreading across thickness because of the fault during manufacture and this situation is described with even (IP-I) and uneven (IP-II) porosity models of Wattanasakulpong and Chaikittiratana (2015) as shown in Fig. 2. Note that, IP-I includes porosity phases with an even distribution of volume fraction over the cross-section i.e. the porosity phases spread uniformly through thickness direction. Besides, IP-II includes porosity phases spread typically around the middle region of the cross-section and the amount of porosity decrease linearly to zero at the top and bottom of the cross section i.e., the porosity phases spread functionally through thickness direction.

The modified rule of mixture for an imperfect FG beam with a porosity volume fraction, $\alpha\,(\alpha << 1)$, can be expressed as (Wattanasakulpong and Chaikittiratana 2015)

$$P = P_m\left(V_m - \frac{\alpha}{2}\right) + P_c\left(V_c - \frac{\alpha}{2}\right) \tag{1}$$

where P_m and P_c are the material properties of metal and the ceramic; V_m and V_c are their volume fractions, respectively.

The relation for the total volume fraction of the metal and ceramic constituents is

$$V_m + V_c = 1 \tag{2}$$

In this study, two different types of functions are considered for volume fraction, i.e., volume fractions of FG beams are described with power law and sigmoid law functions.

2.1 Imperfect P-FGM beams

The volume fraction of the ceramic constituents obeys the following power-law function for P-FGM beams

$$V_c = \left(\frac{z}{h} + \frac{1}{2}\right)^k \tag{3}$$

where k denotes volume fraction index ($0 \le k \le \infty$), and z is the distance from the mid-plane of the FG beam.

The effective material properties of P-FGM beams with even (IP-I) and uneven (IP-II) porosity models can be expressed as follows, respectively (Wattanasakulpong and Chaikittiratana 2015)

$$P = \left(P_c - P_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m - \frac{\alpha}{2}\left(P_c + P_m\right) \tag{4}$$

$$P = \left(P_c - P_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^k + P_m - \frac{\alpha}{2}\left(P_c + P_m\right)\left(1 - \frac{2|z|}{h}\right) \tag{5}$$

Note that, as α = 0 the imperfect FG beam becomes the perfect one. In this case, as k = 0 the FG beam becomes a fully ceramic one, while it turns into fully metal one for k = ∞.

2.2 Imperfect S-FGM beams

The volume fraction of the ceramic constituents obeys the following two power law functions for S-FGM beams

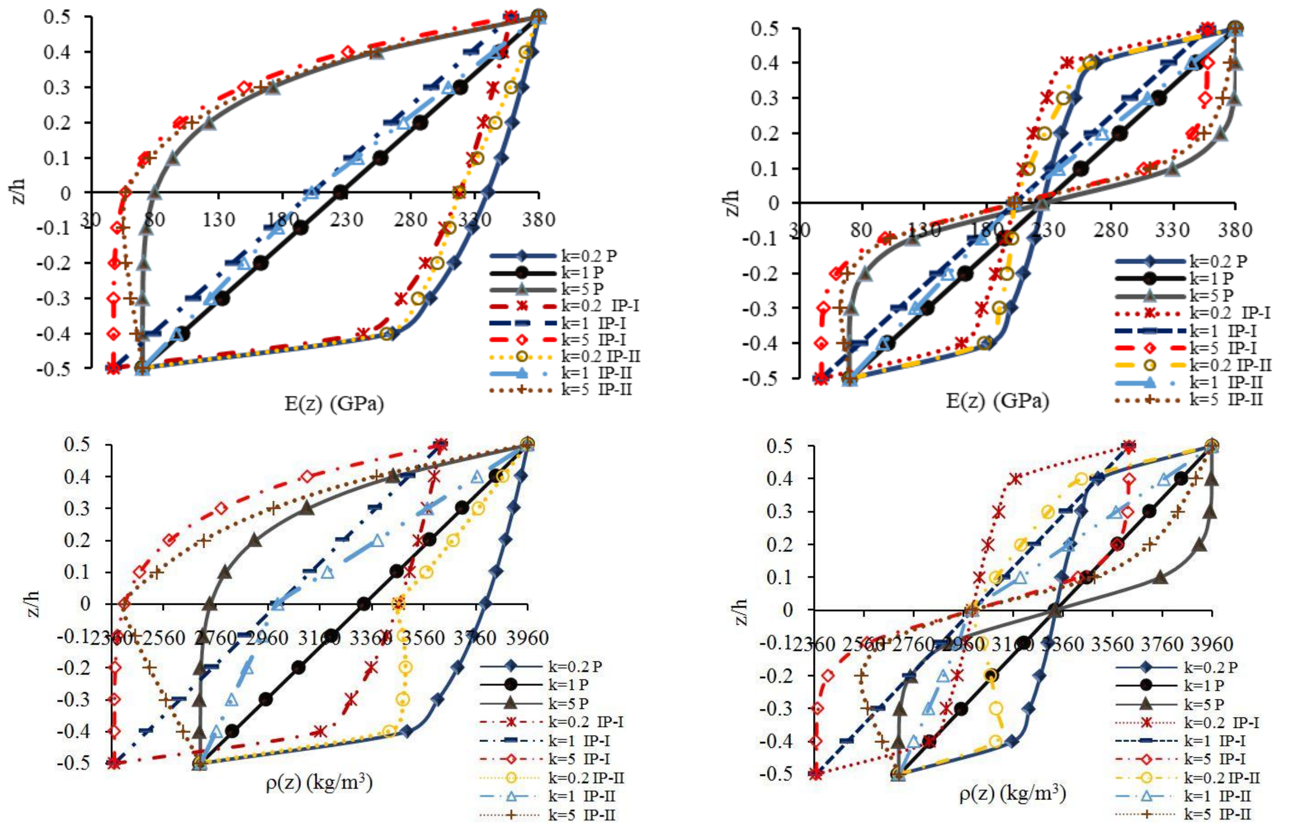
$$\begin{aligned} V_{c_1} &= 1 - \frac{1}{2}\left(1 - 2\frac{z}{h}\right)^k \quad for\ 0 \le z \le \frac{h}{2} \\ V_{c_2} &= \frac{1}{2}\left(1 + 2\frac{z}{h}\right)^k \quad for\ -\frac{h}{2} \le z \le 0 \end{aligned} \tag{6}$$

The effective material properties of S-FGM beams with even (IP-I) and uneven (IP-II) porosity models can be expressed as follows, respectively (Wang and Zu 2018)

$$\begin{aligned} P_1 &= \left(P_c - P_m\right)\left[1 - \frac{1}{2}\left(1 - 2\frac{z}{h}\right)^k\right] + P_m - \frac{\alpha}{2}\left(P_c + P_m\right) \quad for\ 0 \le z \le \frac{h}{2} \\ P_2 &= \left(P_c - P_m\right)\left[\frac{1}{2}\left(1 + 2\frac{z}{h}\right)^k\right] + P_m - \frac{\alpha}{2}\left(P_c + P_m\right) \quad for\ -\frac{h}{2} \le z \le 0 \end{aligned} \tag{7}$$

$$\begin{aligned} P_1 &= \left(P_c - P_m\right)\left[1 - \frac{1}{2}\left(1 - 2\frac{z}{h}\right)^k\right] + P_m - \\ &\quad - \frac{\alpha}{2}\left(P_c + P_m\right)\left(1 - \frac{2|z|}{h}\right) for\ 0 \le z \le \frac{h}{2} \\ P_2 &= \left(P_c - P_m\right)\left[\frac{1}{2}\left(1 + 2\frac{z}{h}\right)^k\right] + P_m - \\ &\quad - \frac{\alpha}{2}\left(P_c + P_m\right)\left(1 - \frac{2|z|}{h}\right) for\ -\frac{h}{2} \le z \le 0 \end{aligned} \tag{8}$$

In this study, unless otherwise indicated, FG beams composed of Aluminum (Al) as metal Alumina (Al_2O_3) as ceramic are considered, whose material properties are E_m = 70 GPa; ρ_m = 2702 kg/m^3; E_c = 380 GPa; ρ_c = 3960 kg/m^3.



(a) (b)

Fig. 3 The variations of Young's modulus and densities of a) P-FGM and b) S-FGM beams with even and uneven porosity distributions across the thickness direction (α = 0.1)

Fig. 3 illustrates the variations of Young's modulus and densities of P-FGM and S-FGM beams with even and uneven porosity models across the thickness direction. Fig. 3 shows that, the variation interval of values of Young's modulus and density of P-FGM is large near the lower surface as $k < 1$, while the related interval is large near the upper surface as $k > 1$. However, the variation interval of values of Young's modulus and density of S-FGM is large near the upper and lower surfaces as $k < 1$, while the related interval is large near the middle surface as $k > 1$. Therefore, the usage of S-FGMs provides more smooth stress distribution then P-FGMs, predominantly for $k > 1$.

Furthermore, the results indicated that perfect (P) beams have the highest values of Young's modulus and densities while imperfect FG beams with even porosity model (IP-I) have the lowest ones; the imperfect FG beams with uneven porosity model (IP-II) have values of Young's modulus and densities in the middle of P and IP-II. Moreover, IP-I and IP-II beams have same values of Young's modulus and densities at midline of the FG beams as well as IP-II and P beams have the same values of Young's modulus and densities at the upper and lower surfaces

3. Derivation of governing equations

Let's suppose the deformation of FG beam in x-z plane; the displacement components with respect to x and z directions are $\bar{u}$, and $\bar{w}$ respectively. Therefore, the displacement field at an arbitrary point of an FG beam can be expressed as

$$\bar{u}(x,z,t) = u(x,t) + z\,\phi(x,t) \tag{9}$$

$$\bar{w}(x,z,t) = w(x,t) \tag{10}$$

here $u(x, t)$ and $w(x, t)$ denote axial and transverse displacements of any point on neutral axis, respectively, ϕ is the rotation of the cross section and t is the time.

Consistent with the displacement field given in Eqs. (9)-(10), the normal and shear strains can be expressed respectively as

$$\varepsilon_x = \frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x} \tag{11}$$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \phi \tag{12}$$

In this study, classical beam theory (CBT) and first order shear deformation beam theory of Timoshenko (FSDBT) with constant shear correction factor are considered.

3.1 Classical beam theory

As CBT neglects the shear strain, γ_{xz} = 0, Eq. (11) transforms to the following form

$$\varepsilon_x = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} \tag{13}$$

Supposing that the material constituents of FG beam obey to Hooke's law, constitutive relation can be written as

$$\sigma_x = \bar{Q}_{11}\varepsilon_x = \bar{Q}_{11}\left(\frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2}\right) \tag{14}$$

here $\bar{Q}_{11}$ is stiffness constant and defined as follow

$$\bar{Q}_{11} = E(z) \tag{15}$$

As Eq. (14) is integrated over the cross-section of the FG beam, the stress resultants in terms of axial force and bending moment are found, respectively

$$N_x = A_{11}\frac{\partial u}{\partial x} - B_{11}\frac{\partial^2 w}{\partial x^2} \tag{16}$$

$$M_x = B_{11}\frac{\partial u}{\partial x} - D_{11}\frac{\partial^2 w}{\partial x^2} \tag{17}$$

where A_{11}, B_{11} and D_{11} are the material stiffness components of FG beam and defined as follows

$$\left(A_{11}, B_{11}, D_{11}\right) = \int_A \bar{Q}_{11}\left(1, z, z^2\right) dA \tag{18}$$

The equilibrium conditions of the beam are

$$Q_x = \frac{\partial M_x}{\partial x} \tag{19}$$

$$\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} = I_0\frac{\partial^2 w}{\partial t^2} \tag{20}$$

here I_0 is the mass moment of inertia of the beam and defined as follow

$$I_0 = \int_A \rho(z) dA \tag{21}$$

Considering the equilibrium conditions of FG beam with above given relations, after applying some mathematical operations and simplifications the following differential equation that governs the free vibration of FG beam is derived in the framework of CBT

$$a_{11}\frac{\partial^4 w}{\partial x^4} + I_0\frac{\partial^2 w}{\partial t^2} = 0 \tag{22}$$

where the following definition apply

$$a_{11} = \left(D_{11} - \frac{B_{11}{}^2}{A_{11}}\right) \tag{23}$$

3.2 First order shear deformation beam theory

The constitutive relations of FG beam in the frame work of FSDBT are

$$\sigma_x = \bar{Q}_{11}\varepsilon_{xx} = \bar{Q}_{11}\left(\frac{\partial u}{\partial x} + z\frac{\partial \phi}{\partial x}\right) \tag{24}$$

$$\tau_{xz} = \bar{Q}_{55}\gamma_{xz} = \bar{Q}_{55}\left(\phi + \frac{\partial w}{\partial x}\right) \tag{25}$$

where $\bar{Q}_{55}$ is shear stiffness and defined as

$$\bar{Q}_{55} = \frac{E(z)}{2(1+\nu)} \tag{26}$$

Stress resultants are

$$N_x = A_{11}\frac{\partial u}{\partial x} + B_{11}\frac{\partial \phi}{\partial x} \tag{27}$$

$$M_x = B_{11}\frac{\partial u}{\partial x} + D_{11}\frac{\partial \phi}{\partial x} \tag{28}$$

$$Q_x = KA_{55}\left(\frac{\partial w}{\partial x} + \phi\right) \tag{29}$$

Here K is shear correction factor and taken to be fixed as 5/6 in the whole paper and A_{55} is the material stiffness component and defined as

$$A_{55} = \int_A \bar{Q}_{55} dA \tag{30}$$

Because of FSDBT includes shear strain and rotary inertia effects, the equilibrium conditions of FG beam are

$$Q_x = \frac{\partial M_x}{\partial x} - b_{11}\frac{\partial^2 \phi}{\partial t^2} \tag{31}$$

$$\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} = I_0\frac{\partial^2 w}{\partial t^2} \tag{32}$$

Considering the equilibrium conditions of the FG beam with above given relations after applying some mathematical operations and simplifications the differential equation that governs the free vibration of FG beam is found in the frame work of FSDBT

$$a_{11}\frac{\partial^4 w_0}{\partial x^4} + I_0\frac{\partial^2 w_0}{\partial t^2} - \left(b_{11} + \frac{a_{11}I_0}{KA_{55}}\right)\frac{\partial^4 w_0}{\partial x^2 \partial t^2} + \frac{b_{11}I_0}{KA_{55}}\frac{\partial^4 w_0}{\partial t^4} = 0 \tag{33}$$

where the following definition apply

$$b_{11} = \left(I_2 - \frac{I_1{}^2}{I_0}\right) \tag{34}$$

4. Solution of governing equations

The solution of the governing Eqs. (22) and (33) can be obtained by separation of variables technique. In this case, one assumes the solution as follow

$$w(x,t) = \varphi(x)q(t) \tag{35}$$

where $\varphi(x)$ is a space-dependent function, and $q(t)$ is temporal function and defined as

$$q(t) = b_1 \cos \omega t + b_2 \sin \omega t \tag{36}$$

where, b_1 and b_2 are constants.

FG beam is assumed to have simply supported boundary conditions in both ends, hence these conditions can be stated in terms of w as

$$w(0,t) = 0, \ w(L,t) = 0 \tag{37}$$

$$\frac{d^2w}{dx^2}(0,t) = 0, \ \frac{d^2w}{dx^2}(L,t) = 0 \tag{38}$$

Substituting Eqs. (35) and (36) into Eq. (33) and then considering the Eqs. (37) and (38) in the resulted equation, the frequency equation is found for FG beam in the framework of FSDBT

$$a_{11}\left(\frac{n\pi}{L}\right)^4 - \omega^2\left\{I_0 + \left[b_{11} + \frac{a_{11}I_0}{KA_{55}}\right]\left(\frac{n\pi}{L}\right)^2\right\} + \frac{b_{11}I_0}{KA_{55}}\omega^4 = 0 \tag{39}$$

where n denotes the number of the mode.

Similarly, substituting Eqs. (35) and (36) into Eq. (22) and then considering the Eqs. (37) and (38) in the gotten equation, the following frequency equation is found for FG beam in the framework of CBT

$$a_{11}\left(\frac{n\pi}{L}\right)^4 - \omega^2 I_0 = 0 \tag{40}$$

5. Numerical results and discussion

5.1 Verification studies

In this section, the numerical results for perfect/ imperfect FG beams with S-S boundary conditions are compared with available results in open literature to verify the accuracy of the present work.Table 1 presents the comparison of the first five natural frequencies of perfect FG beam with varying volume fraction index (k) with the results of Li (2008), Simsek (2010a), Kahya and Turan (2017) in the frame work of FSDBT. The FG beam assumed to be composed of steel and aluminum and the computations are carried out for the following material and beam properties: E_{st} = 210 GPa; ρ_{st} = 7850 kg/m^3; E_{Al} = 70 GPa; ρ_{Al} = 210 GPa, $\upsilon_{St} = \upsilon_{Al}$ = 0.3, L = 0.5 m, h = 0.125 m, $\kappa = 5(1 + \upsilon)/(6 + 5\upsilon)$. As seen from Table 1, the present results are in good agreement with the previously published ones.

Table 2 shows the comparison of non-dimensional fundamental frequencies of perfect FG beam versus varying volume fraction index (k) and span-to-depth ratio, (L/h), with the results of Sina *et al*. (2009), Simsek (2010b), Pradhan and Chakraverty (2014), Cheng and Chang (2017, 2018), Nguyen *et al*. (2017) in the frame work of CBT and FSDBT. The computations are done for the following material properties: E_m = 70 GPa; ρ_m = 2700 kg/m^3; E_c = 380 GPa; ρ_c = 3800 kg/m^3, $v_m = v_c$ = 0.23 and the non-dimensional frequency is defined as $\Omega = \frac{\omega L^2}{h}\sqrt{\frac{I_0}{A_1}}$. It is clear from Table 2, the present results are in a good agreement with the previously published results.

Table 3 shows the comparison of the first three non-dimensional frequencies of perfect FG beams with varying volume fraction index (k) for with the findings of Simsek (2010a), Thai and Vo (2012), Nguyen *et al*. (2013), Pradhan and Chakraverty (2014), Kahya and Turan (2017), Chen and Chang (2018) in the frame work of FSDBT. The computations are done for the following material and beam properties E_m = 70 GPa; ρ_m = 2702 kg/m^3; E_c = 380 GPa; ρ_c = 3960 kg/m^3, $v_m = v_c$ = 0.3, K = 5/6, L/h = 5 and the non-dimensional frequency is defined as $\Omega = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$. Table 3 obviously shows that the present results are in good agreement with the previously published ones.

Table 4 shows comparison of the fundamental non-dimensional frequencies of perfect FG beam with varying volume fraction index (k) with the results of Simsek (2010a), Thai and Vo (2012), Wattanasakulpong and Ungbhakorn (2012), Pradhan and Chakraverty (2014), Al-Rjoub and Hamad (2017) and Chen and Chang (2017) in frame work of CBT. Here following material and beam properties are considered E_m = 70 GPa; ρ_m = 2702 kg/m^3; E_c = 380 GPa; ρ_c = 3960 kg/m^3 and non-dimensional frequency is defined as $\Omega = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$. It is apparent from Table 4 that the present results are in good agreement with the previously published ones.

Table 1 The first five natural frequencies (rad/s) of FG beam versus varying volume fraction index

Number of Mode (n)	$k=0$			$k=1$				$k=\infty$		
	Present	Li (2008)	Kahya and Turan (2017)	Present	Li (2008)	Kahya and Turan (2017)	Simsek (2010a)	Present	Li (2008)	Kahya and Turan (2017)
1	6615.66	6615.66	6742.83	6457.93	6457.93	6574.81	6443.08	6728.89	6728.89	6858.23
2	21904.14	21904.14	227888.88	21603.18	21603.18	22456.85	21470.95	22279.03	22279.03	23178.91
3	40402.57	40402.57	42195.37	40145.42	40145.42	41942.82	39775.55	41094.04	41094.04	42917.53
4	59865.40	59865.40	62259.45	59779.01	59779.01	62243.73	59092.37	60889.98	60889.98	63325.00
5	79534.57	79534.57	82223.70	79686.16	79686.16	82505.87	78638.36	80895.78	80895.78	83630.93

Table 2 The non-dimensional fundamental frequencies of FG beam versus varying volume fraction index and span-to-depth ratio

Theory	Source	k = 0			k = 0.3		
		L/h = 10	L/h = 30	L/h = 100	L/h = 10	L/h = 30	L/h = 100
CBT	Present	2.849	2.849	2.849	2.743	2.743	2.743
	Sina *et al*. (2009)	2.849	2.849	2.849	-	-	-
	Simsek (2010b)	2.837	2.847	2.848	2.731	2.741	2.743
	Pradhan and Chakraverty (2014)	2.837	2.847	2.849	2.768	2.778	2.779
	Cheng and Chang (2017)	-	-	-	-	-	2.743
FSDBT	Present	2.805	2.844	2.849	2.703	2.738	2.742
	Sina *et al*. (2009)	2.797	2.843	2.848	2.695	2.737	2.742
	Simsek (2010b)	2.804	2.843	2.848	2.701	2.738	2.742
	Pradhan and Chakraverty (2014)	2.805	2.844	2.848	2.738	2.774	2.778
	Nguyen et al. (2017)	2.804	2.844	2.849	2.702	2.738	2.742
	Chen and Chang (2018)	-	-	-	2.702	-	-

Table 3 The first three non-dimensional frequencies of FG beam versus varying volume fraction index

Number of mode (n)	Source	k							
		0	0.2	0.5	1	2	5	10	∞
1	Present	5.1525	4.8058	4.4098	3.9948	3.6425	3.4401	3.3190	2.67718
	Simsek (2010a)	5.1525	4.8066	4.4083	3.9902	3.6344	3.4312	3.3134	2.67718
	Thai and Vo (2012)	5.1527	-	4.4107	3.9904	3.6264	3.4012	3.2816	-
	Nguyen et al. (2013)	5.1525	4.8047	4.4075	3.9902	3.6344	3.4312	3.3135	-
	Pradhan and Chakraverty (2014)	5.1546	4.8364	4.5313	4.2566	4.0198	3.7464	3.4886	-
	Kahya and Turan (2017)	5.2219	-	4.4693	4.0497	3.69360	3.4882	3.3643	2.7133
	Chen and Chang (2018)	5.1525	-	4.4079	3.9904	3.6346	3.4315	3.3136	-
2	Present	17.8711	16.7453	15.4443	14.0478	12.7895	11.8949	11.3548	9.2857
	Thai and Vo (2012)	17.8812		15.4588	14.0100	12.6405	11.5431	11.024	-
	Nguyen et al. (2013)	17.8711	16.7393	15.4250	14.0030	12.7120	11.8157	11.3073	-
	Pradhan and Chakraverty (2014)	17.8908	16.7317	15.3306	13.7778	12.3619	11.4822	11.1126	-
	Chen and Chang (2018)	17.8711	-	15.4277	14.0064	12.7179	11.8226	11.3119	-
3	Present	34.1449	32.1236	29.7662	27.1781	24.7137	22.6714	21.4402	17.7414
	Thai and Vo (2012)	34.2097		29.8382	27.0979	24.3152	21.7158	20.5561	-
	Nguyen et al. (2013)	34.1449	32.1098	29.7146	27.0525	24.4970	22.4642	21.3219	-
	Pradhan and Chakraverty (2014)	30.2314	28.8311	27.0804	24.9042	22.3517	19.5600	18.0553	-
	Chen and Chang (2018)	34.1449	-	29.7231	27.0683	24.5242	22.4933	21.3399	-

Finally, the accuracy of present formulations is validated for perfect FG beams with the comparisons given in Tables 1-4. Small discrepancies seem to arise from the used solution method, consideration of Poisson's ratio in the constitutive equation and the value of shear correction factors.

Tables 5 and 6 show comparisons of the first three non-dimensional frequencies of perfect/imperfect FG beams with varying power law exponent (k) with the results of Al-Rjoub and Hamad (2017) in the frame work of CBT and FSDBT without rotary inertia respectively. The computations are done for the following material and beam properties E_m = 70 GPa; ρ_m = 2702 kg/m^3; E_c = 380 GPa; ρ_c = 3960 kg/m^3, v_m = v_c = 0.3, L/h = 10 and the non-dimensional frequency is defined as $\Omega = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$. It is seen from Tables 5 and 6 that the present results are completely the same with the results of Al-Rjoub and Hamad (2017) for k = 0 and k = ∞, however slight differences are existing between the results especially in the fundamental mode for the interval $0 < k < \infty$ as a result of Al-Rjoub and Hamad (2017) was considered the effect of Poisson's ratio in constitutive equation.

Table 4 The fundamental non-dimensional frequency of FG beam versus varying volume fraction index

Source	k						
	0	0.2	0.5	1	2	5	∞
Present	5.483	5.102	4.669	4.221	3.852	3.668	2.849
Simsek (2010a)	5.478	5.098	4.665	4.216	3.847	3.663	2.846
Thai and Vo (2012)	5.4777	—	4.6641	4.2163	3.8472	3.6628	-
Wattanasakulpong and Ungbhakorn (2012)	5.483	5.102	4.669	4.221	3.852	3.668	2.849
Pradhan and Chakraverty (2014)	5.4777	5.1299	4.8024	4.5161	4.2832	4.0252	-
Al-Rjoub and Hamad (2017)	5.483	5.102	4.669	4.221	3.852	3.668	2.849
Chen and Chang (2017)	5.48335	5.10190	4.66896	4.22098	3.85182	3.66747	2.84910

Table 5 The first three non-dimensional frequencies of FG beams versus varying volume fraction index

Type of beam	Number of mode (n)	Source	k						
			0	0.2	0.5	1	2	5	∞
Perfect $\alpha = 0$	1	Present	5.4834	5.1019	4.6690	4.2210	3.8518	3.6675	2.8491
	2		21.9335	20.4076	18.6759	16.8839	15.4073	14.6699	11.3964
	3		49.3503	45.9172	42.0207	37.9889	34.6665	33.0073	25.642
	1	Al-Rjoub and Hamad (2017)	5.4834	5.1351	4.8072	4.5207	4.2877	4.0295	2.8491
	2		21.9335	20.4076	18.6759	16.8839	15.4073	14.6699	11.3964
	3		49.3503	45.9506	42.1617	38.3034	35.1372	33.3944	25.642
Imperfect $\alpha = 0.1$	1	Present	5.5574	5.1449	4.6589	4.1269	3.6478	3.3953	2.5065
	2		22.2297	20.5795	18.6354	16.5076	14.5910	13.5811	10.0262
	3		50.0168	46.3038	41.9297	37.1422	32.8298	30.5576	22.5589
	1	Al-Rjoub and Hamad (2017)	5.5574	5.1837	4.8244	4.4982	4.2144	3.9084	2.5065
	2		22.2297	20.5795	18.6354	16.5076	14.591	13.5811	10.0262
	3		50.0168	46.3428	42.0992	37.5368	33.4594	31.1261	22.5589
Imperfect $\alpha = 0.2$	1	Present	5.6452	5.1961	4.6426	3.9917	3.3244	2.9002	1.9616
	2		22.5807	20.7842	18.5705	15.9667	13.2977	11.6008	7.8463
	3		50.8065	46.7645	41.7837	35.9251	29.9199	26.1017	17.6542
	1	Al-Rjoub and Hamad (2017)	5.6452	5.2419	4.8444	4.4636	4.0897	3.6833	1.9616
	2		22.5807	20.7842	18.5705	15.9667	13.2977	11.6008	7.8463
	3		50.8065	46.8107	41.9915	36.4365	30.8147	27.0427	17.6542

5.2 Illustrative studies

Table 7 shows the variations of values of the first three non-dimensional frequencies of perfect/imperfect P-FGM and S-FGM beams with respect to volume fraction index, k, in the frameworks of CBT and FSDBT for $L/h = 5$ and $\alpha = 0.1$. The results showed that the values of non-dimensional frequencies decrease with the increase of volume fraction index for both perfect/imperfect P-FGM and S-FGM beams, because of the percentage of metal phases that are weaker than ceramic phases become more pronounced. Besides, the effect in question becomes more noticeable for imperfect beams and gets the highest influences for P-FGM and S-FGM beams with even porosity distribution (IP-I). For the small values of volume fraction index ($k < 1$), the values of non-dimensional frequency of P-FGM beams are higher than those for S-FGM while the values of non-dimensional frequency of S-FGM beams are higher than those for P-FGM for high values of volume fraction index ($k > 1$).

The effect of porosity on the values of non-dimensional frequencies increases with the increase of the volume fraction index and becomes more pronounced as the values of volume fraction index is larger than one ($k > 1$). The effect of volume fraction index is more dominant for P-FGM beams in comparison with S-FGM beams. Comparisons of the values of non-dimensional frequencies FG beams which are computed adopting CBT and FSDBT showed that the effect of the volume fraction index on the first three number of the modes stays fixed in CBT, however, the same effect decreases with increasing number of modes in FSDBT for both of perfect/imperfect P-FGM and S-FGM beams.

Table 8 represents the variation of first three non-dimensional frequency values of perfect /imperfect P-FGM and S-FGM beams with respect to span to depth ratio, L/h, for $k = 2$ and $\alpha = 0.2$ in the framework of FSDBT. The

Table 6 The first three non-dimensional frequencies of FG beams versus varying volume fraction index

Type of beam	Number of mode (n)	Source	*k* = 0	0.2	0.5	1	2	5	∞
Perfect $\alpha = 0$	1	Present	5.4143	5.0409	4.6165	4.1761	3.8104	3.6201	2.8132
	2		20.8876	19.481	17.8768	16.1986	14.7743	13.9527	10.853
	3		44.4805	41.5861	38.2698	34.7599	31.6866	29.6743	23.1117
	1	Al-Rjoub and Hamad (2017)	5.4143	5.073	4.75	4.4657	4.2307	3.9669	2.8132
	2		20.8776	19.481	17.8768	16.1986	14.7743	13.9527	10.853
	3		44.4806	41.6108	38.3762	34.9997	32.0433	29.9533	23.1117
Imperfect $\alpha = 0.1$	1	Present	5.4890	5.0851	4.6086	4.0859	3.6124	3.3553	2.4757
	2		21.1925	19.6696	17.8687	15.8794	14.0485	12.9731	9.5584
	3		45.1791	42.0429	38.3205	34.1696	30.2593	27.7128	20.377
	1	Al-Rjoub and Hamad (2017)	5.4875	5.1211	4.7674	4.444	4.1588	3.8463	2.475
	2		21.1697	19.6496	17.8517	15.8654	14.0365	12.9597	9.5481
	3		45.0812	41.6543	38.3746	34.4116	30.691	28.0713	20.3329
Imperfect $\alpha = 0.2$	1	Present	5.5773	5.1374	4.5949	3.9554	3.2973	2.8727	1.938
	2		21.5503	19.891	17.8401	15.4089	12.8791	11.1785	7.4883
	3		45.9926	42.5727	38.3349	33.2719	27.9169	24.0981	15.9815
	1	Al-Rjoub and Hamad (2017)	5.5741	5.1789	4.7876	4.4106	4.0371	3.6248	1.9369
	2		21.504	19.8506	17.8069	15.3834	12.8598	11.1592	7.4722
	3		45.793	42.4318	38.3471	33.5572	28.54	24.7311	15.9121

Table 7 The first three non-dimensional frequencies of FG beams with respect to volume fraction index

Type of beam	Number of mode (n)	Material gradation	Theory: CBT, *k* = 0.2	CBT 0.5	CBT 1	CBT 2	CBT 5	FSDBT, *k* = 0.2	FSDBT 0.5	FSDBT 1	FSDBT 2	FSDBT 5
Perfect	1	P-FGM	5.102	4.669	4.221	3.852	3.667	4.806	4.41	3.995	3.642	3.44
	2		20.408	18.676	16.884	15.407	14.67	16.745	15.444	14.048	12.789	11.895
	3		45.917	42.021	37.989	34.666	33.007	32.124	29.766	27.178	24.714	22.671
	1	S-FGM	4.552	4.419	4.221	3.992	3.786	4.281	4.167	3.995	3.793	3.61
	2		18.209	17.674	16.884	15.967	15.143	14.877	14.548	14.048	13.447	12.888
	3		40.969	39.768	37.989	35.925	34.072	28.469	27.963	27.178	26.212	25.291
Imperfect (IP-I)	1	P-FGM	5.145	4.659	4.127	3.648	3.395	4.852	4.408	3.916	3.463	3.198
	2		20.579	18.635	16.508	14.591	13.581	16.946	15.486	13.838	12.251	11.141
	3		46.304	41.930	37.142	32.83	30.558	32.577	29.936	26.897	23.84	21.381
	1	S-FGM	4.541	4.375	4.127	3.835	3.568	4.276	4.133	3.916	3.657	3.417
	2		18.163	17.499	16.508	15.34	14.271	14.892	14.478	13.838	13.052	12.301
	3		40.867	39.374	37.142	34.514	32.11	28.558	27.915	26.897	25.608	24.34
Imperfect (IP-II)	1	P-FGM	5.170	4.720	4.244	3.839	3.638	4.868	4.456	4.015	3.629	3.409
	2		20.681	18.881	16.976	15.355	14.551	16.948	15.595	14.113	12.741	11.767
	3		46.532	42.482	38.196	34.549	32.741	32.495	30.041	27.296	24.619	22.398
	1	S-FGM	4.608	4.461	4.244	3.991	3.762	4.330	4.204	4.015	3.793	3.589
	2		18.431	17.846	16.976	15.963	15.047	15.021	14.662	14.113	13.447	12.823
	3		41.470	40.152	38.196	35.916	33.856	28.711	28.159	27.296	26.223	25.189

Table 8 The first three non-dimensional frequencies of FG beams with respect to span to depth ratio

Type of beam	Number of mode (n)	Material gradation	L/h				
			5	10	20	50	100
Perfect	1	P-FGM	3.642	3.795	3.837	3.85	3.851
	2		12.789	14.57	15.181	15.37	15.398
	3		24.714	30.881	33.559	34.48	34.619
	1	S-FGM	3.793	3.938	3.978	3.989	3.991
	2		13.447	15.173	15.754	15.932	15.958
	3		26.212	32.31	34.879	35.75	35.881
Imperfect (IP-I)	1	P-FGM	3.174	3.284	3.314	3.323	3.324
	2		11.354	12.694	13.136	13.271	13.291
	3		22.332	27.151	29.126	29.787	29.887
	1	S-FGM	3.452	3.564	3.594	3.603	3.604
	2		12.43	13.807	14.255	14.391	14.41
	3		24.599	29.625	31.639	32.305	32.405
Imperfect (IP-II)	1	P-FGM	3.608	3.761	3.803	3.815	3.817
	2		12.665	14.434	15.043	15.231	15.259
	3		24.474	30.584	33.249	34.168	34.307
	1	S-FGM	3.788	3.932	3.972	3.983	3.985
	2		13.434	15.151	15.729	15.906	15.932
	3		26.209	32.270	34.826	35.692	35.823

Table 9 The first three non-dimensional frequencies of FG beams with respect to porosity volume fraction

Theory	Number of mode (n)	Material gradation	Perfect	Imperfect (IP-I)		Imperfect (IP-II)	
			$\alpha = 0$	$\alpha = 0.1$	$\alpha = 0.2$	$\alpha = 0.1$	$\alpha = 0.2$
CBT	1	P-FGM	3.852	3.648	3.324	3.839	3.817
	2		15.407	14.591	13.298	15.355	15.268
	3		34.666	32.83	29.92	34.549	34.354
	1	S-FGM	3.992	3.835	3.604	3.991	3.985
	2		15.967	15.34	14.417	15.963	15.941
	3		35.925	34.514	32.439	35.916	35.867
FSDBT	1	P-FGM	3.795	3.598	3.284	3.782	3.761
	2		14.57	13.851	12.694	14.518	14.434
	3		30.881	29.465	27.151	30.766	30.584
	1	S-FGM	3.938	3.787	3.564	3.937	3.932
	2		15.173	14.627	13.807	15.170	15.151
	3		32.31	31.25	29.625	32.307	32.270

results demonstrated that the values of non-dimensional frequencies increase with the increment of the span to depth ratio for both perfect and imperfect P-FGM and S-FGM beams. Furthermore, the effect in question becomes less pronounced for imperfect FG beams and gets the lowest influences for P-FGM and S-FGM beams with even porosity distribution (IP-I). The effect of the span to depth ratio becomes more noticeable with the increasing number of modes. The effect of the porosity on the values of non-dimensional frequencies increases with the increment of the span to depth ratio. Besides, the values of non-dimensional frequencies of uneven porosity distribution (IP-II) is higher than those for even distribution (IP-I) and the effect of the porosity on the values of non-dimensional frequencies is more remarkable for even distribution (IP-I) on account of both of P-FGM and S-FGM. The effect of the span to depth ratio on the values of non-dimensional frequency is more dominant for P-FGM beams in comparison with S-FGM beams.

Table 9 presents the variations of first three non-dimensional frequency values of perfect /imperfect P-FGM and S-FGM beams with respect to porosity volume fraction,

α, in the frameworks of CBT and FSDBT for L/h = 10 and k = 2. The results showed that the values of non-dimensional frequencies decrease with the increment of porosity volume fraction for both P-FGM and S-FGM beams. The values of the non-dimensional frequencies of uneven porosity distribution (IP-II) is higher than those for even distribution (IP-I) and the effect of the porosity on the values of the non-dimensional frequencies is more noticeable for even distribution (IP-I) on account of both of P-FGM and S-FGM beams. Furthermore, the effect in the question is more pronounced for P-FGM than S-FGM beams with even and uneven porosity distributions in the frameworks of CBT and FSDBT. However, the effect of porosity volume fraction on the values of the non-dimensional frequencies computed adopting CBT is more noticeable than FSDBT and stays constant for all number of the modes while the related effect decreases with increasing number of the modes for FSDBT.

6. Conclusions

In the present study, the free vibration of beams made of imperfect FGMs including porosities is analyzed. The material properties of the beams are supposed to vary continuously through the thickness direction according to the volume fraction of constituents described with the modified rule of mixture including even and uneven porosity distributions over the cross section. Volume fractions of FG beams are described with P-FGM and S-FGM functions. The governing equations P-FGM and S-FGM beams are derived within the frameworks of CBT and FSDBT. The resulting equations are solved using separation of variables technique and considering simply supported end conditions. The effects of the types of volume fraction function, beam theory and porosity volume fraction, as well as volume fraction index, beam characteristics, porosity volume fraction, on the first three non-dimensional frequencies are examined in detail.

Briefly, the following results are obtained:

- The values of non-dimensional frequencies decrease with the increase of the volume fraction index for both perfect/imperfect P-FGM and S-FGM beams
- The effect of the porosity on the values of non-dimensional frequencies increases with the increase of the volume fraction index and becomes more pronounced as the values of volume fraction index is larger than one
- The effects of variation of volume fraction index, span to depth ratio and porosity volume fraction on the values of the non-dimensional frequency are more dominant for P-FGM beams in comparison with S-FGM beams
- The effect of variation of porosity volume fraction on the values of non-dimensional frequencies computed adopting CBT is more noticeable than FSDBT
- The effect of varying volume fraction index and porosity volume fraction on non-dimensional frequencies becomes more pronounced for imperfect P-FGM and S-FGM beams with even porosity distribution (IP-I)
- The effect of the variation of volume fraction index and porosity volume fraction on the first three frequencies stays fixed according to CBT however the same effect is variable according to FSDBT for both of perfect/imperfect P-FGM and S-FGM beams

Finally, it is concluded that the types of adopted volume fraction function, beam theory and porosity volume fraction, as well as the variations of volume fraction index, span to depth ratio and porosity volume fraction have significant effects on the vibration frequencies of the beams.

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