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Nonlinear vibration of unsymmetrical laminated composite beam on elastic foundation

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Abstract. In this paper, nonlinear vibrations of the unsymmetrical laminated composite beam (LCB) on a nonlinear elastic foundation are studied. The governing equation of the problem is derived by using Galerkin method. Two different end conditions are considered: the simple-simple and the clamped-clamped one. The Hamiltonian Approach (HA) method is adopted and applied for solving of the equation of motion. The advantage of the suggested method is that it does not need any linearization of the problem and the obtained approximate solution has a high accuracy. The method is used for frequency calculation. The frequency of the nonlinear system is compared with the frequency of the linear system. The influence of the parameters of the foundation nonlinearity on the frequency of vibration is considered. The differential equation of vibration is solved also numerically. The analytical and numerical results are compared and is concluded that the difference is negligible. In the paper the new method for error estimation of the analytical solution in comparison to the exact one is developed. The method is based on comparison of the calculation energy and the exact energy of the system. For certain numerical data the accuracy of the approximate frequency of vibration is determined by applying of the suggested method of error estimation. Finally, it has been indicated that the proposed Hamiltonian Approach gives enough accurate result.

Keywords: non-linear vibration; analytical solution; beam vibration; fourth-order Runge-Kutta method

1. Introduction

Composites are materials which are widely used in structural elements and modern technologies, principally, in aerospace and aircraft industry. The greatest advantages of these materials are their high stiffness-to-weight ratio, high strength-to-weight ratio, but also the anisotropy that can be tailored in the desired direction depending on the type of loading. Composite structures have a great potential for reducing the weight. However, the analyses of composite structures is complex, due to the fact that bending-extension coupling exists. The classical beam theory based on the Bernoulli-Euler assumption and also the classical lamination theory are not suitable for modeling of unsymmetrically laminated composite beams (Kapania and Goyal 2002, Wang *et al*. 2013d) with small transverse shear modulus in comparison to the in-plane tensile moduli. In some specific cases, the assumption of negligibility of axial and rotatory inertia is not acceptable. Some papers, in which both axial and rotatory inertia effects, are taken into account (Lenci et al. 2012a, b).

Namely, the transverse shear deformation can be of considerable importance compared to homogenous isotropic materials. To overcome the problem, recently the nonlinear analysis of beams has gained attention for increasing the efficiency of structural design. Thus, in Kapania *et al.* (Kapania and Raciti 1989, Kapania and Goyal 2002) the application of the finite-element method is suggested, where for analysis of the nonlinear vibrations of the symmetrically and unsymmetrically laminated beams the simple one-dimensional finite element is introduced. The beam element has 10 or more degrees of freedom at each of the two nodes: the axial displacement, the transverse deflection and the slope due to bending and shear, the twisting angle, the in-plane shear rotation and their derivatives. Using the model it is concluded that the nonlinear vibrations of unsymmetrically laminated beams have soft spring behavior for certain boundary condition as opposed to a hard spring behavior observed in isotropic and symmetrically laminated beams. In the papers (Singh *et al*. 1991, 1992) the degrees-of-freedom of the one-dimensional finite element is extended to 12, by applying of the classical lamination theory, first order shear deformation theory and higher order deformation theory. The dynamic nonlinear finite element equations are reduced to two second-order ordinary nonlinear differential equations using converged normalized spatial deformations in the positive and negative deflection half-cycles. The modal equations of motion are solved using the direct numerical integration method. In the papers (Baghani *et al*. 2011, Jafari-Talookolaei *et al*. 2011) the free vibrations of unsymmetrically laminated composite beams settled on nonlinear elastic foundation are investigated. The elastic foundation has cubic nonlinearity with shearing

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layer. The equation of motion is a fourth order partial differential equation which is an appropriate way separated into two equations. In Baghani *et al*. (2011) the differential equation is solved by using the variational iteration method, while in Jafari-Talookolaei *et al*. (2011) the homotopy analysis method is applied. In the first approximation the both methods give the same frequency-amplitude relation, while there is a difference in the frequency in the second-order approximation. There some many other analytical methods and nonlinear vibrartions of beams which can be found in literatures (Arikoglu and Ozkol 2005, He 2008, Jamshidi and Ganji 2010, Shen and Mo 2009, Ramana and Prasad 2014, Bambill *et al*. 2013, Clementi et al. 2015, Fang and Zhou 2015, He *et al*. 2013, Yu *et al*. 2012, Lenci *et al*. 2013, 2015, Civalek 2006, 2013, Nguyen and Lee 2015, Babilio 2013, 2014, Wang *et al*. 2013a, b, c, Szekrényes 2015, Basu and Kameswara Rao 2013, Pradhan and Murmu 2009, Sheikholeslami and Ganji 2013, 2015, 2016, Sheikholeslami *et al*. 2016, Cheng *et al*. 2012, Ghasemi and Mohandes 2016, Poloei *et al*. 2017, Shafiei and Setoodeh 2017, Bayat *et al*. 2016, 2017, Bayat and Pakar 2017a, b, Alkayem *et al*. 2017). The aim of this paper is to develop a solving procedure for the free nonlinear vibrations of the unsymmetrical laminated composite beam which will give directly the more appropriate frequency value. The Hamiltonian Approach (HA) method which is already applied for solving free vibrations for the systems with one-degree-of-freedom systems (see He 2010, Bayat *et al*. 2014, Navarro and Cveticanin 2016) is adopted for solving this problem. The paper has seven sections. After the Introduction the physical and mathematical model of the problem is presented. A short introduction to the HA method is given in Section 3. The HA method is adopted for solving equations of free motion of the beam (Section 4). In Section 5, the new error estimation is developed. The method has the aim to prove the accuracy of the approximately calculated frequency in comparison to the unknown exact one. In Section 6, for certain numerical values the frequency of vibration is obtained applying the suggested Hamiltonian approach and the accuracy of the result is proved by comparing the value with the numerically one and also by estimating of the error according to the procedure developed in the paper.

2. Description of the problem

In Fig. 1, the laminated composite beam (LCB) on the nonlinear layer is plotted. The laminated beam is straight and contains the linear and shear layer. The length of the beam is L, its width is b and thickness h. Coordinate along the axis of the beam is $\hat{X}$ and in the direction of the thickness of the beam is $\hat{Z}$. On the beam an axial force $\hat{P}$ acts. It causes transversal-axial coupled vibrations. It is convenient to transform the two equations of motion into only one as is suggested in (Jafari-Talookolaei *et al*. 2011)

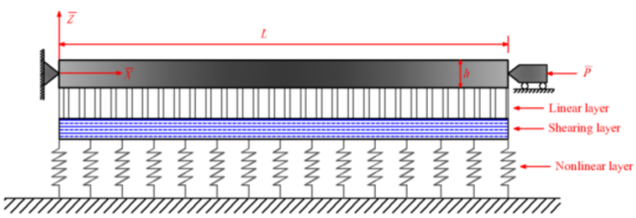

Fig. 1 The LCB with simply supported end conditions

$$\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} P-B\int_0^1\left(\frac{\partial w}{\partial x}\right)^2 dx \\ -\Lambda\left(\frac{\partial w}{\partial x}(1,0)-\frac{\partial w}{\partial x}(0,t)\right)\end{bmatrix}=F_w \quad (1)$$

Where w is the transversal displacement of the beam along the $\hat{Z}$ coordinate, B and Λ are coefficients which are functions of stiffness of the beam and gyration ratio of the cross section and F_w is the foundation force. The load-displacement relationship is

$$F_w=-k_1 w-k_2 w^3+k_S\frac{\partial^2 w}{\partial x^2} \quad (2)$$

Where k_1 and k_2 are linear and nonlinear elastic foundation coefficients and k_S is the coefficient of the shearing layer elastic foundation. Substituting (2) into (1) a dimensionless partial differential equation is obtained

$$\begin{aligned}&\frac{\partial^2 w}{\partial t^2}+\frac{\partial^4 w}{\partial x^4}+\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} P-B\int_0^1\left(\frac{\partial w}{\partial x}\right)^2 dx \\ -\Lambda\left(\frac{\partial w}{\partial x}(1,0)-\frac{\partial w}{\partial x}(0,t)\right)\end{bmatrix} \\ &+k_1 w+k_2 w^3-k_S\frac{\partial^2 w}{\partial x^2}=0\end{aligned} \quad (3)$$

It is worth to say that the coupling stiffness B and therefore Λ differ from zero for the asymmetric laminate.

Assuming $w(x,t)=\phi(x)W(t)$ where $\phi(x)$ is the first eigenmode of the beam and using the Ritz method, the governing equation of motion follows as

$$\begin{aligned}&\frac{d^2W(t)}{dt^2}+\left[\alpha_1+P\alpha_2+\alpha_3+\alpha_4\right]W(t) \\ &+\alpha_5 W(t)^2+(\alpha_6+\alpha_7)W(t)^3=0\end{aligned} \quad (4)$$

Where α_i (i = 1, 2,…7) are presented in Appendix A. (The complete formulation of Eq. (3) can be seen in (Jafari-Talookolaei *et al*. 2011)).

The initial conditions are as follows

$$W(0)=a,\quad \frac{dW(0)}{dt}=0 \quad (5)$$

where a denotes the non-dimensional initial amplitude of oscillation.

Note that, for isotropic and symmetrically laminated beams, the coefficient α_5 of the quadratic term $W(t)^2$ is zero. It causes that the analysis of nonlinear vibrations for unsymmetrically laminated beams significantly differs from that of isotropic and symmetrically laminated beams, as the bending-stretching coupling (the coefficient B_{11}) induces the quadratic term $W(t)^2$.

Post-buckling load–deflection relation of the LCB can be determined from Eq. (4) as

$$P_{NL} = -\frac{\alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 W + (\alpha_6 + \alpha_7) W^2}{\bar{\alpha}_2} \tag{6}$$

Neglecting the contribution of W in the previous Eq. (6), the linear buckling load can be obtained as

$$P_L = P_{cr} = -\frac{\alpha_1 + \alpha_3 + \alpha_4}{\alpha_2} \tag{7}$$

Comparing the buckling load in the nonlinear system with the linear case gives

$$\frac{P_{NL}}{P_L} = 1 + \frac{\alpha_6 + \alpha_5 W + \alpha_7 W^2}{\alpha_1 + \alpha_3 + \alpha_4} \tag{8}$$

Due to nonlinearity the buckling load increases: the higher the nonlinearity of foundation, the higher the value of the buckling force in comparison to the linear case.

3. Hamiltonian Approach adopted for the problem

Hamiltonian Approach proposed by He (2010) is adopted for solving the Eq. (8). The method is based on variation of the Hamiltonian

$$H = \frac{1}{2}\dot{W}^2 + F(W) \tag{9}$$

where $\partial F/\partial W = f(W)$ and

$$f(W) = (\alpha_1 + P\alpha_2 + \alpha_3 + \alpha_4)W + \alpha_5 W^2 + (\alpha_6 + \alpha_7) W^3 \tag{10}$$

Hamiltonian for the conservative system is constant, i.e.,

$$H = H_0 = const. \tag{11}$$

Let us assume the solution of (9) in the form of a trigonometric function

$$W = a\cos(\omega t) \tag{12}$$

Where a is the amplitude and ω is the frequency of vibration.

Substituting (12) into (9) and using (11) it is

$$H = (1/2)a^2\omega^2 \sin^2(\omega t) + F(a\cos(\omega t)) = H_0 \tag{13}$$

The first derivative of (13) for the amplitude a is

$$\frac{\partial H}{\partial a} = a\omega^2 \sin^2(\omega t) + \frac{\partial F}{\partial W}\frac{\partial W}{\partial a} = 0 \tag{14}$$

i.e.,

$$\frac{\partial H}{\partial a} = a\omega^2 \sin^2(\omega t) + f(a\cos(\omega t)\cos(\omega t)) = 0 \tag{15}$$

where $f = \partial F/\partial W$. The assumed solution (12) is not accurate enough. It is the reason that the new approximate form of Hamiltonian is introduced

$$\overline{H} = \int_0^{T/4} H\, dt \tag{16}$$

i.e.,

$$H = 4\frac{\partial \overline{H}}{\partial T} \tag{17}$$

Rewriting (15) with the new function (16) it is

$$\frac{\partial H}{\partial a} = 4\frac{\partial}{\partial a}\left(\frac{\partial \overline{H}}{\partial T}\right) \tag{18}$$

For the period of vibration $T = 2\pi/\omega$ the relation (18) transforms into

$$\frac{\partial}{\partial a}\left(\frac{\partial \overline{H}}{\partial (1/\omega)}\right) = 0 \tag{19}$$

From Eq. (19) we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator.

4. Application of method

Hamiltonian (9) for Eq. (4) is constructed as

$$H = \frac{1}{2}\dot{W}^2 + \frac{1}{2}\beta_1 W^2 + \frac{1}{3}\beta_2 W^3 + \frac{1}{4}\beta_4 W^4 \tag{20}$$

where

$$\beta_1 = [\alpha_1 + P\alpha_2 + \alpha_3 + \alpha_4], \quad \beta_2 = \alpha_5, \quad \beta_3 = (\alpha_6 + \alpha_7) \tag{21}$$

Integrating Eq. (20) with respect to t from 0 to $T/4$, we have

$$\overline{H} = \int_0^{T/4}\left(\frac{1}{2}\dot{W}^2 + \frac{1}{2}\beta_1 W^2 + \frac{1}{3}\beta_2 W^3 + \frac{1}{4}\beta_4 W^4\right)dt \tag{22}$$

Introducing the trial solution (12) into (21) and after integration it is

$$\overline{H} = \frac{\pi}{8}a^2\omega + \frac{1}{\omega}\left(\frac{\pi}{8}\beta_1 a^2 + \frac{2}{9}\beta_2 a^3 + \frac{3\pi}{64}\beta_3 a^4\right) \tag{23}$$

Setting (22) into (19) it is

$$\frac{\partial}{\partial a}\left(\frac{\partial \overline{H}}{\partial (1/\omega)}\right) = \frac{a\pi}{4} - \frac{1}{\omega^2}\left(\frac{\pi}{4}\beta_1 a + \frac{2}{3}\beta_2 a^2 + \frac{3\pi}{16}\beta_3 a^3\right) = 0 \tag{24}$$

If we solve Eq. (23) the approximate frequency of the system is

$$\omega=\sqrt{\beta_1+\frac{8}{3}\frac{\beta_2 a}{\pi}+\frac{3}{4}\beta_3 a^2} \tag{25}$$

By substituting (21) in to (25) we have

$$\omega=\sqrt{\alpha_1+P\alpha_2+\alpha_3+\alpha_4+\frac{8}{3}\frac{\alpha_5 a}{\pi}+\frac{3}{4}(\alpha_6+\alpha_7)a^2} \tag{26}$$

Hence, the approximate solution can be readily obtained

$$W(t)=a\cos\left(\sqrt{\alpha_1+P\alpha_2+\alpha_3+\alpha_4+\frac{8}{3}\frac{\alpha_5 a}{\pi}+\frac{3}{4}(\alpha_6+\alpha_7)a^2}\;t\right) \tag{27}$$

The ratio of the non-linear to linear frequency is

$$\frac{\omega_{NL}}{\omega_L}=\sqrt{1+\frac{8}{3}\frac{\beta_2 a}{\beta_1\pi}+\frac{3\beta_3}{4\beta_1}a^2} \tag{28}$$

Or

$$\frac{\omega_{NL}}{\omega_L}=\sqrt{1+\frac{8}{3}\frac{\alpha_5 a}{(\alpha_1+P\alpha_2+\alpha_3+\alpha_4)\pi}+\frac{3(\alpha_6+\alpha_7)}{4\alpha_1+P\alpha_2+\alpha_3+\alpha_4}a^2} \tag{29}$$

Due to nonlinear properties of fundament the nonlinear frequency of vibration of the beam is higher than that of the beam on the linear foundation.

5. Error estimation

The obtained approximate solution (24) differs from the exact one. Our aim is to determine the accuracy of the obtained result. For this purpose the method for error estimation is developed. The suggested procedure is based on the difference between the exact value of the total energy and the approximate value obtained with assumed trial solution and calculated approximate frequency.

Substituting the trial solution (12) into (20) the approximate value of the Hamiltonian follows as

$$\begin{aligned}H&=\frac{1}{2}a^2\omega^2\sin^2(\omega t)+\frac{1}{2}\beta_1 a^2\cos^2(\omega t)\\&\quad+\frac{1}{3}\beta_2 a^3\cos^3(\omega t)+\frac{1}{4}\beta_3 a^4\cos^4(\omega t)\end{aligned} \tag{30}$$

The relation (27) corresponds to the approximate value of the energy of the system. According to the initial conditions (5) the exact value of the total energy of the system is

$$H_0=\frac{1}{2}\beta_1 a^2+\frac{1}{3}\beta_2 a^3+\frac{1}{4}\beta_3 a^4 \tag{31}$$

The residual between the exact and approximate values of the energy is

$$R=H-H_0 \tag{32}$$

i.e.,

$$\begin{aligned}R&=\frac{1}{2}a^2\omega^2\sin^2(\omega t)-\frac{1}{2}\beta_1 a^2\left(1-\cos^2(\omega t)\right)\\&\quad-\frac{1}{3}\beta_2 a^3\left(1-\cos^3(\omega t)\right)-\frac{1}{4}\beta_3 a^4\left(1-\cos^4(\omega t)\right)\end{aligned} \tag{33}$$

It is evident that the residual is a time periodical function. For simplicity, let us average the residual R over the period T of the time variable functions. The Eq. (30) transforms into

$$\Delta_1=\frac{1}{T}\int_0^T R\,dt \tag{34}$$

i.e.,

$$\begin{aligned}\Delta_1&=\frac{1}{T}\int_0^T\frac{1}{2}a^2\omega^2\sin^2(\omega t)-\frac{1}{2}\beta_1 a^2\left(1-\cos^2(\omega t)\right)\\&\quad-\frac{1}{3}\beta_2 a^3\left(1-\cos^3(\omega t)\right)-\frac{1}{4}\beta_3 a^4\left(1-\cos^4(\omega t)\right)dt\end{aligned} \tag{35}$$

After integration of (32) we have

$$\Delta_1=\frac{1}{4}a^2\omega^2-\frac{1}{4}a^2\left(\beta_1+\frac{4}{3}\beta_2 a+\frac{5}{8}\beta_3 a^2\right) \tag{36}$$

Substituting the value of the approximate frequency (24) and modifying the Eq. (33) we obtain

$$\Delta_1=\frac{1}{4}a^2\left(\frac{1}{8}\beta_3 a^2-\frac{4(\pi-2)}{3\pi}\beta_2 a\right) \tag{37}$$

Comparing the averaged residual of energy with the exact total energy the averaged estimation error follows as

$$\Delta=\frac{\Delta_1}{H_0} \tag{38}$$

i.e.,

$$\Delta=\frac{a\left(3a\pi\beta_3+64\beta_2-32\pi\beta_2\right)}{8\pi\left(6\beta_1+4\beta_2 a+3\beta_3 a^2\right)} \tag{39}$$

Substituting (21) into (39)

$$\Delta=\frac{a\left(3a\pi(\alpha_6+\alpha_7)+64\alpha_5-32\pi\alpha_5\right)}{8\pi\left(6(\alpha_1+P\alpha_2+\alpha_3+\alpha_4)+4\alpha_5 a+3(\alpha_6+\alpha_7)a^2\right)} \tag{40}$$

The error depends on the initial amplitude of vibration and on the parameters of the elastic foundation.

6. Results and discussion.

The Hamiltonian Approach is used to obtain an analytical solution for simply supported and clamped-clamped beams.

From the reference (Lewandowski 1987) for a simply supported beam we had $a=\delta/\sqrt{12}$ and for clamped-

Table 1 Comparison of nonlinear to linear frequency ratio (ω_{NL}/ω_L) for simply-supported beams

δ	a	Hamiltonian Approach	Azrar *et al.* (1999)	Lewandowski (1987)
1	0.2886	1.0897	1.0891	1.0897
2	0.5773	1.3228	1.3177	1.3229
3	0.8660	1.6393	1.6256	1.6394
4	1.1547	2	-	1.9999

Table 2 Comparison of nonlinear to linear frequency ratio (ω_{NL}/ω_L) for Clamped-Clamped Beams

δ	$w_1^*(1/2)$	a	Hamiltonian Approach	Azrar *et al.* (1999)	Lewandowski (1987)
1	1.58815	0.18177	1.0222	1.0222	1.0222
1.5	1.58815	0.27265	1.0494	1.0492	1.0492
2	1.58815	0.36354	1.0862	1.0857	1.0858
2.5	1.58815	0.45442	1.1318	1.1307	1.1308
3	1.58815	0.54531	1.1852	1.1831	1.1832
3.5	1.58815	0.63619	1.2453	1.2420	1.2422
4	1.58815	0.72707	1.3112	1.3064	1.3063
4.5	1.58815	0.81796	1.3822	1.3756	1.3751

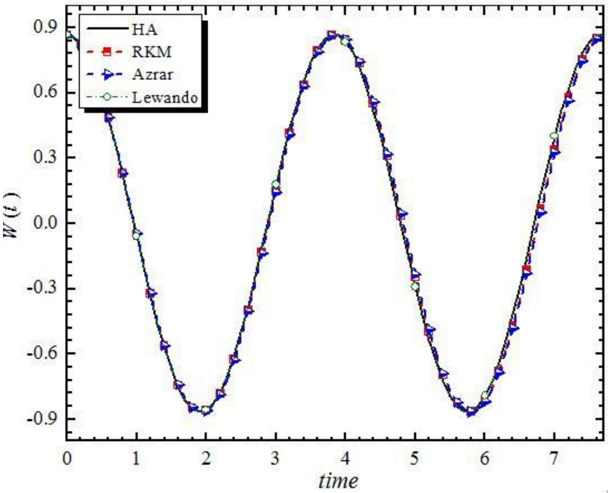

Fig. 2 Comparison of analytical solution of $W(t)$ based on time with the RKM solution for simply supported beam

clamped beam $a = \delta/\sqrt{12}w_1^*(1/2)$ that δ is maximum amplitude parameter and $w_1^*(1/2)$ is first mod of beam in middle of beam. Tables 1 and 2 represent the comparisons of nonlinear to linear frequency ratio (ω_{NL}/ω_L) for Simply-Supported Beam and for the Clamped-Clamped Beams with the Hamiltonian Approach (HA) and the numerical solutions and other researchers results for different parameters of a. Azrar (Azrar *et al.* 1999) and Lewandowski (1987) ignored to consider the mid-plane effect in their study therefore for large amplitude the ratio of nonlinear to linear frequency increases. To show the accuracy of the HA results, Runge-Kutta 4th is used to

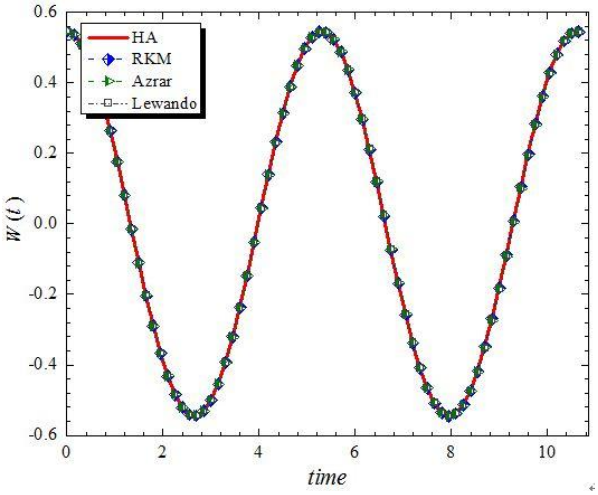

Fig. 3 Comparison of analytical solution of $W(t)$ based on time with the RKM solution for Clamped-Clamped beam

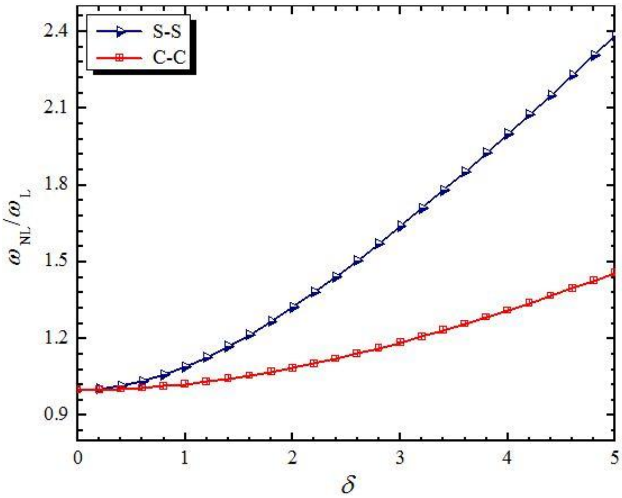

Fig. 4 Nonlinear to linear frequency ratio versus non-dimensional amplitude ratio

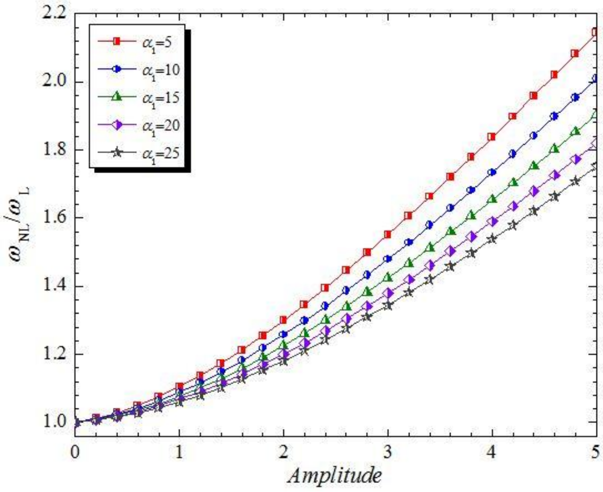

Fig. 5 Influence of α_1 on nonlinear to linear frequency base on amplitude

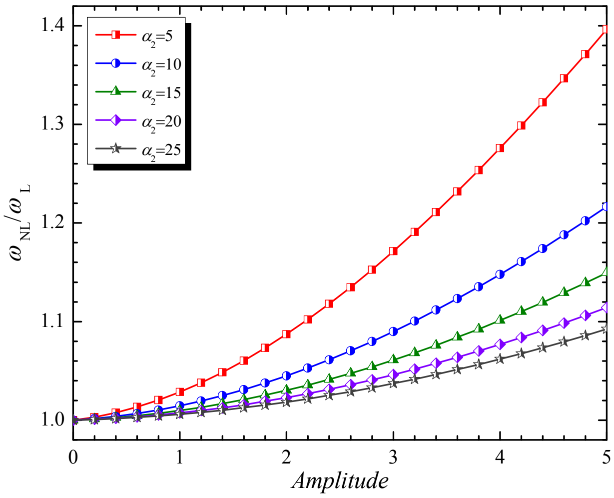

Fig. 6 Influence of α_2 on nonlinear to linear frequency base on amplitude

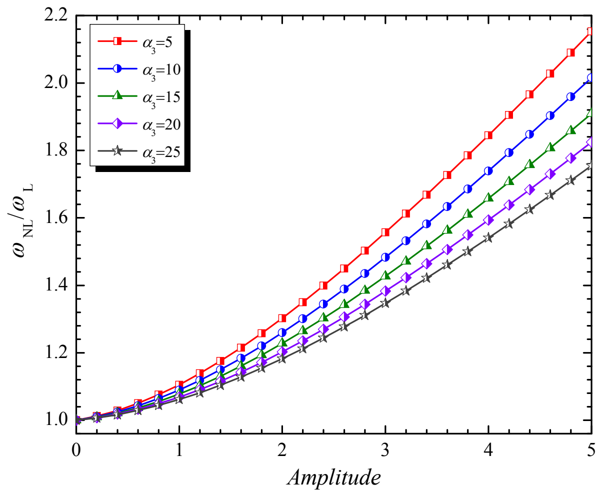

Fig. 7 Influence of α_3 on nonlinear to linear frequency base on amplitude

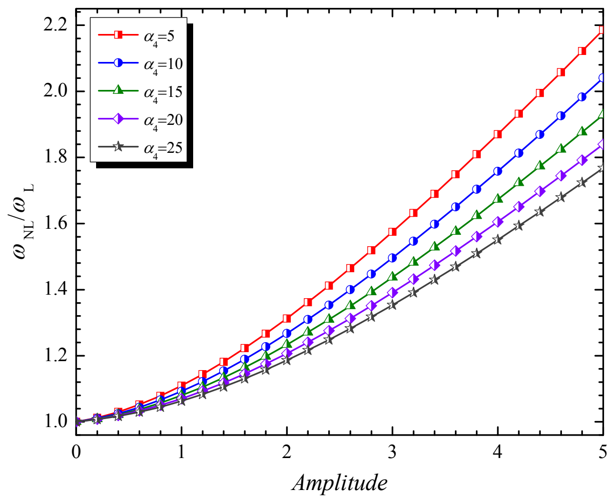

Fig. 8 Influence of α_4 on nonlinear to linear frequency base on amplitude

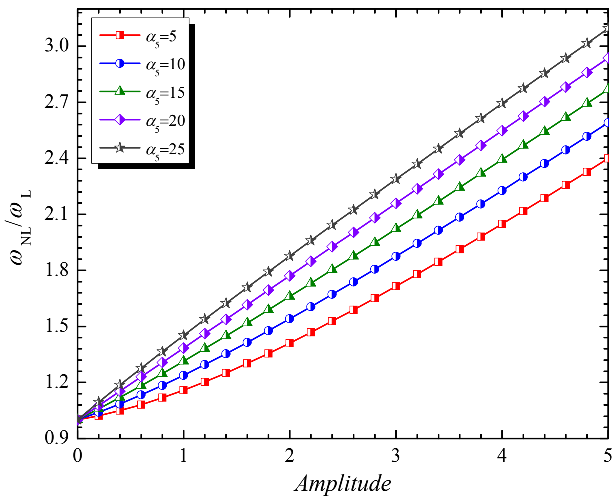

Fig. 9 Influence of α_5 on nonlinear to linear frequency base on amplitude

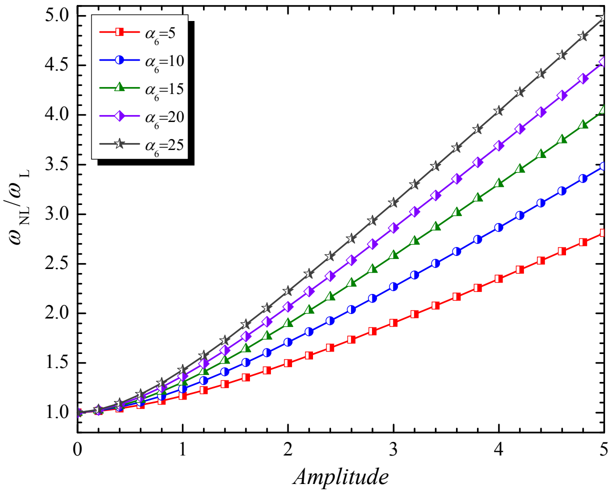

Fig. 10 Influence of α_6 on nonlinear to linear frequency base on amplitude

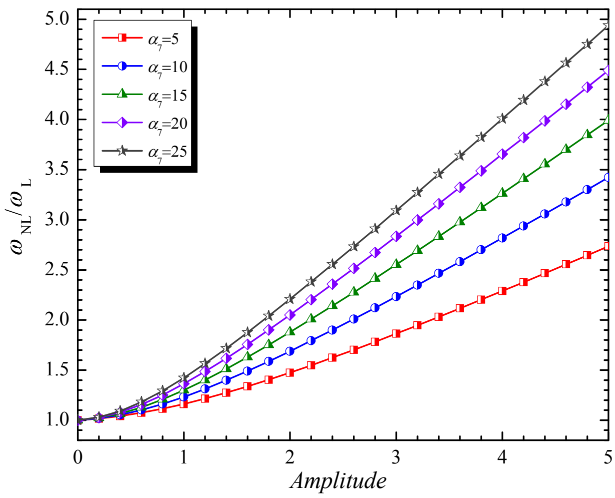

Fig. 11 Influence of α_7 on nonlinear to linear frequency base on amplitude

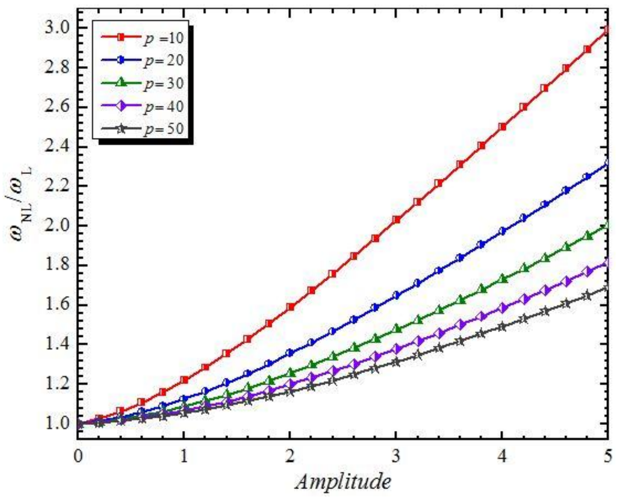

Fig. 12 Influence of p on nonlinear to linear frequency base on amplitude

Table 3 Error estimation of analytical solution

A	$\overline{\alpha_1}$	$\overline{\alpha_2}$	$\overline{\alpha_3}$	$\overline{\alpha_4}$	$\overline{\alpha_5}$	$\overline{\alpha_6}$	$\overline{\alpha_7}$	P	HA	Δ×100
0.5	0.5	0.5	1	3	1.5	2	3	40	5.106	0.404
1	3	5	1.5	2	4	1	0.5	10	7.812	1.461
1.5	8	1	3	1.5	0.5	0.2	2.5	15	5.718	0.638
2	1	1.5	0.5	1	1	1.5	0.2	10	4.929	0.268
2.5	5	0.5	1	0.5	2.5	1	3	20	6.368	0.146
3	3	2.5	2	5	10	5	2	50	14.412	1.786
3.5	5	10	1.5	1	15	1.5	2.5	40	22.109	2.068
4	2.5	5	1.5	2.5	5	2	10	100	25.836	1.162
4.5	2	5	2	3	10	0.5	0.5	80	21.457	2.155
5	5	2	1.5	1.5	8	3	5	30	15.873	1.443

consider the effect of the variation of non-dimensional amplitude ratio versus t for the beam center. Figs. 2 and 3 represent a comparison of analytical solution of $W(t)$ based on time with the numerical solution. From Figs. 2 and 3, the motion of the system is a periodic motion and the amplitude of vibration is a function of the initial conditions. In clamped beams the eigenmodes of them involve hyperbolic component and simply supported beams have only sinusoidal component in their eigenmodes, in this case the HA provides more accurate solution. Fig. 4 shows the effect of non-dimensional amplitude ratio δ on the ratio of non-linear to linear frequency. For better understanding the effects of different parameters on the ratio of nonlinear to linear frequency, it has been considered the following parameters in Figs. 7 to 11: $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$. Fig. 12, shows the effects of different axial loads on the nonlinear to linear frequency ratio based on amplitude of the system.

6.1 Error estimation

For the given numerical data and the calculated value of the frequency of vibration the estimation error is calculated. In the Table 3 the error for various values of initial amplitude of vibration and some values of $\beta 1$, $\beta 2$ and $\beta 3$ (see Eq. (2)) for different values of $\bar{\alpha}_{i=1to7}$ and P are presented in Table 3.

7. Conclusions

In this paper, nonlinear vibrations of the unsymmetrical laminated composite beam (LCB) on a nonlinear elastic foundation were studied analytically. The governing equation was derived and two cases were developed: (1) simple-simple; (2) clamped-clamped. A new approximate analytical approach has been presented and applied to achieve the nonlinear frequency response of the problem. The effects of different important parameters were studied completely and all the results were compare to the numerical one using Runge-Kutta's algorithm. A new method for error estimation of the analytical solution in comparison to the exact one is also developed. Full procedure of this method was presented in detail. Finally, it has been demonstrated that the presented Hamiltonian Approach (HA) gives enough accurate result as it compared with numerical solutions.

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Appendix

In Eq. (1), the physical parameters are given by Jafari-Talookolaei *et al.* (2011)

$$\alpha_1 = \frac{\int_0^1 \phi'''' \phi dx}{\int_0^1 \phi^2 dx} \tag{A1}$$

$$\alpha_2 = \frac{\int_0^1 \phi'' \phi dx}{\int_0^1 \phi^2 dx} \tag{A2}$$

$$\alpha_3 = k_1 \tag{A3}$$

$$\bar{\alpha}_4 = -k_s \frac{\int_0^1 \phi'' \phi dx}{\int_0^1 \phi^2 dx} \tag{A4}$$

$$\alpha_5 = -\Lambda\left(\phi'(1) - \phi'(0)\right) \frac{\int_0^1 \phi'' \phi dx}{\int_0^1 \phi^2 dx} \tag{A5}$$

$$\alpha_6 = -k_2 \frac{\int_0^1 \phi^4 \phi dx}{\int_0^1 \phi^2 dx} \tag{A6}$$

$$\alpha_7 = -B \frac{\int_0^1 \phi'' \phi dx \int_0^1 \phi'^2 dx}{\int_0^1 \phi^2 dx} \tag{A7}$$

in which

$$\begin{gathered} P = \frac{\hat{P}\bar{t}^2}{mL^2}, \quad B = \frac{bA_{11} r^2 \bar{t}^2}{2mL^4}, \\ \Lambda = \frac{bB_{11} r \bar{t}^2}{mL^4}, \quad \bar{t} = \frac{t}{\hat{t}}, \quad r = \sqrt{(I/A)} \end{gathered} \tag{A8}$$

where $\phi(x)$ is the first eigenmode of the beam and $\hat{k}_1, \hat{k}_2$ are linear and nonlinear elastic foundation coefficients, respectively and $\hat{k}_s$ is the coefficient of shearing layer in elastic foundation as follow

$$\begin{gathered} k_1 = \frac{\hat{k}_1 L^4}{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)}, \qquad k_2 = \frac{\hat{k}_2 r^2 L^4}{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)}, \\ k_s = \frac{\hat{k} L^2}{b\left(D_{11} - \frac{B_{11}^2}{A_{11}}\right)} \end{gathered} \tag{A9}$$

The stiffness coefficients A_{11}, B_{11} and D_{11} are given as follows

$$\begin{gathered} A_{11} = \sum_{k=1}^{n} \bar{Q}_{11}^{(k)} \left(\hat{Z}_k - \hat{Z}_{k-1}\right), \qquad B_{11} = \frac{1}{2}\sum_{k=1}^{n} \bar{Q}_{11}^{(k)} \left(\hat{Z}_k^2 - \hat{Z}_{k-1}^2\right), \\ D_{11} = \frac{1}{3}\sum_{k=1}^{n} \bar{Q}_{11}^{(k)} \left(\hat{Z}_k^3 - \hat{Z}_{k-1}^3\right) \end{gathered} \tag{A10}$$

where $\bar{Q}_{11}^{(k)}$ is the stiffness transformed to the $\hat{x}$ direction, $\hat{Z}_k$ is the height of the kth layer and n is the number of laminas.