Steel and Composite Structures, *Vol. 30, No. 1 (2019) 57-67*
DOI: https://doi.org/10.12989/scs.2019.30.1.057

Deflection of axially functionally graded rectangular plates by Green's function method

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(*Received September 1, 2018, Revised January 1, 2019, Accepted January 14, 2019*)

Abstract. This paper deals with the static analysis of axially functionally graded rectangular plates. It is assumed that the flexural rigidity of the plate varies exponentially along one of the plate's in-plane dimensions. Both an analytical approach and a numerical method are utilized to solve the problem. The analytical solution is obtained by using the Green's function method. To employ this approach, the adjoint boundary value problem is established. Then, exact solutions for deflection of the plate for different boundary conditions are found. In another way, a finite element formulation for the problem is developed. In order to demonstrate the validity of the Authors' formulation, the results obtained via both mentioned schemes are compared with each other for functionally graded plates and with results of previously published works for homogeneous plates. The effect of plate parameters on the response of the plate is also investigated. To remind the research background, a brief review on the application of Green's function method in plates' analysis and functionally graded plates is also presented.

Keywords: Green's function method; deflection; axially functionally graded rectangular plate; exact solution; finite element formulation

1. Introduction

Rectangular plates are important structures, which are utilized in many branches of engineering such as civil, mechanical and aerospace engineering (Altunsaray and Bayer 2014, Uysal and Güven 2015, Ebrahimi and Habibi 2016, Wang *et al*. 2017). After the introduction of functionally graded materials in 1984 (Chegenizadeh *et al*. 2014), analysis of structures made up of such materials has been under the attention of many researchers (Kiani and Eslami 2013, Uysal and Kremzer 2015, Mansouri and Shariyat 2015, Bouguenina *et al*. 2015, Rezaiee-Pajand *et al*. 2018a, b). Moreover, as mentioned by Ravasoo (2014), exponentially graded materials are widely employed in many important areas as coatings and interfacial regions for the purpose of reducing residual and thermal stresses and increasing the bounding stress. Therefore, due to great importance of such materials, many researchers have analyzed exponentially graded beams (Mazzei and Scott 2013, Li *et al*. 2013, Kukla and Rychlewska 2013, Tang *et al*. 2014, Rezaiee-Pajand and Hozhabrossadati 2016, Chen *et al*. 2017) and plates (Zenkour 2012, Chakraverty and Pradhan 2014, Helal and Shi 2014, Fekrar *et al*. 2014, Yin *et al*. 2017). For the analysis of such structures, different approaches have been employed. Among them, a well-known scheme is the Green's function method (GFM).

Next section gives a brief review on the application of GFM in the analysis of homogeneous and functionally graded plates (FGP). This review shows that the bending of axially functionally graded rectangular plates with exponential law and different boundary conditions have not been considered by this method yet. This article obtains the solution of this problem via two different techniques. An exact solution is derived by GFM. The well-known finite element method (FEM) provides a numerical solution. To verify both presented approaches, their results are compared with each other for FGP and with available results in the literatures for homogeneous plates. It is observed that they are in excellent agreement.

2. Brief review

Green's function method has been under attention of many researchers. Excellent textbooks are available regarding this approach in the literature (Korn and Korn 1968, Greenberg 1971, Haberman 1987, Duffy 2001, Riley *et al*. 2006). Moreover, a great deal of plate structures has been solved by this analytical scheme. Stanisic and Laffayeta (1979) used GFM to find the exact solution for thin rectangular plates with one edge clamped and the other ones simply supported. On their paper, the results for deflection and moments were presented. The results were verified by finite element solution. Excellent agreement between both ways indicates the solution accuracy.

Nicholson and Bergman (1985) showed the efficacy of the modal series representation for GF related to vibrating continuous structures. Comparison of the results obtained by the modal series representation with available results indicated their approach accuracy. Kerr and El-Sibaie

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http://www.techno-press.org/?journal=scs&subpage=6
ISSN: 1229-9367 (Print), 1598-6233 (Online)

(1989) presented the Green function (GF) for infinite elastic plate on the different type of the foundations. Winkler, Pasternak and Kerr's theories for foundation modeling were utilized. In this research, GF in each case was obtained. The method of images was employed to generate closed-form solution for semi-infinite and quarter plates with simply supported boundaries. Bergman *et al*. (1993) analyzed the free vibration of Levy's plates coupled with substructures. After formulating the problem, the corresponding GF was found. Using the properties of these functions, the natural frequencies and mode shapes of the plate with different combination of boundary conditions were also obtained. Kukla (1998) dealt with the free vibration analysis of a double-plate structure. This system was made up of two rectangular orthotropic Levy plates. GFM was obtained to find the solution.

Sun (2001) takes advantage of the well-known Fourier transform to derive the elastodynamic Green's function of a rectangular plate on viscoelastic foundation. The problem was solved using GFM. He obtained the solutions for plate under impulse and harmonic line loads. Moreover, as a special case, the static response of the plate under point load was found, as well.

Bending, buckling and vibration of Levy's plates on two-parameter foundation were investigated by Lam *et al*. (2002). Three foundation models, including Winkler, Pasternak and generalized foundation were taken into account in this investigation. Using GF formulation, exact solutions were derived for such plates.

Free axisymmetric vibration of annular plates with was formulated by Kukla and Szewczyk (2007). The plate has elastic or rigid intermediate circular supports. In their formulation of the problems, it was assumed that the plates have outer and inner free edges. Green's function corresponding to this case was obtained. The frequency equations and natural frequencies for plates with two circular rigid or elastic ring supports were found. An analytical method for deriving GF of circular and annular plates was presented by Chen *et al*. (2009). Degenerate kernels, null-field integral equations and Fourier's series were employed to find the problem solution. The proposed solution was verified against available ones, and the results were compared with the finite element results.

In order to investigate the free vibration of clamped thin plates, GFM was employed by Li and Yuan (2012). Two examples demonstrated the validity of the proposed solution. Zur (2015) used GF to investigate the free vibration problem of circular plates with different boundary conditions. The problem was solved for non-uniform plates with variable thickness. His proposed solution took advantage of Neumann's power series to find the exact natural frequencies of the plate. The effects of plate parameters such as Poisson's ratio, variable thickness and boundary conditions were examined. Soon after, Zur (2016) extended his formulation to deal with the free vibration of annular plates. In this study, Green's functions along with Neumann's power series were employed to find the frequency equations and natural frequencies of annular plates with nonlinear variable thickness under different boundary conditions.

It is interesting to point out that plates made up of functionally graded materials have been analyzed by many researchers. Lal and Sharma (2004) considered the axisymmetric free vibration of non-homogeneous polar orthotropic annular plates of exponentially varying thickness. A three-dimensional elasticity solution for a simply supported rectangular plate subjected to the transverse loading was developed by Kashtalyan (2004).

Exact solutions for functionally graded thick plates on Pasternak foundation based on three-dimensional theory of elasticity were presented by Huang *et al*. (2008). The state space approach was used to solve this problem.

The differential transform technique was employed for free vibration and modal stress analyses of two-directional functionally graded circular plates resting on two parameters elastic foundation by Shariyat and Alipour (2011). Gupta *et al*. (2014) analyzed the forced vibration of a non-homogeneous rectangular plate of variable thickness. In another study, Lal and Alhawat (2015a) solved the vibration problem of functionally graded circular plates of linearly varying thickness by means of the differential transform method (DTM). The critical buckling loads were also computed for plates with clamped and simply supported edges on Winkler's foundation. Uysal and Güven (2016) investigated the shear buckling of adhesively bonded plates including an orthotropic material. Later, Lal and Ahlawat (2015b) dealt with the axisymmetric vibration and buckling of functionally graded circular plates. A semi-analytical formulation, namely DTM, was used to solve the problem. Uysal (2016) considered stability of functionally graded polymeric thin-walled hemispherical shells. In another study, the buckling of a functionally graded truncated shell under external displacement-dependent pressure was studied by Khayat *et al*. (2017).

This review clearly shows that researchers have paid less attention to the analysis of rectangular plate deflection with axially functionally graded material. Some studies have assumed that material properties vary in the thickness direction, but not in the in-plane directions. In this article, axially functionally graded rectangular plates (AFGRP) are analyzed by using the GFM. This analytical scheme is employed and the problem is solved for a selection of boundary conditions. Furthermore, the finite element method (FEM) is utilized to determine the structural responses. Findings indicate good agreements of the both techniques with each other for functionally graded plates and with some available solutions for homogeneous plates.

3. Governing boundary value problem

In this section, the boundary value problem, which governs the static behavior of functionally graded rectangular plates, is presented. Fig. 1 shows this structure. The Cartesian xyz reference system will be used for the problem formulation, as indicated in Fig. 1. The dimensions of the plate in x, y and z directions are assumed as a, b and h, respectively. A partial differential equation, with variable coefficients and eight boundary conditions, represents the structural behavior. This governing differential equation of

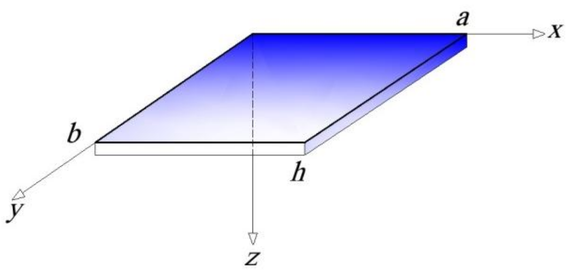


Fig. 1 The AFGRP under study

bending of thin rectangular plates with variable flexural rigidity has the following form (Szilard 2004)

$$\frac{\partial^2}{\partial x^2}\left\{D(x,y)\left[\frac{\partial^2 w(x,y)}{\partial x^2}+\nu\frac{\partial^2 w(x,y)}{\partial y^2}\right]\right\} +2(1-\nu)\frac{\partial^2}{\partial x\partial y}\left[D(x,y)\frac{\partial^2 w(x,y)}{\partial x\partial y}\right] +\frac{\partial^2}{\partial y^2}\left\{D(x,y)\left[\frac{\partial^2 w(x,y)}{\partial y^2}+\nu\frac{\partial^2 w(x,y)}{\partial x^2}\right]\right\}=q(x,y) \quad (1)$$

where $w(x, y)$ indicates the deflection function of the plate in the z-direction, $D(x, y)$ is the flexural rigidity of the plate, $q(x, y)$ demonstrates the lateral loads applied on the plate, and ν denotes Poisson's ratio. Herein, the variable flexural rigidity is due to the use of axially functionally graded materials. Eq. (1) may be written as

$$\nabla^2\left(D\nabla^2 w\right)-(1-\nu)\left(\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-2\frac{\partial^2 D}{\partial x\partial y}\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial x^2}\right)=q(x,y) \quad (2)$$

It is assumed that the flexural rigidity of the plate exponentially varies in the y direction, namely, $E(x, y) = E_0e^{\beta y}$. Herein, E_0 is the elasticity modulus at (0, 0) and β is a constant. In this case, the function $D(x, y)$ is expressed as

$$D(x,y)=D_0e^{\frac{\alpha y}{b}} \quad (3)$$

in which $D_0 = E_0h^3/12(1-\nu^2)$ In this relation, Substituting Eq. (3) into Eq. (2) gives

$$D_0e^{\beta y}\frac{\partial^4 w}{\partial x^4}+D_0e^{\beta y}\frac{\partial^4 w}{\partial x^2\partial y^2}+D_0e^{\beta y}\frac{\partial^4 w}{\partial x^2\partial y^2} +2\beta D_0e^{\beta y}\frac{\partial^3 w}{\partial x^2\partial y}+\beta^2 D_0e^{\beta y}\frac{\partial^2 w}{\partial x^2}+D_0e^{\beta y}\frac{\partial^4 w}{\partial y^4} +2\beta D_0e^{\beta y}\frac{\partial^3 w}{\partial y^3}+\beta^2 D_0e^{\beta y}\frac{\partial^2 w}{\partial y^2}-(1-\nu)\beta^2 D_0e^{\beta y}\frac{\partial^2 w}{\partial x^2}=q(x,y) \quad (4)$$

Eq. (4) can be simplified as

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+2\beta\frac{\partial^3 w}{\partial x^2\partial y} +2\beta\frac{\partial^3 w}{\partial y^3}+\beta^2\frac{\partial^2 w}{\partial y^2}+\nu\beta^2\frac{\partial^2 w}{\partial x^2}=Q(x,y) \quad (5)$$

where $Q(x, y) = q(x, y)/D_0e^{\beta y}$. This equation governs the bending of the studied AFGRP. At this stage, the boundary conditions of the plate are specified. It is assumed that the two opposite edges of the plate at $x = 0$ and $x = a$ are simply supported. Moreover, for the other opposite edges at $y = 0$ and $y = b$ three different boundary conditions, including clamped, simply supported and free are assumed. Therefore, the corresponding boundary conditions of the plate at edges $x = 0$ and $x = a$ are of the next shape

$$w\big|_{x=0}=0 \quad ; \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=0}=0 \qquad w\big|_{x=a}=0 \quad ; \quad \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=a}=0 \quad (6)$$

The boundary conditions at edges $y = 0$ and $y = b$ are For the clamped edge

$$w\big|_{y=0}=0 \quad ; \quad \left.\frac{\partial w}{\partial y}\right|_{y=0}=0 \qquad w\big|_{y=b}=0 \quad ; \quad \left.\frac{\partial w}{\partial y}\right|_{y=b}=0 \quad (7)$$

For the simply supported edge

$$w\big|_{y=0}=0 \quad ; \quad \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=0}=0 \qquad w\big|_{y=b}=0 \quad ; \quad \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=b}=0 \quad (8)$$

For the free edge

$$m_y\big|_{y=0}=0 \quad ; \quad V_y\big|_{y=0}=0 \qquad m_y\big|_{y=b}=0 \quad ; \quad V_y\big|_{y=b}=0 \quad (9)$$

The bending moments and shear forces of the plate at y direction are defined as below

$$m_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right) \qquad V_y=-D\left[\frac{\partial^3 w}{\partial y^3}+(2-\nu)\frac{\partial^3 w}{\partial x^2\partial y}\right] \quad (10)$$

Finally, the partial differential Eq. (5) along with eight boundary conditions Eqs. (6)-(9) forms the governing boundary value problem of the AFGRP. In the next section, this problem is solved by means of GFM.

4. Problem solution

In this part, an analytical approach is presented for the solution the governing boundary value problem. The solution uses the well-known GFM, and exact solution will be at hand. Employing the Levy method, the deflection function of the plate is assumed as

$$w(x,y)=\sum_{m=1}^{\infty}F(y)\sin\frac{m\pi x}{a} \tag{11}$$

It is evident that this solution satisfies the assumed simply supported boundary conditions in the x direction. The right-hand side of Eq. (5), i.e., $Q(x, y)$, can be expanded as

$$Q(x,y)=\sum_{m=1}^{\infty}q_m(y)\sin\frac{m\pi x}{a} \tag{12}$$

in which

$$q_m(y)=\frac{2}{a}\int_0^a\frac{q(x,y)}{D_0e^{\beta y}}\sin\frac{m\pi x}{a}dx \tag{13}$$

By using the Mathematica software, Eqs. (11)-(12) are utilized for calculating the needed derivatives in Eq. (5). Substituting them into Eq. (5) gives

$$\begin{aligned}&\sum_{m=1}^{\infty}\left[\left(\frac{m\pi}{a}\right)^4F(y)-2\left(\frac{m\pi}{a}\right)^2\frac{d^2F(y)}{dy^2}+\frac{d^4F(y)}{dy^4}\right.\\&-2\beta\left(\frac{m\pi}{a}\right)^2\frac{dF(y)}{dy}+2\beta\frac{d^3F(y)}{dy^3}+\beta^2\frac{d^2F(y)}{dy^2}\\&\left.-\nu\beta^2\left(\frac{m\pi}{a}\right)^2F(y)\right]\sin\frac{m\pi x}{a}==\sum_{m=1}^{\infty}q_m(y)\sin\frac{m\pi x}{a}\end{aligned} \tag{14}$$

Equating the coefficients of $\sin\frac{m\pi x}{a}$ for both sides of Eq. (14) yields

$$\begin{aligned}&\frac{d^4F(y)}{dy^4}+2\beta\frac{d^3F(y)}{dy^3}+\left[\beta^2-2\left(\frac{m\pi}{a}\right)^2\right]\frac{d^2F(y)}{dy^2}\\&-2\beta\left(\frac{m\pi}{a}\right)^2\frac{dF(y)}{dy}+\left[\left(\frac{m\pi}{a}\right)^4-\nu\beta^2\left(\frac{m\pi}{a}\right)^2\right]F(y)=q_m(y)\end{aligned} \tag{15}$$

This equation is solved via GFM. Employing this approach, the function $F(y)$ is found as

$$F(y)=\int_0^bG(\eta;y)q_m(\eta)d\eta \tag{16}$$

in which $G(\eta; y)$ is the corresponding GF of Eq. (15). In the following section, GF of the problem is derived. Finding the function $F(y)$, the transverse deflection of the plate is obtained by using Eq. (11). Utilizing Eq. (11), the boundary conditions at y direction becomes:

For the clamped edge

$$\begin{aligned}&\sum_{m=1}^{\infty}F(0)\sin\frac{m\pi x}{a}=0 \quad ; \quad \sum_{m=1}^{\infty}\left.\frac{dF}{dy}\right|_{y=0}\sin\frac{m\pi x}{a}=0\\&\sum_{m=1}^{\infty}F(b)\sin\frac{m\pi x}{a}=0 \quad ; \quad \sum_{m=1}^{\infty}\left.\frac{dF}{dy}\right|_{y=b}\sin\frac{m\pi x}{a}=0\end{aligned} \tag{17}$$

These relations hold only

$$\begin{aligned}&F(0)=0 \quad ; \quad \left.\frac{dF}{dy}\right|_{y=0}=0\\&F(b)=0 \quad ; \quad \left.\frac{dF}{dy}\right|_{y=b}=0\end{aligned} \tag{18}$$

For the simply supported edge

$$\begin{aligned}&\sum_{m=1}^{\infty}F(0)\sin\frac{m\pi x}{a}=0 \quad ; \quad \sum_{m=1}^{\infty}\left.\frac{d^2F}{dy^2}\right|_{y=0}\sin\frac{m\pi x}{a}=0\\&\sum_{m=1}^{\infty}F(b)\sin\frac{m\pi x}{a}=0 \quad ; \quad \sum_{m=1}^{\infty}\left.\frac{d^2F}{dy^2}\right|_{y=b}\sin\frac{m\pi x}{a}=0\end{aligned} \tag{19}$$

or

$$\begin{aligned}&F(0)=0 \quad ; \quad \left.\frac{d^2F}{dy^2}\right|_{y=0}=0\\&F(b)=0 \quad ; \quad \left.\frac{d^2F}{dy^2}\right|_{y=b}=0\end{aligned} \tag{20}$$

For the free edge

$$\begin{aligned}&-D\sum_{m=1}^{\infty}\left[\frac{d^2F(y)}{dy^2}-\nu\left(\frac{m\pi}{a}\right)^2F(y)\right]_{y=0}\sin\frac{m\pi x}{a}=0\\&-D\sum_{m=1}^{\infty}\left[\frac{d^3F(y)}{dy^3}-(2-\nu)\left(\frac{m\pi}{a}\right)^2\frac{dF(y)}{dy}\right]_{y=0}\sin\frac{m\pi x}{a}=0\\&-D\sum_{m=1}^{\infty}\left[\frac{d^2F(y)}{dy^2}-\nu\left(\frac{m\pi}{a}\right)^2F(y)\right]_{y=b}\sin\frac{m\pi x}{a}=0\\&-D\sum_{m=1}^{\infty}\left[\frac{d^3F(y)}{dy^3}-(2-\nu)\left(\frac{m\pi}{a}\right)^2\frac{dF(y)}{dy}\right]_{y=b}\sin\frac{m\pi x}{a}=0\end{aligned} \tag{21}$$

or

$$\begin{aligned}&\left[\frac{d^2F(y)}{dy^2}-\nu\left(\frac{m\pi}{a}\right)^2F(y)\right]_{y=0}=0\\&\left[\frac{d^3F(y)}{dy^3}-(2-\nu)\left(\frac{m\pi}{a}\right)^2\frac{dF(y)}{dy}\right]_{y=0}=0\\&\left[\frac{d^2F(y)}{dy^2}-\nu\left(\frac{m\pi}{a}\right)^2F(y)\right]_{y=b}=0\\&\left[\frac{d^3F(y)}{dy^3}-(2-\nu)\left(\frac{m\pi}{a}\right)^2\frac{dF(y)}{dy}\right]_{y=b}=0\end{aligned} \tag{22}$$

5. Green's function

Eq. (15) may be written in the following form

$$\mathbf{L}F(y)=q_m(y) \tag{23}$$

where $\mathbf{L}$ is a linear differential operator acting on $F(y)$. It has the next shape

$$\begin{aligned}\mathbf{L}=&\frac{d^4}{dy^4}+2\beta\frac{d^3}{dy^3}+\left[\beta^2-2\left(\frac{m\pi}{a}\right)^2\right]\frac{d^2}{dy^2}\\&-2\beta\left(\frac{m\pi}{a}\right)^2\frac{d}{dy}+\left[\left(\frac{m\pi}{a}\right)^4-\nu\beta^2\left(\frac{m\pi}{a}\right)^2\right]\end{aligned} \tag{24}$$

In order to find GF of the problem, the succeeding problem must be solved (Hozhabrossadati *et al*. 2015)

$$\mathbf{L}^{*}G(\eta;y)=\delta(\eta-y) \tag{25}$$

In this relation, $\mathbf{L}^{*}$ is the adjoint operator of $\mathbf{L}$ and $\delta(\eta - y)$ is the Dirac delta function. To find $\mathbf{L}^{*}$, the following equality is used

$$<\mathbf{L}u,v>=<u,\mathbf{L}^{*}v> \tag{26}$$

in which $<\mathbf{L}u, v>$ is the inner product of $\mathbf{L}u$ $\mathbf{L}u$ and v. Moreover, u and v are arbitrary differentiable functions. For real functions, it is defined as

$$<\mathbf{L}u,v>=\int_{c}^{d} v\,\mathbf{L}u\,dy \tag{27}$$

Herein, c and d are two arbitrary points, which show the interval of the problem. Therefore, Eq. (26) becomes

$$\int_{c}^{d} v\,\mathbf{L}u\,dy=\int_{c}^{d} u\,\mathbf{L}^{*}v\,dy \tag{28}$$

Assuming $u = F_m$ and $v = G$, the adjoint operator $\mathbf{L}^{*}$ is found using Eq. (28) as follows

$$\int_{0}^{b} G\,\mathbf{L}F\,dy=\int_{0}^{b} F\,\mathbf{L}^{*}G\,dy \tag{29}$$

Substituting Eq. (24), the left-hand side of this equality becomes

$$\begin{aligned}&\int_{0}^{b} G\left\{\frac{d^{4}F}{dy^{4}}+2\beta\frac{d^{3}F}{dy^{3}}+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\frac{d^{2}F}{dy^{2}}\right.\\&\left.-2\beta\left(\frac{m\pi}{a}\right)^{2}\frac{dF}{dy}+\left[\left(\frac{m\pi}{a}\right)^{4}-\nu\beta^{2}\left(\frac{m\pi}{a}\right)^{2}\right]F\right\}dy\end{aligned} \tag{30}$$

Using integration by parts, one can write the next relations

$$\begin{aligned}&\int_{0}^{b} G\frac{d^{4}F}{dy^{4}}dy=\left[G\frac{d^{3}F}{dy^{3}}-\frac{dG}{dy}\frac{d^{2}F}{dy^{2}}+\frac{d^{2}G}{dy^{2}}\frac{dF}{dy}-\frac{d^{3}G}{dy^{3}}F\right]_{0}^{b}\\&\qquad+\int_{0}^{b} F\frac{d^{4}G}{dy^{4}}dy\\&\int_{0}^{b} 2\beta G\frac{d^{3}F}{dy^{3}}dy=2\beta\left\{\left[G\frac{d^{2}F}{dy^{2}}-\frac{dG}{dy}\frac{dF}{dy}+\frac{d^{2}G}{dy^{2}}F\right]_{0}^{b}\right.\\&\qquad\left.-\int_{0}^{b} F\frac{d^{3}G}{dy^{3}}dy\right\}\\&\int_{0}^{b}\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]G\frac{d^{2}F(y)}{dy^{2}}dy=\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\\&\qquad\times\left\{\left[G\frac{dF}{dy}-\frac{dG}{dy}F\right]_{0}^{b}+\int_{0}^{b} F\frac{d^{2}G}{dy^{2}}dy\right\}\\&\int_{0}^{b} 2\beta\left(\frac{m\pi}{a}\right)^{2}G\frac{dF(y)}{dy}dy=2\beta\left(\frac{m\pi}{a}\right)^{2}\left\{\left[G\frac{dF}{dy}\right]_{0}^{b}-\int_{0}^{b} F\frac{dG}{dy}dy\right\}\end{aligned} \tag{31}$$

Substituting these equalities in Eq. (30), one obtains

$$\begin{aligned}&\int_{a}^{b} G\left\{\frac{d^{4}F}{dy^{4}}+2\beta\frac{d^{3}F}{dy^{3}}+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\frac{d^{2}F}{dy^{2}}-2\beta\left(\frac{m\pi}{a}\right)^{2}\frac{dF}{dy}\right.\\&\left.+\left[\left(\frac{m\pi}{a}\right)^{4}-\nu\beta^{2}\left(\frac{m\pi}{a}\right)^{2}\right]F\right\}dy=\left\{G\frac{d^{3}F}{dy^{3}}-\frac{dG}{dy}\frac{d^{2}F}{dy^{2}}\right.\\&\left.+\frac{d^{2}G}{dy^{2}}\frac{dF}{dy}-\frac{d^{3}G}{dy^{3}}F+2\beta\left[G\frac{d^{2}F}{dy^{2}}-\frac{dG}{dy}\frac{dF}{dy}+\frac{d^{2}G}{dy^{2}}F\right]\right.\\&\left.+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\left[G\frac{dF}{dy}-\frac{dG}{dy}F\right]-2\beta\left(\frac{m\pi}{a}\right)^{2}\left[G\frac{dF}{dy}\right]\right\}_{0}^{b}\\&+\int_{0}^{b} F\left\{\frac{d^{4}G}{dy^{4}}-2\beta\frac{d^{3}G}{dy^{3}}+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\frac{d^{2}G}{dy^{2}}\right.\\&\left.+2\beta\left(\frac{m\pi}{a}\right)^{2}\frac{dG}{dy}+\left[\left(\frac{m\pi}{a}\right)^{4}-\nu\beta^{2}\left(\frac{m\pi}{a}\right)^{2}\right]G\right\}dy\end{aligned} \tag{32}$$

from which

$$\begin{aligned}\mathbf{L}^{*}=&\frac{d^{4}}{dy^{4}}-2\beta\frac{d^{3}}{dy^{3}}+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\frac{d^{2}}{dy^{2}}\\&+2\beta\left(\frac{m\pi}{a}\right)^{2}\frac{d}{dy}+\left[\left(\frac{m\pi}{a}\right)^{4}-\nu\beta^{2}\left(\frac{m\pi}{a}\right)^{2}\right]\end{aligned} \tag{33}$$

It can be observed that the differential operator of the problem is not self-adjoint, i.e., $\mathbf{L}\neq\mathbf{L}^{*}$. The expressions in square brackets can be removed. To do so, they are expanded in the below form (Hozhabrossadati and Aftabi Sani 2016)

$$\begin{aligned}&G(b)F'''(b)-G'(b)F''(b)+G''(b)F'(b)-G'''(b)F(b)\\&+2\beta\left[G(b)F''(b)-G'(b)F'(b)+G''(b)F(b)\right]\\&+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\left[G(b)F'(b)-G'(b)F(b)\right]\\&-2\beta\left(\frac{m\pi}{a}\right)^{2}\left[G(b)F'(b)\right]-G(0)F'''(0)+G'(0)F''(0)-G''(0)F'(0)\\&+G'''(0)F(0)-2\beta\left[G(0)F''(0)-G'(0)F'(0)+G''(0)F(0)\right]\\&-\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]\left[G(0)F'(0)-G'(0)F(0)\right]\\&+2\beta\left(\frac{m\pi}{a}\right)^{2}\left[G(0)F'(0)\right]\end{aligned} \tag{34}$$

or

$$\begin{aligned}&G(b)F'''(b)+\left[-G'(b)+2\beta G(b)\right]F''(b)\\&+\left\{G''(b)-2\beta G'(b)+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]G(b)-2\beta\left(\frac{m\pi}{a}\right)^{2}G(b)\right\}F'(b)\\&+\left[-G'''(b)+2\beta G''(b)-\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]G'(b)\right]F(b)\\&-G(0)F'''(0)+\left[G'(0)-2\beta G(0)\right]F''(0)\\&+\left\{-G''(b)+2\beta G'(b)-\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]G(b)+2\beta\left(\frac{m\pi}{a}\right)^{2}G(b)\right\}F'(0)\\&+\left[G'''(b)-2\beta G''(b)+\left[\beta^{2}-2\left(\frac{m\pi}{a}\right)^{2}\right]G'(b)\right]F(0)\end{aligned} \tag{35}$$

Herein, these expressions are removed for the clamped-clamped boundary conditions. Substituting the pertinent boundary conditions of the clamped-clamped edges, given by Eq. (18), Eq. (35) becomes

$$\begin{aligned}&G(b)F'''(b)+\left[-G'(b)+2\beta G(b)\right]F''(b)\\&-G(0)F'''(0)+\left[G'(0)-2\beta G(0)\right]F''(0)\end{aligned} \tag{36}$$

Since the values of $F_m''(b)$, $F_m'''(0)$, $F_m''(0)$ are not known, their coefficients must vanish

$$\begin{aligned}&G(b)=0\\&-G'(b)+2\beta G(b)=0\\&-G(0)=0\\&G'(0)-2\beta G(0)=0\end{aligned} \tag{37}$$

from which

$$\begin{aligned}&G(b)=0 \quad ; \quad G'(b)=0\\&G(0)=0 \quad ; \quad G'(0)=0\end{aligned} \tag{38}$$

These conditions are exactly those of the clamped edge. For other edge conditions, i.e., the simply supported and free edges, the same results are found. The boundary conditions for simply supported edges take the following form

$$\begin{aligned}&G(0)=0 \quad ; \quad G''(0)=0\\&G(b)=0 \quad ; \quad G''(b)=0\end{aligned} \tag{39}$$

For the free edges, one can obtain

$$\begin{aligned}&G''(0)-\nu\left(\frac{m\pi}{a}\right)^2 G(0)=0\\&G'''(0)-(2-\nu)\left(\frac{m\pi}{a}\right)^2 G'(0)=0\\&G''(b)-\nu\left(\frac{m\pi}{a}\right)^2 G(b)=0\\&G'''(b)-(2-\nu)\left(\frac{m\pi}{a}\right)^2 G'(b)=0\end{aligned} \tag{40}$$

At this stage, GF for each plate under study is derived. In view of Eq. (25), it is the solution of the next differential equation

$$\begin{aligned}&\frac{d^4G}{d\eta^4}-2\beta\frac{d^3FG}{d\eta^3}+\left[\beta^2-2\left(\frac{m\pi}{a}\right)^2\right]\frac{d^2G}{d\eta^2}\\&+2\beta\left(\frac{m\pi}{a}\right)^2\frac{dG}{d\eta}+\left[\left(\frac{m\pi}{a}\right)^4-\nu\beta^2\left(\frac{m\pi}{a}\right)^2\right]G=\delta(\eta-y)\end{aligned} \tag{41}$$

which is subjected to the pertinent boundary and continuity conditions of the problem. These conditions are (Hozhabrossadati and Aftabi Sani 2015)

(1) All four appropriate boundary conditions of the plate. The boundary conditions for the clamped-clamped, simply supported-simply supported and free-free edges are given by Eqs. (38), (39) and (40), respectively.

(2) The continuity of GF and its derivatives up to $n-2$ order at $\eta = y$. In the mathematical form

$$\begin{aligned}&G(y^+)=G(y^-)\\&G'(y^+)=G'(y^-)\\&G''(y^+)=G''(y^-)\end{aligned} \tag{42}$$

where a prime shows differentiation with respect to η.

(3) The jump discontinuity of the $n-1$ derivative of Green's function at $\eta = y$

$$G'''(y^+)-G'''(y^-)=1 \tag{43}$$

The Green function is assumed as

$$G(y;\eta)=\begin{cases}C_1e^{n_1\eta}+C_2e^{n_2\eta}+C_3e^{n_3\eta}+C_4e^{n_4\eta} & 0\le\eta\le y\\ C_5e^{n_1\eta}+C_6e^{n_2\eta}+C_7e^{n_3\eta}+C_8e^{n_4\eta} & y\le\eta\le b\end{cases} \tag{44}$$

in which n_1, n_2, n_3 and n_4 are the solutions of the corresponding characteristic equation of Eq. (41)

$$\begin{aligned}&n^4-2\beta n^3+\left[\beta^2-2\left(\frac{m\pi}{a}\right)^2\right]n^2\\&+2\beta\left(\frac{m\pi}{a}\right)^2 n+\left[\left(\frac{m\pi}{a}\right)^4-\nu\beta^2\left(\frac{m\pi}{a}\right)^2\right]=0\end{aligned} \tag{45}$$

The roots of this quartic equation are as follows

$$\begin{aligned}&n_1=\frac{a^2\beta-a\sqrt{4m^2\pi^2+a^2\beta^2-4am\pi\beta\sqrt{\nu}}}{2a^2}\\&n_2=\frac{a^2\beta+a\sqrt{4m^2\pi^2+a^2\beta^2-4am\pi\beta\sqrt{\nu}}}{2a^2}\\&n_3=\frac{a^2\beta-a\sqrt{4m^2\pi^2+a^2\beta^2+4am\pi\beta\sqrt{\nu}}}{2a^2}\\&n_4=\frac{a^2\beta+a\sqrt{4m^2\pi^2+a^2\beta^2+4am\pi\beta\sqrt{\nu}}}{2a^2}\end{aligned} \tag{46}$$

Applying the eight pertinent boundary and compatibility conditions of the problem in each case, the eight unknown constants $C_1 - C_8$ are found, and corresponding Green's function is in hand. Since this function is very complicated and long, it is not presented here for the sake of brevity. After finding Green's function, the function $F(y)$ is obtained using Eq. (16).

6. Special case

In this section, GFs of homogeneous rectangular plates, as a special case of AFGRP, are found. In this case, the governing differential equation has the next shape

$$\nabla^4 w(x,y)=\frac{q(x,y)}{D} \tag{47}$$

Using the same procedure as Sections 4 and 5, GF is the solution of the subsequent equation

$$\frac{d^4G}{d\eta^4}-2\left(\frac{m\pi}{a}\right)^2\frac{d^2G}{d\eta^2}+\left(\frac{m\pi}{a}\right)^4 G=\delta(\eta-y) \tag{48}$$

It is interesting to point out that the differential operator of the problem in this case is self-adjoint. Utilizing the characteristic equation, GF takes the below form

$$\begin{aligned}
&G(y;\eta)=\begin{cases} g_1(y;\eta) & 0\le\eta\le y\\ g_2(y;\eta) & y\le\eta\le b\end{cases}\\
&g_1(y;\eta)=C_1\sinh\varphi\eta+C_2\cosh\varphi\eta+\varphi\eta\left(C_3\sinh\varphi\eta+C_4\cosh\varphi\eta\right)\\
&g_2(y;\eta)=C_5\sinh\varphi\eta+C_6\cosh\varphi\eta+\varphi\eta\left(C_7\sinh\varphi\eta+C_8\cosh\varphi\eta\right)
\end{aligned} \tag{49}$$

where $\phi = m\,\pi\,/\,a$. Herein, GF for the simply supported-simply supported edges, as an example, is presented

$$\begin{aligned}
&g_1(y;\eta)=\frac{\sinh\phi y\,\left[\eta\phi\cosh\phi\eta(\coth\phi b-\coth\phi y)\right]}{2\phi^3}\\
&+\frac{\left[(b-y)\phi+\coth\phi y+\coth\phi b(-1-b\phi\coth\phi b+y\,\phi\coth\phi y)\right]}{2\phi^3}\\
&\times\sinh\phi\eta\sinh\phi y\\
&g_2(y;\eta)=-\frac{\sinh\phi y\,(\eta\phi+\coth\phi b+b\phi\,\mathrm{csch}^2\phi b)\sinh\phi\eta}{2\phi^3}\\
&+\frac{\left[\cosh\phi\eta(1+\eta\phi\coth\phi b)-y\,\phi\cosh\phi y\times\mathrm{csch}\phi b\times\sinh[(b-\eta)\phi]\right]}{2\phi^3}\\
&\times\sinh\phi y
\end{aligned} \tag{50}$$

7. Numerical results

The present section investigates the deflection of the plate. Three plates with different boundary conditions are analyzed. The boundary conditions SSSS, SCSC and SFSF are shown in Fig. 2. Herein, S denotes simply supported, C clamped and F free edges. It is informative to mention that a finite element code has been developed by authors for the solution of the problem at hand. Four-node twelve-degree-of-freedom bending elements are used in the analysis. The basic relations of this solution are taken from Rao (2011). The plates are regularly divided into finite elements. For square plates, a 20×20 mesh is utilized. For rectangular plates with b/a = 1.5 and b/a = 2, meshes of 20×30 and 20×40 are employed, respectively. In all analyses, the value v = 0.3 is used. At first, for the sake of verification of the proposed solution, the special case of homogeneous rectangular plate is considered. To simulate this plate, the coefficient β = 0 is set equal to zero. It is assumed that the plate is under uniformly distributed load, i.e., $q(x, y) = q_0$. Presented in Table 1 are the values of the central deflection of such a plate. For simplicity, the amounts of $\overline{w}=Dw/qa^4$ are given. It can be observed that the proposed results by both GFM, and FEM are in excellent agreement with those of Szilard (2004). This fact indicates the accuracy of authors' solution. For another check, a rectangular homogeneous plate with two opposite edges simply supported, and the other ones clamped (SCSC), is investigated. The plate is under the uniformly distributed load, i.e., $q(x, y) = q_0$. It should be reminded that this problem was solved by Timoshenko and Woinowsky

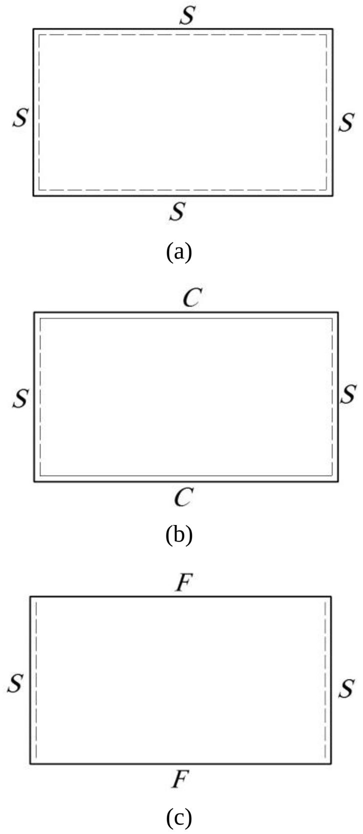


Fig. 2 The rectangular plates with different boundary conditions

Table 1 The values of the central deflection of SSSS homogeneous rectangular plate ($\overline{w}=Dw/qa^4$) under uniformly distributed load

Method	b/a		
	1	1.5	2
GFM	0.00406	0.00772	0.01013
FEM	0.00407	0.00773	0.01014
Szilard (2004)	0.00406	0.00772	0.01013

Krieger (1959). Table 2 shows the amounts of the central deflection of this plate. Once again, the excellent agreement of the proposed results, with those of Timoshenko and Woinowsky-Krieger (1959), demonstrates the accuracy of the suggested solutions. The third case is devoted to a plate with four edges simply supported (SSSS) and under a hydrostatic load, i.e., $q(x, y) = q_0x/a$. Table 3 gives the values of the central deflection of the plate for three different aspect ratios. The suggested amounts are compared with those of Szilard (2004). Once again, the accuracy of the proposed findings is obvious.

After verifying the proposed solutions against the available results, the plate made up of functionally graded materials is analyzed. First, a plate with four simply supported edges (SSSS) is taken into account. Table 4 gives the values of the deflection for the center of the plate for different amounts of β under the uniform distributed load.

Table 2 The values of the central deflection of SCSC homogeneous rectangular plate ($\bar{w} = Dw/qa^4$) under uniformly distributed load

Method	b/a		
	1	1/1.5	1/2
GFM	0.00192	0.00532	0.00844
FEM	0.00192	0.00534	0.00845
Timoshenko (1959)*	0.00192	0.00531	0.00844

* These values are from Timoshenko and Woinowsky-Krieger (1959)

Table 3 The values of the central deflection of SSSS homogeneous rectangular plate ($\bar{w} = Dw/qa^4$) under hydrostatic load

Method	b/a		
	1	1.5	2
GFM	0.00203	0.00076	0.00031
Szilard (2004)	0.00203	0.00076	0.00031

Furthermore, Fig. 3 represents the effect of β on the deflection of the plate for b/a = 1 and b/a = 2. It is interesting to note that the suggested values by GFM are obtained using only six terms of the series. This fact shows the efficiency and accuracy of the method. Based on the findings, two points are of interest: (1) the gradient index, i.e., β, has strong influence on the deflection of the plates

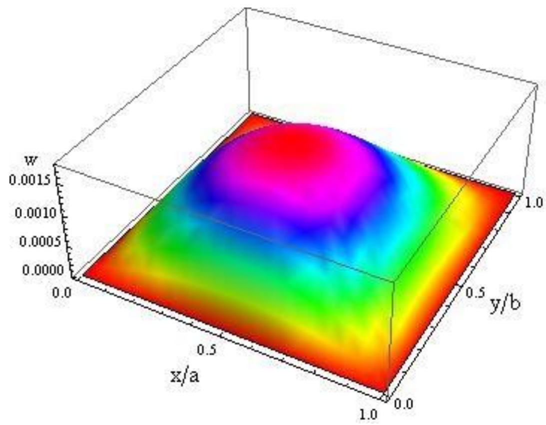
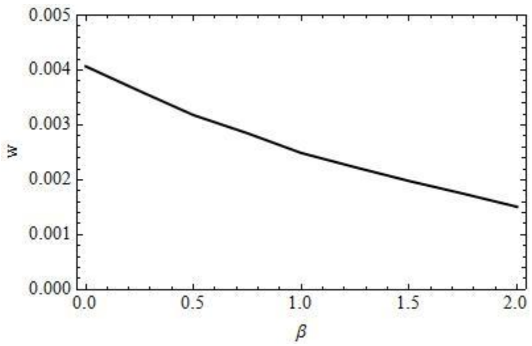
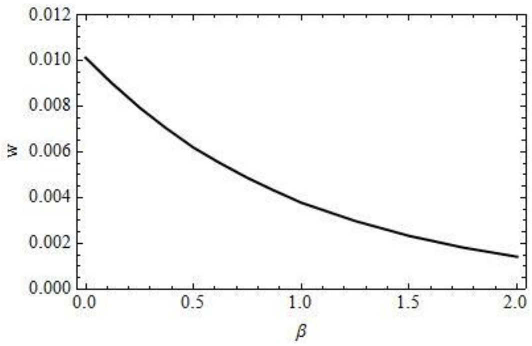


Fig. 4 The deflection of FGM SSSS plate for β = 2

and the deflection tremendously decreases as β increases. For instance, the value of $\bar{w}$ decreases from 0.00616 to 0.00141 when β increases from 0.5 to 2 for the rectangular plate with b/a = 2. (2) as the gradient index increases, the ratio of the deflection of the plate with b/a = 2 to the deflection of the plate with b/a = 1 decreases. This ratio is 2.48 for homogeneous plate (β = 0) while it is 0.94 for β = 2. Interestingly, the deflection of the square plate is greater than that of the rectangular plate with b/a = 2. This fact indicates the enormous importance of the functionally graded materials. One can adjust the value of β such that the desired deflection is obtained. Fig. 4 indicates the deflection of the square plate for β = 0. Moreover, Figs. 5 and 6 represent the deflection of center lines of the plate parallel to x and y axes, respectively. It can be clearly seen from



(a)



(b)

Fig. 3 The effect of β on the central deflection of the plate; (a) b/a = 1; (b) b/a = 2

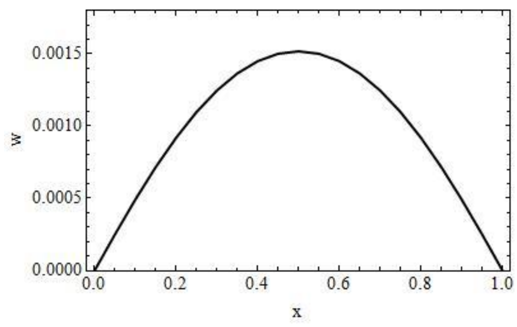


Fig. 5 The deflection of center line of SSSS plate parallel to x direction

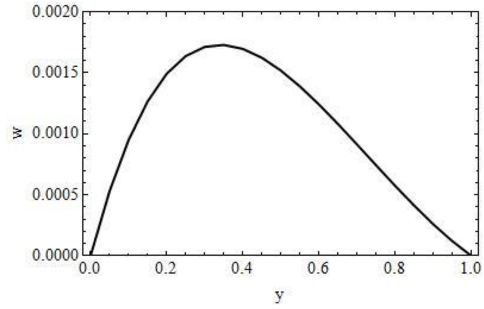


Fig. 6 The deflection of center line of SSSS plate parallel to y direction

Table 4 The values of the central deflection of SSSS inhomogeneous rectangular plate ($\bar{w} = Dw/qa^4$)*

β	b/a		
	1	1.5	2
0.5	0.00317	0.00531	0.00616
	(0.00318)	(0.00535)	(0.00619)
1	0.00247	0.00366	0.00375
	(0.00249)	(0.00370)	(0.00378)
2	0.00152	0.00175	0.00141
	(0.00151)	(0.00174)	(0.00142)

* The values within the parentheses are from finite element analysis

Fig. 6 that the maximum deflection in the y direction occurs in the first-half part of the plate.

In this part, a plate with two opposite edges simply supported, and the others clamped (SCSC), is investigated. Inserted in Table 5 are the values of the deflection of the center point of the plate under the uniform distributed load for different aspect ratio and gradient index. Besides, Fig. 7 demonstrates the variation of the central deflection of two plates with respect to β. Once again, six terms are utilized to find the outcomes from GFM. Similar to SSSS plate, the gradient index has great influence on the deflection of the plates. Moreover, this effect is more considerable for rectangular plate with aspect ratio b/a = 2 than the square plate. It is worth mentioning that Table 6 studies the convergence of the proposed GFM. The central deflection values are worked out for β = 2 and different aspect ratio. The number of terms is indicated by N. It is observed that the solution rapidly converges after a few terms. This fact shows the efficiency of the proposed exact solution.

Finally, the finite element solution is examined. Table 7 gives the values of the central deflection of the inhomogeneous square plate under uniformly distributed load for different amounts of gradient index and number of elements. It is assumed that two opposite edges of the plate are simply supported, and the others are free. The convergence of Authors' results indicates the accuracy of the solution. Moreover, Fig. 8 shows the displacement of the plate under uniformly distributed load.

Table 5 The values of the central deflection of SCSC inhomogeneous rectangular plate ($\bar{w} = Dw/qa^4$)*

β	b/a		
	1	1.5	2
0.5	0.00150	0.00365	0.00512
	(0.00150)	(0.00366)	(0.00513)
1	0.00116	0.00250	0.00310
	(0.00116)	(0.00251)	(0.00310)
2	0.000689	0.00116	0.00113
	(0.000692)	(0.00116)	(0.00113)

* The values within the parentheses are from finite element analysis

Table 6 The convergence study for the central deflection of SCSC inhomogeneous rectangular plate ($\bar{w} = Dw/qa^4$) for β = 2

N	b/a		
	1	1.5	2
1	0.000705	0.0011676	0.0011325
3	0.000687	0.0011556	0.0011251
5	0.000689	0.0011565	0.0011255
7	0.000689	0.0011564	0.0011256
9	0.000689	0.0011564	0.0011256
11	0.000689	0.0011564	0.0011256

Table 7 The values of the central deflection of SFSF inhomogeneous square plate ($\bar{w} = Dw/qa^4$)

β	Number of elements				GFM
	16	64	256	1024	
0.5	0.010733	0.010469	0.010326	0.010212	0.010211
1	0.00882	0.00835	0.00811	0.00799	0.00798
2	0.00584	0.00522	0.00492	0.0487	0.0481

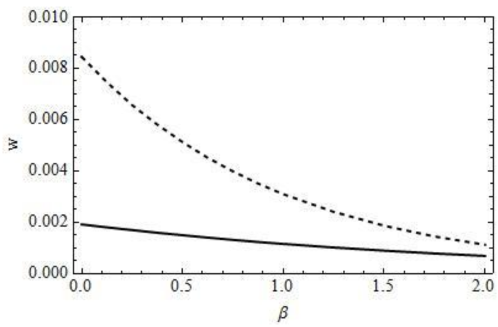


Fig. 7 The effect of β on the central deflection of the plate; b/a = 1 (solid); b/a = 2 (dashed)

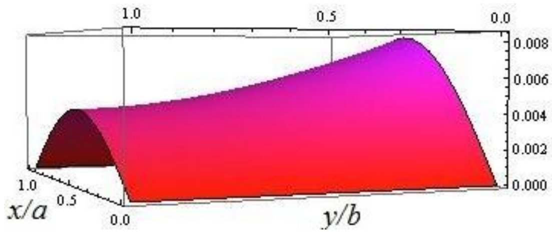


Fig. 8 The deflection of SFSF plate for β = 2

8. Conclusions

First, a brief review on the application of GFM in the analysis of homogeneous and functionally graded plates was presented. After that, static analysis of inhomogeneous rectangular plates was performed in this article. The flexural rigidity of the plate varied in the y-direction in an exponential manner. The well-known Green's function formulation was employed, and exact solution was obtained. The adjoint problem was established, and corresponding GF of the problem was derived. The effect of inhomogeneity of the material on the deflection of the plate was thoroughly investigated. In contrast to homogeneous plates, it was shown that using functionally graded material can reduce the deflection of the plate in such a way that a rectangular plate with aspect ratio $b/a = 2$ has smaller deflection than a square plate. This property can help the practical engineers and researchers to obtain the desired deflection. To validate Authors' formulation, the proposed solution was verified against some available results for homogeneous plates and finite element outcomes for axially functionally graded plates.

References

Altunsaray, E. and Bayer, I. (2014), "Buckling of symmetrically laminated quasi-isotropic thin rectangular plate", *Steel Compos. Struct., Int. J.*, **17**(3), 305-320.

Bergman, L.A., Hall, J.K., Lueschen, G.G.G. and McFarland, D.M. (1993), "Dynamic Green's functions for Levy plates", *J. Sound Vib.*, **162**, 281-310.

Bouguenina, O., Belakhdar, K., Tounsi, A. and Adda, E.A. (2015), "Numerical analysis of FGM plates with variable thickness subjected to thermal buckling", *Steel Compos. Struct., Int. J.*, **19**(3), 679-695.

Chakraverty, S. and Pradhan, K.K. (2014), "Free vibration of exponential functionally graded rectangular plates in thermal environment with general boundary conditions", *Aerosp. Sci. Technol.*, **36**, 132-156.

Chegenizadeh, A., Ghadimi, B., Nikraz, H. and Simsek, M. (2014), "A novel two-dimensional approach to modelling functionally graded beams resting on a soil medium", *Struct. Eng. Mech., Int. J.*, **51**(5), 727-741.

Chen, J.T., Liao, H.Z. and Lee, W.M. (2009), "An analytical approach for the Green's functions of biharmonic problems with circular and annular domains", *J. Mech.*, **25**, 59-74.

Chen, D.Q., Sun, D.L. and Li, X.F. (2017), "Surface effects on resonance frequencies of axiaaly functionally graded Timoshenko nanocantilevers with attached nanoparticle", *Compos. Struct.*, **173**, 116-126.

Duffy, D.J. (2001), *Green's Functions with Applications*, CRC Press, USA.

Ebrahimi, F. and Habibi, S. (2016), "Deflection and vibration analysis of higher-order shear deformable compositionally graded porous plate", *Steel Compos. Struct., Int. J.*, **20**(1), 205-225.

Fekrar, A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2014), "A new five unknown refined theory based on neutral surface position for bending analysis of exponential graded plates", *Meccanica*, **49**, 795-810.

Gray, L.J., Kaplan, T., Richardson, J.D. and Paulino, G.H. (2003), "Green's functions and boundary integral analysis for exponentially graded materials: heat conduction", *J. Appl. Mech.*, **70**, 543-549.

Greenberg, M. (1971), *Application of Green's Functions in Science and Engineering*, Prentice Hall, USA.

Gupta, A.K., Saini, M., Singh, S. and Kumar, R. (2014), "Forced vibrations of non-homogeneous rectangular plate of linearly varying thickness", *J. Vib. Control*, **20**, 876-884.

Haberman, R. (1987), *Elementary Applied Partial Differential Equations with Fourier Series and Boundary Value Problems*, Second Edition, Prentice Hall, USA.

Helal, W.M.K. and Shi, D. (2014), "Optimum material gradient for functionally graded rectangular plate with finite element method", *Indian J. Mater. Sci.*, **2014**, 501935.

Hozhabrossadati, S.M. and Aftabi Sani, A. (2015), "Deformation of Euler-Bernoulli beams by means of modified Green's function: Application of Fredholm Alternative Theorem", *Mechanics Based Design of Structures and Machines*, **43**, 277-293.

Hozhabrossadati, S.M. and Aftabi Sani, A. (2016), "Application of Green's function for constructing influence lines", *Journal of Engineering Mechanics ASCE*, **142**, 04015097.

Hozhabrossadati, S.M., Aftabi Sani, A., Mehri, B. and Mofid, M. (2015), "Green's function for uniform Euler-Bernoulli beams at resonant condition: Introduction of Fredholm Alternative Theorem", *Appl. Math. Model.*, **39**, 3366-3379.

Huang, Z.Y., Lu, C.F. and Chen, W.Q. (2008), "Benchmark solutions for functionally graded thick plates resting on Winkler-Pasternak elastic foundations", *Compos. Struct.*, **85**, 95-104.

Kashtalyan, M. (2004), "Three-dimensional elasticity solution for bending of functionally graded rectangular plates", *Eur. J. Mech. A/Solids*, **23**, 853-864.

Kerr, A.D. and El-Sibaie, M.A. (1989), "Green's functions for continuously supported plates", *J. Appl. Math. Phys.*, **40**, 15-38.

Khayat, M., Poorveis, D. and Moradi, S. (2017), "Buckling analysis of functionally graded truncated conical shells under external displacement-dependent pressure", *Steel Compos. Struct., Int. J.*, **23**, 1-16.

Kiani, Y. and Eslami, M.R. (2013), "An exact solution for thermal buckling of annular FGM plates on an elastic medium", *Composites Part B*, **45**, 101-110.

Korn, G.A. and Korn, T.M. (1968), *Mathematical Handbook for Scientists and Engineers*, Second Edition, McGraw-Hill, USA.

Kukla, S. (1998), "Application of Green's function in free vibration analysis of a system of line connected rectangular plates", *J. Sound Vib.*, **217**, 1-15.

Kukla, S. and Rychlewska, J. (2013), "Free vibration analysis of functionally graded beams", *J. Appl. Math. Computat. Mech.*, **12**, 39-44.

Kukla, S. and Szewczyk, M. (2007), "Frequency analysis of annular plates with elastic concentric supports by Green's function method", *J. Sound Vib.*, **300**, 387-393.

Lal, R. and Alhawat, N. (2015a), "Buckling and vibration of functionally graded non-uniform circular plates resting on Winkler foundation", *Latin Am. J. Solids Struct.*, **12**, 2231-2258.

Lal, R. and Ahlawat, N. (2015b), "Axisymmetric vibration and buckling analysis of functionally graded circular plates via differential transform method", *Eur. J. Mech. A/Solids*, **52**, 85-94.

Lal, R. and Sharma, S. (2004), "Axisymmetric vibrations of non-homogeneous polar orthotropic annular plates of variable thickness", *J. Sound Vib.*, **272**, 245-265.

Lam, K.Y., Wang, C.M. and He, X.Q. (2000), "Canonical exact solutions for Levy-plates on two-parameter foundation using Green's function", *Eng. Struct.*, **22**, 364-378.

Li, S. and Yuan, H. (2012), "Green quasifunction method for free vibration of clamped thin plates", *Acta Mechanica Solida Sinica*, **25**, 37-45.

Li, X.F., Kang, Y.A. and Wu, J.X. (2013), "Exact frequency equation of free vibration of exponentially functionally graded beams", *Appl. Acoust.*, **74**, 413-420.

Liu, Y., Xiao, J. and Shu, D. (2014), "Free vibration of exponential functionally graded beams with single delamination", *Procedia Eng.*, **75**, 164-168.

Mansouri, M.H. and Shariyat, M. (2015), "Biaxial Thermo-mechanical buckling of orthotropic auxetic FGM plates with temperature and moisture dependent material properties on elastic foundations", *Compos. Part B*, **83**, 88-104.

Mazzei, Jr. A.J. and Scott, R.A. (2013), "On the effects of non-homogeneous materials on the vibrations and static stability of tapered shafts", *J. Vib. Control*, **19**, 771-786.

Nicholson, J.W. and Bergman, L.A. (1985), "On the efficacy of the modal series representation for the Green's functions of vibrating continuous structures", *J. Sound Vib*., **98**, 299-304.

Rao, S. (2011), *The Finite Element Method in Engineering*, Fifth Edition, Elsevier, USA.

Ravasoo, A. (2014), "Interaction of bursts in exponentially graded materials characterized by parametric plots", *Wave Motion*, **51**, 758-767.

Rezaiee-Pajand, M. and Hozhabrossadati, S.M. (2016), "Analytical and numerical method for free vibration of double-axially functionally graded beams", *Compos. Struct*., **152**, 488-498.

Rezaiee-Pajand, M., Masoodi, A. and Alepaighambar, A. (2018a), "Lateral-torsional buckling of functionally graded tapered I-beams considering lateral bracing", *Steel Compos. Struct., Int. J*., **28**(4), 403-414.

Rezaiee-Pajand, M., Masoodi, A. and Arabi, E. (2018b), "Geometrically nonlinear analysis of FG doubly-curved and hyperbolical shells via laminated by new element", *Steel Compos. Struct., Int. J*., **28**(3), 389-401.

Riley, K.F., Hobson, M.P. and Bence, S.J. (2006), *Mathematical Methods for Physics and Engineering*, Third Edition, Cambridge University Press, UK.

Shariyat, M. and Alipour, M.M. (2011), "Differential transform vibration and modal stress analyses of circular plates made of two-directional functionally graded materials resting on elastic foundations", *Arch. Appl. Mech*., **81**, 1289-1306.

Stanisic, M.M. and Laffayeta, W. (1979), "On the response of thin elastic plate by means of Green's functions", *Ingenieur-Archiv*, **48**, 279-288.

Sun, L. (2001), "Time-harmonic elastodynamic Green functions of plates for line loads", *J. Sound Vib*., **246**, 337-348.

Szilard, R. (2004), *Theories and Applications of Plate Structures*, John Wiley and Sons, NJ, USA.

Tang, A.Y., Wu, J.X., Li, X.F. and Lee, K.Y. (2014), "Exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beam", *Int. J. Mech. Sci*., **89**, 1-11.

Timoshenko, S. and Woinowsky-Krieger, S. (1959), *Theory of Plates and Shells*, Second Edition, McGraw-Hill, Singapore.

Uysal, M.U. (2016), "Buckling behaviours of functionally graded polymeric thin-walled hemispherical shells", *Steel Compos. Struct., Int. J*., **21**(4), 849-862.

Uysal, M.U. and Güven, U. (2015), "Buckling of functional graded polymeric sandwich panel under different load cases", *Compos. Struct*., **121**, 182-196.

Uysal, M.U. and Güven, U. (2016), "A bonded plate having orthotropic inclusion in the adhesive layer under in-plane shear loading", *J. Adhesion*, **92**, 214-235.

Uysal, M.U. and Kremzer, M. (2015), "Buckling behaviour of short cylindrical functionally gradient polymeric materials", *Acta Physica Polonica A*, **127**, 1355-1357.

Wang, X., Liang, X. and Jin, C. (2017), "Accurate dynamic analysis of functionally graded beams under a moving point load", *Mech. Based Des. Struct. Mach*., **45**, 76-91.

Yin, S., Yu, T., Bui, T.Q., Zheng, X. and Yi, G. (2017), "Rotation-free isogeometric analysis of functionally graded thin plates considering in-plane material inhomogeneity", *Thin-Wall. Struct*., **119**, 385-395.

Zenkour, A.M. (2012), "Hygrothermal of exponentially graded rectangular plates", *J. Mech. Mater. Struct*., **7**, 687-700.

Zur, K.K. (2015), "Green's functions in frequency analysis of circular thin plates of variable thickness", *J. Theor. Appl. Mech*., **53**, 873-884.

Zur, K.K. (2016), "Green's functions in frequency analysis of thin annular plates with nonlinear variable thickness", *Appl. Math. Model*., **40**, 3601-3619.

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