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Nonlocal strain gradient 3D elasticity theory for anisotropic spherical nanoparticles

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Abstract. In this paper, three-dimensional (3D) elasticity theory in conjunction with nonlocal strain gradient theory (NSGT) is developed for mechanical analysis of anisotropic nanoparticles. The present model incorporates two scale coefficients to examine the mechanical characteristics much accurately. All the elastic constants are considered and assumed to be the functions of (r, θ, φ), so all kind of anisotropic structures can be modeled. Moreover, all types of functionally graded spherical structures can be investigated. To justify our model, our results for the radial vibration of spherical nanoparticles are compared with experimental results available in the literature and great agreement is achieved. Next, several examples of the radial vibration and wave propagation in spherical nanoparticles including nonlocal strain gradient parameters are presented for more than 10 different anisotropic nanoparticles. From the best knowledge of authors, it is the first time that 3D elasticity theory and NSGT are used together with no approximation to derive the governing equations in the spherical coordinate. Moreover, up to now, the NSGT has not been used for spherical anisotropic nanoparticles. It is also the first time that all the 36 elastic constants as functions of (r, θ, φ) are considered for anisotropic and functionally graded nanostructures including size effects. According to the lack of any common approximations in the displacement field or in elastic constant, present theory can be assumed as a benchmark for future works.

Keywords: three-dimensional elasticity theory; nonlocal strain gradient theory; spherical coordinate; anisotropic material; nanoparticles

1. Introduction

Anisotropic materials are the property of being directionally dependent, which refers to distinct properties in different directions. Unlike isotropic materials that have material properties identical in all directions, anisotropic material's physical and mechanical properties such as (Young's Modulus, conductivity, absorbance, tensile strength, etc.) change with direction along the object. An example of the anisotropic material is the light coming through a polarizer. Another is composites and wood. Anisotropic nanostructures have been employed in many parts of nano-electro-mechanical systems (NEMSs, e.g., nanogenerator, nanoresonator, chemical sensors, light-emitting diodes, etc.). Up to now, several types of researches have been done on the anisotropic structures (Hamidi *et al*. 2015, Bourada *et al*. 2016, Houari *et al*. 2016, Benahmed *et al*. 2017, Shahsavari *et al*. 2018b). To refer, time-resolved experiments proposed by (Voisin *et al*. 2000), low-frequency Raman scattering by (Shukla and Kumar, 2011) and ultrafast pump–probe spectroscopy presented by (Ruijgrok *et al*. 2012). Also, (Mock *et al*. 2017) investigated the frequency dependence of four independent $CdWO_4$ Cartesian dielectric function tensor elements by generalized spectroscopic ellipsometry within mid-infrared and far-infrared spectral regions. Also, Single crystal surfaces cut under different angles from a bulk crystal, (010) and (001), were studied. A formula for the Raman scattering intensity as a function of incoming and outgoing polarization and the Raman tensor viewed through birefringent crystal (calcite) was presented by (Grundmann *et al*. 2016). Also, the authors discussed the general form of the dielectric function of anisotropic crystals based on individual dipole oscillators for phonon and electronic resonance. In recent years, directional dependence in non-isotropic structures has been also well studied theoretically by several researchers. Wave steering effects in anisotropic composite structures based on a finite element scheme was presented by (Chronopoulos 2017). Also, a structure of arbitrary anisotropy, layering, and geometric complexity was modeled through Finite Elements coupled to a periodic structure wave scheme. In addition, a generic approach for efficiently computing the angular sensitivity of the wave slowness for each wave type, direction and frequency was presented. (Ziane *et al*. 2013) studied the free vibration of anisotropic structures on the basis of first-order shear deformation theory (FSDT). (Mousavi *et al*. 2016) presented the analysis of centrosymmetric anisotropic plate structures based on Reddy's third-order shear deformable plate theory with considering strain gradient elasticity. Moreover, it was obtained that the gradient theory provides the ability to include the size effects in anisotropic plate

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structures. Finally, analytical solutions were introduced for the buckling and bending and of orthotropic Kirchhoff plates. Vibration and buckling behavior of thick orthotropic plates and laminates considering the simply supported boundary conditions was investigated by (Srinivas and Rao 1970). Also, three-dimensional, linear, small deformation theory of elasticity solution was examined for the vibration, bending and buckling of thick orthotropic rectangular plates and laminates considering simply supported boundary conditions. In addition, all the nine elastic constants of orthotropy are taken into account. Theories for composite, multilayered, anisotropic plates and shells were investigated by (Carrera 2002) including the complicating effects that have been introduced by anisotropic behavior and layered constructions.

Size effect is an interesting topic due to the current applications in modern technology include a variety of length scales from a few centimeters to a few nanometers (Zbib and Aifantis 2003). The classical continuum theory cannot model nanostructures including small size effect. So different size-dependent theories, such as micropolar theory (Eringen 1967), nonlocal elasticity theory (Eringen and Edelen 1972), surface elasticity (Gurtin *et al*. 1998), strain gradient theory (Aifantis 1999), the modified couples stress theory (Yang *et al*. 2002) and the nonlocal strain gradient theory (Askes and Aifantis 2009), were presented. In recent years, nonlocal elasticity, strain gradient elasticity and nonlocal strain gradient elasticity theories have been largely used for the modeling of different nanostructures (Bağdatlı 2015, Chaht *et al*. 2015, Zenkour and Abouelregal 2015, Li *et al*. 2016a, Karami and Janghorban 2016, Sobhy 2017, Ebrahimi and Barati 2017b, Shen *et al*. 2017, Şimşek 2016, Karami *et al*. 2017a, Shahsavari *et al*. 2017, 2018a, Sahmani and Aghdam 2017, Shahsavari and Janghorban 2017, Li *et al*. 2017, Mehralian *et al*. 2017, Karami *et al*. 2018a, e, Jandaghian and Rahmani 2017).

It has been recently shown that nonlocal differential elasticity based model maybe ill-posed. Of course, due to the simplification of the nonlocal differential elasticity, many works have been focused on the size-dependent behaviors based on the nonlocal differential models. More recently, it is shown that the nonlocal differential and integral elasticity based models may be not equivalent to each other. (Zhu and Li 2017d) presented a nonlocal integral model to study the twisting static behaviors of through-radius FG nanotubes via Eringen's nonlocal integral elasticity. The authors have shown that in comparison to the widely-used nonlocal differential model in the literature, the nonlocal integral model developed there was self-consistent and well-posed. Longitudinal and torsional dynamic problems for small-scaled rods were modeled by utilizing an integral formula of two-phase nonlocal theory by (Zhu and Li 2017b). Among the non-continuum theories, the nonlocal strain gradient theory proposed by (Askes and Aifantis 2009) is preferable to considering the size effect as it involves two material length parameters. (Askes and Aifantis 2011) presented different formats of gradient elasticity and their capability in static and dynamic applications. Moreover, it was observed that the removal of singularities in statics and dynamics, as well as the size-dependent mechanical response predicted by gradient elasticity. Analysis of resonance frequencies of FG micro and nanoplates based on the nonlocal elasticity and strain gradient theory is performed by (Nami and Janghorban 2014). They used nonlocal and strain gradient theories separately, and concluded that these theories have different mechanisms in analysis of nanoplates. (Li *et al*. 2015) investigated the wave propagation of FG nanobeams based on the nonlocal strain gradient theory, in which the stress accounts for not only the nonlocal elastic stress field but also the strain gradients stress field. A size-dependent Timoshenko beam model, which accounts for through-thickness power-law variation of a two-constituent functionally graded (FG) material, was derived in the framework of the nonlocal strain gradient theory by (Li *et al*. 2016b). The longitudinal dynamic problem of a size-dependent elasticity rod was formulated by utilizing an integral form of nonlocal strain gradient theory by (Zhu and Li 2017c). In another study, a size-dependent integral elasticity model was developed for a small-scaled rod in tension based on the nonlocal strain gradient theory by (Zhu and Li 2017a). (Karami *et al*. 2017b) investigated the in-plane magnetic field effect on the wave propagation of rectangular FG nanoplates based on a refined plate theory and nonlocal strain gradient theory. Wave analysis of porous FG nanoplates under in-plane magnetic field effect via nonlocal strain gradient theory and second-order shear deformation plate theory were studied by (Karami *et al*. 2018d). A size-dependent Euler–Bernoulli beam model was formulated and devoted to investigating the scaling effect on the post-buckling behaviors of (FG) nanobeams with the von Kármán geometric nonlinearity based on the nonlocal strain gradient theory by (Li and Hu 2017). (Farajpour *et al*. 2016) proposed a higher-order nonlocal strain gradient plate model for buckling of orthotropic nanoplates subjected to thermal effect. Moreover, the effects of various scale parameters together on the buckling behavior of graphene sheets were presented in numerical results. (Ebrahimi and Barati 2017a) studied the hygrothermal effects on vibration characteristics of (FG) viscoelastic nanobeams embedded in viscoelastic foundation based on nonlocal strain gradient elasticity theory. That modeling of nanobeam was carried out via a higher order refined beam theory. The governing equations of nonlocal strain gradient viscoelastic nanobeam were obtained by using Hamilton's principle. More recently, in order to demonstrate the effectiveness of the nonlocal strain gradient theory in nanostructures analysis, (Karami *et al*. 2018b) investigated the wave propagation of graphene via a second-order shear deformation theory in conjunction with nonlocal strain gradient theory. In their analysis, the results have shown good agreement with the experimental data, and in another study, (Karami *et al*. 2018c) studied the hygrothermal wave propagation in viscoelastic graphene under in-plane magnetic field based on nonlocal strain gradient theory. The results for all wave numbers improved by adding an extra nonlocal parameter into nonlocal strain gradient theory.

In this paper, radial vibration and wave propagation of anisotropic nanoparticles are investigated based on nonlocal strain gradient elasticity theory and three dimensional elasticity theory. This comprehensive theory with no approximation in displacements has the ability to study different models such as size-dependent structures, monoclinic and triclinic materials and multi-directional functionally graded materials. Present theory has only two length scale parameters which seem to be accurate and somehow simple for various problems. In order to show the accuracy of present model, our results for the radial vibration of anisotropic nanoparticles are verified with experimental results and great agreement is achieved.

2. Review of nonlocal strain gradient theory

It is well known that conventional nonlocal elasticity considers long range interaction between atoms without considering strain gradient influence. Developed nonlocal strain gradient model (Lim *et al*. 2015) incorporates the nonlocality of stress field as well as strain gradients by assuming the stress field in the following form

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \frac{\partial \sigma_{ij}^{(1)}}{\partial x} \tag{1}$$

here the classical stress $\sigma_{xx}^{(0)}$ and the higher-order stress $\sigma_{xx}^{(1)}$ are related to strain ε_{xx} and strain gradient $\varepsilon_{xx,x}$, respectively and are defined as

$$\sigma_{ij}^{0} = \int_V \alpha_0\left(x', x, e_0 a\right) C_{ijkl} : \varepsilon'_{kl}\left(x'\right) dV \tag{2}$$

$$\sigma_{ij} = \sigma_{ij}^{(0)} - \frac{\partial \sigma_{ij}^{(1)}}{\partial x} \tag{3}$$

where C_{ijkl} are the elastic constants, ε'_{kl} is the nonlocal strain tensor, $\nabla\varepsilon'_{kl,m}$ is the strain gradient tensor, e_0a and e_1a are nonlocal parameters which regards the influence of the nonlocal elastic stress field and l material characteristic parameter (or strain gradient parameter) and introduces the influence of higher order strain gradient stress field. The nonlocal parameters e_0a and e_1a in the above nonlocal functions can be determined by matching the wave dispersion relation from experimental data or atomic lattice dynamics. When the nonlocal functions α_0 (x, x', e_0a) and α_1 (x, x', e_1a) satisfy the developed conditions by Eringen, the constitutive relation can be stated as

$$\begin{aligned} &\left[1-(e_1 a)^2\nabla^2\right]\left[1-(e_0 a)^2\nabla^2\right]\sigma_{ij} = C_{ijkl}\left[1-(e_1 a)^2\nabla^2\right]\varepsilon_{kl} \\ &-C_{ijkl}\, l^2\left[1-(e_0 a)^2\nabla^2\right]\nabla^2\varepsilon_{kl} \end{aligned} \tag{4}$$

The linear nonlocal differential operator which is written as follows is applied to the both sides of Eq. (1), the operator can be defined as

$$L_i = 1-\left(e_i a\right)^2\nabla^2 \qquad \text{for } \mathrm{i} = 0,1 \tag{5}$$

in above relation ∇^2 is the Laplacian operator in spherical coordinate and can be defined as

$$\begin{aligned} \nabla^2 \equiv &\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) \\ &+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \end{aligned} \tag{6}$$

Now considering the terms of order $O(\nabla^2)$ and supposing $e_1 = e_0 = e$, the general constitutive relation in Eq. (4) can be rewritten as

$$\left[1-(e)^2\nabla^2\right]\sigma_{ij} = C_{ijkl}\left[1-l^2\nabla^2\right]\varepsilon_{kl} \tag{7}$$

3. Fundamental equations

According to 3D elasticity theory, the displacement components in spherical coordinates (r, θ, ϕ) and the time can be expressed by (Sadd 2009)

$$\begin{aligned} u_r &= u_r\left(r,\theta,\phi,t\right) \\ u_\theta &= u_\theta\left(r,\theta,\phi,t\right) \\ u_\phi &= u_\phi\left(r,\theta,\phi,t\right) \end{aligned} \tag{8}$$

Non-zero strains of the suggested model can be expressed as follows

$$\begin{aligned} \varepsilon_{rr} &= \frac{\partial u_r}{\partial r}, \\ \varepsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right), \\ \varepsilon_{\phi\phi} &= \frac{1}{r\sin(\theta)}\left(\frac{\partial u\phi}{\partial\phi} + \sin(\theta)u_r + \cos(\theta)u_\theta\right), \\ \gamma_{r\theta} &= \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right), \\ \gamma_{\theta\phi} &= \frac{1}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial u_\theta}{\partial\phi} + \frac{\partial u_\phi}{\partial\theta} - \cot(\theta)u_\phi\right), \\ \gamma_{r\phi} &= \left(\frac{1}{r\sin(\theta)}\frac{\partial u_r}{\partial\phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \end{aligned} \tag{9}$$

Substituting above strains in Eq. (1), following relations are achieved

$$\left(1-\mu\nabla^2\right)\begin{Bmatrix}\sigma_{rr}\\ \sigma_{\theta\theta}\\ \sigma_{\phi\phi}\\ \sigma_{r\theta}\\ \sigma_{\theta\phi}\\ \sigma_{r\phi}\end{Bmatrix} = \left(1-\eta\nabla^2\right)\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{pmatrix}\begin{Bmatrix}\varepsilon_{rr}\\ \varepsilon_{\theta\theta}\\ \varepsilon_{\phi\phi}\\ \gamma_{r\theta}\\ \gamma_{\theta\phi}\\ \gamma_{r\phi}\end{Bmatrix} \tag{10}$$

where nonlocal parameter $\mu = (e_0a)^2$ and gradient parameter $\eta = l^2$. e_0 is a material constant and a is the internal characteristic length. The value of the coefficient e_0 depends on the crystal structure in lattice dynamics and the nature of the physics under investigation.

The spherical coordinate system is shown in Fig. 1, and in spherical coordinates (r, θ, ϕ) the equations of motion are (Sadd 2009)

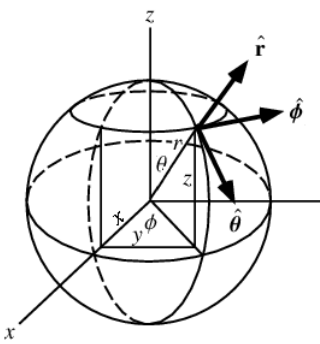

Fig. 1 Spherical coordinate system

$$\begin{aligned}&\frac{\sigma_{rr}}{\partial r}+\frac{1}{r}\frac{\sigma_{r\theta}}{\partial\theta}+\frac{1}{r}\left(2\sigma_{rr}-\sigma_{\theta\theta}-\sigma_{\phi\phi}+\sigma_{r\theta}\cot\theta\right)\\&+\frac{1}{r\sin\theta}\frac{\sigma_{r\phi}}{\partial\phi}+F_r=\rho\frac{\partial^2u_r}{\partial t^2}\end{aligned}\tag{11}$$

$$\begin{aligned}&\frac{\sigma_{r\theta}}{\partial r}+\frac{1}{r}\frac{\sigma_{\theta\theta}}{\partial\theta}+\frac{1}{r}\left(\left(\sigma_{\theta\theta}-\sigma_{\phi\phi}\right)\cot\theta+3\sigma_{r\theta}\right)\\&+\frac{1}{r\sin\theta}\frac{\sigma_{\theta\phi}}{\partial\phi}+F_\theta=\rho\frac{\partial^2u_\theta}{\partial t^2}\end{aligned}\tag{12}$$

$$\begin{aligned}&\frac{\sigma_{r\phi}}{\partial r}+\frac{1}{r}\frac{\sigma_{\theta\phi}}{\partial\theta}+\frac{1}{r}\left(2\sigma_{\theta\phi}\cot\theta+3\sigma_{r\phi}\right)\\&+\frac{1}{r\sin\theta}\frac{\sigma_{\phi\phi}}{\partial\phi}+F_\phi=\rho\frac{\partial^2u_\phi}{\partial t^2}\end{aligned}\tag{13}$$

where σ, u, ρ and (F_r, F_θ, F_ϕ) denote stress, displacement components, density and body forces, respectively in the spherical coordinate.

Generally, in spherical coordinates (r, θ, ϕ) as commonly used in physics: radial distance r, polar angle θ (theta) and azimuthal angle ϕ (phi).

The governing equations of spherical anisotropic nanoparticle in terms of displacements are obtained by inserting Eqs. (7) and (8) into Eqs. (12)-(12) as follows

$$\begin{aligned}&\left[\frac{\partial C_{11}}{\partial r}+\frac{1}{r}\frac{\partial C_{41}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{61}}{\partial\phi}+\frac{1}{r}\left(2C_{11}-C_{21}-C_{31}+C_{41}\cot\theta\right)\right]\left[X_{rr}-\eta Y_{rr}\right]\\&+[C_{11}]\left[\frac{\partial}{\partial r}\left(X_{rr}-\eta Y_{rr}\right)\right]+\left[\frac{1}{r}C_{41}\right]\left[\frac{\partial}{\partial\theta}\left(X_{rr}-\eta Y_{rr}\right)\right]+\left[\frac{1}{r\sin\theta}C_{61}\right]\left[\frac{\partial}{\partial\phi}\left(X_{rr}-\eta Y_{rr}\right)\right]\\&+\left[\frac{\partial C_{12}}{\partial r}+\frac{1}{r}\frac{\partial C_{42}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{62}}{\partial\phi}+\frac{1}{r}\left(2C_{12}-C_{22}-C_{32}+C_{42}\cot\theta\right)\right]\left[X_{\theta\theta}-\eta Y_{\theta\theta}\right]\\&+[C_{12}]\left[\frac{\partial}{\partial r}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]+\left[\frac{1}{r}C_{42}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]+\left[\frac{1}{r\sin\theta}C_{62}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]\\&+\left[\frac{\partial C_{13}}{\partial r}+\frac{1}{r}\frac{\partial C_{43}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{63}}{\partial\phi}+\frac{1}{r}\left(C_{13}-C_{23}-C_{33}+C_{43}\cot\theta\right)\right]\left[X_{\phi\phi}-\eta Y_{\phi\phi}\right]\\&+[C_{13}]\left[\frac{\partial}{\partial r}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]+\left[\frac{1}{r}C_{43}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{63}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]\\&+\left[\frac{\partial C_{14}}{\partial r}+\frac{1}{r}\frac{\partial C_{44}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{64}}{\partial\phi}+\frac{1}{r}\left(C_{14}-C_{24}-C_{34}+C_{44}\cot\theta\right)\right]\left[X_{r\theta}-\eta Y_{r\theta}\right]\\&+[C_{14}]\left[\frac{\partial}{\partial r}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]+\left[\frac{1}{r}C_{44}\right]\left[\frac{\partial}{\partial\theta}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]+\left[\frac{1}{r\sin\theta}C_{64}\right]\left[\frac{\partial}{\partial\phi}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]\\&+\left[\frac{\partial C_{15}}{\partial r}+\frac{1}{r}\frac{\partial C_{45}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{65}}{\partial\phi}+\frac{1}{r}\left(C_{15}-C_{25}-C_{35}+C_{45}\cot\theta\right)\right]\left[X_{\theta\phi}-\eta Y_{\theta\phi}\right]\\&+[C_{15}]\left[\frac{\partial}{\partial r}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]+\left[\frac{1}{r}C_{45}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{65}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]\\&+\left[\frac{\partial C_{16}}{\partial r}+\frac{1}{r}\frac{\partial C_{46}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{66}}{\partial\phi}+\frac{1}{r}\left(C_{16}-C_{26}-C_{36}+C_{46}\cot\theta\right)\right]\left[X_{r\phi}-\eta Y_{r\phi}\right]\\&+[C_{16}]\left[\frac{\partial}{\partial r}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]+\left[\frac{1}{r}C_{46}\right]\left[\frac{\partial}{\partial\theta}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{66}\right]\left[\frac{\partial}{\partial\phi}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]\\&=\rho\frac{\partial^2u_r}{\partial t^2}-F_r-\mu\nabla^2\left(\rho\frac{\partial^2u_r}{\partial t^2}-F_r\right)\end{aligned}\tag{14}$$

$$\begin{aligned}&\left[\frac{\partial C_{41}}{\partial r}+\frac{1}{r}\frac{\partial C_{21}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{51}}{\partial\phi}+\frac{1}{r}\left(\left(C_{21}-C_{31}\right)\cot\theta+3C_{41}\right)\right]\left[X_{rr}-\eta Y_{rr}\right]\\&+[C_{41}]\left[\frac{\partial}{\partial r}\left(X_{rr}-\eta Y_{rr}\right)\right]+\left[\frac{1}{r}C_{21}\right]\left[\frac{\partial}{\partial\theta}\left(X_{rr}-\eta Y_{rr}\right)\right]+\left[\frac{1}{r\sin\theta}C_{51}\right]\left[\frac{\partial}{\partial\phi}\left(X_{rr}-\eta Y_{rr}\right)\right]\\&+\left[\frac{\partial C_{42}}{\partial r}+\frac{1}{r}\frac{\partial C_{22}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{52}}{\partial\phi}+\frac{1}{r}\left(\left(C_{22}-C_{32}\right)\cot\theta+3C_{42}\right)\right]\left[X_{\theta\theta}-\eta Y_{\theta\theta}\right]\\&+[C_{42}]\left[\frac{\partial}{\partial r}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]+\left[\frac{1}{r}C_{22}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]+\left[\frac{1}{r\sin\theta}C_{52}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]\\&+\left[\frac{\partial C_{43}}{\partial r}+\frac{1}{r}\frac{\partial C_{23}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{53}}{\partial\phi}+\frac{1}{r}\left(\left(C_{23}-C_{33}\right)\cot\theta+3C_{43}\right)\right]\left[X_{\phi\phi}-\eta Y_{\phi\phi}\right]\\&+[C_{43}]\left[\frac{\partial}{\partial r}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]+\left[\frac{1}{r}C_{23}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{53}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]\\&+\left[\frac{\partial C_{44}}{\partial r}+\frac{1}{r}\frac{\partial C_{24}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{54}}{\partial\phi}+\frac{1}{r}\left(\left(C_{24}-C_{34}\right)\cot\theta+3C_{44}\right)\right]\left[X_{r\theta}-\eta Y_{r\theta}\right]\\&+[C_{44}]\left[\frac{\partial}{\partial r}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]+\left[\frac{1}{r}C_{24}\right]\left[\frac{\partial}{\partial\theta}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]+\left[\frac{1}{r\sin\theta}C_{54}\right]\left[\frac{\partial}{\partial\phi}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]\\&+\left[\frac{\partial C_{45}}{\partial r}+\frac{1}{r}\frac{\partial C_{25}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{55}}{\partial\phi}+\frac{1}{r}\left(\left(C_{25}-C_{35}\right)\cot\theta+3C_{45}\right)\right]\left[X_{\theta\phi}-\eta Y_{\theta\phi}\right]\\&+[C_{45}]\left[\frac{\partial}{\partial r}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]+\left[\frac{1}{r}C_{25}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{55}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]\\&+\left[\frac{\partial C_{46}}{\partial r}+\frac{1}{r}\frac{\partial C_{26}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{56}}{\partial\phi}+\frac{1}{r}\left(\left(C_{26}-C_{36}\right)\cot\theta+3C_{46}\right)\right]\left[X_{r\phi}-\eta Y_{r\phi}\right]\\&+[C_{46}]\left[\frac{\partial}{\partial r}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]+\left[\frac{1}{r}C_{26}\right]\left[\frac{\partial}{\partial\theta}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{56}\right]\left[\frac{\partial}{\partial\phi}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]\\&=\rho\frac{\partial^2u_\theta}{\partial t^2}-F_\theta-\mu\nabla^2\left(\rho\frac{\partial^2u_\theta}{\partial t^2}-F_\theta\right)\end{aligned}\tag{15}$$

$$\begin{aligned}&\left[\frac{\partial C_{61}}{\partial r}+\frac{1}{r}\frac{\partial C_{51}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{31}}{\partial\phi}+\frac{1}{r}\left(2C_{51}\cot\theta+3C_{61}\right)\right]\left[X_{rr}-\eta Y_{rr}\right]\\&+[C_{61}]\left[\frac{\partial}{\partial r}\left(X_{rr}-\eta Y_{rr}\right)\right]+\left[\frac{1}{r}C_{51}\right]\left[\frac{\partial}{\partial\theta}\left(X_{rr}-\eta Y_{rr}\right)\right]+\left[\frac{1}{r\sin\theta}C_{31}\right]\left[\frac{\partial}{\partial\phi}\left(X_{rr}-\eta Y_{rr}\right)\right]\\&+\left[\frac{\partial C_{62}}{\partial r}+\frac{1}{r}\frac{\partial C_{52}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{32}}{\partial\phi}+\frac{1}{r}\left(2C_{52}\cot\theta+3C_{62}\right)\right]\left[X_{\theta\theta}-\eta Y_{\theta\theta}\right]\\&+[C_{62}]\left[\frac{\partial}{\partial r}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]+\left[\frac{1}{r}C_{52}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]+\left[\frac{1}{r\sin\theta}C_{32}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\theta\theta}-\eta Y_{\theta\theta}\right)\right]\\&+\left[\frac{\partial C_{63}}{\partial r}+\frac{1}{r}\frac{\partial C_{53}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{33}}{\partial\phi}+\frac{1}{r}\left(2C_{53}\cot\theta+3C_{63}\right)\right]\left[X_{\phi\phi}-\eta Y_{\phi\phi}\right]\\&+[C_{63}]\left[\frac{\partial}{\partial r}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]+\left[\frac{1}{r}C_{53}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{33}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\phi\phi}-\eta Y_{\phi\phi}\right)\right]\\&+\left[\frac{\partial C_{64}}{\partial r}+\frac{1}{r}\frac{\partial C_{54}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{34}}{\partial\phi}+\frac{1}{r}\left(2C_{54}\cot\theta+3C_{64}\right)\right]\left[X_{r\theta}-\eta Y_{r\theta}\right]\\&+[C_{64}]\left[\frac{\partial}{\partial r}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]+\left[\frac{1}{r}C_{54}\right]\left[\frac{\partial}{\partial\theta}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]+\left[\frac{1}{r\sin\theta}C_{34}\right]\left[\frac{\partial}{\partial\phi}\left(X_{r\theta}-\eta Y_{r\theta}\right)\right]\\&+\left[\frac{\partial C_{65}}{\partial r}+\frac{1}{r}\frac{\partial C_{55}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{35}}{\partial\phi}+\frac{1}{r}\left(2C_{55}\cot\theta+3C_{65}\right)\right]\left[X_{\theta\phi}-\eta Y_{\theta\phi}\right]\\&+[C_{65}]\left[\frac{\partial}{\partial r}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]+\left[\frac{1}{r}C_{55}\right]\left[\frac{\partial}{\partial\theta}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{35}\right]\left[\frac{\partial}{\partial\phi}\left(X_{\theta\phi}-\eta Y_{\theta\phi}\right)\right]\\&+\left[\frac{\partial C_{66}}{\partial r}+\frac{1}{r}\frac{\partial C_{56}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial C_{36}}{\partial\phi}+\frac{1}{r}\left(2C_{56}\cot\theta+3C_{66}\right)\right]\left[X_{r\phi}-\eta Y_{r\phi}\right]\\&+[C_{66}]\left[\frac{\partial}{\partial r}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]+\left[\frac{1}{r}C_{56}\right]\left[\frac{\partial}{\partial\theta}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]+\left[\frac{1}{r\sin\theta}C_{36}\right]\left[\frac{\partial}{\partial\phi}\left(X_{r\phi}-\eta Y_{r\phi}\right)\right]\\&=\rho\frac{\partial^2u_\phi}{\partial t^2}-F_\phi-\mu\nabla^2\left(\rho\frac{\partial^2u_\phi}{\partial t^2}-F_\phi\right)\end{aligned}\tag{16}$$

where X_{ij} and Y_{ij} are defined in the appendix.

The relations obtained above can be used for modeling of all kind of anisotropic structures in spherical coordinate such as trigonal, monoclinic, hexagonal and triclinic materials. Moreover, these equations can support multi-directional functionally graded materials including size effects.

In the next section, after presenting a verification with experimental results for spherical nanoparticles, numerical results for the radial vibration and wave propagation of more than 10 different anisotropic nanoparticles are shown including nonlocal and gradient parameters.

4. Results and discussions

One of the important applications in the wave propagation analysis is the calculation of elastic constants

Table 1 Material properties of different anisotropic nanoparticles (Teodosiu 1982, Ghavanloo and Fazelzadeh 2013), ($C_{13} = C_{12}$, $C_{33} = C_{22}$)

Material	Chemical formula	Density (kg/m^3)	Elastic constants (GPa)			
			C_{11}	C_{12}	C_{23}	C_{22}
Cubic crystallinity						
Aluminium	Al	2700	106.43	60.35	60.35	106.43
Argon	Ar	1771	5.29	1.35	1.35	5.29
Carbon	C	3515	1079	124	124	1079
Germanium	Ge	5313	128.35	0.4823	0.4823	128.35
Gold	Au	19283	192.44	162.98	162.98	192.44
Silicon	Si	2331	165.78	63.94	63.94	165.78
Silver	Ag	10500	123.99	93.67	93.67	123.99
Thorium	Th	11700	75.30	48.90	48.90	75.30
Hexagonal crystallinity						
Cadmium selenide	CdSe	5655	83.55	39.30	45.16	70.46
Titanium	Ti	4506	52.80	29.00	35.40	40.80
Zinc sulfide	$\alpha-$ZnS	4090	139.60	45.50	58.50	123.40
Tetragonal crystallinity						
Rutile	TiO_2	4260	483.95	149.57	177.96	271.43
Tin	$\beta-$Sn	7265	88.00	37.40	58.50	72.00
Trigonal crystallinity						
Hematite	Fe_2O_3	5240	227.30	15.42	54.64	242.43

for materials. In this paper, we can see the connection between the wave propagation and the elastic constants. In order to provide results for the wave propagation problem, the elastic constants are considered as input information, and the frequencies, as well as phase velocities, are outputs. Also, MATLAB software is used to calculate the outputs. It is worth noting that in the results, the natural frequencies of anisotropic nanoparticles are given.

This section is devoted to explore the influence of nonlocal parameter on the radial vibrational, and strain gradient parameter on the wave propagation characteristic of anisotropic nanoparticles. In order to simplify the formulations, the components of displacements in the spherical coordinates system can be defined as $u = u(r, t)$ which seems to be good approximation for the following problems. The accuracy of this approximation will be discussed in the following section with experimental results. It is mentioned that the elastic constants of more than 10 different anisotropic nanoparticles such as (Aluminium, Carbon, Thorium, Tin, Titanium, Zinc sulfide, Argon, Cadmium selenide, Germanium, Gold, Rutile, Hematite, Silicon, Silver) used in present paper are given in Table 1 which can be find here (Teodosiu 1982, Ghavanloo and Fazelzadeh 2013).

4.1 Radial vibration of spherical nanoparticles with considering nonlocal parameter

The radial vibration of several anisotropic spherical nanoparticles in radial direction are studied in this section. The nanoparticles are modeled as a solid sphere with radial deformations as mentioned in the previous section. Our formulations are also simplified by ignoring the gradient parameter and considering just the nonlocality. To solve the radial vibration of nanoparticles, it is assumed a harmonic variation for the displacement with respect to the time, which is common in many cases, as follow

$$u(r,t) = U(r)\exp(i\omega t) \tag{17}$$

here ω denotes the angular frequency defined by $\omega = 2\pi f$. Considering Eq. (17), the governing equations obtained in the prior section and holding on mentioned above approximations, we achieve to the following equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \left(\frac{B_1}{r^2} + B_2^2\right)U = 0 \tag{18}$$

in which

$$B_1 = \frac{c_{12} + c_{13} - c_{22} - c_{33} - 2c_{23}}{c_{11} - \rho\mu^2\omega^2} \tag{19}$$

$$B_2^2 = \frac{\rho\omega^2}{c_{11} - \rho\mu^2\omega^2} \tag{20}$$

Eq. (18) shows a Bessel equation which the general solution of that defined by

$$U_r = \frac{1}{\sqrt{r}}\left[A_1 J_v\left(B_2 r\right) + A_2 Y_v\left(B_2 r\right)\right] \tag{21}$$

Table 2 Fundamental radial frequencies for the anisotropic spherical nanoparticles

Material	Diameter (nm)	ω (cm^{-1})			Reference
		Present		Previous studies	
		e_0a = 0.0 (nm)	e_0a = 0.1 (nm)		
Germanium (Ge)	3.38	40.3062	40.0940	40.25	Experimental [1]
Gold (Au)	5.8	17.5555	17.5024	18.39	MD simulation [2]
	11.5	8.8541	8.8450	9.37	MD simulation [2]
	20.2	5.0407	5.0391	5.37	MD simulation [2]
Silicon (Si)	6.8	34.5213	34.4152	34.90	Experimental [3]
Silver (Ag)	3.0	36.0835	35.7652	34.00	Experimental [4]
	3.4	31.8384	31.5731	34.00	Experimental [4]
	4.0	27.0627	26.9035	27.60	Experimental [4]
	9.8	11.0460	11.0361	11.00	Experimental [5]
Cadmium selenide (Cdse)	4.4	24.5273	24.4742	28.40	Experimental [6]

(1) Ref: (Combe *et al*. 2007), (2) Ref: (Ng and Chang 2011), (3) Ref: (Saviot *et al*. 2004),
(4) Ref: (Mankad *et al*. 2012), (5) Ref: (Portales *et al*. 2001), (6) Ref: (Gupta *et al*. 2009)

in which A_1 and A_2 are unknown constants, $v = 0.5\sqrt{1-4B_1}$, and J_v and Y_v denotes Bessel functions of first and second kinds of order v, respectively. Note that displacement need to remain finite at the center of nanoparticle, hence we set $A_2 = 0$ in order to remove the infinite value of $Y_v(B_2 r)/\sqrt{r}$ when $r = 0$. Nevertheless, the resultant equation is

$$U_r = A_1 \frac{J_v\left(B_2 r\right)}{\sqrt{r}} \tag{22}$$

For the case of stress-free boundary condition, $\sigma_{rr} = 0$ at external radius R and therefore

$$\left.\frac{\partial U}{\partial r}\right|_{r=R} = -\frac{c_{12}+c_{13}}{c_{11}}\frac{U\left(R\right)}{R} \tag{23}$$

Substituting resultant equation (Eq. (22)) into stress-free boundary condition (Eq. (23)), the equation of frequency is obtained as follow

$$J_v\left(\xi\right)\left[v-\frac{1}{2}+\frac{c_{12}+c_{13}}{c_{11}}\right]-\xi J_{v+1}\left(\xi\right)=0 \tag{24}$$

where $\xi = B_2 r$.

Solving Eq. (24), leads to natural frequencies of the nanoparticle. It is important to note that in natural frequency the lowest frequency belongs to the breathing mode which is related to the characterization of the nanoparticles due to the Raman spectroscopy.

To prove the accuracy of the suggested model, by omitting the gradient parameter (η), the numerical results are verified with some reported experimental results for spherical nanoparticles with cubic, hexagonal, tetragonal and trigonal symmetric in Table 2. In this table, the fundamental radial frequencies for five different nanoparticles from low-frequency Raman spectra (Combe *et al*. 2007, Saviot *et al*. 2004, Mankad *et al*. 2012, Portales *et al*. 2001, Gupta *et al*. 2009) and molecular dynamics (MD) simulation (Ng and Chang 2011) are tabulated. It can be seen that the results obtained by the existing methodology are in great agreement with the results presented in the literature. From this table it is also found that the nonlocality doesn't have any important effect in this example and can be neglected. After confirming the existing solution, the present method is used to study the different anisotropic nanoparticles.

In Table 3, radial vibration of nanoparticles with cubic crystallinity is investigated at d = 10 nm. It is observable that as nonlocal parameter increases, the value of frequency

Table 3 Vibration of four different nanoparticles versus variations of nonlocal parameters

Material	Radial frequencies (THz)				
	e_0a = 0.05 (nm)	e_0a = 0.15 (nm)	e_0a = 0.2 (nm)	e_0a = 0.25 (nm)	e_0a = 0.3 (nm)
	Cubic crystallinity				
Aluminium	1.2239	1.2111	1.2016	1.1889	1.1761
Carbon	1.2680	1.2648	1.2616	1.2584	1.2552
Gold	0.3070	0.3056	0.3054	0.3038	0.3022
Silver	0.3234	0.3155	0.3075	0.2996	0.2992

Table 4 The effects of gradient parameter and wave number on the wave frequency in spherical nanoparticles ($\omega = \omega \times 10^{13}$)

Material	Wave frequency (Rad/Sec)					
	$k_r = 5\times10^8$			$k_r = 1\times10^9$		
	$l = 0.0$	$l = 0.5$	$l = 1.0$	$l = 0.0$	$l = 0.5$	$l = 1.0$
	Cubic crystallinity					
Aluminium	0.3900	0.4049	0.4473	0.6738	0.7624	0.9848
Argon	0.1074	0.1112	0.1222	0.1855	0.2093	0.2693
Carbon	1.0884	1.1261	1.2349	1.8803	2.1186	2.7218
Germanium	0.3053	0.3165	0.3486	0.5275	0.5958	0.7678
Gold	0.1962	0.2042	0.2266	0.3390	0.3846	0.4985
Silicon	0.5239	0.5432	0.5983	0.9050	1.0223	1.3177
Silver	0.2135	0.2220	0.2459	0.3688	0.4180	0.5412
Thorium	0.1576	0.1637	0.1811	0.2723	0.3083	0.3986
	Hexagonal crystallinity					
Cadmium selenide	0.2265	0.2360	0.2627	0.3844	0.4420	0.5830
Titanium	0.1972	0.2060	0.2306	0.3307	0.3842	0.5128
Zinc sulfide	0.3505	0.3642	0.4032	0.5964	0.6813	0.8915
	Tetragonal crystallinity					
Rutile	0.5597	0.5885	0.6679	0.9105	1.0921	1.5118
Tin	0.2514	0.2584	0.2789	0.4382	0.4802	0.5914
	Trigonal crystallinity					
Hematite	0.4253	0.4388	0.4782	0.7312	0.8180	1.0401

Table 5 The effects of gradient parameter and wave number on the phase velocity in spherical nanoparticles ($\omega / k_r = \omega / k_r \times 10^4$)

Material	Phase velocity					
	$k_r = 5\times10^8$			$k_r = 1\times10^9$		
	$l = 0.0$	$l = 0.5$	$l = 1.0$	$l = 0.0$	$l = 0.5$	$l = 1.0$
	Cubic crystallinity					
Aluminium	0.7800	0.8098	0.8946	0.6738	0.7624	0.9848
Argon	0.2148	0.2224	0.2444	0.1855	0.2093	0.2693
Carbon	2.1768	2.2522	2.4698	1.8803	2.1186	2.7218
Germanium	0.6106	0.6330	0.6972	0.5275	0.5958	0.7678
Gold	0.3924	0.4084	0.4532	0.3390	0.3846	0.4985
Silicon	1.0478	1.0864	1.1966	0.9050	1.0223	1.3177
Silver	0.4270	0.4440	0.4918	0.3688	0.4180	0.5412
Thorium	0.3152	0.3274	0.3622	0.2723	0.3083	0.3986
	Hexagonal crystallinity					
Cadmium selenide	0.4530	0.4720	0.5254	0.3844	0.4420	0.5830
Titanium	0.3944	0.4120	0.4612	0.3307	0.3842	0.5128
Zinc sulfide	0.7010	0.7284	0.8064	0.5964	0.6813	0.8915
	Tetragonal crystallinity					
Rutile	1.1194	1.1770	1.3358	0.9105	1.0921	1.5118
Tin	0.5028	0.5168	0.5578	0.4382	0.4802	0.5914
	Trigonal crystallinity					
Hematite	0.8505	0.8777	0.9563	0.7312	0.8180	1.0401

reduces. This may occur according to the reduction of structural rigidity of particle because of the surface compression generated by the nonlocal interactions of atoms.

Table 6 The effects of spherical nanoparticles diameter and wave number on the wave frequency ($\omega = \omega \times 10^{13}$)

Material	Wave frequency (Rad/Sec)					
	$k_r = 5\times10^8$			$k_r = 1\times10^9$		
	d = 5.0	d = 15.0	d = 20.0	d = 5.0	d = 15.0	d = 20.0
Cubic crystallinity						
Aluminium	0.6030	0.4010	0.3812	1.1800	0.9345	0.9149
Argon	0.1634	0.1020	0.1046	0.3198	0.2563	0.2513
Carbon	1.6453	1.1119	1.0593	3.2195	2.5945	2.5453
Germanium	0.4676	0.3131	0.2979	0.9151	0.7280	0.7153
Gold	0.3076	0.2026	0.1923	0.6021	0.4717	0.4612
Silicon	0.8027	0.5373	0.5112	1.5708	1.2526	1.2274
Silver	0.3331	0.2201	0.2090	0.6520	0.5125	0.5014
Thorium	0.2446	0.1622	0.1542	0.4788	0.3780	0.3699
Hexagonal crystallinity						
Cadmium selenide	0.3559	0.2337	0.2210	0.7025	0.5515	0.5391
Titanium	0.3146	0.2040	0.1921	0.6221	0.4835	0.4720
Zinc sulfide	0.5449	0.3597	0.3407	1.0686	0.8451	0.8270
Tetragonal crystallinity						
Rutile	0.9070	0.5850	0.5461	1.8417	1.4201	1.3835
Tin	0.3802	0.2514	0.2401	0.7030	0.5641	0.5536
Trigonal crystallinity						
Hematite	0.6445	0.4296	0.4090	1.2318	0.9914	0.9726

Table 7 The effects of geometrical dimensions and wave number on the phase velocities ($\omega / k_r = \omega / k_r \times 10^4$)

Material	Phase velocity					
	$k_r = 5\times10^8$			$k_r = 1\times10^9$		
	d = 5.0	d = 15.0	d = 20.0	d = 5.0	d = 15.0	d = 20.0
Cubic crystallinity						
Aluminium	1.2060	0.8020	0.7624	1.1800	0.9345	0.9149
Argon	0.3268	0.2040	0.2092	0.3198	0.2563	0.2513
Carbon	3.2906	2.2238	2.1186	3.2195	2.5945	2.5453
Germanium	0.9352	0.6262	0.5958	0.9151	0.7280	0.7153
Gold	0.6152	0.4052	0.3846	0.6021	0.4717	0.4612
Silicon	1.6054	1.0746	1.0224	1.5708	1.2526	1.2274
Silver	0.6662	0.4402	0.4180	0.6520	0.5125	0.5014
Thorium	0.4892	0.3244	0.3084	0.4788	0.3780	0.3699
Hexagonal crystallinity						
Cadmium selenide	0.7118	0.4674	0.4420	0.7025	0.5515	0.5391
Titanium	0.6292	0.4080	0.3842	0.6221	0.4835	0.4720
Zinc sulfide	1.0898	0.7194	0.6814	1.0686	0.8451	0.8270
Tetragonal crystallinity						
Rutile	1.8140	1.1700	1.0922	1.8417	1.4201	1.3835
Tin	0.7604	0.5028	0.4802	0.7030	0.5641	0.5536
Trigonal crystallinity						
Hematite	1.2890	0.8592	0.8180	1.2318	0.9914	0.9726

4.2 Wave propagation in spherical nanoparticles with considering gradient parameter

In this section, wave propagation analysis of different anisotropic nanoparticles in spherical coordinates is investigated. Approximations used in above section are also considered with one exception. To capture the small-scale effects, the gradient parameter is included and the nonlocality is ignored. For studying wave propagation, it is assumed that the waves are not reached the boundary conditions, well-known as bulk waves with the application in non-destructive tests. So following example will be discussed without considering the boundary conditions (simply supported, free, clamped, etc.) similar to many other studies on macro and nanostructures. For this purpose, the displacements in the radial direction are assumed as follow

$$u(r,t) = A\exp\left[i\left(k_r r - \omega t\right)\right] \tag{25}$$

where A is the coefficients of wave amplitude, k_r is the wave numbers of wave propagation along radial direction, and ω is the frequency. Substituting Eq. (25) into governing Eqs. (14)-(16) with considering our approximations, including one length scale parameter (gradient parameter), a closed-form solution for the frequencies versus wave numbers, known as dispersion relation, are achieved in the following form

$$\omega = \sqrt{\frac{\psi_1 R^2 k_r^4 - i\psi_2 R k_r^3 - \left(c_{22} * R^2 + \psi_3\right)k_r^2 + i\psi_4 R k_r + \psi_5}{-\rho R^2}} \tag{26}$$

where

$$\begin{aligned} &\psi_1 = -\eta c_{11}, \psi_2 = -4\eta c_{11}, \\ &\psi_3 = -2\eta c_{11} + \eta c_{12} + \eta c_{13} + \eta c_{22} + \eta c_{23} + \eta c_{33}, \\ &\psi_4 = 2c_{11}, \psi_5 = c_{12} + c_{13} - c_{22} - 2c_{23} - c_{33}. \end{aligned} \tag{27}$$

Phase velocity can be calculated using the obtained frequency as well as wave number, as follow

$$c_p = \frac{\omega}{k_r} \tag{28}$$

Next, the frequencies and phase velocities for different anisotropic nanoparticles such as Cubic crystallinity, Hexagonal crystallinity, Tetragonal crystallinity, Trigonal crystallinity are provided in Tables 4-7.

In Table 4, the variations of wave frequencies for different anisotropic spherical nanoparticles with respect to various gradient parameters are shown at d = 10 nm. Entirely, the wave frequency rises for the all anisotropic nanoparticles as the gradient parameter and wave number grows. This result shows the differences between the behaviors of models based on nonlocality and strain gradient. From this table, it can be concluded that the wave frequencies of nanoparticles are not sensitive with the variations of gradient parameter at small wave numbers but in higher wave numbers, the variations of wave frequencies are more noticeable. Furthermore, it can be seen that for small wave numbers, the Hematite has the highest wave frequency, independent of the values of gradient parameter but with increasing the wave numbers, this no longer occurs. Furthermore, the Carbon has the highest wave frequency at $k_r = 1\times10^9$ and length scale parameter between 0.0 to 1.0 nm. Additionally, in this investigation, it is observed that the Argon has the lowest wave frequency for different gradient parameters and wave numbers for cubic crystallinity.

In Table 5, the variation of phase velocities for different anisotropic spherical nanoparticles with respect to various gradient parameters are shown at d = 10 nm. The phase velocities have a direct relation with gradient parameter but an inverse relation with wave number. It is noteworthy that for high values of wave numbers, the phase velocity difference becomes more significant for each value of gradient parameter. Also, it can be concluded that the effects of gradient parameter on the increase of the phase velocities are more than the influences of wave number in reduction of them. For example, the phase velocity of Hematite with trigonal crystallinity firstly, with increases the wave number will decrease, but at length scale parameter $l = 1\times10^{-9}$ the phase velocity increases as the wave number grows. Additionally, in this investigation, it is observed that the Carbon has the highest phase velocity in the various gradient parameter and wave number for cubic crystallinity.

The variations of wave frequencies for different anisotropic spherical nanoparticles with respect to variations of diameter are shown at length scale parameter $l = 1\times10^{-9}$ in Table 6. It is seen that for all anisotropic nanoparticles, all the wave frequencies reduce as the diameter of nanoparticle grows. Moreover, it should be noted that this decreasing trend is more obvious in higher wave numbers. Furthermore, it is very important to mention that the effect of nanoparticles diameter changes in different nanostructures are different. Also, it can be concluded that most changes of wave frequencies are occurred when the diameter of Carbon nanoparticle is increasing.

In Table 7, the trend of phase velocities for different anisotropic spherical nanoparticles with respect to variations of geometrical dimension are examined at length scale parameter $l = 1\times10^{-9}$. It is shown that the phase velocities reduce for all of the anisotropic nanoparticles as the diameter of nanoparticle and wave number grow. Also, it can be concluded that the effects of increasing the diameter in decreasing the phase velocities are more obvious at the higher wave numbers. Furthermore, again it is obtained that the lowest and highest value of phase velocities are related to cubic crystallinity.

5. Conclusions

This paper was concerned with the modeling, vibration, and wave propagation analysis of anisotropic nanoparticles according to the nonlocal strain gradient theory. The proposed generalized theory introduced two scale parameters for the prediction of mechanics of nanoparticles much accurately. The formulation of spherical nanoparticle was based on a three dimensional elasticity theory. To verify our model, our results for the radial vibration of spherical nanoparticles were compared with experimental results and great agreement was achieved. Several numerical examples with considering different parameters such as geometrical dimension and material properties were discussed on the radial vibration and wave propagation of spherical nanoparticles. From the best knowledge of authors, it was the first time that three-dimensional elasticity theory and nonlocal strain gradient theory were used together with no approximation to derive the governing equations in spherical coordinate. According to the numerical results of the described study, the following conclusions are notable,

- The magnitude of radial frequencies reduces by increasing nonlocal parameter, especially at lower values of the radius.
- It is indicated that with an increase of strain gradient parameters, the anisotropic nanoparticle becomes stiffer and the wave frequency enlarges.
- It is seen that the influence of the radius of nanoparticles on wave characteristics of anisotropic nanoparticles is significant for higher values of wave number.
- The wave number possesses increasing and decreasing effects on the wave frequencies and phase velocities of nanoparticles, respectively.

References

Aifantis, E.C. (1999), "Strain gradient interpretation of size effects", *Int. J. Fract.*, **95**(1-4), 299.

Askes, H. and Aifantis, E.C. (2009), "Gradient elasticity and flexural wave dispersion in carbon nanotubes", *Phys. Rev. B*, **80**(19), 195412.

Askes, H. and Aifantis, E.C. (2011), "Gradient elasticity in statics and dynamics: an overview of formulations, length scale identification procedures, finite element implementations and new results", *Int. J. Solids Struct.*, **48**(13), 1962-1990.

Bağdatlı, S.M. (2015), "Non-linear transverse vibrations of tensioned nanobeams using nonlocal beam theory", *Struct. Eng. Mech., Int. J.*, **55**(2), 281-298.

Benahmed, A., Houari, M.S.A., Benyoucef, S., Belakhdar, K. and Tounsi, A. (2017), "A novel quasi-3D hyperbolic shear deformation theory for functionally graded thick rectangular plates on elastic foundation", *Geomech. Eng., Int. J.*, **12**(1), 9-34.

Bourada, F., Amara, K. and Tounsi, A. (2016), "Buckling analysis of isotropic and orthotropic plates using a novel four variable refined plate theory", *Steel Compos. Struct., Int. J.*, **21**(6), 1287-1306.

Carrera, E. (2002), "Theories and finite elements for multilayered, anisotropic, composite plates and shells", *Arch. Computat. Method. Eng.*, **9**(2), 87-140.

Chaht, F.L., Kaci, A., Houari, M.S.A., Tounsi, A., Bég, O.A. and Mahmoud, S. (2015), "Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect", *Steel Compos. Struct., Int. J.*, **18**(2), 425-442.

Chronopoulos, D. (2017), "Wave steering effects in anisotropic composite structures: Direct calculation of the energy skew angle through a finite element scheme", *Ultrasonics*, **73**, 43-48.

Combe, N., Huntzinger, J.R. and Mlayah, A. (2007), "Vibrations of quantum dots and light scattering properties: Atomistic versus continuous models", *Phys. Rev. B*, **76**(20), 205425.

Ebrahimi, F. and Barati, M.R. (2017a), "Hygrothermal effects on vibration characteristics of viscoelastic FG nanobeams based on nonlocal strain gradient theory", *Compos. Struct.*, **159**, 433-444.

Ebrahimi, F. and Barati, M.R. (2017b), "Vibration analysis of embedded size dependent FG nanobeams based on third-order shear deformation beam theory", *Struct. Eng. Mech., Int. J.*, **61**(6), 721-736.

Eringen, A.C. (1967), "Theory of micropolar plates", *Zeitschrift für angewandte Mathematik und Physik ZAMP*, **18**(1), 12-30.

Eringen, A.C. and Edelen, D. (1972), "On nonlocal elasticity", *Int. J. Eng. Sci.*, **10**(3), 233-248.

Farajpour, A., Yazdi, M.H., Rastgoo, A. and Mohammadi, M. (2016), "A higher-order nonlocal strain gradient plate model for buckling of orthotropic nanoplates in thermal environment", *Acta Mechanica*, **227**(7), 1849-1867.

Ghavanloo, E. and Fazelzadeh, S.A. (2013), "Radial vibration of free anisotropic nanoparticles based on nonlocal continuum mechanics", *Nanotechnology*, **24**(7), 075702.

Grundmann, M., Sturm, C., Kranert, C., Richter, S., Schmidt-Grund, R., Deparis, C. and Zúñiga-Pérez, J. (2016), "Optically anisotropic media: New approaches to the dielectric function, singular axes, microcavity modes and Raman scattering intensities", *Physica Status Solidi (RRL)-Rapid Research Letters*, **11**(1).

Gupta, S.K., Sahoo, S., Jha, P.K., Arora, A. and Azhniuk, Y. (2009), "Observation of torsional mode in CdS1− xSex nanoparticles in a borosilicate glass", *J. Appl. Phys.*, **106**(2), 024307.

Gurtin, M., Weissmüller, J. and Larche, F. (1998), "A general theory of curved deformable interfaces in solids at equilibrium", *Philosophical Magazine A*, **78**(5), 1093-1109.

Hamidi, A., Houari, M.S.A., Mahmoud, S. and Tounsi, A. (2015), "A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates", *Steel Compos. Struct., Int. J.*, **18**(1), 235-253.

Houari, M.S.A., Tounsi, A., Bessaim, A. and Mahmoud, S. (2016), "A new simple three-unknown sinusoidal shear deformation theory for functionally graded plates", *Steel Compos. Struct., Int. J.*, **22**(2), 257-276.

Jandaghian, A.A. and Rahmani, O. (2017), "Vibration analysis of FG nanobeams based on third-order shear deformation theory under various boundary conditions", *Steel Compos. Struct., Int. J.*, **25**(1), 67-78.

Karami, B. and Janghorban, M. (2016), "Effect of magnetic field on the wave propagation in nanoplates based on strain gradient theory with one parameter and two-variable refined plate theory", *Modern Physics Letters B*, **30**(36), 1650421.

Karami, B., Janghorban, M. and Tounsi, A. (2017a), "Effects of triaxial magnetic field on the anisotropic nanoplates", *Steel Compos. Struct., Int. J.*, **25**(3), 361-374.

Karami, B., Shahsavari, D. and Janghorban, M. (2017b), "Wave propagation analysis in functionally graded (FG) nanoplates under in-plane magnetic field based on nonlocal strain gradient theory and four variable refined plate theory", *Mech. Adv. Mater. Struct.*, 1-11.

Karami, B., Janghorban, M. and Li, L. (2018a), "On guided wave propagation in fully clamped porous functionally graded nanoplates", *Acta Astronautica*, **143**, 380-390.

Karami, B., Shahsavari, D., Janghorban, M. and Li, L. (2018b), "Wave dispersion of mounted graphene with initial stress", *Thin-Wall. Struct.*, **122**, 102-111.

Karami, B., Shahsavari, D. and Li, L. (2018c), "Hygrothermal wave propagation in viscoelastic graphene under in-plane magnetic field based on nonlocal strain gradient theory", *Physica E; Low-dimens. Syst. Nanostruct.*, **97**, 317-327.

Karami, B., Shahsavari, D. and Li, L. (2018d), "Temperature-dependent flexural wave propagation in nanoplate-type porous heterogenous material subjected to in-plane magnetic field", *J. Therm. Stress.*, **41**(4), 483-499.

Karami, B., Shahsavari, D., Li, L., Karami, M. and Janghorban, M. (2018e), "Thermal buckling of embedded sandwich piezoelectric nanoplates with functionally graded core by a nonlocal second-order shear deformation theory", *Proceedings of the Institution of Mechanical Engineers, Part C; Journal of*

Mechanical Engineering Science.

Li, L. and Hu, Y. (2017), "Post-buckling analysis of functionally graded nanobeams incorporating nonlocal stress and microstructure-dependent strain gradient effects", *Int. J. Mech. Sci.*, **120**, 159-170.

Li, L., Hu, Y. and Ling, L. (2015), "Flexural wave propagation in small-scaled functionally graded beams via a nonlocal strain gradient theory", *Compos. Struct.*, **133**, 1079-1092.

Li, L., Hu, Y. and Ling, L. (2016a), "Wave propagation in viscoelastic single-walled carbon nanotubes with surface effect under magnetic field based on nonlocal strain gradient theory", *Physica E; Low-dimens. Syst. Nanostruct.*, **75**, 118-124.

Li, L., Li, X. and Hu, Y. (2016b), "Free vibration analysis of nonlocal strain gradient beams made of functionally graded material", *Int. J. Eng. Sci.*, **102**, 77-92.

Li, X., Li, L., Hu, Y., Ding, Z. and Deng, W. (2017), "Bending, buckling and vibration of axially functionally graded beams based on nonlocal strain gradient theory", *Compos. Struct.*, **165**, 250-265.

Lim, C., Zhang, G. and Reddy, J. (2015), "A higher-order nonlocal elasticity and strain gradient theory and its applications in wave propagation", *J. Mech. Phys. Solids*, **78**, 298-313.

Mankad, V., Mishra, K., Gupta, S.K., Ravindran, T. and Jha, P.K. (2012), "Low frequency Raman scattering from confined acoustic phonons in freestanding silver nanoparticles", *Vibrational Spectroscopy*, **61**, 183-187.

Mehralian, F., Beni, Y.T. and Zeverdejani, M.K. (2017), "Calibration of nonlocal strain gradient shell model for buckling analysis of nanotubes using molecular dynamics simulations", *Physica B; Condensed Matter*.

Mock, A., Korlacki, R., Knight, S. and Schubert, M. (2017), "Anisotropy, phonon modes, and lattice anharmonicity from dielectric function tensor analysis of monoclinic cadmium tungstate", *arXiv preprint arXiv;1701.00813*.

Mousavi, S., Reddy, J. and Romanoff, J. (2016), "Analysis of anisotropic gradient elastic shear deformable plates", *Acta Mechanica*, **227**(12), 3639-3656.

Nami, M.R. and Janghorban, M. (2014), "Resonance behavior of FG rectangular micro/nano plate based on nonlocal elasticity theory and strain gradient theory with one gradient constant", *Compos. Struct.*, **111**, 349-353.

Ng, M.-Y. and Chang, Y.-C. (2011), "Laser-induced breathing modes in metallic nanoparticles: a symmetric molecular dynamics study", *J. Chem. Phys.*, **134**(9), 094116.

Portales, H., Saviot, L., Duval, E., Fujii, M., Hayashi, S., Del Fatti, N. and Vallée, F. (2001), "Resonant Raman scattering by breathing modes of metal nanoparticles", *J. Chem. Phys.*, **115**(8), 3444-3447.

Ruijgrok, P.V., Zijlstra, P., Tchebotareva, A.L. and Orrit, M. (2012), "Damping of acoustic vibrations of single gold nanoparticles optically trapped in water", *Nano Letters*, **12**(2), 1063-1069.

Sadd, M.H. (2009), *Elasticity; Theory, Applications, and Numerics*, Academic Press.

Sahmani, S. and Aghdam, M. (2017), "Nonlinear instability of axially loaded functionally graded multilayer graphene platelet-reinforced nanoshells based on nonlocal strain gradient elasticity theory", *Int. J. Mech. Sci.*, **131**, 95-106.

Saviot, L., Murray, D.B. and De Lucas, M.D.C.M. (2004), "Vibrations of free and embedded anisotropic elastic spheres: Application to low-frequency Raman scattering of silicon nanoparticles in silica", *Phys. Rev. B*, **69**(11), 113402.

Shahsavari, D. and Janghorban, M. (2017), "Bending and shearing responses for dynamic analysis of single-layer graphene sheets under moving load", *J. Brazil. Soc. Mech. Sci. Eng.*, **39**(10), 3849-3861.

Shahsavari, D., Karami, B., Janghorban, M. and Li, L. (2017), "Dynamic characteristics of viscoelastic nanoplates under moving load embedded within visco-Pasternak substrate and hygrothermal environment", *Mater. Res. Express*, **4**(8), 085013.

Shahsavari, D., Karami, B. and Mansouri, S. (2018a), "Shear buckling of single layer graphene sheets in hygrothermal environment resting on elastic foundation based on different nonlocal strain gradient theories", *Eur. J. Mech.-A/Solids*, **67**, 200-214.

Shahsavari, D., Shahsavari, M., Li, L. and Karami, B. (2018b), "A novel quasi-3D hyperbolic theory for free vibration of FG plates with porosities resting on Winkler/Pasternak/Kerr foundation", *Aerosp. Sci. Technol.*, **72**, 134-149.

Shen, J., Li, C., Fan, X. and Jung, C. (2017), "Dynamics of silicon nanobeams with axial motion subjected to transverse and longitudinal loads considering nonlocal and surface effects", *Smart Struct. Syst., Int. J.*, **19**(1), 105-113.

Shukla, A. and Kumar, V. (2011), "Low-frequency Raman scattering from silicon nanostructures", *J. Appl. Phys.*, **110**(6), 064317.

Şimşek, M. (2016), "Nonlinear free vibration of a functionally graded nanobeam using nonlocal strain gradient theory and a novel Hamiltonian approach", *Int. J. Eng. Sci.*, **105**, 12-27.

Sobhy, M. (2017), "Hygro-thermo-mechanical vibration and buckling of exponentially graded nanoplates resting on elastic foundations via nonlocal elasticity theory", *Struct. Eng. Mech., Int. J.*, **63**(3), 401-415.

Srinivas, S. and Rao, A. (1970), "Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates", *Int. J. Solids Struct.res*, **6**(11), 1463-1481.

Teodosiu, C. (1982), *The Elastic Field of Point Defects*, Springer.

Voisin, C., Del Fatti, N., Christofilos, D. and Vallée, F. (2000), "Time-resolved investigation of the vibrational dynamics of metal nanoparticles", *Appl. Surf. Sci.*, **164**(1), 131-139.

Yang, F., Chong, A., Lam, D.C. and Tong, P. (2002), "Couple stress based strain gradient theory for elasticity", *Int. J. Solids Struct.*, **39**(10), 2731-2743.

Zbib, H. and Aifantis, E. (2003), "Size effects and length scales in gradient plasticity and dislocation dynamics", *Scripta Materialia*, **48**(2), 155-160.

Zenkour, A.M. and Abouelregal, A.E. (2015), "Thermoelastic interaction in functionally graded nanobeams subjected to time-dependent heat flux", *Steel Compos. Struct., Int. J.*, **18**(4), 909-924.

Zhu, X. and Li, L. (2017a), "Closed form solution for a nonlocal strain gradient rod in tension", *Int. J. Eng. Sci.*, **119**, 16-28.

Zhu, X. and Li, L. (2017b), "Longitudinal and torsional vibrations of size-dependent rods via nonlocal integral elasticity", *Int. J. Mech. Sci.*, **133**, 639-650.

Zhu, X. and Li, L. (2017c), "On longitudinal dynamics of nanorods", *Int. J. Eng. Sci.*, **120**, 129-145.

Zhu, X. and Li, L. (2017d), "Twisting statics of functionally graded nanotubes using Eringen's nonlocal integral model", *Compos. Struct.*, **178**, 87-96.

Ziane, N., Meftah, S.A., Belhadj, H.A. and Tounsi, A. (2013), "Free vibration analysis of thin and thick-walled FGM box beams", *Int. J. Mech. Sci.*, **66**, 273-282.

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Appendix

According to the complex governing equations for nonlocal strain gradient three-dimensional elasticity theory in spherical coordinate most of the parameters are in the appendix. Here, $X_{ij} = \varepsilon_{ij}$ and $Y_{ij} = \nabla^2\varepsilon_{ij}$, $(i, j = r, \theta, \phi)$.

$$X_{rr} = \frac{\partial u_r}{\partial r} \tag{A1}$$

$$X_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) \tag{A2}$$

$$X_{\phi\phi} = \frac{1}{r\sin(\theta)}\left(\frac{\partial u \phi}{\partial \phi} + \sin(\theta)u_r + \cos(\theta)u_\theta\right) \tag{A3}$$

$$X_{r\theta} = \left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{A4}$$

$$X_{\theta\phi} = \frac{1}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial u_\theta}{\partial \phi} + \frac{\partial u_\phi}{\partial \theta} - \cot(\theta)u_\phi\right) \tag{A5}$$

$$X_{r\phi} = \left(\frac{1}{r\sin(\theta)}\frac{\partial u_r}{\partial \phi} + \frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r}\right) \tag{A6}$$

$$Y_{rr} = \frac{2}{r}\frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^3 u_r}{\partial r^3} + \frac{1}{r^2}\left\{\cot\theta\frac{\partial^2 u_r}{\partial r\partial\theta} + \frac{\partial^3 u_r}{\partial r\partial\theta^2}\right\} + \frac{1}{r^2\sin^2\theta}\frac{\partial^3 u_r}{\partial r\partial\phi^2} \tag{A7}$$

$$Y_{\theta\theta} = \frac{1}{r}\left\{\frac{\partial^3 u_\theta}{\partial r^2\partial\theta} + \frac{\partial^2 u_r}{\partial r^2}\right\} + \frac{1}{r^3}\left\{\cot\theta\left(\frac{\partial^2 u_\theta}{\partial\theta^2} + \frac{\partial u_r}{\partial\theta}\right) + \left(\frac{\partial^3 u_\theta}{\partial\theta^3} + \frac{\partial^2 u_r}{\partial\theta^2}\right)\right\} + \frac{1}{r^3\sin^2\theta}\left\{\frac{\partial^3 u_\theta}{\partial\theta\partial\phi^2} + \frac{\partial^2 u_r}{\partial\phi^2}\right\} \tag{A8}$$

$$\begin{aligned} Y_{\phi\phi} = {} & \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial r^2\partial\phi} + \frac{\partial^2 u_r}{\partial r^2} + \frac{\partial^2 u_\theta}{\partial r^2}\cot\theta\right\} - \frac{1}{r^3}\left\{\cot\theta\left(\frac{1}{\sin\theta}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi} + u_r\cot\theta + \cot\theta\frac{\partial u_\theta}{\partial\theta} - u_\theta + \frac{\partial u_r}{\partial\theta}\right)\right. \\ & \left. + \frac{1+\cot^2\theta}{\sin\theta}\left(\frac{\partial u\phi}{\partial\phi} + u_r\sin\theta + u_\theta\cos\theta\right) + \left(\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\theta^2\partial\phi}2\frac{\partial u_r}{\partial\theta}\cot\theta - u_r + \frac{\partial^2 u_\theta}{\partial\theta^2}\cot\theta + \frac{\partial^2 u_r}{\partial\theta^2} - 2\frac{\partial u_\theta}{\partial\theta} - u_\theta\cot\theta\right)\right\} \\ & + \frac{1}{r^3\sin^2\theta}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\phi^3} + \frac{\partial^2 u_r}{\partial\phi^2} + \frac{\partial^2 u_\theta}{\partial\phi^2}\cot\theta\right\} \end{aligned} \tag{A9}$$

$$\begin{aligned} Y_{r\theta} = {} & \frac{1}{r}\left\{\frac{\partial^3 u_r}{\partial r^2\partial\theta} + 2\frac{\partial^2 u_\theta}{\partial r^2} + r\frac{\partial^3 u_\theta}{\partial r^3} - \frac{\partial^2 u_\theta}{\partial r^2}\right\} + \frac{1}{r^2}\left\{\cot\theta\left(\frac{1}{r}\frac{\partial^2 u_r}{\partial\theta^2} + \frac{\partial^2 u_\theta}{\partial r\partial\theta} - \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) + \left(\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta^3} + \frac{\partial^3 u_\theta}{\partial r\partial\theta^2} - \frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta^2}\right)\right\} \\ & + \frac{1}{r^2\sin^2\theta}\left\{\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta\partial\phi^2} + \frac{\partial^3 u_\theta}{\partial r\partial\phi^2} - \frac{1}{r}\frac{\partial^2 u_\theta}{\partial\phi^2}\right\} \end{aligned} \tag{A10}$$

$$\begin{aligned} Y_{\theta\phi} = {} & \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\theta}{\partial r^2\partial\phi} + \frac{\partial^3 u_\phi}{\partial r^2\partial\theta} - \frac{\partial^2 u_\phi}{\partial r^2}\cot\theta\right\} + \frac{1}{r^3}\left\{\left(\frac{1+\cot^2\theta}{\sin\theta}\right)\frac{\partial u_\phi}{\partial\phi} - \frac{\cot\theta}{\sin\theta}\frac{\partial^2 u_\theta}{\partial\theta\partial\phi} + \frac{1}{\sin\theta}\frac{\partial^3 u_\theta}{\partial\theta^2\partial\phi}\right. \\ & \left. + \left(2+\cot^2\theta\right)\frac{\partial u_\phi}{\partial\theta} + \left(3\cot\theta - \cot^3\theta\right)u_\phi + \frac{\partial^3 u_\phi}{\partial\theta^3}\right\} + \frac{1}{r^3\sin^2\theta}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\theta}{\partial\phi^3} + \frac{\partial^3 u_\phi}{\partial\theta\partial\phi^2} - \frac{\partial^2 u_\phi}{\partial\phi^2}\cot\theta\right\} \end{aligned} \tag{A11}$$

$$\begin{aligned} Y_{r\phi} = {} & \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_r}{\partial r^2\partial\phi} + 2\frac{\partial^2 u_\phi}{\partial r^2} + r\frac{\partial^3 u_\phi}{\partial r^3} - \frac{\partial^2 u_\phi}{\partial r^2}\right\} + \frac{1}{r}\left\{\left(\frac{1+\cot^2\theta}{r\sin\theta}\right)\frac{\partial u_r}{\partial\phi} - \frac{\cos\theta}{r}\frac{\partial^2 u_r}{\partial\theta\partial\phi} + \frac{1}{r\sin\theta}\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}\right. \\ & \left. + \cot\theta\frac{\partial^2 u_r}{\partial r\partial\theta} + \frac{\partial^3 u_\phi}{\partial r\partial\theta^2} - \frac{\cot\theta}{r}\frac{\partial u_\phi}{\partial\theta} - \frac{1}{r}\frac{\partial^2 u_\phi}{\partial\theta^2}\right\} + \frac{1}{r^2\sin^2\theta}\left\{\frac{1}{r\sin\theta}\frac{\partial^3 u_r}{\partial\phi^3} + \frac{\partial^3 u_\phi}{\partial r\partial\phi^2} - \frac{1}{r}\frac{\partial^2 u_\phi}{\partial\phi^2}\right\} \end{aligned} \tag{A12}$$

$$\frac{\partial}{\partial r}(X_{rr}) = \frac{\partial^2 u_r}{\partial r^2} \tag{A13}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{rr}\right)=&-\frac{2}{r^2}\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial^3 u_r}{\partial r^3}+\frac{\partial^4 u_r}{\partial r^4}-\frac{2}{r^3}\left(\cot\theta\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{\partial^3 u_r}{\partial r\partial\theta^2}\right)\\&+\frac{1}{r^2}\left\{\cot\theta\frac{\partial^3 u_r}{\partial r^2\partial\theta}+\frac{\partial^4 u_r}{\partial r^2\partial\theta^2}\right\}-\frac{2}{r^3\sin^2\theta}\frac{\partial^3 u_r}{\partial r\partial\phi^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^4 u_r}{\partial r^2\partial\phi^2}\end{aligned}\tag{A14}$$

$$\frac{\partial}{\partial r}\left(X_{\theta\theta}\right)=-\frac{1}{r^2}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)+\frac{1}{r}\left(\frac{\partial^2 u_\theta}{\partial r\partial\theta}+\frac{\partial u_r}{\partial r}\right)\tag{A15}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{\theta\theta}\right)=&-\frac{1}{r^2}\left\{\frac{\partial^3 u_\theta}{\partial r^2\partial\theta}+\frac{\partial^2 u_r}{\partial r^2}\right\}+\frac{1}{r}\left\{\frac{\partial^4 u_\theta}{\partial r^3\partial\theta}+\frac{\partial^3 u_r}{\partial r^3}\right\}-\frac{3}{r^4}\left\{\cot\theta\left(\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial u_r}{\partial\theta}\right)+\left(\frac{\partial^3 u_\theta}{\partial\theta^3}+\frac{\partial^2 u_r}{\partial\theta^2}\right)\right\}\\&+\frac{1}{r^3}\left\{\cot\theta\left(\frac{\partial^3 u_\theta}{\partial r\partial\theta^2}+\frac{\partial^2 u_r}{\partial r\partial\theta}\right)+\left(\frac{\partial^4 u_\theta}{\partial r\partial\theta^3}+\frac{\partial^3 u_r}{\partial r\partial\theta^2}\right)\right\}-\frac{3}{r^4\sin^2\theta}\left(\frac{\partial^3 u_\theta}{\partial\theta\partial\phi^2}+\frac{\partial^2 u_r}{\partial\phi^2}\right)+\frac{1}{r^3\sin^2\theta}\left(\frac{\partial^4 u_\theta}{\partial r\partial\theta\partial\phi^2}+\frac{\partial^3 u_r}{\partial r\partial\phi^2}\right)\Big\}\end{aligned}\tag{A16}$$

$$\frac{\partial}{\partial r}\left(X_{\phi\phi}\right)=-\frac{1}{r^2\sin\theta}\left(\frac{\partial u_\phi}{\partial\phi}+u_r\sin\theta+u_\theta\cos\theta\right)+\frac{1}{r\sin\theta}\left(\frac{\partial^2 u_\phi}{\partial r\partial\phi}+\frac{\partial u_r}{\partial r}\sin\theta+\frac{\partial u_\theta}{\partial r}\cos\theta\right)\tag{A17}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{\phi\phi}\right)=&-\frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial r^2\partial\phi}+\frac{\partial^2 u_r}{\partial r^2}+\frac{\partial^2 u_\theta}{\partial r^2}\cot\theta\right\}+\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial r^3\partial\phi}+\frac{\partial^3 u_r}{\partial r^3}+\frac{\partial^3 u_\theta}{\partial r^3}\cot\theta\right\}\\&+\frac{3}{r^4}\left\{\cot\theta\left(\frac{1}{\sin\theta}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}+u_r\cot\theta+\frac{\partial u_\theta}{\partial\theta}\cot\theta-u_\theta+\frac{\partial u_r}{\partial\theta}\right)+\frac{1+\cot^2\theta}{\sin\theta}\left(\frac{\partial u_\phi}{\partial\phi}+u_r\sin\theta+u_\theta\cos\theta\right)\right.\\&\left.+\left(\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\theta^2\partial\phi}+2\cot\theta\frac{\partial u_r}{\partial\theta}-u_r+\frac{\partial^2 u_\theta}{\partial\theta^2}\cot\theta+\frac{\partial^2 u_r}{\partial\theta^2}-2\frac{\partial u_\theta}{\partial\theta}-u_\theta\cot\theta\right)\right\}-\frac{1}{r^3}\left\{\cot\theta\left(\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial r\partial\theta\partial\phi}\right.\right.\\&\left.+\frac{\partial u_r}{\partial r}\cot\theta+\frac{\partial^2 u_\theta}{\partial r\partial\theta}\cot\theta-\frac{\partial u_\theta}{\partial r}+\frac{\partial^2 u_r}{\partial r\partial\theta}\right)+\left(\frac{1+\cot^2\theta}{\sin\theta}\right)\left(\frac{\partial^2 u_\phi}{\partial r\partial\phi}+\frac{\partial u_r}{\partial r}\sin\theta+\frac{\partial u_\theta}{\partial r}\cos\theta\right)\\&\left.+\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial r\partial\theta^2\partial\phi}+2\cot\theta\frac{\partial^2 u_r}{\partial r\partial\theta}-\frac{\partial u_r}{\partial r}+\frac{\partial^3 u_\theta}{\partial r\partial\theta^2}\cot\theta+\frac{\partial^3 u_r}{\partial r\partial\theta^2}-2\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{\partial u_\theta}{\partial r}\cot\theta\right\}\\&-\frac{3}{r^4\sin^2\theta}\left(\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\phi^3}+\frac{\partial^2 u_r}{\partial\phi^2}+\frac{\partial^2 u_\theta}{\partial\phi^2}\cot\theta\right)+\frac{1}{r^3\sin^2\theta}\left(\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial r\partial\phi^3}+\frac{\partial^3 u_r}{\partial r\partial\phi^2}+\frac{\partial^3 u_\theta}{\partial r\partial\phi^2}\cot\theta\right)\end{aligned}\tag{A18}$$

$$\frac{\partial}{\partial r}\left(X_{r\theta}\right)=-\frac{1}{r^2}\frac{\partial u_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r^2}u_\theta-\frac{1}{r}\frac{\partial u_\theta}{\partial r}\tag{A19}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{r\theta}\right)=&-\frac{1}{r^2}\left\{\frac{\partial^3 u_r}{\partial r^2\partial\theta}+2\frac{\partial^2 u_\theta}{\partial r^2}+r\frac{\partial^3 u_\theta}{\partial r^3}-\frac{\partial^2 u_\theta}{\partial r^2}\right\}+\frac{1}{r}\left\{\frac{\partial^4 u_r}{\partial r^3\partial\theta}+2\frac{\partial^3 u_\theta}{\partial r^3}+\frac{\partial^3 u_\theta}{\partial r^3}+r\frac{\partial^4 u_\theta}{\partial r^4}-\frac{\partial^3 u_\theta}{\partial r^3}\right\}\\&-\frac{2}{r^3}\left\{\cot\theta\left(\frac{1}{r}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right)+\left(\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta^3}+\frac{\partial^3 u_\theta}{\partial r\partial\theta^2}-\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta^2}\right)\right\}\\&+\frac{1}{r^2}\left\{\cot\theta\left(-\frac{1}{r^2}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{1}{r}\frac{\partial^3 u_r}{\partial r\partial\theta^2}+\frac{\partial^3 u_\theta}{\partial r^2\partial\theta}+\frac{1}{r^2}\frac{\partial u_\theta}{\partial\theta}-\frac{1}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta}\right)-\frac{1}{r^2}\frac{\partial^3 u_r}{\partial\theta^3}+\frac{1}{r}\frac{\partial^4 u_r}{\partial r\partial\theta^3}+\frac{\partial^4 u_\theta}{\partial r^2\partial\theta^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\theta^2}-\frac{1}{r}\frac{\partial^3 u_\theta}{\partial r\partial\theta^2}\right\}\\&-\frac{2}{r^3\sin^2\theta}\left\{\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta\partial\phi^2}+\frac{\partial^3 u_\theta}{\partial r\partial\phi^2}-\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\phi^2}\right\}+\frac{1}{r^2\sin^2\theta}\left\{\frac{1}{r^2}\frac{\partial^3 u_r}{\partial\theta\partial\phi^2}+\frac{1}{r}\frac{\partial^4 u_r}{\partial r\partial\theta\partial\phi^2}+\frac{\partial^4 u_\theta}{\partial r^2\partial\phi^2}+\frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial\phi^2}-\frac{1}{r}\frac{\partial^3 u_\theta}{\partial r\partial\phi^2}\right\}\end{aligned}\tag{A20}$$

$$\frac{\partial}{\partial r}\left(X_{\theta\phi}\right)=-\frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{\partial u_\phi}{\partial\theta}-u_\phi\cot\theta\right\}+\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^2 u_\theta}{\partial r\partial\phi}+\frac{\partial^2 u_\phi}{\partial r\partial\theta}-\frac{\partial u_\phi}{\partial r}\cot\theta\right\}\tag{A21}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{\theta\phi}\right)=&-\frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\theta}{\partial r^2\partial\phi}+\frac{\partial^3 u_\phi}{\partial r^2\partial\theta}-\frac{\partial^2 u_\phi}{\partial r^2}\cot\theta\right\}+\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_\theta}{\partial r^3\partial\phi}+\frac{\partial^4 u_\phi}{\partial r^3\partial\theta}-\frac{\partial^3 u_\phi}{\partial r^3}\cot\theta\right\}\\&-\frac{3}{r^4}\left\{\left(\frac{1+\cot^2\theta}{\sin\theta}\right)\frac{\partial u_\phi}{\partial\phi}-\frac{\cot\theta}{\sin\theta}\frac{\partial^2 u_\theta}{\partial\theta\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^3 u_\theta}{\partial\theta^2\partial\phi}+\left(2+\cot^2\theta\right)\frac{\partial u_\phi}{\partial\theta}+\left(3\cot\theta-\cot^3\theta\right)u_\phi+\frac{\partial^3 u_\phi}{\partial\theta^3}\right\}\\&+\frac{1}{r^3}\left\{\left(\frac{1+\cot^2\theta}{\sin\theta}\right)\frac{\partial^2 u_\phi}{\partial r\partial\phi}-\frac{\cot\theta}{\sin\theta}\frac{\partial^3 u_\theta}{\partial r\partial\theta\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^4 u_\theta}{\partial r\partial\theta^2\partial\phi}+\left(2+\cot^2\theta\right)\frac{\partial^2 u_\phi}{\partial r\partial\theta}+\left(3\cot\theta-\cot^3\theta\right)u_\phi+\frac{\partial^4 u_\phi}{\partial r\partial\theta^3}\right\}\\&-\frac{3}{r^4\sin^2\theta}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_\theta}{\partial\phi^3}+\frac{\partial^3 u_\phi}{\partial\theta\partial\phi^2}-\frac{\partial^2 u_\phi}{\partial\phi^2}\cot\theta\right\}+\frac{1}{r^3\sin^2\theta}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_\theta}{\partial r\partial\phi^3}+\frac{\partial^4 u_\phi}{\partial r\partial\theta\partial\phi^2}-\frac{\partial^3 u_\phi}{\partial r\partial\phi^2}\cot\theta\right\}\end{aligned}\tag{A22}$$

$$\frac{\partial}{\partial r}\left(X_{r\phi}\right)=-\frac{1}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial r\partial\phi}+\frac{\partial^2 u_\phi}{\partial r^2}+\frac{1}{r^2}u_\phi-\frac{1}{r}\frac{\partial u_\phi}{\partial r} \tag{A23}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{r\phi}\right)=&-\frac{1}{r^2}\left\{\frac{1}{\sin\theta}\frac{\partial^3 u_r}{\partial r^2\partial\phi}+\frac{\partial^2 u_\phi}{\partial r^2}+r\frac{\partial^3 u_\phi}{\partial r^3}\right\}+\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_r}{\partial r^3\partial\phi}+2\frac{\partial^3 u_\phi}{\partial r^3}+r\frac{\partial^4 u_\phi}{\partial r^4}\right\}\\
&-\frac{1}{r^2}\left\{\frac{1+\cot^2\theta}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{\cos\theta}{r}\frac{\partial^2 u_r}{\partial\theta\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}+\frac{\partial^2 u_\phi}{\partial r\partial\theta}\cot\theta+\frac{\partial^3 u_\phi}{\partial r\partial\theta^2}-\frac{\cot\theta}{r}\frac{\partial u_\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^2 u_\phi}{\partial\theta^2}\right\}\\
&+\frac{1}{r}\left\{-\frac{1+\cot^2\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\phi}+\frac{1+\cot^2\theta}{r\sin\theta}\frac{\partial^2 u_r}{\partial r\partial\phi}+\frac{\cos\theta}{r^2}\frac{\partial^2 u_r}{\partial\theta\partial\phi}-\frac{\cos\theta}{r}\frac{\partial^3 u_r}{\partial r\partial\theta\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}\right.\\
&\left.+\frac{1}{r\sin\theta}\frac{\partial^4 u_r}{\partial r\partial\theta^2\partial\phi}+\frac{\partial^3 u_r}{\partial r^2\partial\theta}\cot\theta+\frac{\partial^4 u_\phi}{\partial r^2\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_\phi}{\partial\theta}-\frac{\cot\theta}{r}\frac{\partial^2 u_\phi}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial^2 u_\phi}{\partial\theta^2}\;\frac{1}{r}\frac{\partial^3 u_\phi}{\partial r\partial\theta^2}\right\}\end{aligned} \tag{A24}$$

$$\frac{\partial}{\partial\theta}\left(X_{rr}\right)=\frac{\partial^2 u_r}{\partial r\partial\theta} \tag{A25}$$

$$\frac{\partial}{\partial\theta}\left(Y_{rr}\right)=\frac{2}{r}\frac{\partial^3 u_r}{\partial r^2\partial\theta}+\frac{\partial^4 u_r}{\partial r^3\partial\theta}+\frac{1}{r^2}\left(\left(-1-\cot^2\theta\right)\frac{\partial^2 u_r}{\partial r\partial\theta}+\cot\theta\frac{\partial^3 u_r}{\partial r\partial\theta^2}+\frac{\partial^4 u_r}{\partial r\partial\theta^3}\right)-\frac{2\cos\theta}{r^2\sin^3\theta}\frac{\partial^3 u_r}{\partial r\partial\phi^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^4 u_r}{\partial r\partial\theta\partial\phi^2} \tag{A26}$$

$$\frac{\partial}{\partial\theta}\left(X_{\theta\theta}\right)=\frac{1}{r}\left(\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial u_r}{\partial\theta}\right) \tag{A27}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(Y_{\theta\theta}\right)=&\frac{1}{r}\left\{\frac{\partial^4 u_\theta}{\partial r^2\partial\theta^2}+\frac{\partial^3 u_r}{\partial r^2\partial\theta}\right\}+\frac{1}{r^3}\left\{\left(-1-\cot^2\theta\right)\left(\frac{\partial^2 u_\theta}{\partial\theta^2}+\frac{\partial u_r}{\partial\theta}\right)+\cot\theta\left(\frac{\partial^3 u_\theta}{\partial\theta^3}+\frac{\partial^2 u_r}{\partial\theta^2}\right)+\frac{\partial^4 u_\theta}{\partial\theta^4}+\frac{\partial^3 u_r}{\partial\theta^3}\right\}\\
&-\frac{2\cos\theta}{r^3\sin^3\theta}\left(\frac{\partial^3 u_\theta}{\partial\theta\partial\phi^2}+\frac{\partial^2 u_r}{\partial\phi^2}\right)+\frac{1}{r^3\sin^2\theta}\left(\frac{\partial^4 u_\theta}{\partial\theta^2\partial\phi^2}+\frac{\partial^3 u_r}{\partial\theta\partial\phi^2}\right)\end{aligned} \tag{A28}$$

$$\frac{\partial}{\partial\theta}\left(X_{\phi\phi}\right)=-\frac{\cos\theta}{r\sin^2\theta}\left(\frac{\partial u_\phi}{\partial\phi}+u_r\sin\theta+u_\theta\cos\theta\right)+\frac{1}{r\sin\theta}\left(\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}+\frac{\partial u_r}{\partial\theta}\sin\theta+\cos\theta u_r+\frac{\partial u_\theta}{\partial\theta}\cos\theta-\sin\theta u_\theta\right) \tag{A29}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(Y_{\phi\phi}\right)=&\frac{1}{r}\left\{\frac{\partial^3 u_r}{\partial r^2\partial\theta}-\frac{\cos\theta}{\sin^2\theta}\frac{\partial^3 u_\phi}{\partial r^2\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial r^2\partial\theta\partial\phi}+\frac{\partial^3 u_\theta}{\partial r^2\partial\theta}+\frac{\partial^3 u_\theta}{\partial r^2\partial\theta}\cot\theta-\left(1+\cot^2\theta\right)\frac{\partial^2 u_\theta}{\partial r^2}\right\}\\
&-\frac{1}{r^3}\left\{\left(-1-\cot^2\theta\right)\left(\frac{1}{\sin\theta}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}+u_r\cot\theta+\cot\theta\frac{\partial u_\theta}{\partial\theta}-u_\theta+\frac{\partial u_r}{\partial\theta}\right)\right.\\
&+\cot\theta\left(-\frac{\cos\theta}{\sin^2\theta}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\theta^2\partial\phi}+\frac{\partial u_r}{\partial\theta}\cot\theta-\left(1+\cot^2\theta\right)u_r-\left(1+\cot^2\theta\right)\frac{\partial u_\theta}{\partial\theta}+\cot\theta\frac{\partial^2 u_\theta}{\partial\theta^2}-\frac{\partial u_\theta}{\partial\theta}+\frac{\partial^2 u_r}{\partial\theta^2}\right)\\
&-\left(\frac{3\cot\theta}{\sin\theta}+\frac{3\cot^3\theta}{\sin\theta}\right)\left(\frac{\partial u_\phi}{\partial\phi}+u_r\sin\theta+u_\theta\cos\theta\right)+\frac{1+\cot^2\theta}{\sin\theta}\left(\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}+\frac{\partial u_r}{\partial\theta}\sin\theta+u_r\cos\theta+\frac{\partial u_\theta}{\partial\theta}\cos\theta-\sin\theta u_\theta\right)\\
&-\frac{\cos\theta}{\sin^2\theta}\frac{\partial^3 u_\phi}{\partial\theta^2\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial\theta^3\partial\phi}-2\left(1+\cot^2\theta\right)\frac{\partial u_r}{\partial\theta}+2\cot\theta\frac{\partial^2 u_r}{\partial\theta^2}-\frac{\partial u_r}{\partial\theta}+\frac{\partial^3 u_\theta}{\partial\theta^3}\cot\theta-\left(1+\cot^2\theta\right)\frac{\partial^2 u_\theta}{\partial\theta^2}\\
&\left.+\frac{\partial^3 u_r}{\partial\theta^3}-2\frac{\partial^2 u_\theta}{\partial\theta^2}-\frac{\partial u_\theta}{\partial\theta}\cot\theta-\left(1+\cot^2\theta\right)u_\theta\right\}-\frac{2\cos\theta}{r^3\sin^3\theta}\left(\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\phi^3}+\frac{\partial^2 u_r}{\partial\phi^2}+\frac{\partial^2 u_\theta}{\partial\phi^2}\cot\theta\right)\\
&+\frac{1}{r^3\sin^2\theta}\left(-\frac{\cos\theta}{\sin^2\theta}\frac{\partial^3 u_\phi}{\partial\phi^3}+\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial\theta\partial\phi^3}+\frac{\partial^3 u_r}{\partial\theta\partial\phi^2}+\frac{\partial^3 u_\theta}{\partial\theta\partial\phi^2}\cot\theta-\left(1+\cot^2\theta\right)\frac{\partial^2 u_\theta}{\partial\phi^2}\right)\end{aligned} \tag{A30}$$

$$\frac{\partial}{\partial\theta}\left(X_{r\theta}\right)=\frac{1}{r}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} \tag{A31}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(Y_{r\theta}\right)=&\frac{1}{r}\left(\frac{\partial^4 u_r}{\partial r^2\partial\theta^2}+2\frac{\partial^3 u_\theta}{\partial r^2\partial\theta}+r\frac{\partial^4 u_\theta}{\partial r^3\partial\theta}-\frac{\partial^3 u_\theta}{\partial r^2\partial\theta}\right)+\frac{1}{r^2}\left\{\left(-1-\cot^2\theta\right)\left(\frac{1}{r}\frac{\partial^2 u_r}{\partial\theta^2}+\frac{\partial^2 u_\theta}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right).\right.\\
&\left.+\cot\theta\left(\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta^3}+\frac{\partial^3 u_\theta}{\partial r\partial\theta^2}-\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta^2}\right)+\left(\frac{1}{r}\frac{\partial^4 u_r}{\partial\theta^4}\frac{\partial^4 u_\theta}{\partial r\partial\theta^3}-\frac{1}{r}\frac{\partial^3 u_\theta}{\partial\theta^3}\right)\right\}\\
&-\frac{2\cos\theta}{r^2\sin^2\theta}\left(\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta\partial\phi^2}+\frac{\partial^3 u_\theta}{\partial r\partial\phi^2}-\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\phi^2}\right)+\frac{1}{r^2\sin^2\theta}\left\{\frac{1}{r}\frac{\partial^4 u_r}{\partial\theta^2\partial\phi^2}+\frac{\partial^4 u_\theta}{\partial r\partial\theta\partial\phi^2}-\frac{1}{r}\frac{\partial^3 u_\theta}{\partial\theta\partial\phi^2}\right\}\end{aligned} \tag{A32}$$

$$\frac{\partial}{\partial\theta}\left(X_{\theta\phi}\right)=\frac{1}{r}\left(-\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^2 u_\theta}{\partial\theta\partial\phi}+\frac{\partial^2 u_\phi}{\partial\theta^2}-\frac{\partial u_\phi}{\partial\theta}\cot\theta+\left(1+\cot^2\theta\right)u_\phi\right)\tag{A33}$$

$$\frac{\partial}{\partial\theta}\left(X_{r\phi}\right)=-\frac{\cos\theta}{r\sin^2\theta}\frac{\partial u_r}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial\theta\partial\phi}+\frac{\partial^2 u_\phi}{\partial r\partial\theta}-\frac{1}{r}\frac{\partial u_\phi}{\partial\theta}\tag{A34}$$

$$\begin{aligned}\frac{\partial}{\partial r}\left(Y_{rr}\right)=&-\frac{2}{r^2}\frac{\partial^2 u_r}{\partial r^2}+\frac{2}{r}\frac{\partial^3 u_r}{\partial r^3}+\frac{\partial^4 u_r}{\partial r^4}-\frac{2}{r^3}\left(\cot\theta\frac{\partial^2 u_r}{\partial r\partial\theta}+\frac{\partial^3 u_r}{\partial r\partial\theta^2}\right)+\frac{1}{r^2}\left\{\cot\theta\frac{\partial^3 u_r}{\partial r^2\partial\theta}+\frac{\partial^4 u_r}{\partial r^2\partial\theta^2}\right\}\\&-\frac{2}{r^3\sin^2\theta}\frac{\partial^3 u_r}{\partial r\partial\phi^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^4 u_r}{\partial r^2\partial\phi^2}\end{aligned}\tag{A35}$$

$$\begin{aligned}\frac{\partial}{\partial\theta}\left(Y_{r\phi}\right)=&\frac{1}{r}\left\{-\frac{\cos\theta}{\sin^2\theta}\frac{\partial^3 u_r}{\partial r^2\partial\phi}+\frac{1}{\sin\theta}\frac{\partial^4 u_r}{\partial r^2\partial\theta\partial\phi}+2\frac{\partial^3 u_\phi}{\partial r^2\partial\theta}+r\frac{\partial^4 u_\phi}{\partial r^3\partial\theta}-\frac{\partial^3 u_\phi}{\partial r^2\partial\theta}\right\}\\&+\frac{1}{r}\left\{\left(-\frac{3\cot\theta}{\sin\theta}-\frac{3\cot^3\theta}{\sin\theta}\right)\frac{\partial u_r}{\partial\phi}+\left(\frac{1+\cot^2\theta}{r\sin\theta}\right)\frac{\partial^2 u_r}{\partial\theta\partial\phi}+\frac{\sin\theta}{r}\frac{\partial^2 u_r}{\partial\theta\partial\phi}-\frac{\cos\theta}{r}\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}\right.\\&\left.-\frac{\cos\theta}{r\sin^2\theta}\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^4 u_r}{\partial\theta^3\partial\phi}-\left(1+\cot^2\theta\right)\frac{\partial^2 u_r}{\partial r\partial\theta}+\cot\theta\frac{\partial^3 u_r}{\partial r\partial\theta^2}+\frac{\partial^4 u_\phi}{\partial r\partial\theta^3}-\frac{\cot\theta}{r}\frac{\partial^2 u_\phi}{\partial\theta^2}+\left(\frac{1+\cot^2\theta}{r}\right)\frac{\partial u_\phi}{\partial\theta}-\frac{1}{r}\frac{\partial^3 u_\phi}{\partial\theta^3}\right\}\\&-\frac{2\cos\theta}{r^2\sin^2\theta}\left(\frac{1}{r\sin\theta}\frac{\partial^3 u_r}{\partial\phi^3}+\frac{\partial^3 u_\phi}{\partial r\partial\phi^2}-\frac{1}{r}\frac{\partial^2 u_\phi}{\partial\phi^2}\right)+\frac{1}{r^2\sin^2\theta}\left(-\frac{\cos\theta}{r\sin^2\theta}\frac{\partial^3 u_r}{\partial\phi^3}+\frac{1}{r\sin\theta}\frac{\partial^4 u_r}{\partial\theta\partial\phi^3}+\frac{\partial^4 u_\phi}{\partial r\partial\theta\partial\phi^2}-\frac{1}{r}\frac{\partial^3 u_\phi}{\partial\theta\partial\phi^2}\right)\end{aligned}\tag{A36}$$

$$\frac{\partial}{\partial\phi}\left(X_{rr}\right)=\frac{\partial^2 u_r}{\partial r\partial\phi}\tag{A37}$$

$$\frac{\partial}{\partial\phi}\left(Y_{rr}\right)=\frac{2}{r}\frac{\partial^3 u_r}{\partial r^2\partial\phi}+\frac{\partial^4 u_r}{\partial r^3\partial\phi}+\frac{1}{r^2}\left(\cot\theta\frac{\partial^3 u_r}{\partial r\partial\theta\partial\phi}+\frac{\partial^4 u_r}{\partial r\partial\theta^2\partial\phi}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^4 u_r}{\partial r\partial\phi^3}\tag{A38}$$

$$\frac{\partial}{\partial\phi}\left(X_{\theta\theta}\right)=\frac{1}{r}\left(\frac{\partial^2 u_\theta}{\partial\theta\partial\phi}+\frac{\partial u_r}{\partial\phi}\right)\tag{A39}$$

$$\frac{\partial}{\partial\phi}\left(Y_{\theta\theta}\right)=\frac{1}{r}\left\{\frac{\partial^4 u_\theta}{\partial r^2\partial\theta\partial\phi}+\frac{\partial^3 u_r}{\partial r^2\partial\phi}\right\}+\frac{1}{r^3}\left\{\cot\theta\left(\frac{\partial^3 u_\theta}{\partial\theta^2\partial\phi}+\frac{\partial^2 u_r}{\partial\theta\partial\phi}\right)+\frac{\partial^4 u_\theta}{\partial\theta^3\partial\phi}+\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}\right\}+\frac{1}{r^3\sin^2\theta}\left(\frac{\partial^4 u_\theta}{\partial\theta\partial\phi^3}+\frac{\partial^3 u_r}{\partial\phi^3}\right)\tag{A40}$$

$$\frac{\partial}{\partial\phi}\left(X_{\phi\phi}\right)=\frac{1}{r\sin\theta}\left(\frac{\partial^2 u_\phi}{\partial\phi^2}+\frac{\partial u_r}{\partial\phi}\sin\theta+\frac{\partial u_\theta}{\partial\phi}\cos\theta\right)\tag{A41}$$

$$\begin{aligned}\frac{\partial}{\partial\phi}\left(Y_{\phi\phi}\right)=&\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial r^2\partial\phi^2}+\frac{\partial^3 u_r}{\partial r^2\partial\phi}+\frac{\partial^3 u_\theta}{\partial r^2\partial\phi}\cot\theta\right\}-\frac{1}{r^3}\left\{\cot\theta\left(\frac{1}{\sin\theta}\frac{\partial^3 u_\phi}{\partial\theta\partial\phi^2}+\frac{\partial u_r}{\partial\phi}\cot\theta+\frac{\partial^2 u_\theta}{\partial\theta\partial\phi}\cot\theta-\frac{\partial u_\theta}{\partial\phi}+\frac{\partial^2 u_r}{\partial\theta\partial\phi}\right)\right.\\&\left.+\frac{1+\cot^2\theta}{\sin\theta}\left(\frac{\partial^2 u_\phi}{\partial\phi^2}+\frac{\partial u_r}{\partial\phi}\sin\theta+\frac{\partial u_\theta}{\partial\phi}\cos\theta\right)+\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial\theta^2\partial\phi^2}+2\cot\theta\frac{\partial^2 u_r}{\partial\theta\partial\phi}-\frac{\partial u_r}{\partial\phi}+\frac{\partial^3 u_\theta}{\partial\theta^2\partial\phi}\cot\theta+\frac{\partial^3 u_r}{\partial\theta^2\partial\phi}-2\frac{\partial^2 u_\theta}{\partial\theta\partial\phi}-\frac{\partial u_\theta}{\partial\phi}\cot\theta\right\}\\&+\frac{1}{r^3\sin^2\theta}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_\phi}{\partial\phi^4}+\frac{\partial^3 u_r}{\partial\phi^3}+\frac{\partial^3 u_\theta}{\partial\phi^3}\cot\theta\right\}\end{aligned}\tag{A42}$$

$$\frac{\partial}{\partial\phi}\left(X_{r\theta}\right)=\frac{1}{r}\frac{\partial^2 u_r}{\partial\theta\partial\phi}+\frac{\partial^2 u_\theta}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial u_\theta}{\partial\phi}\tag{A43}$$

$$\begin{aligned}\frac{\partial}{\partial\phi}\left(Y_{r\theta}\right)=&\frac{1}{r}\left(\frac{\partial^4 u_r}{\partial r^2\partial\theta\partial\phi}+2\frac{\partial^3 u_\theta}{\partial r^2\partial\phi}+r\frac{\partial^4 u_\theta}{\partial r^3\partial\phi}-\frac{\partial^3 u_\theta}{\partial r^2\partial\phi}\right)+\frac{1}{r^2}\left\{\cot\theta\left(\frac{1}{r}\frac{\partial^3 u_r}{\partial\theta\partial\phi}+\frac{\partial^3 u_\theta}{\partial r\partial\theta\partial\phi}-\frac{1}{r}\frac{\partial^2 u_\theta}{\partial\theta\partial\phi}\right)\right.\\&\left.+\left(\frac{1}{r}\frac{\partial^4 u_r}{\partial\theta^3\partial\phi}+\frac{\partial^4 u_\theta}{\partial r\partial\theta^2\partial\phi}-\frac{1}{r}\frac{\partial^3 u_\theta}{\partial\theta^2\partial\phi}\right)\right\}+\frac{1}{r^2\sin^2\theta}\left\{\frac{1}{r}\frac{\partial^4 u_r}{\partial\theta\partial\phi^3}+\frac{\partial^4 u_\theta}{\partial r\partial\phi^3}-\frac{1}{r}\frac{\partial^3 u_\theta}{\partial\phi^3}\right\}\end{aligned}\tag{A44}$$

$$\frac{\partial}{\partial\phi}\left(X_{\theta\phi}\right)=\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial^2 u_\theta}{\partial\phi^2}+\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}-\frac{\partial u_\phi}{\partial\phi}\cot\theta\right)\tag{A45}$$

$$\frac{\partial}{\partial\phi}\left(Y_{\theta\phi}\right)=\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_\theta}{\partial r^2\partial\phi^2}+\frac{\partial^4 u_\phi}{\partial r^2\partial\theta\partial\phi}-\frac{\partial^3 u_\phi}{\partial r^2\partial\phi}\cot\theta\right\}+\frac{1}{r^3}\left\{\left(\frac{1+\cot^2\theta}{\sin\theta}\right)\frac{\partial^2 u_\phi}{\partial\phi^2}-\frac{\cot\theta}{\sin\theta}\frac{\partial^3 u_\theta}{\partial\theta\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial^4 u_\theta}{\partial\theta^2\partial\phi^2}\right.$$
$$\left.+\left(2+\cot^2\theta\right)\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}+\left(3\cot\theta-\cot^3\theta\right)\frac{\partial u_\phi}{\partial\phi}+\frac{\partial^4 u_\phi}{\partial\theta^3\partial\phi}\right\}+\frac{1}{r^3\sin^2\theta}\left(\frac{1}{\sin\theta}\frac{\partial^4 u_\theta}{\partial\phi^4}+\frac{\partial^4 u_\phi}{\partial\theta\partial\phi^3}-\frac{\partial^3 u_\phi}{\partial\phi^3}\cot\theta\right) \tag{A46}$$

$$\frac{\partial}{\partial\phi}\left(X_{r\phi}\right)=\frac{1}{r\sin\theta}\frac{\partial^2 u_r}{\partial\phi^2}+\frac{\partial^2 u_\phi}{\partial r\partial\phi}-\frac{1}{r}\frac{\partial u_\phi}{\partial\phi} \tag{A47}$$

$$\frac{\partial}{\partial\phi}\left(Y_{r\phi}\right)=\frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{\partial^4 u_r}{\partial r^2\partial\phi^2}+2\frac{\partial^3 u_\phi}{\partial r^2\partial\phi}+r\frac{\partial^4 u_\phi}{\partial r^3\partial\phi}-\frac{\partial^3 u_\phi}{\partial r^2\partial\phi}\right\}+\frac{1}{r}\left\{\left(\frac{1+\cot^2\theta}{r\sin\theta}\right)\frac{\partial^2 u_r}{\partial\phi^2}-\frac{\cos\theta}{r}\frac{\partial^3 u_r}{\partial\theta\partial\phi^2}+\frac{1}{r\sin\theta}\frac{\partial^4 u_r}{\partial\theta^2\partial\phi^2}\right.$$
$$\left.+\cot\theta\frac{\partial^3 u_r}{\partial r\partial\theta\partial\phi}+\frac{\partial^4 u_\phi}{\partial r\partial\theta^2\partial\phi}-\frac{\cot\theta}{r}\frac{\partial^2 u_\phi}{\partial\theta\partial\phi}-\frac{1}{r}\frac{\partial^3 u_\phi}{\partial\theta^2\partial\phi}\right\}+\frac{1}{r^2\sin^2\theta}\left(\frac{1}{r\sin\theta}\frac{\partial^4 u_r}{\partial\phi^4}+\frac{\partial^4 u_\phi}{\partial r\partial\phi^3}-\frac{1}{r}\frac{\partial^3 u_\phi}{\partial\phi^3}\right) \tag{A48}$$