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Stability of five layer sandwich beams – a nonlinear hypothesis

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Abstract. The paper is devoted to the stability analysis of a simply supported five layer sandwich beam. The beam consists of five layers: two metal faces, the metal foam core and two binding layers between faces and the core. The main goal is to elaborate a mathematical and numerical model of this beam. The beam is subjected to an axial compression. The nonlinear hypothesis of deformation of the cross section of the beam is formulated. Based on the Hamilton's principle the system of four stability equations is obtained. This system is approximately solved. Applying the Bubnov-Galerkin's method gives an ordinary differential equation of motion. The equation is then numerically processed. The equilibrium paths for a static and dynamic load are derived and the influence of the binding layers is considered. The main goal of the paper is an analytical description including the influence of binding layers on stability, especially on critical load, static and dynamic paths. Analytical solutions, in particular mathematical model are verified numerically and the results are compared with those obtained in experiments.

Keywords: five layer beam; binding layers; shear effect; equilibrium paths; critical load

1. Introduction

Sandwich structures with a core made of metal foam are the subject of contemporary research, as is evident in numerous publications. Because of their excellent properties, for example the good acoustic absorption, high impact and heat resistance, vibration reduction and easy assembly these structures are widely used in automotive, aerospace, rail and shipbuilding industry. Allen (1969) described the bases of the theory of sandwich structures. Ashby *et al.* (2000) described the mechanical properties of metal foams. Banhart (2001) provided a comprehensive description of various manufacturing processes of metal foams and porous metallic structures. Altenbach and Eremeyev (2011) described various plate theories used in thin-walled structures made of foams. Małachowski *et al.* (2012) presented the experimental investigations and numerical modelling of closed-cell aluminium alloy foam (Alporas). Thai and Choi (2013) proposed a simple first order shear deformation theory for laminated composite plates. Altenbach *et al.* (2015) analysed three-layer laminates with thin and soft core layer with the use of the first order shear deformation plate theory. Belica *et al.* (2011) presented an analysis of the dynamic stability of a metal foam circular cylindrical shell subjected to combined loads. Jasion *et al.* (2012), Jasion and Magnucki (2012) studied analytically, numerically and experimentally the global and local buckling of the face sheets of sandwich beams. Jasion and Magnucki (2013) analysed the local buckling problem of sandwich beams under pure bending. Magnucki *et al.* (2006) carried out analytical investigations of bending and buckling of a rectangular plate made of a porous material. Magnucka-Blandzi (2009) presented a theoretical study on dynamic stability of a metal foam circular plate. Magnucka-Blandzi and Magnucki (2007) optimized the sandwich beam with metal foam core under strength and stability constrains. Magnucki *et al.* (2013) studied three-layer beams with corrugated core subjected to compression and four point bending. Magnucki *et al.* (2013), Smyczynski and Magnucka-Blandzi (2016, 2018) presented the strength analysis of a simply supported five layer sandwich beams with a metal foam core. Kim *et al.* (2013) studied a dynamic stability behavior of the shear-flexible composite beams based on finite element model using Hermitian beam elements. Magnucka-Blandzi (2011) compared the results of vibration problem of a sandwich beams for the three different modified Timoshenko hypotheses of deformation. Paczos *et al.* (2016) studied experimentally and numerically a problem of elastic three-point bending of five-layered trapezoidal beams. Farkas and Jarmai (1998) compared the optimized versions of a three and a five layers sandwich beams in terms of strength and minimum material cost. Grygorowicz *et al.* (2015) analytically and numerically studied elastic buckling of a three-layered beam with variable mechanical properties of the core. Pawlus (2007, 2011) presented the computational results of critical loads calculations of annular three-layered plates with a soft core. Lee and Fan (1996) elaborated the mathematical model of finite element with shear effect, and analyzed numerically bending and vibration of composite sandwich plates. Yang *et al.* (2012) analyzed the dynamic stability for composite laminated beams with delaminations. Yu *et al.* (2003) investigated experimentally the response and failure of dynamically loaded sandwich beams with an aluminum foam core. Mohanty *et al.* (2012) presented the

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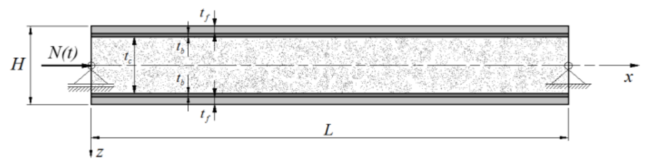

Fig. 1 Scheme of five layer beam subjected to an axial force

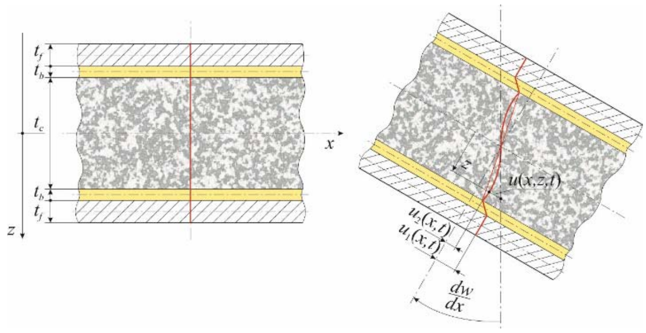

Fig. 2 The field of displacement – a nonlinear hypothesis

evaluation of static and dynamic behavior of functionally graded Timoshenko beams. Zenkour (2005) presented the sinusoidal shear deformation plate theory, the buckling and free vibrations problems of the simply supported functionally graded sandwich plate.

The paper presents the stability analysis of a simply supported five layer sandwich beam, which consists of: two thin faces (aluminium sheets) of a thickness t_f, one core (aluminium foam) of a thickness t_c and two thin binding layers (e.g., glue) of a thickness t_b. Each type of layer of the beam has different mechanical properties. The beam has the length L, the width b and the depth H. The beam carries a compressive axial load $N(t)$ varying in time as shown in Fig. 1. This load is assumed as follows

$$N\left(t\right)=N_0\frac{t}{t_0}, \tag{1}$$

where $N_0 = N_{cr}$ - critical load, t_0 - time base.

The mathematical and numerical model is elaborated. Most of the work known from the literature omits the binding layers. The main goal of the paper is among others examine the influence of binding layers on stability of a five layer sandwich beam. In the literature there are not many papers on analytical studies of stability of sandwich beams. Most of them concern the sandwich plates and numerical investigations. In the paper the shear effect is taken into account, not only in the main core of the beam, but in the binding layers too. This has been realized by formulating a nonlinear hypothesis of deformation of the beam's cross section which is a generalization of the classical "broken-line" hypothesis presented for example in (Jasion and Magnucki 2013, Magnucki *et al*. 2014). The proposed hypothesis also generalize the nonlinear hypotheses in (Magnucka-Blandzi and Magnucki 2007, Magnucka-Blandzi 2011), which are devoted to three layer beams. Carrera and Brischetto (2009) presented a survey of the theories for the analysis of sandwich plates. Based on extensive research they concluded that the classical lamination theory (CLT) and the first order shear deformation theory (FSDT) cannot be effectively used for the analysis of sandwich structures.

In the paper the binding layers are taken into account, so the considered beam is five layer one. In the above mentioned papers the binding layers are omitted. Finally the analytical results (critical loads) are verified by numerical (ANSYS) and experimental ones. Moreover the equilibrium paths for static and dynamic loads are determined. A novelty of study is the analytical description of stability of five-layer sandwich beams including two thin binding layers between facings and a core.

2. Nonlinear hypothesis of deformation of the flat cross section of the beam

The field of displacement for the flat cross section of the five layer beam, is presented in Fig. 2. Assuming the nonlinear hypothesis the shear effect is taken into account.

The longitudinal displacements are formulated as follows:

(1) for the upper face

$$\begin{gathered}-\left(1/2+x_1+x_2\right)\le\zeta\le-\left(1/2+x_1\right)\\ u\left(x,\zeta,t\right)=-t_c\left[\zeta\frac{\partial w}{\partial x}+\psi_1\left(x,t\right)\right],\end{gathered} \tag{2}$$

(2) for the upper binding layer

$$\begin{gathered}-\left(1/2+x_1\right)\le\zeta\le-1/2\\ u\left(x,\zeta,t\right)=-t_c\left\{\zeta\frac{\partial w}{\partial x}+\psi_2\left(x,t\right)-\right.\\ \left.+\frac{1}{x_1}\left(\zeta+\frac{1}{2}\right)\left[\psi_1\left(x,t\right)-\psi_2\left(x,t\right)\right]\right\},\end{gathered} \tag{3}$$

(3) for the core

$$\begin{gathered}-1/2\le\zeta\le1/2\\ u\left(x,\zeta,t\right)=-t_c\left\{\zeta\left[\frac{\partial w}{\partial x}-2\psi_2\left(x,t\right)\right]+\right.\\ \left.+\frac{1}{2\pi}\psi_3\left(x,t\right)\sin\left(2\pi\zeta\right)\right\},\end{gathered} \tag{4}$$

(4) for the lower binding layer

$$\begin{gathered}1/2\le\zeta\le1/2+x_1\\ u\left(x,\zeta,t\right)=-t_c\left\{\zeta\frac{\partial w}{\partial x}-\psi_2\left(x,t\right)-\right.\\ \left.+\frac{1}{x_1}\left(\zeta-\frac{1}{2}\right)\left[\psi_1\left(x,t\right)-\psi_2\left(x,t\right)\right]\right\},\end{gathered} \tag{5}$$

(5) for the lower face

$$\begin{gathered}1/2+x_1\le\zeta\le1/2+x_1+x_2\\ u\left(x,\zeta,t\right)=-t_c\left[\zeta\frac{\partial w}{\partial x}-\psi_1\left(x,t\right)\right],\end{gathered} \tag{6}$$

where

$x_1 = t_b/t_c$, $x_2 = t_f/t_c$ – dimensionless parameters,

$\zeta = z / t_c$ – dimensionless coordinate,

$\psi_1(x, t) = u_1(x, t) / t_c$, $\psi_2(x, t) = u_2(x, t) / t_c$, $\psi_3(x, t)$ – dimensionless functions of displacement, which determine the field of displacements.

If $\psi_3 \equiv 0$ then the proposed nonlinear hypothesis becomes the broken line hypothesis. So the assumed hypothesis is a generalization of the classical one described in (Magnucki *et al*. 2013, Smyczynski and Magnucka-Blandzi 2015).

Strains of the layers of the five layer beam are defined assuming nonlinear relations between strains and displacements, as follows

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z},$$

so

(1) for the upper face

$$-\left(1/2 + x_1 + x_2\right) \le \zeta \le -\left(1/2 + x_1\right)$$

$$\varepsilon_x = -t_c\left[\zeta\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi_1}{\partial x}\right] + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad (7)$$

$$\gamma_{xz} = 0,$$

(2) for the upper binding layer

$$-\left(1/2 + x_1\right) \le \zeta \le -1/2$$

$$\varepsilon_x = -t_c\left[\zeta\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi_2}{\partial x} - \right.$$

$$\left. + \frac{1}{x_1}\left(\zeta + \frac{1}{2}\right)\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial x}\right)\right] + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad (8)$$

$$\gamma_{xz} = \frac{1}{x_1}\left(\psi_1 - \psi_2\right),$$

(3) for the core

$$-1/2 \le \zeta \le 1/2$$

$$\varepsilon_x = -t_c\left[\zeta\left(\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial \psi_2}{\partial x}\right) + \right.$$

$$\left. + \frac{1}{2\pi}\frac{\partial \psi_3}{\partial x}\sin\left(2\pi\zeta\right)\right] + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad (9)$$

$$\gamma_{xz} = 2\psi_2\left(x,t\right) - \psi_3\left(x,t\right)\cos\left(2\pi\zeta\right),$$

(4) for the lower binding layer

$$1/2 \le \zeta \le 1/2 + x_1$$

$$\varepsilon_x = -t_c\left[\zeta\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi_2}{\partial x} - \right.$$

$$\left. + \frac{1}{x_1}\left(\zeta - \frac{1}{2}\right)\left(\frac{\partial \psi_1}{\partial x} - \frac{\partial \psi_2}{\partial x}\right)\right] + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad (10)$$

$$\gamma_{xz} = \frac{1}{x_1}\left(\psi_1 - \psi_2\right),$$

(5) for the lower face

$$1/2 + x_1 \le \zeta \le 1/2 + x_1 + x_2$$

$$\varepsilon_x = -t_c\left[\zeta\frac{\partial^2 w}{\partial x^2} - \frac{\partial \psi_1}{\partial x}\right] + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad (11)$$

$$\gamma_{xz} = 0.$$

Stresses in all layers of the beam, according to Hooke's law, for individual layers are

$$\sigma_x = E\varepsilon_x, \quad \tau_{xz} = G\gamma_{xz}. \quad (12)$$

3. Equations of equilibrium

Basing on the Hamilton's principle

$$\delta\int_{t_1}^{t_2}\left(T - U_\varepsilon + W\right)dt = 0, \quad (13)$$

the system of four stability equations was obtained. The kinetic energy is

$$T = \frac{1}{2}bt_c\int_0^L\left(2\rho_f x_2 + 2\rho_b x_1 + \rho_c\right)\left(\frac{\partial w}{\partial t}\right)^2 dx. \quad (14)$$

The potential energy of the elastic strain of the beam is

$$U_\varepsilon = \frac{1}{2}\iiint_V(\varepsilon_x\sigma_x + \gamma_{xz}\tau_{xz})dV. \quad (15)$$

The work W of the external compressive load $N(t)$ is the following

$$W = \frac{1}{2}\int_0^L N\left(t\right)\left(\frac{\partial w}{\partial x}\right)^2 dx. \quad (16)$$

The system of four equations of motion takes the following form

$$\begin{cases} a_{11}\dfrac{\partial^4 w}{\partial x^4}-a_{12}\dfrac{\partial^3\psi_1}{\partial x^3}-a_{13}\dfrac{\partial^3\psi_2}{\partial x^3}+\\ +a_{14}\dfrac{\partial^3\psi_3}{\partial x^3}+\dfrac{1}{t_c^2}a_{15}\left(\dfrac{\partial w}{\partial x}\right)^2\dfrac{\partial^2 w}{\partial x^2}+\\ +\dfrac{\rho_c}{t_c^2E_c}a_{16}\dfrac{\partial^2 w}{\partial t^2}=-\dfrac{1}{bt_c^3E_c}N(t)\dfrac{\partial^2 w}{\partial x^2},\\ a_{21}\dfrac{\partial^3 w}{\partial x^3}-a_{22}\dfrac{\partial^2\psi_1}{\partial x^2}-a_{23}\dfrac{\partial^2\psi_2}{\partial x^2}+\\ +\dfrac{1}{t_c^2}a_{24}\left[\psi_1(x,t)-\psi_2(x,t)\right]=0,\\ a_{31}\dfrac{\partial^3 w}{\partial x^3}-a_{32}\dfrac{\partial^2\psi_1}{\partial x^2}-a_{33}\dfrac{\partial^2\psi_2}{\partial x^2}+\\ +a_{34}\dfrac{\partial^2\psi_3}{\partial x^2}+\dfrac{1}{t_c^2}\left(-a_{24}\psi_1(x,t)+\right.\\ \left.+a_{24}\psi_2(x,t)+a_{35}\psi_2(x,t)\right)=0,\\ -a_{41}\dfrac{\partial^3 w}{\partial x^3}+a_{42}\dfrac{\partial^2\psi_2}{\partial x^2}-a_{43}\dfrac{\partial^2\psi_3}{\partial x^2}+\\ +\dfrac{1}{t_c^2}a_{44}\psi_3(x,t)=0, \end{cases} \tag{17}$$

where

$$a_{11}=2\alpha_1e_2+2\alpha_3e_1+\tfrac{1}{12},$$
$$a_{12}=2\alpha_2e_2+\tfrac{1}{6}x_1\left(3+4x_1\right)e_1,$$
$$a_{13}=\tfrac{1}{6}\left[1+\left(3+2x_1\right)x_1e_1\right],$$
$$a_{14}=1/4\pi^2,$$
$$a_{15}=3\left(x_2e_2+\tfrac{1}{2}+x_1e_1\right),$$
$$a_{16}=2x_2\rho_2+2x_1\rho_1+1,$$
$$a_{21}=a_{12},$$
$$a_{22}=2\left(x_2e_2+\tfrac{1}{3}x_1e_1\right),$$
$$a_{23}=\tfrac{1}{3}x_1e_1,$$
$$a_{24}=e_1/x_1\left(1+\nu_b\right),$$
$$a_{31}=a_{13},$$
$$a_{32}=a_{23},$$
$$a_{33}=\tfrac{1}{3}\left(1+2x_1e_1\right),$$
$$a_{34}=1/2\pi^2,$$
$$a_{35}=2/\left(1+\nu_c\right),$$
$$a_{36}=a_{24}+2/\left(1+\nu_c\right),$$
$$a_{41}=a_{14},$$
$$a_{42}=a_{34},$$
$$a_{43}=1/8\pi^2,$$
$$a_{44}=1/\left(4+4\nu_c\right),$$
$$\alpha_3=\tfrac{1}{12}x_1\left(4x_1^2+6x_1+3\right),$$
$$\alpha_1=\tfrac{1}{12}x_2\left(12x_1^2+12x_1x_2+12x_1+4x_2^2+6x_2+3\right),$$
$$\alpha_2=\tfrac{1}{2}x_2\left(2x_1+x_2+1\right),$$
$$\alpha_4=\tfrac{1}{2}x_1\left(1+x_1\right),$$
$$e_1=E_b/E_c,$$
$$e_2=E_f/E_c,$$
$$\rho_1=\rho_b/\rho_c,$$
$$\rho_2=\rho_f/\rho_c.$$

4. Stability of the five layer beam

4.1 Analytical solution

Unknown functions w, ψ_1, ψ_2 and ψ_3 are assumed in the following forms

$$w(x,t)=w_a(t)\sin\frac{\pi x}{L}, \tag{18}$$

$$\psi_1(x,t)=\psi_{a1}(t)\cos\frac{\pi x}{L},\quad \psi_2(x,t)=\psi_{a2}(t)\cos\frac{\pi x}{L},\quad \psi_3(x,t)=\psi_{a3}(t)\cos\frac{\pi x}{L}. \tag{19}$$

The functions ψ_1, ψ_2 and ψ_3 describe a transverse shear effect and transverse forces. Each of the above functions (18)-(19) identically satisfy boundary conditions

$$\begin{gathered} w(x,t)\big|_{x=0}=0,\qquad w(x,t)\big|_{x=L}=0,\\ \left.\frac{\partial\psi_1}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\psi_2}{\partial x}\right|_{x=0}=0,\quad \left.\frac{\partial\psi_3}{\partial x}\right|_{x=0}=0,\\ \left.\frac{\partial\psi_1}{\partial x}\right|_{x=L}=0,\quad \left.\frac{\partial\psi_2}{\partial x}\right|_{x=L}=0,\quad \left.\frac{\partial\psi_3}{\partial x}\right|_{x=L}=0,\\ M_b(x,t)\big|_{x=0}=0,\qquad M_b(x,t)\big|_{x=L}=0, \end{gathered} \tag{20}$$

where $M_b(x, t) = N(t) \cdot w(x)$ is a bending moment.

Moreover dimensionless functions (19) satisfy conditions

$$\left.\frac{\partial\psi_1}{\partial x}\right|_{x=\frac{L}{2}}=0,\quad \left.\frac{\partial\psi_2}{\partial x}\right|_{x=\frac{L}{2}}=0,\quad \left.\frac{\partial\psi_3}{\partial x}\right|_{x=\frac{L}{2}}=0. \tag{21}$$

Substituting these assumed functions (18) and (19) into the system of equilibrium (17) gives

$$\begin{cases} a_{11}\dfrac{\pi}{L}w_a(t)-a_{12}\psi_{a1}(t)-a_{13}\psi_{a2}(t)+ \\ +a_{14}\psi_{a3}(t)+a_{15}\dfrac{\pi}{L}\dfrac{1}{t_c^2}w_a^3\cos^2\dfrac{\pi x}{L}+ \\ +\dfrac{\rho_c}{t_c^2E_c}\left(\dfrac{L}{\pi}\right)^3a_{16}\dfrac{d^2w_a}{dt^2}=\dfrac{L}{\pi bt_c^3E_c}N(t)w_a(t), \\ -a_{12}\dfrac{w_a(t)}{L}\pi+a_{22}\psi_{a1}(t)+a_{23}\psi_{a2}(t)+ \\ +\left(\dfrac{L}{t_c}\right)^2\dfrac{1}{\pi^2}a_{24}\left[\psi_{a1}(t)-\psi_{a2}(t)\right]=0, \\ -a_{13}\dfrac{w_a(t)}{L}\pi+a_{23}\psi_{a1}(t)+a_{33}\psi_{a2}(t)- \\ +a_{34}\psi_{a3}(t)+\left(\dfrac{L}{t_c}\right)^2\dfrac{1}{\pi^2}\left[-a_{24}\psi_{a1}(t)+\right. \\ \left.+a_{24}\psi_{a2}(t)+a_{35}\psi_{a2}(t)\right]=0, \\ a_{14}\dfrac{w_a(t)}{L}\pi-a_{34}\psi_{a2}(t)+a_{43}\psi_{a3}(t)+ \\ +\left(\dfrac{L}{t_c}\right)^2\dfrac{1}{\pi^2}a_{44}\psi_{a3}(t)=0. \end{cases} \tag{22}$$

From the second, third and fourth algebraic equations of the system (22) unknown functions ψ_{a1}, ψ_{a2} and ψ_{a3} could be calculated, namely

$$\begin{aligned} \psi_{a1}(t)&=\tilde{\psi}_{a1}\frac{w_a(t)}{L}\pi, \\ \psi_{a2}(t)&=\tilde{\psi}_{a2}\frac{w_a(t)}{L}\pi, \\ \psi_{a3}(t)&=\tilde{\psi}_{a3}\frac{w_a(t)}{L}\pi, \end{aligned} \tag{23}$$

where

$$\tilde{\psi}_{a1}=\frac{W_{\psi_{a1}}}{W},\quad \tilde{\psi}_{a2}=\frac{W_{\psi_{a2}}}{W},\quad \tilde{\psi}_{a3}=\frac{W_{\psi_{a3}}}{W},$$

And W, $W\psi_{a1}$, $W\psi_{a2}$ and $W\psi_{a3}$ are the following determinants

$$W=\begin{vmatrix} a_{22}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{23}-\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & 0 \\ a_{23}-\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{33}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2(a_{24}+a_{35}) & -a_{34} \\ 0 & -a_{34} & a_{43}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{44} \end{vmatrix}$$

$$W_{\psi_{a1}}=\begin{vmatrix} a_{12} & a_{23}-\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & 0 \\ a_{13} & a_{33}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2(a_{24}+a_{35}) & -a_{34} \\ -a_{14} & -a_{34} & a_{43}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{44} \end{vmatrix}$$

$$W_{\psi_{a2}}=\begin{vmatrix} a_{22}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{12} & 0 \\ a_{23}-\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{13} & -a_{34} \\ 0 & -a_{14} & a_{43}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{44} \end{vmatrix}$$

$$W_{\psi_{a3}}=\begin{vmatrix} a_{22}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{23}-\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{12} \\ a_{23}-\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2a_{24} & a_{33}+\dfrac{1}{\pi^2}\left(\dfrac{L}{t_c}\right)^2(a_{24}+a_{35}) & a_{13} \\ 0 & -a_{34} & -a_{14} \end{vmatrix}$$

Substituting functions (23) into the first equation of the system (22) and the use of the Bubnov-Galerkin’s method gives the following equation of motion

$$\begin{aligned} &\left(\frac{L}{t_c}\right)^2\frac{L^2\rho_c}{\pi^4E_c}a_{16}\frac{d^2w_a}{dt^2}+\frac{1}{3}\frac{1}{t_c^2}a_{15}w_a^3(t)+ \\ &+\left[a_{11}-a_{12}\tilde{\psi}_{a1}-a_{13}\tilde{\psi}_{a2}+a_{14}\tilde{\psi}_{a3}-\right. \\ &\left.+\left(\frac{L}{t_c}\right)^2\frac{1}{\pi^2bt_cE_c}N(t)\right]w_a(t)=0. \end{aligned} \tag{24}$$

From Eq. (24) the static critical load could be obtained

$$\begin{aligned} N_{cr}&=\pi^2bt_cE_c\left(\frac{t_c}{L}\right)^2 \\ &\left(a_{11}-a_{12}\tilde{\psi}_{a1}-+a_{13}\tilde{\psi}_{a2}+a_{14}\tilde{\psi}_{a3}\right). \end{aligned} \tag{25}$$

Additionally, from the above equation, the equilibrium path for a static load may be determined

$$N(w_a)=N_{cr}+\frac{1}{3}\left(\frac{\pi}{L}\right)^2bt_cE_ca_{15}w_a^2. \tag{26}$$

4.2 Numerical calculations

In the numerical calculations the Runge-Kutta's method was used, and a small deflection was assumed, which is necessary to use this method. The amplitude of deflection of the beam is shown in Figs. 3 and 4 (the equilibrium paths), for the below assumed parameters: thickness of the faces t_f = 1 mm, thickness of the core t_c = 17.8 mm, thickness of the binding layers t_b = 0.1 mm, Young's modulus of the faces E_f = 65600 MPa, mass density of the faces ρ_f = 2.7×10^{-6} kg/mm^3, mass density of the core ρ_c = 2.7×10^{-7} kg/mm^3, mass density of the binding layers ρ_b = 1×10^{-6} kg/mm^3, the length L = 800 mm, the width b = 50 mm, Poisson's ratios $\nu_c = \nu_b$ = 0.3, time base t_0 = 3s and the compressive load (1).

The Fig. 3(a) shows equilibrium paths for various Young's moduli of the binding layers <200 MPa, 500 MPa, 1000 MPa, 1500 MPa> with constant Young's modulus of the core E_c = 1200 MPa. The Fig. 3(b) shows equilibrium paths for various Young's moduli of the core <200 MPa, 500 MPa, 1000 MPa, 1200 MPa> with constant Young's modulus of the binding layers E_b = 1500 MPa. It can be seen, that binding layers have not a significant impact on static equilibrium paths. Graphs for various mechanical properties of the binding layers or of the core are very close to each other (overlaping).

The Fig. 4 shows the static and dynamic paths. The parabola corresponds the equilibrium path for a static load. The curve oscillating around this path is the dynamic equilibrium path, which is the solution of the differential Eq. (24) obtained by Runge-Kutta's method.

According to the Eq. (25), critical loads have been analytically calculated for various Young's moduli of the binding layers < 50 MPa, 100 MPa, 250 MPa, 500 MPa, 1000 MPa, 1500 MPa > with constant Young's modulus of the core E_c = 1200 MPa, and for various thicknesses of binding layers < 0.1 mm, 0.2 mm, 0.3 mm, 0.4 mm, 0.5 mm> with constant total thickness H = 20 mm.

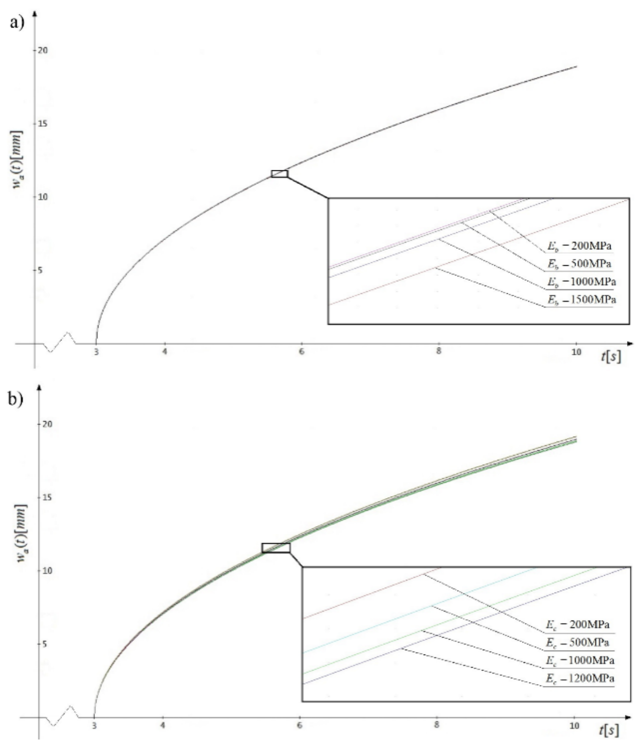

Fig. 3 Equilibrium paths for a static load for few different Young's modulus of binding layers (a); and of a core (b)

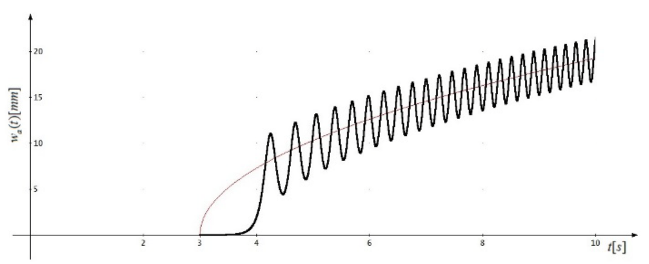

Fig. 4 Equilibrium paths

Table 1 Critical loads for various Young's moduli of the binding layers

E_b [MPa]	50	10	250	500	1000	1500
$N_{cr}^{(Analit)}$ [N]	9335	9359	9374	98382	9390	9397
$N_{cr}^{(FEM)}$ [N]	9358	9375	9388	9397	9407	9414
Relative error	0.25%	0.17%	0.15%	0.16%	0.18%	0.18%

Table 2 Critical loads for various thicknesses of binding layers

h_b [MPa]	0.1	0.2	0.3	0.4	0.5
$N_{cr}^{(Analit)}$ [N]	9397	9401	9405	9408	9412
$N_{cr}^{(FEM)}$ [N]	9414	9419	9423	9427	9431
Relative error	0.18%	0.19%	0.19%	0.20%	0.20%

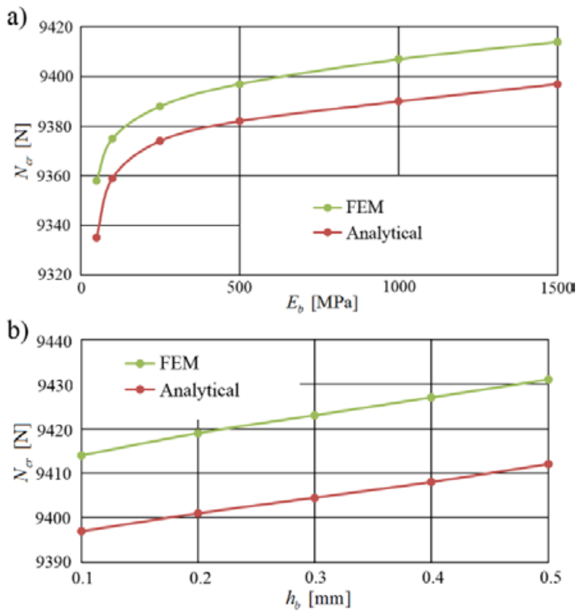

Fig. 5 Critical loads: (a) for various Young's modulus; (b) for various thickness of binding layers

Moreover, critical loads obtained analytically are verified numerically (a description of FEM model is presented in Section 5.1). Critical loads obtained analytically and numerically are shown in Tables 1 and 2 and in Fig. 5.

In line with increasing Young's moduli of the binding layers, the critical load also increases (by 0.7%). The increase in thickness of the binding layers causes only slight growth of critical load (by 0.2%). Therefore, it may be seen, that the change in mechanical properties or a thickness of binding layers has not significant impact on critical loads. Furthermore, discrepancies between the results obtained analytically and numerically do not exceed 1%.

5. Stability – verification of the results

The dimensions of a cross-section of the beam are as follows: the width b = 50.2 mm the total thickness H = 19.68 mm thickness of the faces t_f = 1 mm. Particular layers of the beam was glued together - thickness of the binding layers t_b = 0.1 mm. The distance between supported ends of the beam was 933 mm. The material constants for the aluminium alloy of the faces is E_f = 65600 MPa, and for the binding glue layers: E_b = 1500 MPa. The Young's modulus for the aluminium foam core, based on the experimental tests described in details in the monograph by (Magnucki *et al*. 2012), is E_c = 216 MPa.

5.1 FEM analysis

The finite element model of the five layer sandwich beam has been elaborated. Eight- noded 3D brick elements (SOLID185) have been used to model the core and two binding layers. Two faces have been modelled with the use of four-noded 2D shell elements (SHELL181). The faces were offset from the binding layers about half of the thickness. Between particular layers the tie conditions have been imposed. The entire beam has been modelled. According to the analytical model the discretized model of the beam was simply supported on the two outer edges of the beam.

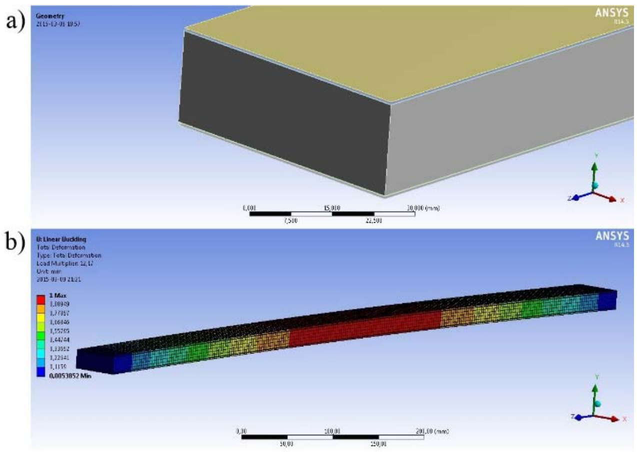

Fig. 6 Scheme of: (a) the FEM model of a sandwich beam; (b) results of buckling analysis

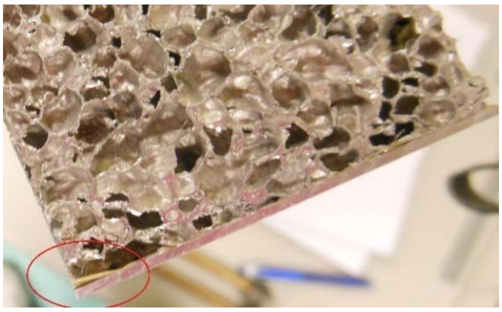

Fig. 7 Cross section of the tested beam

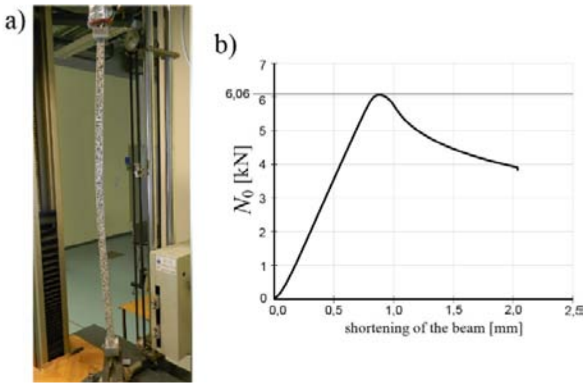

Fig. 8 Test stand: (a) buckled beam; (b) results

As to the axially compressed beam the linear buckling analysis has been performed. The results of this analysis are shown in Fig. 6.

5.2 Experimental investigations

In order to verify analytical and numerical model, the experimental test has been performed. In the experimental investigations a sandwich beam with a metal foam core was axially compressed. The aluminium faces were glued to the metal foam core with glue of the thickness t_b = 0.1 mm. A part of this glue layer is quite well visible on Fig. 7.

The test stand is shown in Fig. 8(a). The dimensions of the beam as well as the material properties were the same as in the example considered in the subsection 5.1.

Axial compression process of a beam was recorded – the axial displacement-shortening and the axial force N_0 have been measured. The obtained results are given in Fig. 8b in the relation between the axial load and the shortening of the beam.

5.3 Comparison of the results

The buckling analysis has been carried out for axially compressed beam. The critical loads have been obtained with the use of three methods: analytically, numerically (FEM, ANSYS) and experimentally. The dimensions of the beam as well as the material properties were the same for each method and had values as in the example considered in the subsection 5.1. The critical load obtained analytically is $N_{0,CR}^{(Analit\,)} = 6.10\ kN$. The critical load obtained numerically (FEM) is $N_{0,CR}^{(FEM)} = 6.09\ kN$, and the critical load obtained

Table 3 Critical loads

Method	Analytical	FEM	Experimental
$N_{0,CR}$ [kN]	6.10	6.09	6.06
ε [%]	–	0.2	0.7

experimentally is $N_{0,CR}^{(Exp)} = 6.06\ kN$. The relative error has been calculated according to the formula

$$\varepsilon = \frac{\left| N_{0,CR}^{(Analit)} - N_{0,CR}^{(i)} \right|}{N_{0,CR}^{(Analit)}} \cdot 100\%, \tag{27}$$

where

$$i = FEM, Exp.$$

A good agreement can be seen between the critical loads obtained from these three methods – the differences are less than 1%.

6. Conclusions

The study is devoted to the analysis of dynamic stability of a simply supported five later sandwich beam under compression. The main goal is to elaborate a mathematical model of the beam in which the binding layers will be treated as separate layers. This way the influence of the thickness and the mechanical properties of the binding layers can be investigated what is usually omitted when sandwich structures are analyzed. A nonlinear hypothesis has been assumed to describe the deformation of the flat cross section of the beam. Such approach allowed obtaining a formula with which the stability problem of the five layered beam can be solved. The stability problem is mathematically solved.

Based on the studies there are very small discrepancies in equilibrium paths for different values of Young's modulus of the core and binding layers.

The analytical results were compared with this obtained numerically (FEM) and experimentally. A good agreement can be seen between the critical loads obtained from these three methods – the differences are less than 1%.

The proposed hypothesis for the five-layer beam can be generalized and adapted to sandwich plates.

The proposed method enables the derivation of the system of equation of motion and determination of the critical loads and the static equilibrium paths (the one formula for critical loads, and quadratic function describing the load dependence with respect to maximum deflection). By this method, the equation of motion was also obtained (the differential equation of the second order). The solution of this equation is a dynamic equilibrium path.

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References

Allen, H.G. (1969), *Analysis and Design of Structural Sandwich Panels*, Pergamon Press, Oxford, London, Edinburgh, New York, Toronto, Sydney, Paris, Braunschweig.

Altenbach, H. and Eremeyev, V. (2011), *Thin-walled Structures Made of Foams*.

Altenbach, H., Eremeyev, V.A. and Naumenko, K. (2015), "On the use of the first order shear deformation plate theory for the analysis of three-layer plates with thin soft core layer", *ZAMM – J. Appl. Math. Mech. / Zeitschrift für Angewandte Mathematik und Mechanik*, **95**(10), 1004-1011.

Ashby, M.F., Evans, A.G., Fleck, N.A., Gibson, L.J. and Hutchinson, J.W. (2000), *Metal Foams: A Design Guide*, Butterworth Heinemann, Boston, MA, USA.

Banhart, J. (2001), "Manufacture, characterisation and application of cellular metals and metal foams", *Prog. Mater. Sci.*, **46**(6), 559-632.

Belica, T., Malinowski, M. and Magnucki, K. (2011), "Dynamic stability of an isotropic metal foam cylindrical shell subjected to external pressure and axial compression", *J. Appl. Mech.-Trans. ASME*, **78**(4), 8-20.

Carrera, E. and Brischetto, S. (2009), "A survey with numerical assessment of classical and refined theories for the analysis of sandwich plates", *Appl. Mech. Rev.*, **62**(1), 17.

Farkas, J. and Jarmai, K. (1998), "Minimum material cost design of five-layer sandwich beams", *Struct. Optimiz.*, **15**(3-4), 215-220.

Grygorowicz, M., Magnucki, K. and Malinowski, M. (2015), "Elastic buckling of a sandwich beam with variable mechanical properties of the core", *Thin-Wall. Struct.*, **87**, 127-132.

Jasion, P. and Magnucki, K. (2012), "Face wrinkling of sandwich beams under pure bending", *J. Theor. Appl. Mech.*, **50**(4), 933-941.

Jasion, P. and Magnucki, K. (2013), "Global buckling of a sandwich column with metal foam core", *Journal of Sandwich Structures & Materials*. **15**(6), 718-732.

Jasion, P., Magnucka-Blandzi, E., Szyc, W. and Magnucki, K. (2012), "Global and local buckling of sandwich circular and beam-rectangular plates with metal foam core", *Thin-Wall. Struct.*, **61**, 154-161.

Kim, N., Jeon, C. and Lee, J. (2013), "Dynamic stability analysis of shear-flexible composite beams", *Arch. Appl. Mech.*, **83**(5), 685-707.

Lee, L.J. and Fan, Y.J. (1996), "Bending and vibration analysis of composite sandwich plates", *Comput. Struct.*, **60**(1), 103-112.

Magnucka-Blandzi, E. (2009), "Dynamic stability of a metal foam circular plate", *J. Theor. Appl. Mech.*, **47**(2), 421-433.

Magnucka-Blandzi, E. (2011), "Dynamic stability and static stress state of a sandwich beam with a metal foam core using three modified Timoshenko hypotheses", *Mech. Adv. Mater. Struct.*, **18**(2), 147-158.

Magnucka-Blandzi, E. and Magnucki, K. (2007), "Effective design of a sandwich beam with a metal foam core", *Thin-Walled Struct.*, **45**(4), 432-438.

Magnucki, K., Malinowski, M. and Kasprzak, J. (2006), "Bending and buckling of a rectangular porous plate", *Steel Compos. Struct., Int. J.*, **6**(4), 319-333.

Magnucki, K., Szyc, W. and (red) (2012), *Wytrzymałość i stateczność belek i płyt trójwarstwowych z rdzeniem z pianki aluminiowej*, Wydawnictwo Politechniki Poznańskiej, Poznań.

Magnucki, K., Jasion, P., Krus, M., Kuligowski, P. and Wittenbeck, L. (2013), "Strength and buckling of sandwich beams with corrugated core", *J. Theor. Appl. Mech.*, **51**(1), 15-24.

Magnucki, K., Jasion, P., Szyc, W. and Smyczynski, M.J. (2014), "Strength and buckling of a sandwich beam with thin binding layers between faces and a metal foam core", *Steel Compos. Struct., Int. J.*, **16**(3), 325-337.

Magnucki, K., Smyczynski, M. and Jasion, P. (2013), "Deflection and strength of a sandwich beam with thin binding layers between faces and a core", *Arch. Mech.*, **65**(4), 301-311.

Małachowski, J., Klasztorny, M., Dziewulski, P., Nycz, D. and Gotowicki, P. (2012), "Experimental investigations and modelling of Alporas aluminium foam", *Model. Eng.*, **42**, 97-112. [In Polish]

Mohanty, S., Dash, R. and Rout, T. (2012), "Static and dynamic stability analysis of a functionally graded Timoshenko beam", *Int. J. Struct. Stab. Dyn.*, **12**(4).

Paczos, P., Wasilewicz, P. and Magnucka-Blandzi, E. (2016), "Experimental and numerical investigations of five-layered trapezoidal beams", *Compos. Struct.*, **145**, 129-141.

Pawlus, D. (2007), "Critical static loads calculations in finite element method of three-layered annular plates", *Arch. Civil Mech. Eng.*, **7**(1), 21-33.

Pawlus, D. (2011), "Critical loads calculations of annular three-layered plates with soft elastic or viscoelastic core", *Arch. Civil Mech. Eng.*, **11**(4), 993-1009.

Smyczynski, M.J. and Magnucka-Blandzi, E. (2015), "Static and dynamic stability of an axially compressed five-layer sandwich beam", *Thin-Wall. Struct.*, **90**, 23-30.

Smyczynski, M.J. and Magnucka-Blandzi, E. (2016), "Strength of the three layer beam with two binding layers", *Arch. Civil Eng.*, **62**(3), 189-206.

Smyczynski, M.J. and Magnucka-Blandzi, E. (2018), "The three-point bending of a sandwich beam with two binding layers – Comparison of two nonlinear hypotheses", *Compos. Struct.*, **183**, 96-102.

Thai, H.-T. and Choi, D.-H. (2013), "A simple first-order shear deformation theory for laminated composite plates", *Compos. Struct.*, **106**, 754-763.

Yang, J., Chen, D., Ning, M., Yang, W. and Li, Q. (2012), "Analysis of dynamic stability for composite laminated beam with delamination", *Progress Ind. Civil Eng., Pts. 1-5*. **204-208**, 3074-3077.

Yu, J., Wang, X., Wei, Z. and Wang, E. (2003), "Deformation and failure mechanism of dynamically loaded sandwich beams with aluminum-foam core", *Int. J. Impact Eng.*, **28**(3), 331-347.

Zenkour, A.M. (2005), "A comprehensive analysis of functionally graded sandwich plates: Part 2 - Buckling and free vibration", *Int. J. Solids Struct.*, **42**(18-19), 5243-5258.

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