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Analytical determination of shear correction factor for Timoshenko beam model

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Abstract. Timoshenko beam model is widely exploited in the literature to examine the mechanical behavior of stubby beam-like components. Timoshenko beam theory is well-known to require the shear correction factor in order to recognize the non-uniform shear distribution at a section. While a variety of shear correction factors are appeared in the literature so far, there is still no consensus on the most appropriate form of the shear correction factor. The Saint-Venant's flexure problem is first revisited in the frame work of the classical theory of elasticity and a highly accurate approximate closed-form solution is presented employing the extended Kantorovich method. The resulted approximate solution for the elasticity field is then employed to introduce two shear correction factors consistent with the Cowper's and energy approaches. The mathematical form of the proposed shear correction factors are then simplified and compared with the results available in the literature over an extended range of Poisson's and aspect ratios. The proposed shear correction factors do not exhibit implausible issue of negative values and do not result in numerical instabilities too. Based on the comprehensive discussion on the shear correction factors, a piecewise definition of shear correction factor is introduced for rectangular cross-sections having excellent agreement with the numerical results in the literature for both shallow and deep cross-sections.

Keywords: Saint-Venant's flexure problem; shear correction factors; Timoshenko beam theory; semi-inverse method; Extended Kantorovich method

1. Introduction

In 1921, the shear correction factor was introduced by Timoshenko (1921) in his first-order shear deformation beam model to demonstrate the fact that shear strains and stresses are not uniformly distributed over a cross-section (Cowper 1966). Since then, Timoshenko beam model is extensively employed in the literature to analyze the mechanical behavior of short and thick beam-like components (Tagrara *et al*. 2015, Zemirline *et al*. 2015, Ahouel *et al*. 2016, Mohammadimehr and Shahedi 2016, Zhou *et al*. 2016, Mirjavadi *et al*. 2017, Rahmani *et al*. 2017, 2018, Akbas 2018, Kourehli *et al*. 2018). While a variety of correction factors have been achieved employing different scientific and rational computing methods, there has been no consensus on the exact value of the shear correction factor (Dong *et al*. 2010, Romano *et al*. 2012, Steinboeck *et al*. 2013).

The shear correction factor can be determined utilizing experimental (Kaneko 1975), analytical (Stephen and Levinson 1979, Stephen 1980, Levinson 1981, Renton 1991, Hutchinson 2001, Chan *et al*. 2011, Kennedy 2011), semi-analytical finite element (Dong *et al*. 2010, 2013) and numerical methods (Schramm *et al*. 1994, Gruttmann and Wagner 2001, Lepe *et al*. 2014, Balduzzi *et al*. 2017). Due to the numerous researches conducted in this field, an extensive literature review is not within the scope of the present study. A comprehensive review of the state of art may be found in (Kaneko 1975) and more recently in (Kennedy *et al*. 2011, Steinboeck *et al*. 2013).

Three approaches are generally used for analytical determination of the shear correction factor. The first approach is based on matching the beam vibration frequency with exact solutions of elasticity theory, employed by Timoshenko (1922), Hutchinson (1981) and Chan *et al*. (2011) to obtain the shear correction factor. The usual approach entails matching the high frequency spectrums of vibrating beams to a few known exact results for beam vibrations, or making use of approximation procedures and simplifying assumptions within the linear theory of elasticity (Levinson 1981). The second approach is to consider the difference between the average shear strain and actual shear strain utilizing exact solutions of the theory of elasticity, employed by Cowper (1966) to obtain the shear correction factor. The results of Saint-Venant's flexure problem (Barretta 2012, Romano *et al*. 2012, Barretta 2013a, b, 2014, Faghidian 2016) are normally used in this approach to obtain the shear correction factor (Faghidian 2017). The third one is to utilize an energy-

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consistent approach to achieve the shear correction factor by matching exact shear strain energy of the theory of elasticity with the work done by shear forces in accordance with the Timoshenko beam theory. Renton (1991, 1997) and Pie and Schulz (1999) employed the energy-consistent approach to obtain the shear correction factor.

Based on the elasticity solution of the Saint-Venant's flexure problem (Saint-Venant 1856), Cowper (1966) presented the modified equilibrium equations of the Timoshenko beam theory along with the shear correction factor. A generalization of Cowper's approach for arbitrary non-symmetric cross-sections is presented by Mason and Herrmann (1968). For symmetric cross-sections, the general form of the energetic shear factor is given in the work of Renton (1997). The Lower bound for the energy consistent shear factor is provided by Favata *et al*. (2010). While Cowper's approach yields the same shear correction factor for various rectangular cross-sections, the shear correction factors introduced in the literature are demonstrated to depend on the aspect ratio of rectangular cross-section (Stephen and Levinson 1979, Stephen 1980, Renton 1991, Pai and Schulz 1999, Hutchinson 2001, Stephen 2001, Faghidian 2017).

For the shear correction factors various expressions have been given in the literature which depend on the Poisson's ratio and aspect ratio of the beam cross-section, but there is no good agreement between these results particularly for the shallow rectangular cross-sections (Dong *et al*. 2010). Furthermore, some of the proposed shear correction factor for rectangular cross-sections are shown to be numerically instable and may result in physically unacceptable negative numerical values (Dong *et al*. 2010, Faghidian 2017).

In the present study, the shear correction factor for a beam with rectangular cross-section is analytically determined based on the Cowper's and energy-consistent approaches. The differential and boundary conditions of static equilibrium in accordance to the Saint-Venant's flexure problem are first solved employing the Extended Kantorovich Method. The resulted displacement and stress fields are then employed in the Cowper's and energy-consistent approaches, respectively, to obtain highly accurate approximate closed-form solution for shear correction factor of rectangular cross-sections. The proposed mathematical form of the shear correction factors are then simplified for both shallow and deep rectangular cross-sections and demonstrated to be in an excellent agreement with the numerical results available in the literature.

2. Saint-Venant's flexure problem

Since most definitions of shear coefficients based on the Saint-Venant's elasticity solution (Saint-Venant 1856) are considered to be exact, Saint-Venant's elasticity solution has vast applications for determination of shear correction factors in analytical approaches (Stephen and Levinson 1979). Though the Saint-Venant's elasticity problem has been thoroughly studied in classical theory of elasticity (Love 1944, Sokolnikoff 1956, Timoshenko and Goodier 1970), yet it is still the main subject of many recent studies

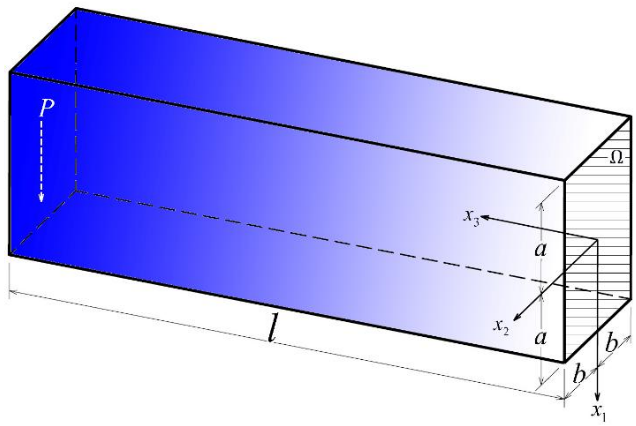

Fig. 1 A tip-loaded rectangular cross-section beam

(Barretta and Barretta 2010, Barretta 2012, 2013a, b, c, 2014, Zarmehi *et al*. 2011, Romano *et al*. 2012, Barretta and Diaco 2013, Ecsedi 2009, 2013, Baksa and Ecsedi 2009, Ecsedi and Baksa 2010, 2011, 2014, 2016).

A linear elastic, isotropic and homogeneous beam of length l and rectangular cross-section Ω under the effect of shear force P is considered, as shown in Fig. 1.

Based on the well-known flexural moment of $M_2 = P(l - x_3)$, the normal stress t_{33} can be expressed as (Iesan 2009)

$$t_{33} = \psi_2 x_1 \tag{1}$$

where the constant $\psi_2 = -M_2/I$, and I denotes the second moment of inertia of the cross-section about x_2-axis. Since the three components of stress field t_{11}, t_{12} and t_{22} are assumed to be zero, the differential condition of static equilibrium for the shear stresses is given by

$$t_{13,1} + t_{23,2} = \psi_{2,3} x_1 \tag{2}$$

Accordingly, the shear stresses t_{13} and t_{23} in terms of stress function φ write as (Timoshenko and Goodier 1970)

$$\begin{aligned} t_{13} &= \varphi_{,2} + \tfrac{\psi_{2,3}}{2}(a^2 - x_1^2) \\ t_{23} &= -\varphi_{,1} \end{aligned} \tag{3}$$

where the stress function φ should satisfy the stress-free boundary conditions of static equilibrium. The governing Poisson's equation for the flexure of the rectangular cross-section beam can be determined as a result of imposing the compatibility equations

$$\Delta\varphi = \varphi_{,11} + \varphi_{,22} = (\frac{\nu}{1+\nu}\psi_{2,3})x_2 \tag{4}$$

subject to the stress-free boundary condition of $\varphi = 0$ along the edges of the rectangular cross-section. Also, ν is the Poisson's ratio of linear elastic materials. As a result of solving the Poisson's Eq. (4) while employing Eq. (3), the stress field components can be achieved. The displacement field associated with the stress field may be conveniently determined employing a standard technique of elasticity theory (Timoshenko and Goodier 1970).

3. Extended Kantorovich method

The extended Kantorovich method, introduced by Kerr (1968), is an effective variational method to obtain highly accurate approximate closed-form solutions for structural problems (Aghdam *et al*. 2012, Liu *et al*. 2014, Huang and Kim 2015, Joodaky and Joodaky 2015, Ebrahimi and Beni 2016, Rostami *et al*. 2016, Kumari *et al*. 2016, 2017, Kumari and Shakya 2017). The extended Kantorovich method is usually employed with the weighted residual technique (Reddy 2017) to convert the governing partial differential equation to ordinary differential equations. Accurate approximated solutions with a fast convergence can be achieved by iterative solving of the resulted ordinary differential equations (Dym and Shames 2013).

The stress function can be assumed to be expressed as coordinate functions of x_1 and x_2 as

$$\varphi_{ij}(x_1,x_2) = (\frac{\nu}{1+\nu}\psi_{2,3})\Gamma_i(x_1)\Upsilon_j(x_2) \tag{5}$$

where Γ_i and Υ_j are unknown coordinate functions yet to be determined by solving the associated ordinary differential equations through the i^{th} and j^{th} iterations, correspondingly. The stress-free boundary conditions in terms of Γ_i and Υ_j may be written as

$$\begin{aligned}\varphi_{ij}(\pm a,x_2) = \Gamma_i(\pm a) = 0\\ \varphi_{ij}(x_1,\pm b) = \Upsilon_j(\pm b) = 0\end{aligned} \tag{6}$$

Consistent with the Galerkin method, the residual of approximation should be orthogonal to a set of linearly independent weight functions. Therefore, the governing Poisson's Eq. (4) can be expressed as

$$\int_{-b}^{b}\int_{-a}^{a}\left(\Delta\varphi_{ij} - \frac{\nu}{1+\nu}\psi_{2,3}x_2\right)\delta\varphi_{ij}\,dx_1dx_2 = 0 \tag{7}$$

The initial guess for coordinate function Γ_0 may be assumed as the subsequent function satisfying the stress-free boundary conditions

$$\Gamma_0(x_1) = \left(1-\cos\left(\frac{2\pi x_1}{a}\right)\right)^2 \tag{8}$$

As the first step in the iteration procedure, replacing the initial guess for coordinate function Γ_0 in Eq. (7) while utilizing the fundamental lemma of variational calculus, the governing equation of coordinate function Υ_1 can be cast into the following differential equation

$$-35a^2\frac{d^2\Upsilon_1}{dx^2}(x_2) + 80\pi^2\Upsilon_1(x_2) + 12a^2x_2 = 0 \tag{9}$$

in conjunction with the stress-free boundary conditions Eq. $(6)_2$. Accordingly, the stress function φ_{01} can be determined by substituting the resulted closed-form solution of Υ_1 together with the initial guess of Γ_0 as

$$\begin{aligned}&\varphi_{01}(x_1,x_2)\\ &= \left(\frac{\nu}{1+\nu}\psi_{2,3}\right)\left(1-\cos\left(\frac{2\pi x_1}{a}\right)\right)^2\\ &\left(-x_2\lambda_1 + \frac{b\lambda_1\left(e^{(b+x_2)\lambda_2} - e^{(b-x_2)\lambda_2}\right)}{e^{2b\lambda_2}-1}\right)\end{aligned} \tag{10}$$

$$\lambda_1 = \frac{3a^2}{20\pi^2}, \qquad \lambda_2 = \frac{4\pi}{\sqrt{7}a}$$

To express the solution form of the stress function independent of the choice of initial guess, the iteration procedure will be continued by introducing the achieved coordinate function Υ_1 to the Eq. (7). Consequently, the stress function φ_{11} can be determined after some straightforward mathematics

$$\begin{aligned}\varphi_{11}(x_1,x_2) = &\left(\frac{\nu}{1+\nu}\psi_{2,3}\right)\\ &\left(\mu_1 - \frac{\mu_1\left(e^{(a+x_1)\mu_2} + e^{(a-x_1)\mu_2}\right)}{(e^{2a\mu_2}+1)}\right)\\ &-x_2\lambda_1 + \frac{b\lambda_1\left(e^{(b+x_2)\lambda_2} - e^{(b-x_2)\lambda_2}\right)}{e^{2b\lambda_2}-1}\end{aligned}$$

$$\mu_1 = \frac{2\begin{pmatrix}3+3b\lambda_2+b^2{\lambda_2}^2-\left(6+2b^2{\lambda_2}^2\right)e^{2b\lambda_2}\\ +\left(3-3b\lambda_2+b^2{\lambda_2}^2\right)e^{4b\lambda_2}\end{pmatrix}}{3{\lambda_2}^2\lambda_1\begin{pmatrix}-2-b\lambda_2+\left(4+4b^2{\lambda_2}^2\right)e^{2b\lambda_2}\\ -(2-b\lambda_2)e^{4b\lambda_2}\end{pmatrix}} \tag{11}$$

$$\begin{aligned}&\mu_2\\ &= \left(\frac{3{\lambda_2}^2\begin{pmatrix}-2-b\lambda_2+\left(4+4b^2{\lambda_2}^2\right)e^{2b\lambda_2}\\ -(2-b\lambda_2)e^{4b\lambda_2}\end{pmatrix}}{\begin{matrix}12+9b\lambda_2+2b^2{\lambda_2}^2-\left(24+16b6^2{\lambda_2}^2\right)e^{2b\lambda_2}\\ +\left(12-9b\lambda_2+2b^2{\lambda_2}^2\right)e^{4b\lambda_2}\end{matrix}}\right)^{\frac{1}{2}}\end{aligned}$$

The shear stress components t_{13} and t_{23} can be obtained by substituting the resulted stress function φ in Eq. (3). Accordingly, the non-zero components of stress field write as

$$\begin{aligned}t_{13} = &\frac{\psi_{2,3}}{2}\Bigg(a^2 - x_1^2 + \frac{2\lambda_1\nu}{1+\nu}\\ &\left(\mu_1 - \frac{\left(e^{\mu_2(a-x_1)} + e^{\mu_2(a+x_1)}\right)\mu_1}{1+e^{2a\mu_2}}\right)\\ &\left(-1 + \frac{b\lambda_2\left(e^{\lambda_2(b-x_2)} + e^{\lambda_2(b+x_2)}\right)}{-1+e^{2b\lambda_2}}\right)\Bigg)\\ t_{23} = &\frac{\nu}{1+\nu}\psi_{2,3}\mu_1\mu_2\left(\frac{e^{\mu_2(a+x_1)} - e^{\mu_2(a-x_1)}}{1+e^{2a\mu_2}}\right)\\ &\left(\frac{b\lambda_1\left(e^{\lambda_2(b+x_2)} - e^{\lambda_2(b-x_2)}\right)}{-1+e^{2b\lambda_2}} - x_2\lambda_1\right)\\ t_{33} = &\psi_2 x_1\end{aligned} \tag{12}$$

To determine the displacement field associated with the stress distribution Eq. (12), the strain components are first obtained adopting generalized Hooke's law, and consequently, the strain-displacement equations are integrated as

$$
\begin{aligned}
u_1 &= -\frac{\nu}{E}\psi_2\left(\frac{{x_1}^2-{x_2}^2}{2}+\frac{{x_3}^2(3l-x_3)}{6\,\nu(l-x_3)}-a^2\right)\\
u_2 &= -\frac{\nu}{E}\psi_2 x_1 x_2\\
u_3 &= \frac{-x_1}{E}\psi_{2,3}\Bigg(lx_3-\frac{{x_3}^2}{2}-a^2-\nu {x_2}^2\\
&\quad -\nu\left(\frac{{x_1}^2}{6}-\frac{{x_2}^2}{2}\right)+\frac{1}{3}{x_1}^2(1+\nu)\\
&\quad -\frac{\nu\mu_1\mu_2}{x_1}\left(\frac{e^{\mu_2(a+x_1)}-e^{\mu_2(a-x_1)}}{1+e^{2a\mu_2}}\right)\\
&\quad \left(\frac{2b\lambda_1\left(-e^{\lambda_2(b-x_2)}+e^{\lambda_2(b+x_2)}\right)}{(-1+e^{2b\lambda_2})\lambda_2}-{x_2}^2\lambda_1\right)\Bigg)
\end{aligned}
\tag{13}
$$

where E designates Young's modulus.

The displacement and stress fields will be utilized in Section 4 to achieve the accurate approximate closed-form solution for shear correction factor of rectangular cross-sections.

4. Shear correction factor

4.1 Cowper's approach

According to the method proposed by Cowper (1966) for obtaining the shear correction factor, first a residual displacement u_3^{res} is introduced as the difference between actual and average displacement

$$
u_3^{res} = u_3 - \frac{1}{A}\iint_{\Omega} u_3 dx_1 dx_2 - \frac{x_1}{I}\iint_{\Omega} x_1 u_3 dx_1 dx_2 \tag{14}
$$

where A is the area of rectangular cross-section. The shear coefficient K is subsequently proposed by Cowper (1966) via reconciling the shear rotations from the averaged displacement with shear rotations from Timoshenko technical beam model as

$$
K = \frac{P}{\iint_{\Omega}(t_{13}-\frac{E}{2(1+\nu)}\frac{\partial u_3^{res}}{\partial x_1})dx_1 dx_2} \tag{15}
$$

Utilizing the series solution of the Saint-Venant's flexure problem in accordance with the classical theory of elasticity, Cowper (1966) introduced the shear correction factor for the rectangular cross-section

$$
K_{Cowper} = \frac{10(1+\nu)}{12+11\nu} \tag{16}
$$

where K_{Cowper} is noticeably independent of the geometry of the rectangular cross-section.

Introduction of the displacement field Eq. (13) in the definition of residual displacement Eq. (14) and substitution of the result together with the shear stress t_{13} in the Cowper's formula yields the shear correction factor for rectangular cross-section

$$
\begin{aligned}
K_C &= \frac{10(1+\nu)}{12+11\nu+\nu\big(\chi(\xi)\big)}\\
\chi(\xi) &= \gamma_2(\xi)\left(\frac{(3+\gamma_1(\xi)^2)\left(e^{2\gamma_1(\xi)}-1\right)}{\gamma_1(\xi)(1+e^{2\gamma_1(\xi)})}-3\right)\\
\gamma_1(\xi) &= \left(\frac{\frac{48}{7}\pi^2\left(e^{\frac{8\pi}{\sqrt{7}\xi}}\left(-2+\frac{4\pi}{\sqrt{7}\xi}\right)+\frac{64\pi^2}{7\xi^2}+4\right) - e^{\frac{-8\pi}{\sqrt{7}\xi}}\left(2+\frac{4\pi}{\sqrt{7}\xi}\right)}{-\frac{256\pi^2}{7\xi^2}+e^{\frac{8\pi}{\sqrt{7}\xi}}\left(-\frac{36\pi}{\sqrt{7}\xi}+12+\frac{32\pi^2}{7\xi^2}\right) -24+e^{\frac{-8\pi}{\sqrt{7}\xi}}\left(\frac{32\pi^2}{7\xi^2}+12+\frac{36\pi}{\sqrt{7}\xi}\right)}\right)^{\frac{1}{2}}\\
\gamma_2(\xi) &= \frac{\left(-6-\frac{32\pi^2}{7\xi^2}+e^{\frac{-8\pi}{\sqrt{7}\xi}}\left(\frac{16\pi^2}{7\xi^2}+\frac{12\pi}{\sqrt{7}\xi}+3\right) + e^{\frac{8\pi}{\sqrt{7}\xi}}\left(3-\frac{12\pi}{\sqrt{7}\xi}+\frac{16\pi^2}{7\xi^2}\right)\right)}{4+\frac{64\pi^2}{7\xi^2}-e^{\frac{-8\pi}{\sqrt{7}\xi}}\left(2+\frac{\pi}{\sqrt{7}\xi}\right)+e^{\frac{8\pi}{\sqrt{7}\xi}}\left(-2+\frac{4\pi}{\sqrt{7}\xi}\right)}\\
&\quad \left(-\frac{245}{64\pi^4}+\frac{35}{24\pi^2\xi^2}\right)
\end{aligned}
\tag{17}
$$

where $\xi = a/b$ is the aspect ratio of the rectangular cross-section. The intricate mathematical form of the proposed shear correction factor can be simplified employing the Taylor series expansion while ignoring the higher-order terms

$$
\widetilde{K_C} = \frac{10(1+\nu)}{12+11\nu+\nu\left(\frac{8}{9}\xi^{-\frac{3}{2}}\right)} \tag{18}
$$

4.2 Energy-consistent approach

As first discussed by Renton (1991) and Pai and Schulz (1999) and then modified by Faghidian (2017), the energy-consistent shear correction factor can be determined by reconciling the shear strain energy from the elasticity solution with the shear strain energy of the beam from Timoshenko technical beam model as

$$
K = \frac{P^2}{A}\frac{1}{\iint_{\Omega}(t_{13}^2+t_{23}^2)dx_1 dx_2} \tag{19}
$$

Employing the series solution of the Saint-Venant's flexure problem according to the classical theory of elasticity, Renton (1991) introduced the shear correction factor for the rectangular cross-section

$$
K_{Renton} = \left(\frac{6}{5}+\left(\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{144\left(\frac{1}{\xi}\right)^4}{\left(\pi^3 n(2m+1)\right)^2\left(\left(\frac{2m+1}{2\xi}\right)^2+n^2\right)}\right)\left(\frac{\nu}{1+\nu}\right)^2\right)^{-1} \tag{2}
$$

Introducing the distribution of shear stress components t_{13} and t_{23} in the Renton's formula results in the shear correction factor for rectangular cross-section

$$K_R = \left(\frac{6}{5} + \kappa(\xi)\left(\frac{\nu}{1+\nu}\right)^2\right)^{-1}$$

$$\kappa(\xi) = \frac{147\left(\gamma_2(\xi)\right)^2 \left(\begin{array}{c} 48\xi^2\pi^2\left(5 + 8e^{2\gamma_1(\xi)} - e^{4\gamma_1(\xi)}\right) \\ \left(-2 - \frac{4\pi}{\sqrt{7}\xi} + e^{\frac{8\pi}{\sqrt{7}\xi}}\left(2 + \frac{4\pi}{\sqrt{7}\xi} + \frac{64\pi^2}{7\xi^2}\right)\right) \\ -4\gamma_1(\xi)\left(\begin{array}{c} 1 - e^{4\gamma_1(\xi)} \\ +4\gamma_1(\xi)e^{2\gamma_1(\xi)} \end{array}\right) \\ \left(\begin{array}{c} 8\pi^2\left(1 - 8e^{\frac{8\pi}{\sqrt{7}\xi}} + e^{\frac{16\pi}{\sqrt{7}\xi}}\right) \\ -9\sqrt{7}\pi\left(-1 + e^{\frac{16\pi}{\sqrt{7}\xi}}\right) \\ +21\xi^2\left(-1 + e^{\frac{8\pi}{\sqrt{7}\xi}}\right)^2 \end{array}\right) \end{array} \right)}{\left(\begin{array}{c} \left(\frac{280\pi}{3\xi} - \frac{245\xi}{\pi}\right) \\ \left(1 + e^{2\gamma_1(\xi)}\right)\left(1 - e^{\frac{8\pi}{\sqrt{7}\xi}}\right) \end{array} \right)^2} \tag{21}$$

Once more, the intricate mathematical form of the proposed shear correction factor can be simplified employing the Taylor series expansion while ignoring the higher-order terms

$$\widetilde{K}_R = \left(\frac{6}{5} + \left(\frac{12}{19}\frac{1}{\xi^3}\right)\left(\frac{\nu}{1+\nu}\right)^2\right)^{-1} \tag{22}$$

5. Results and discussion

5.1 Shear correction factor based on Cowper's approach

The proposed shear correction factor based on Cowper's approach K_C Eq. (17) together with the simplified shear correction factor for shallow section $\widetilde{K}_C$ Eq. (18) are compared with the Cowper's formula K_{Cowper} Eq. (16) in Fig. 2. While the Poisson's ratio ν is assumed to range in the set $\{0,0.1,0.2,0.3\}$, the aspect ratio ξ is ranging in the interval $]0.1,10[$. As it can be observed from Fig. 2, both shear correction factors K_C and $\widetilde{K}_C$ have an excellent convergence to the Cowper's formula over an extended range of Poisson's and aspect ratios.

It can be also demonstrated that for zero Poisson's ratio, both shear correction factors K_C and $\widetilde{K}_C$ and Cowper's formula coincide on the well-known value of 5/6. Furthermore, both shear correction factors K_C and $\widetilde{K}_C$ approach the Cowper's formula for deep sections with $\xi \gg 1$.

Based on the recently introduced stress field for Saint-Venant's flexure problem with uniaxial symmetric cross-sections (Faghidian 2016), Faghidian (2017) proposed a unified formulation of Cowper's formula for uniaxial symmetric cross-sections. The innovative unified formulation of Cowper's formula for rectangular cross-sections is given by (Faghidian 2017)

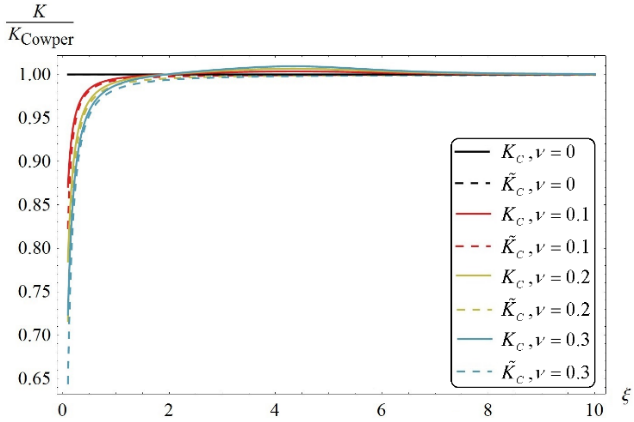

Fig. 2 Comparison of the introduced shear correction factors with Cowper's formula

$$K^C_{Faghidian} = \frac{10(1+\nu)}{12 + 11\nu - \frac{2\nu}{1+2\xi^2}} \tag{23}$$

The comparison of the proposed shear correction factors K_C and $\widetilde{K}_C$ with the unified formulation of Cowper's formula is illustrated in Fig. 2 for the same range of Poisson's and aspect ratios. Again, both shear correction factors K_C and $\widetilde{K}_C$ have an excellent fast convergence to the unified formulation of Cowper's formula over an extended range of Poisson's and aspect ratios. Notably, both shear correction factors K_C and $\widetilde{K}_C$ underestimate the Cowper's formula and unified formulation of Cowper's formula.

5.2 Energy-consistent shear correction factor

The numerical behavior of the introduced shear correction factor based on the energy approach K_R Eq. (21) in addition to the simplified shear correction factor for shallow section $\widetilde{K}_R$ Eq. (22) are illustrated in Fig. 4 and compared to Renton's formula K_{Renton} Eq. (20). The comparison is made for the Poisson's ratio ν ranging in the set $\{0,0.1,0.2,0.3\}$ and the aspect ratio ξ ranging in the interval $]0.1,10[$. Also, to numerically determine the shear

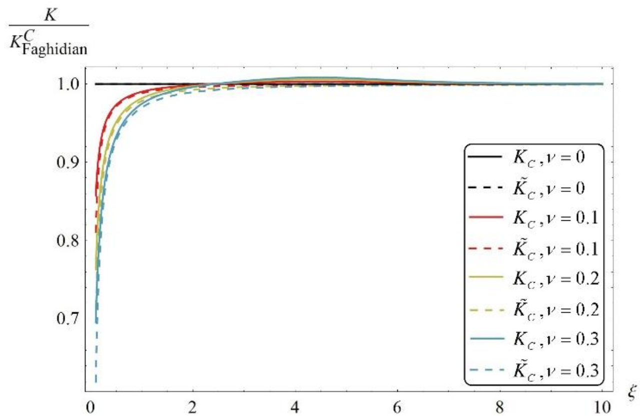

Fig. 3 Comparison of the introduced shear correction factors with unified formulation of Cowper's formula

correction factor K_{Renton} , 10^4 terms of the series solution are used.

Faghidian (2017) also introduced a novel elasticity-based displacement field for a Timoshenko beam theory and developed an energy-consistent variational scheme employing the Reissner principle. Subsequently, the definition of shear deformation factor in energy-consistent approaches (Renton 1991, Pai and Schulz 1999) is modified and the subsequent energy-consistent shear correction factor for rectangular cross-sections is introduced (Faghidian 2017)

$$K^R_{Faghidian} = \frac{20(1+\nu)^2}{24+\nu\left(45+\frac{5}{\xi^2}-\frac{8}{1+2\xi^2}\right)+\nu^2\left(21+\frac{58+\frac{39}{\xi^2}}{(1+2\xi^2)^2}\right)} \tag{24}$$

The introduced energy-consistent shear correction factors K_R and $\widetilde{K}_R$ are also compared to the Faghidian's formula $K^R_{Faghidian}$ in Fig. 5 for the same range of Poisson's and aspect ratios. Both energy-consistent shear correction factors K_R and $\widetilde{K}_R$ have an excellent fast convergence to the Renton's and Faghidian's formula over an extended range of Poisson's ratios. Nonetheless, both shear correction factors K_R and $\widetilde{K}_R$ overestimate the Renton's and Faghidian's formula and a fairly acceptable discrepancy from the Renton's and Faghidian's formula can be observed for shallow sections with $\xi \ll 1$. The accuracy of the proposed energy-consistent shear correction factors for small values of aspect ratios will be discussed later based on the results of semi-analytical finite element method (Dong *et al*. 2010). Once more, both shear correction factors K_R and $\widetilde{K}_R$ coincide on the well-known value of 5/6 for zero Poisson's ratio.

Another important shear deformation factor introduced by (Stephen and Levinson 1979) and revisited by (Hutchinson 2001) can be also written for a rectangular cross-section as

$$K_{Hutchinson} = \frac{5(1+\nu)}{6+5\nu-90\sum_{n=1}^{\infty}\frac{\nu^2(n\pi\xi-\mathrm{Tanh}[n\pi\xi])}{(n\pi\xi)^5(1+\nu)}} \tag{25}$$

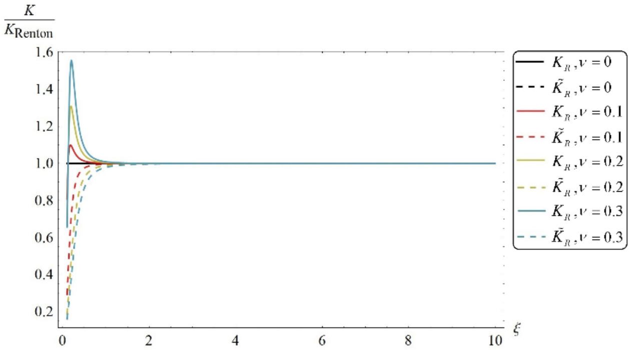

Fig. 4 Comparison of the proposed energy-consistent shear correction factors with Renton's formula

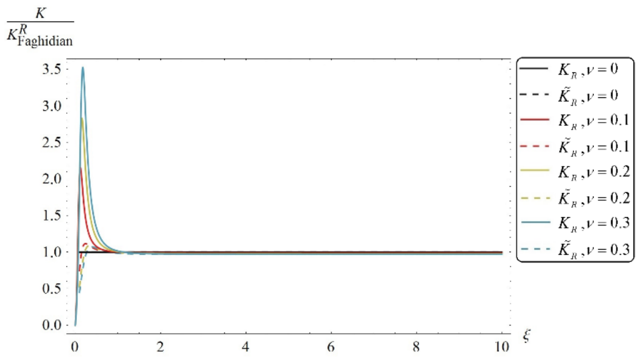

Fig. 5 Comparison of the proposed energy-consistent shear correction factors with Faghidian's formula

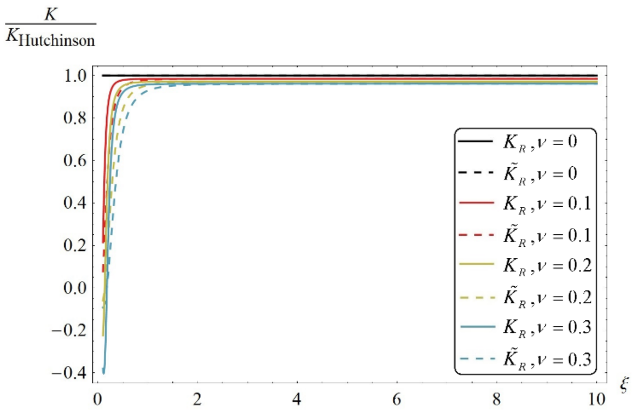

Fig. 6 Comparison of the proposed energy-consistent shear correction factors with Hutchinson's formula

The introduced energy-consistent shear correction factors K_R and $\widetilde{K}_R$ are furthermore compared to the Hutchinson's formula $K_{Hutchinson}$ in Fig. 6 for the same range of Poisson's and aspect ratios. Also, to numerically determine the shear correction factor $K_{Hutchinson}$, 10^4 terms of the series solution are used. It is clearly deduced from Fig. 6 that a deviation between the shear correction factors exists for shallow rectangular sections where employing the Hutchinson's formula yields in negative values for the shear correction factors. In addition to the implausible issue of a negative shear, Hutchinson's formula exhibit numerical instabilities for shallow sections. The implausible negative values of shear correction factor, which indeed violates the first law of thermodynamics, was first raised by Stephen (2001) and then discussed by Dong *et al*. (2010) and Faghidian (2017). Furthermore, the shear correction factors have up to 4% differences in $\nu = 0.3$ for deep sections with $\xi \gg 1$.

As illustrated, there is no consensus on the value of shear correction factor of rectangular cross-section particularly for shallow sections. Therefore, the proposed and available shear correction factors in the literature are tabulated in Table 1 and compared with the results of semi-analytical finite element method of Dong *et al*. (2010). The proposed shear correction factors are demonstrated to have the best agreement with the semi-analytical finite element results of Dong *et al*. (2010) in all ranges of the aspect ratios including both shallow and deep cross-sections. As

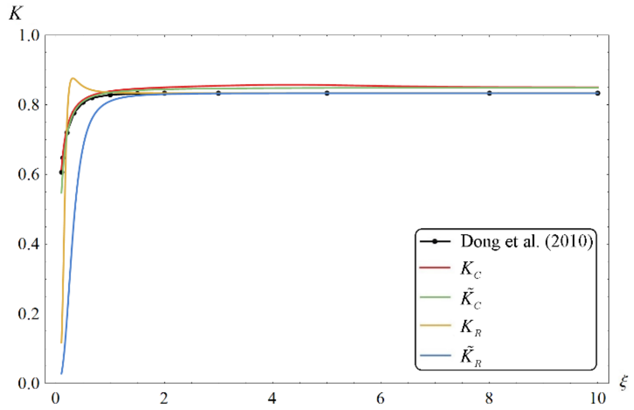

Fig. 7 Comparison of the proposed shear correction factors with numerical results of Dong *et al*. (2010)

noticeably inferred from Table 1, the proposed shear correction factors based on Cowper's approach K_C and $\tilde{K}_C$ are in excellent agreement with the numerical results of Dong *et al*. (2010) for shallow sections. Additionally, the introduced shear correction factors based on the energy approach K_R and $\tilde{K}_R$ are remarkably consistent with the numerical results of Dong *et al*. (2010) for deep sections. The numerical behavior of introduced simplified shear correction factors $\tilde{K}_C$ and $\tilde{K}_R$ are also exhibited in Fig. 7 as a function of the aspect ratio for Poisson's ratio $\nu = 0.3$ in comparison to the numerical results of Dong *et al*. (2010). As a result, to integrate the comprehensive discussions and comparisons made in the present study, the compound shear correction factor is proposed for rectangular cross-sections to be employed for all ranges of aspects ratios

$$K = \begin{cases} \tilde{K}_C = \dfrac{10(1+\nu)}{12 + 11\nu + \nu\left(\frac{8}{9}\xi^{-\frac{3}{2}}\right)}, & \xi < 1 \\ \tilde{K}_R = \left(\dfrac{6}{5} + \left(\dfrac{12}{19}\dfrac{1}{\xi^3}\right)\left(\dfrac{\nu}{1+\nu}\right)^2\right)^{-1}, & \xi > 1 \end{cases} \tag{26}$$

6. Conclusions

Due to the importance of Timoshenko beam theory in mechanical analysis of short and thick beams with rectangular cross-sections, the issue of shear correction factor is thoroughly examined in the present study. The Saint-Venant's flexure problem is first revisited in the framework of the theory of elasticity and a highly accurate approximate closed-form solution for the stress function is presented employing the Extended Kantorovich Method.

The resulted elasticity field is then compared to the technical Timoshenko beam model, and subsequently, a shear correction factor based on Cowper's approach is proposed and simplified for both the shallow and deep sections. The approximated elasticity field is furthermore exploited in the determination of the shear strain energy, and then, the energy-consistent shear correction factor is introduced and simplified for both the shallow and deep sections. The introduced shear correction factors are comprehensively examined and compared to the available results in the literature.

Both shear correction factors is demonstrated to coincide on the well-known value of 5/6 for zero Poisson's ratio. While the simplified shear correction factor based on Cowper's approach is in excellent agreement with the numerical values reported in the literature for shallow sections, the simplified energy-consistent shear correction factor is remarkably consistent with the numerical results reported in the literature for deep sections.

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Table 1 Numerical values of shear correction factor for rectangular cross-section with Poisson's ratio of 0.3

	Shear correction factor								
ξ	K_{Cowper}	K_C	$\tilde{K}_C$	$K^C_{Faghidian}$	K_{Renton}	K_R	$\tilde{K}_R$	$K^R_{Faghidian}$	Dong *et al.* (2010)
10	0.850	0.850	0.849	0.850	0.833	0.833	0.833	0.858	0.833
8	0.850	0.850	0.849	0.850	0.833	0.833	0.833	0.858	0.833
5	0.850	0.857	0.848	0.850	0.833	0.833	0.833	0.857	0.833
3	0.850	0.854	0.847	0.851	0.833	0.833	0.832	0.856	0.833
2	0.850	0.850	0.844	0.853	0.833	0.833	0.830	0.854	0.833
1.5	0.850	0.846	0.842	0.856	0.832	0.833	0.826	0.848	0.832
1	0.850	0.839	0.835	0.861	0.828	0.835	0.811	0.823	0.829
0.667	0.850	0.827	0.832	0.868	0.813	0.842	0.761	0.748	0.820
0.5	0.850	0.814	0.810	0.872	0.784	0.853	0.681	0.645	0.807
0.333	0.850	0.786	0.780	0.878	0.694	0.878	0.474	0.446	0.777
0.2	0.850	0.731	0.711	0.882	0.478	0.744	0.185	0.218	0.720
0.125	0.850	0.657	0.609	0.883	0.256	0.258	0.054	0.097	0.648
0.1	0.850	0.615	0.548	0.884	0.179	0.117	0.029	0.064	0.607

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