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Free vibration analysis of a three-layered microbeam based on strain gradient theory and three-unknown shear and normal deformation theory

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Abstract. Free vibration analysis of a three-layered microbeam including an elastic micro-core and two piezo-magnetic face-sheets resting on Pasternak's foundation are studied in this paper. Strain gradient theory is used for size-dependent modeling of microbeam. In addition, three-unknown shear and normal deformations theory is employed for description of displacement field. Hamilton's principle is used for derivation of the governing equations of motion in electro-magneto-mechanical loads. Three micro-length-scale parameters based on strain gradient theory are employed for prediction of vibrational characteristics of structure in micro-scale. The results show that increase of three micro-length-scale parameters leads to significant increase of three natural frequencies especially for increase of second micro-length-scale parameter. This result is according to this fact that stiffness of a micro-scale structure is increased with increase of micro-length-scale parameters.

Keywords: free vibration analysis; strain gradient theory; micro-length-scale parameters; three unknown shear and normal deformation theory; natural frequencies

1. Introduction

A comprehensive vibration or elastic analysis of various structures such as the beams or plates needs a three dimensional formulation of the problem without any simplification. We know that three dimensional analysis of a structure with exact boundary conditions and presentation of a comprehensive analytical solution for governing equations is not possible in general state. To overcome this problem and to reach a method for solution of the problems, some simplified theories such Euler-Bernoulli, Timoshenko, Kirchhoff, Mindlin and etc. have been proposed for beams and plates. Some of these theories such Euler-Bernoulli and Kirchhoff theories ignore in plane shear and out of plane normal deformation and another such as Timoshenko and Mindlin theories ignore out of plane normal deformation. In this paper, we employ three-unknown shear and normal deformation theory for size-dependent vibration analysis of a three-layered piezo-magnetic microbeam based on strain gradient theory. a comprehensive literature review about three-unknown shear and normal deformation theory and size-dependent theories are presented to show necessity of this research.

Shimpi and Patel (2006) employed a new two-variable refined theory for analysis of orthotropic plates. They mentioned that employing this theory leads to only two governing equations, which are completely uncoupled for static analysis, and are only inertially coupled for dynamic analysis. In addition, parabolic variation of shear stress was considered across the thickness direction to ignore need to shear stress correction factor. Schnabl *et al*. (2007) presented an analytical solution for linear two-layer beams with different material and geometric characteristics. The effect of transverse shear deformation was studied on the static and dynamic responses of beam. In addition influence of various non dimensional material properties and geometric parameters was studied on the results of beam. Arefi and Rahimi (2010, 2011, 2012a, b and 2014), Khoshgoftar *et al*. (2011), Rahimi *et al*. (2012) and Arefi and Khoshgoftar (2014) studied electro elastic analysis of functionally graded piezoelectric cylinder, sphere and plate subjected to electric and magnetic potentials. Arefi (2014) presented a comprehensive study on the electro-magneto-elastic analysis of functionally graded shell of revolution based on curvilinear coordinate system and tensor analysis. The effect of electric and magnetic fields was studied on the results of system. Zhang *et al*. (2015) presented a four-unknown shear deformation theory to develop a functionally graded cylindrical microshell model and they introduced three material length scale parameters by using the strain gradient elasticity theory.

Thai and Vo (2013) presented bending, buckling, and vibration of functionally graded plates based on new sinusoidal shear deformation theory without using shear correction factor. They assumed that material properties are graded along the thickness direction based on power law distribution of the volume fraction of the constituents. Thai

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and Kim (2013) presented bending and free vibration analysis of functionally graded plates based on new higher-order shear deformation theory using Hamilton's principle. Bousahla *et al*. (2014) introduced trigonometric higher-order theory for static analysis of functionally graded plates including the stretching effect. The governing equations were derived using principle of virtual work and employing the concept of neutral surface. The Navier-type analytical solution was employed for functionally graded plate subjected to transverse load for simply supported boundary conditions. Hebali *et al*. (2014) studied free vibration analysis of a functionally graded plate based on quasi-three-dimensional (3D) hyperbolic shear deformation theory. They mentioned total transverse displacement into bending, shear, and thickness stretching parts. They showed importance of these parts in analysis via comparison with previous references.

Yahia *et al*. (2015) studied wave propagation characteristics of functionally graded porous plates based on various higher-order shear deformation plate theories. To account porosities in functionally graded materials, some modifications were considered on the rule of mixture. The effects of the volume fraction distributions and porosity volume fraction were considered on the wave propagation characteristics of functionally graded plate. Bourada *et al*. (2015) studied bending and free vibration analysis of functionally graded beams. A simple and refined trigonometric higher-order beam theory was employed by authors to include thickness stretching effect. They mentioned that employing the refined trigonometric higher-order beam theory does not need to shear stress correction factor. In addition, the concept of neutral surface was employed to derive governing equations of motion. Thermo-elastic analysis of multilayered cross-ply laminates and angle-ply sandwich plates resting on Pasternak's foundation subjected to sinusoidal temperature distribution was studied by Zenkour (2015) by considering transverse shear strains. Bounouara *et al*. (2016) studied free vibration analysis of functionally graded nano plates resting on elastic foundation. Zeroth-order shear deformation plate theory was used for kinematic description of nano plate. Due to variation of the transverse shear strains across the thickness of the nano plate, there was no need for account of shear stress correction factor. Mori-Tanaka homogenization scheme was used for gradation of material properties.

Adim *et al*. (2016) employed a refined shear deformation theory for static, buckling, and free vibration of orthotropic laminated composite plates. Parabolic distribution was assumed for transverse shear stress to satisfy condition of zero shear stress and no need to shear stress correction factor. The bending, free vibration and buckling loads of composite plate were calculated in terms of various parameters. Bennoun *et al*. (2016) developed a new five-variable refined plate theory to derive governing equations of motion of a functionally graded sandwich plate. Effect of gradation of core and face sheets was studied on the free vibration responses of plate. To show accuracy and correctness of obtained results, a comparison with literature including three dimensional elasticity formulation was performed. A quasi-3D theory including influence of normal and shear deformations and anisotropy coupling was employed by Vo *et al*. (2017) for bending analysis of a sandwich beam. Influence of important parameters such fibre angle, lay-up and span-to-height ratio was studied on the distribution of displacements and stresses.

Li *et al*. (2011) employed nonlocal elasticity theory to investigate resonance and stability for the transverse vibrations of a nanobeam subjected to a variable initial axial force, including axial tension and axial compression. Li (2014a, b) studied torsion vibration of cylindrical nanostructures and carbon nanotube based on higher-order stress and two nonlocal models. Nonlocal stress gradient theory was employed by Liu *et al*. (2017) to dynamic analysis and stabilities of axially moving nano-beams with time-dependent velocity. He mentioned that the natural frequencies are increased with increase of nonlocal parameter.

Li *et al*. (2015) studied longitudinal dynamic behaviors of some common one-dimensional nanostructures such as nanorods and nanotubes based on the hardening nonlocal approach. The numerical results were presented to show nonlocal longitudinal vibration responses under various boundary conditions including soft and hard constraints. Shen and Li (2017) used modified semi-continuum Euler beam model for bending analysis of micro/nano-beam. Principle of minimum potential energy was used to derive governing equations.

Zhu and Li (2017) studied nonlocal static analysis of functionally graded nanotubes using Eringen's nonlocal integral elasticity. In spite of others papers in nanoscale that used nonlocal differential model, this paper used nonlocal integral model for calculating the twisting static behavior of functionally graded nanotubes. Wave propagation analysis of a viscoelastic SWCNTs subjected to magnetic field with surface effect was studied by Li *et al*. (2016a) based on nonlocal strain gradient theory. It was mentioned that phase velocity was increased by increase of damping parameter, surface effect and magnetic field. Free vibration analysis of a size dependent Timoshenko beam made of functionally graded materials was studied by Li *et al*. (2016b) based on nonlocal strain gradient theory. They reached to novel conclusions through change of material characteristic parameter larger or smaller than the nonlocal parameter.

Strain gradient theory was studied by authors to investigate vibration and bending analysis of micro-structures subjected to applied electric and magnetic potentials (Ansari *et al*. 2013, Arefi and Zenkour 2017a, b, c, d, Li 2013, Zhang *et al*. 2015, Mohammadimehr *et al*. 2016, Şimşek 2016, Li *et al*. 2017). Also, nonlocal elasticity Eringen's theory was used for thermo-magneto-electro-elastic analysis of nano structures using various classic and advanced theories (Kaghazian *et al*. 2017, Arefi 2016a, b, Arefi and Zenkour 2016a, b, Zenkour and Arefi 2017). In addition, there have been a number of size-dependent models developed for the vibration analysis of microbeams based on the modified couple stress theory (Akgöz and Civalek 2013, Tang *et al*. 2014, Ghadiri and Shafiei 2016, Sourki and Hoseini 2016).

Literature review has been completed above. It shows that although various analyses about shear deformation theory and micro scale problems have been performed, combination of strain gradient theory and normal and shear deformation theory cannot be observed in various studies. Vibration analysis of a three-layered microbeam is studied in this paper based on strain gradient theory and three-unknown shear and normal deformation theory. Hamilton's principle is used for derivation of the governing equations of motion. Effect of three micro-length-scale parameters is studied on the responses of sandwich microbeam. the results of this study can be used for design and manufacturing of micro-electro mechanical systems including sensor and actuator components.

2. Basic relations

In this paper, free vibration responses of a three-layered piezo-magnetic shear deformable microbeam are studied. Strain gradient theory and three unknown shear and normal deformation theory are employed for formulation of the problem. The assumptions for our problem are presented as follows:

(1) Three-unknown shear and normal deformations theory was used in this paper. Based on this theory, the normal strain along z direction is considered.
(2) The axial deformation of mid-surface is ignored.
(3) No discontinuity is considered between core and face-sheets. Based on this assumption, the displacement field is assumed continuous along the thickness direction.
(4) The solution was proposed for simply-supported boundary conditions.

Based on this theory, two longitudinal and transverse deformations $u(x,z)$ and $w(x,z)$ of microbeam are expressed as (Arefi and Zenkour 2016a)

$$\begin{aligned} u(x,z,t) &= \delta_1 u_0(x,t) - z\frac{\mathrm{d}w_1(x,t)}{\mathrm{d}x} - \delta_2\Phi_1(z)\frac{\mathrm{d}w_2(x,t)}{\mathrm{d}x}, \\ w(x,z,t) &= w_1(x,t) + \delta_2 w_2(x,t) + \delta_2\Phi_2(z)w_3(x,t), \end{aligned} \tag{1}$$

where u is axial displacement of mid-plane; w_1 and w_2 are the bending and shear components of the transverse displacement, and w_3 is an additional function of x that shows higher order transverse deflection. In this study and based on assumed displacement field, the following functions may be supposed as (Arefi and Zenkour 2016a)

$$\Phi_1(z) = z - \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right), \quad \Phi_2(z) = \cos\left(\frac{\pi z}{h}\right), \quad \delta_1 = 0, \quad \delta_2 = 1. \tag{2}$$

The linear strain-displacement relations are used to derive strain components as follows

$$\begin{aligned} \varepsilon_{xx} &= \delta_1\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} - \delta_2\Phi_1\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2}, \\ \varepsilon_{zz} &= \delta_2\Phi_2' w_3, \quad \gamma_{xz} = \delta_2\Phi_2\frac{\mathrm{d}}{\mathrm{d}x}(w_2 + w_3). \end{aligned} \tag{3}$$

The stress-strain relations for isotropic core are expressed as

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{zz}\\ \sigma_{xz}\end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0\\ c_{13} & c_{33} & 0\\ 0 & 0 & c_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \gamma_{xz}\end{Bmatrix}, \tag{4}$$

where c_{ij} are stiffness coefficients which are expressed as

$$c_{11} = c_{33} = \frac{E}{1-\nu^2}, \quad c_{13} = \frac{E\nu}{1-\nu^2}, \quad c_{55} = \frac{E}{2(1+\nu)}, \tag{5}$$

in which E and ν are Young's modulus and Poisson's ratio respectively. The constitutive relations piezo-magnetic face sheets are expressed as (Arefi and Zenkour 2017a)

$$\begin{Bmatrix}\sigma_1\\ \sigma_3\\ \sigma_5\end{Bmatrix}^p = \begin{bmatrix} c_{11} & c_{13} & 0\\ c_{13} & c_{33} & 0\\ 0 & 0 & c_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{zz}\\ \gamma_{xz}\end{Bmatrix} - \begin{bmatrix} 0 & 0 & e_{13}\\ 0 & 0 & e_{23}\\ e_{31} & 0 & 0\end{bmatrix}\begin{Bmatrix}E_1\\ 0\\ E_3\end{Bmatrix} - \begin{bmatrix} 0 & 0 & q_{13}\\ 0 & 0 & q_{23}\\ e_{31} & 0 & 0\end{bmatrix}\begin{Bmatrix}H_1\\ 0\\ H_3\end{Bmatrix}, \tag{6}$$

in which e_{ij} are the piezoelectric coefficients, q_{ij} are piezomagnetic coefficients, E_i and H_i are the components of electric and magnetic fields, respectively. The electric and magnetic fields (E_i and H_i) are expressed as (Arefi and Zenkour 2017a)

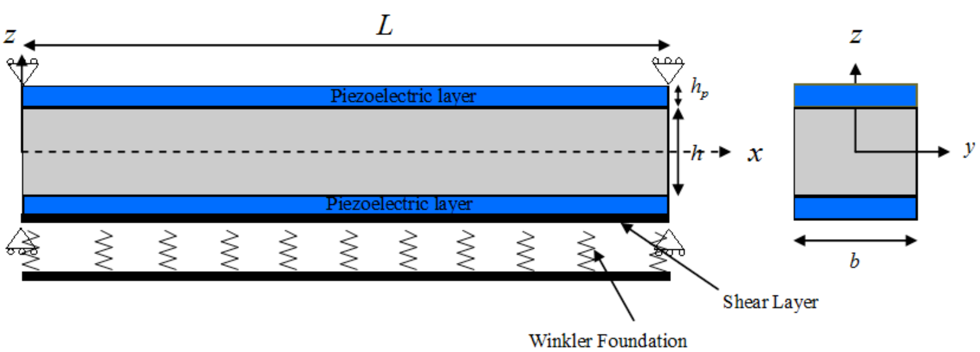

Fig. 1 The schematic of a sandwich microbeam subjected to applied voltage

$$\begin{Bmatrix}\{D_1,B_1\}\\\{D_3,B_3\}\end{Bmatrix} = \begin{bmatrix}0 & 0 & \{e_{15},q_{15}\}\\\{e_{13},q_{13}\} & \{e_{13},q_{13}\} & 0\end{bmatrix}\begin{Bmatrix}\varepsilon_1-\alpha T\\\varepsilon_3-\alpha T\\\varepsilon_5\end{Bmatrix} - \begin{bmatrix}\{\epsilon_{11},m_{11}\} & 0\\0 & \{\epsilon_{33},m_{33}\}\end{bmatrix}\begin{Bmatrix}E_1\\E_3\end{Bmatrix} - \begin{bmatrix}\{m_{11},\mu_{11}\} & 0\\0 & \{m_{33},\mu_{33}\}\end{bmatrix}\begin{Bmatrix}H_1\\H_3\end{Bmatrix}, \quad (7)$$

where ϵ_{ii}, m_{ii} and μ_{ij} are the dielectric, electromagnetic and magnetic coefficients, respectively. The electric and magnetic fields are expressed as

$$\{E_1,E_3,H_1,H_3\} = -\left\{\frac{\partial\bar\psi}{\partial x},\frac{\partial\bar\psi}{\partial z},\frac{\partial\bar\phi}{\partial x},\frac{\partial\bar\phi}{\partial z}\right\}, \quad (8)$$

in which $\bar\psi$ and $\bar\phi$ are the electric and magnetic potentials. For a nanobeam excited with initial electric and magnetic potentials, we can assume electric and magnetic potentials as follows

$$\bar\psi(x,z,t) = \frac{2\check z}{h_p}\psi_0 - \psi(x,t)\cos\left(\frac{\pi\check z}{h_p}\right),$$
$$\bar\phi(x,z,t) = \frac{2\check z}{h_p}\phi_0 - \phi(x,t)\cos\left(\frac{\pi\check z}{h_p}\right), \quad (9)$$

in which for simplification, $\check z = z \pm \frac{h_e}{2} \pm \frac{h_p}{2}$, ψ_0,ϕ_0 are applied electric and magnetic potentials and $\psi(x,t)$, $\phi(x,t)$ are electric and magnetic potentials along the longitudinal directions. For the electric and magnetic potentials, the following electric and magnetic field can be derived as

$$\{E_1,H_1\} = -\left\{\frac{\partial\bar\psi}{\partial x},\frac{\partial\bar\phi}{\partial x}\right\} = \left\{\frac{\partial\psi}{\partial x},\frac{\partial\phi}{\partial x}\right\}\cos\left(\frac{\pi\check z}{h}\right),$$
$$\{E_3,H_3\} = -\left\{\frac{\partial\bar\psi}{\partial z},\frac{\partial\bar\phi}{\partial z}\right\} = -\frac{2}{h_p}\{\psi_0,\phi_0\} - \frac{\pi}{h_p}\{\psi,\phi\}\sin\left(\frac{\pi\check z}{h}\right). \quad (10)$$

Substitution of strain components into stress relations of core leads to

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left(\delta_1\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2} - \delta_2\Phi_1\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2} + \nu\delta_2\Phi_2^{'}w_3\right), \quad (11)$$

$$\sigma_{zz} = \frac{E}{1-\nu^2}\left[\delta_2\Phi_2^{'}w_3 - +\nu\left(\delta_1\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2} - \delta_2\Phi_1\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}\right)\right], \quad (12)$$

$$\tau_{xz} = \frac{E}{2(1+\nu)}\delta_2\Phi_2\left(\frac{\mathrm{d}w_2}{\mathrm{d}x} + \frac{\mathrm{d}w_3}{\mathrm{d}x}\right). \quad (13)$$

Also, the stress components of piezo-magnetic face-sheets are expressed as

$$\sigma_{xx}^p = c_{11}\left(\delta_1\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2} - \delta_2\Phi_1\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}\right) + c_{13}\delta_2 v^{'} m\Phi_2^{'}w_3 + e_{13}\left[\frac{2\psi_0}{h_p} + \frac{\pi}{h_p}\psi\sin\left(\frac{\pi\check z}{h_p}\right)\right] + q_{13}\left[\frac{2\phi_0}{h_p} + \frac{\pi}{h_p}\phi\sin\left(\frac{\pi\check z}{h}\right)\right], \quad (14)$$

$$\sigma_{zz}^p = c_{13}\left(\delta_1\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2} - \delta_2\Phi_1\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}\right) + c_{33}\delta_2\Phi_2^{'}w_3 + e_{23}\left[\frac{2\psi_0}{h_p} + \frac{\pi}{h_p}\psi\sin\left(\frac{\pi\check z}{h_p}\right)\right] + q_{23}\left[\frac{2\phi_0}{h_p} + \frac{\pi}{h_p}\phi\sin\left(\frac{\pi\check z}{h}\right)\right], \quad (15)$$

$$\tau_{xz}^p = c_{55}\delta_2\Phi_2\left(\frac{\mathrm{d}w_2}{\mathrm{d}x} + \frac{\mathrm{d}w_3}{\mathrm{d}x}\right) - e_{31}\frac{\partial\psi}{\partial x}\cos\left(\frac{\pi\check z}{h_p}\right) - q_{31}\frac{\partial\phi}{\partial x}\cos\left(\frac{\pi\check z}{h_p}\right). \quad (16)$$

The electric displacement and magnetic induction are derived as

$$\{D_1,B_1\} = \{e_{15},q_{15}\}\delta_2\Phi_2\left(\frac{\mathrm{d}w_2}{\mathrm{d}x} + \frac{\mathrm{d}w_3}{\mathrm{d}x}\right) - \{\epsilon_{11},m_{11}\}\frac{\partial\psi}{\partial x}\cos\left(\frac{\pi\check z}{h_p}\right) - \{m_{11},\mu_{11}\}\frac{\partial\phi}{\partial x}\cos\left(\frac{\pi\check z}{h_p}\right), \quad (17)$$

$$\{D_3,B_3\} = \{e_{13},q_{13}\}\left(\delta_1\frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2} - \delta_2\Phi_1\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2} + \delta_2\Phi_2^{'}w_3\right) + \{\epsilon_{33},m_{33}\}\left[\frac{2\psi_0}{h_p} + \frac{\pi}{h_p}\psi\sin\left(\frac{\pi\check z}{h_p}\right)\right] + \{m_{33},\mu_{33}\}\left[\frac{2\phi_0}{h_p} - \frac{\pi}{h}\phi\sin\left(\frac{\pi\check z}{h_p}\right)\right]. \quad (18)$$

After completion of magneto-electro-elastic relations, in this stage we can start formulation of strain gradient theory. The basic relations for strain gradient theory are expressed as follows (Wang *et al*. 2010, Lam *et al*. 2003)

$$p_i = 2\mu l_0^2\gamma_i, \quad \tau_{ijk} = 2\mu l_1^2\eta_{ijk}, \quad m_{ij} = 2\mu l_2^2\chi_{ij}, \quad (19)$$

in which p_i are stress couples, τ_{ijk} and m_{ij} are higher-order stress tensors, η_{ijk} and χ_{ij} are deviatoric stretch gradient tensor and symmetric gradient rotation tensor and γ_i is dilatation gradient tensor. In addition, μ is the bulk shear modulus and l_0, l_1 and l_2 are three micro-length-scale parameters. In Eq. (21), the deviatoric stretch gradient tensor, symmetric gradient rotation tensor and corresponding dilatation are defined as (Wang *et al*. 2010)

$$\gamma_i = \frac{\partial \varepsilon_{kk}}{\partial x_i},$$
$$\eta_{ijk} = \eta_{ijk}^s - \frac{1}{5}\left(\delta_{ij}\eta_{mmk}^s + \delta_{jk}\eta_{mmi}^s + \delta_{ki}\eta_{mmj}^s\right), \quad (20)$$
$$\chi_{ij} = \frac{1}{4}\left(\epsilon_{ipq}\frac{\partial \varepsilon_{qj}}{\partial x_p} + \epsilon_{jpq}\frac{\partial \varepsilon_{qi}}{\partial x_p}\right),$$

in which $\eta_{ijk}^s = \frac{1}{3}\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial^2 u_j}{\partial x_k \partial x_i} + \frac{\partial^2 u_k}{\partial x_i \partial x_j}\right)$, ϵ_{ijk} is the permutation symbol and δ_{ij} is Kronecker delta.

By using the displacement field defined in Eq. (3), the nonzero higher-order strains η_{ijk}^s and summation of them are derived as

$$\eta_{111}^s = \delta_1 \frac{d^2 u_0}{dx^2} - x_3 \frac{d^3 w_1}{dx^3} - \delta_2 \Phi_1(z)\frac{d^3 w_2}{dx^3},$$
$$\eta_{113}^s = \eta_{131}^s = \eta_{311}^s = \frac{1}{3}\left\{-\frac{d^2 w_1}{dx^2} + \delta_2\left[\Phi_2 \frac{d^2 w_3}{dx^2} + (1 - 2\Phi_1')\frac{d^2 w_2}{dx^2}\right]\right\},$$
$$\eta_{133}^s = \eta_{313}^s = \eta_{331}^s = \frac{1}{3}\delta_2\left(2\Phi_2' \frac{dw_3}{dx} - \Phi_1'' \frac{dw_2}{dx}\right),$$
$$\eta_{333}^s = \delta_2 \Phi_2'' w_3, \quad (21)$$
$$\eta_{mm1}^s = \eta_{111}^s + \eta_{221}^s + \eta_{331}^s = \delta_1 \frac{d^2 u_0}{dx^2} - x_3 \frac{d^3 w_1}{dx^3} - \delta_2\left[\Phi_1 \frac{d^3 w_2}{dx^3} + \frac{1}{3}\left(2\Phi_2' \frac{dw_3}{dx} - \Phi_1'' \frac{dw_2}{dx}\right)\right],$$
$$\eta_{mm3}^s = \eta_{113}^s + \eta_{223}^s + \eta_{333}^s = \frac{1}{3}\left\{-\frac{d^2 w_1}{dx^2} + \delta_2\left[\Phi_2 \frac{d^2 w_3}{dx^2} + (1 - 2\Phi_1')\frac{d^2 w_2}{dx^2}\right]\right\} + \delta_2 \Phi_2'' w_3.$$

The elements of dilatation gradient tensor γ_i are derived as

$$\gamma_1 = \delta_1 \frac{d^2 u_0}{dx^2} - z\frac{d^3 w_1}{dx^3} - \delta_2 \Phi_1 \frac{d^3 w_2}{dx^3} + \delta_2 \Phi_2' \frac{dw_3}{dx},$$
$$\gamma_3 = -\frac{d^2 w_1}{dx^2} - \delta_2 \Phi_1' \frac{d^2 w_2}{dx^2} + \delta_2 \Phi_2'' w_3, \quad (22)$$

and finally all none-zero components of deviatoric stretch gradient tensor and symmetric gradient rotation tensor (η_{ijk}, χ_{ij}) are derived as

$$\eta_{111} = \frac{2}{5}\delta_1 \frac{d^2 u_0}{dx^2} - \frac{2}{5}z\frac{d^3 w_1}{dx^3} - \frac{2}{5}\delta_2 \Phi_1 \frac{d^3 w_2}{dx^3} - \frac{2}{5}\delta_2 \Phi_2' \frac{dw_3}{dx} + \frac{1}{5}\delta_2 \Phi_2'' \frac{dw_2}{dx}, \quad (23a)$$

$$\eta_{111} = \frac{2}{5}\delta_1 \frac{d^2 u_0}{dx^2} - \frac{2}{5}z\frac{d^3 w_1}{dx^3} - \frac{2}{5}\delta_2 \Phi_1 \frac{d^3 w_2}{dx^3} - \frac{2}{5}\delta_2 \Phi_2' \frac{dw_3}{dx} + \frac{1}{5}\delta_2 \Phi_2'' \frac{dw_2}{dx}, \quad (23b)$$

$$\eta_{133} = -\frac{1}{5}\delta_1 \frac{d^2 u_0}{dx^2} + \frac{1}{5}z\frac{d^3 w_1}{dx^3} + \frac{1}{5}\delta_2 \Phi_1 \frac{d^3 w_2}{dx^3} + \frac{8}{15}\delta_2 \Phi_2' \frac{dw_3}{dx} - \frac{4}{15}\delta_2 \Phi_1'' \frac{dw_2}{dx}, \quad (23c)$$

$$\eta_{113} = -\frac{4}{15}\frac{d^2 w_1}{dx^2} + \frac{4}{15}\delta_2 \Phi_2 \frac{d^2 w_3}{dx^2} + \frac{4}{15}\delta_2(1 - 2\Phi_1')\frac{d^2 w_2}{dx^2} - \frac{1}{5}\delta_2 \Phi_2'' w_3, \quad (23d)$$

$$\chi_{12} = -\frac{1}{4}\left[\frac{d^2 w_1}{dx^2} + \delta_2(\Phi_1' + \Phi_2)\frac{d^2 w_2}{dx^2} + \delta_2 \Phi_2 \frac{d^2 w_3}{dx^2}\right]. \quad (23e)$$

After derivation of required terms for γ_i, χ_{ij} and η_{ijk}, the corresponding stress components p_i, m_{ij} and τ_{ijk} are expressed as

$$p_1 = 2\mu l_0^2\left(\delta_1 \frac{d^2 u_0}{dx^2} - z\frac{d^3 w_1}{dx^3} - \delta_2 \Phi_1 \frac{d^3 w_2}{dx^3} + \delta_2 \Phi_2' \frac{dw_3}{dx}\right), \quad (24a)$$

$$p_3 = 2\mu l_0^2\left(-\frac{d^2 w_1}{dx^2} - \delta_2 \Phi_1' \frac{d^2 w_2}{dx^2} + \delta_2 \Phi_2'' w_3\right), \quad (24b)$$

$$\tau_{111} = \frac{4}{5}\mu l_1^2\left[\delta_1 \frac{d^2 u_0}{dx^2} - z\frac{d^3 w_1}{dx^3} - \delta_2\left(\Phi_1 \frac{d^3 w_2}{dx^3} - \Phi_2' \frac{dw_3}{dx} + \frac{1}{2}\Phi_1'' \frac{dw_2}{dx}\right)\right], \quad (24c)$$

$$\tau_{333} = \frac{2}{5}\mu l_1^2\left\{\frac{d^2 w_1}{dx^2} - \delta_2\left[\Phi_2 \frac{d^2 w_3}{dx^2} - (1 - 2\Phi_1')\frac{d^2 w_2}{dx^2} + 2\Phi_2'' w_3\right]\right\}, \quad (24d)$$

$$\tau_{133} = \tau_{313} = \tau_{331} = \frac{2}{5}\mu l_1^2\left[-\delta_1 \frac{d^2 u_0}{dx^2} + z\frac{d^3 w_1}{dx^3} + \delta_2\left(\Phi_1 \frac{d^3 w_2}{dx^3} + \frac{8}{3}\Phi_2' \frac{dw_3}{dx} - \frac{4}{3}\Phi_1'' \frac{dw_2}{dx}\right)\right], \quad (24e)$$

$$\tau_{133} = \tau_{313} = \tau_{331} = \frac{2}{5}\mu l_1^2\left[-\delta_1 \frac{d^2 u_0}{dx^2} + z\frac{d^3 w_1}{dx^3} + \delta_2\left(\Phi_1 \frac{d^3 w_2}{dx^3} + \frac{8}{3}\Phi_2' \frac{dw_3}{dx} - \frac{4}{3}\Phi_1'' \frac{dw_2}{dx}\right)\right], \quad (24f)$$

$$m_{12} = -\frac{1}{2}\mu l_2^2\left[\frac{\mathrm{d}^2 w_1}{\mathrm{d}x^2} + \delta_2(\Phi_1' + \Phi_2)\frac{\mathrm{d}^2 w_2}{\mathrm{d}x^2} + \delta_2\Phi_2\frac{\mathrm{d}^2 w_3}{\mathrm{d}x^2}\right]. \tag{24g}$$

After derivation of the required expressions for stresses, electric displacements and higher-order stresses corresponding to strain gradient theory, the dynamic version of principle of virtual work $\delta\int(U - W - T)\mathrm{d}t = 0$ may be employed for derivation of the governing equations of magneto-electro-elastic bending as follows

$$\delta U = \int\begin{pmatrix} p_i\delta\gamma_i + \sigma_{ij}\,\delta\varepsilon_{ij} + \tau_{ijk}\,\delta\eta_{ijk} \\ +m_{ij}\,\delta\chi_{ij} - D_i\delta E_i - B_i\delta H_i \end{pmatrix}\mathrm{d}V,$$

$$\delta W = \int b\left(q\delta w|_{z=\frac{h}{2}} - R_f\,\delta w|_{z=-\frac{h}{2}}\right)\mathrm{d}x, \tag{25}$$

$$R_f = (K_1 - K_2\nabla^2)w,$$

$$\delta T = \iint \rho[\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}]\mathrm{d}z\mathrm{d}x,$$

in which R_f is reaction of Pasternak's foundation. In addition, U is strain energy, W is the work performed by external works and T is the kinetic energy. In addition, ρ denotes material density, K_1 and K_2 are direct and shear parameters of Pasternak's foundation. Substitution of variation of strains $\delta\varepsilon_{ij}$, dilatation gradient tensor $\delta\gamma_i$, deviatoric stretch gradient tensor $\delta\eta_{ijk}$, symmetric rotation gradient tensor $\delta\chi_{ij}$, electric field δE_i and magnetic field δH_i into variation form of energy equation and integration by part on the derived equations. After completion of required terms of strain energy and energy due to external works, the final governing equations of magneto-electro-elastic bending are derived as

$$\delta u_0:\ \frac{\mathrm{d}^2 N_1}{\mathrm{d}x^2} - \frac{\mathrm{d}N_{11}}{dx} + \frac{2}{5}\frac{\mathrm{d}^2 N_{111}}{\mathrm{d}x^2} - \frac{3}{5}\frac{\mathrm{d}^2 N_{133}}{\mathrm{d}x^2} = -B_1\ddot{u}_0 + B_2\frac{\mathrm{d}\ddot{w}_2}{\mathrm{d}x} + B_3\frac{\mathrm{d}\ddot{w}_1}{\mathrm{d}x}, \tag{26a}$$

$$\delta w_1:\ \frac{\mathrm{d}^3 S_1}{\mathrm{d}x^3} - \frac{\mathrm{d}^2 F_3}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 S_{11}}{\mathrm{d}x^2} - \frac{\mathrm{d}I_{13}}{\mathrm{d}x} - \frac{1}{5}\frac{\mathrm{d}J_{111}}{\mathrm{d}x} + \frac{2}{5}\frac{d^3 S_{111}}{\mathrm{d}x^3} - \frac{1}{5}\frac{\mathrm{d}^2 L_{333}}{\mathrm{d}x^2} - \frac{3}{5}\frac{\mathrm{d}^3 S_{133}}{\mathrm{d}x^3} + \frac{5}{5}\frac{\mathrm{d}J_{133}}{\mathrm{d}x} + \frac{4}{5}\frac{\mathrm{d}^2 L_{113}}{\mathrm{d}x_1^2} \tag{26b}$$

$$-\frac{1}{2}\frac{\mathrm{d}^2 O_{12}}{\mathrm{d}x^2} = (q - R_f)\delta_2 - B_3\frac{\mathrm{d}\ddot{u}_0}{\mathrm{d}x} + B_5\frac{\mathrm{d}^2\ddot{w}_2}{\mathrm{d}x^2} + B_6\frac{\mathrm{d}^2\ddot{w}_1}{\mathrm{d}x^2} - B_8\ddot{w}_2 - B_{10}\,\ddot{w}_1 - B_{11}\ddot{w}_3\,,$$

$$\delta w_2:\ \frac{\mathrm{d}^3 M_1}{\mathrm{d}x^3} - \frac{\mathrm{d}^2 N_3}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 M_{11}}{\mathrm{d}x^2} + \frac{2}{5}\frac{\mathrm{d}^3 M_{111}}{\mathrm{d}x^3} + \frac{1}{5}\frac{\mathrm{d}^2 N_{333}}{\mathrm{d}x_1^2} - \frac{3}{5}\frac{\mathrm{d}^3 M_{133}}{\mathrm{d}x^3} - \frac{12}{15}\frac{\mathrm{d}^2 N_{113}}{\mathrm{d}x^2} - \frac{1}{2}\frac{\mathrm{d}^2 M_{12}}{\mathrm{d}x^2}$$
$$= (q - R_f)\delta w_2 - B_2\frac{\mathrm{d}\ddot{u}_0}{\mathrm{d}x} + B_4\frac{\mathrm{d}^2\ddot{w}_2}{\mathrm{d}x^2} + B_5\frac{\mathrm{d}^2\ddot{w}_1}{\mathrm{d}x^2} - B_7\ddot{w}_2 - B_8\ddot{w}_1 - B_9\ddot{w}_3\,, \tag{26c}$$

$$\delta w_3:\ -\frac{\mathrm{d}Q_1}{\mathrm{d}x} + G_3 + Q_{33} - \frac{\mathrm{d}I_{13}}{\mathrm{d}x} + \frac{2}{5}\frac{\mathrm{d}Q_{111}}{\mathrm{d}x} - \frac{1}{5}\frac{\mathrm{d}^2 I_{333}}{\mathrm{d}x^2} + \frac{2}{5}G_{333} - \frac{8}{5}\frac{\mathrm{d}Q_{133}}{\mathrm{d}x} + \frac{4}{5}\frac{\mathrm{d}^2 I_{113}}{\mathrm{d}x^2} - \frac{3}{5}G_{113} \tag{26d}$$

$$-\frac{1}{2}\frac{d^2 I_{12}}{dx^2} = \delta_2\left[q\Phi_2\big|_{z=\frac{h}{2}} - R_f\Phi_2\big|_{z=-\frac{h}{2}}\right] - B_9\ddot{w}_2 - B_{11}\,\ddot{w}_1 - B_{12}\ddot{w}_3,$$

$$\delta\psi:\ \frac{\mathrm{d}\bar{D}_1}{\mathrm{d}x} + \bar{D}_3 = 0, \tag{26e}$$

$$\delta\phi:\ \frac{\mathrm{d}\bar{B}_1}{\mathrm{d}x_1} + \bar{B}_3 = 0, \tag{26f}$$

in which the resultant components are defined as

$$\{N_1, M_1, S_1, Q_1\} = \int p_1\{\delta_1, \delta_1 z, \delta_2\Phi_1, \delta_2\Phi_2'\}\mathrm{d}z,$$

$$\{N_3, F_3, G_3\} = \int p_3\{1, \delta_2\Phi_1', \delta_2\Phi_2''\}\mathrm{d}z,$$

$$\{N_{11}, M_{11}, S_{11}\} = \int \sigma_{11}\{\delta_1, \delta_1 z, \delta_2\Phi_1\}\mathrm{d}z,$$

$$\{I_{13}, Q_{33}\} = \int \delta_2\{\Phi_2\sigma_{13}, \Phi_2'\sigma_{33}\}\mathrm{d}z,$$

$$\{N_{111}, M_{111}, S_{111}, Q_{111}, J_{111}\} = \int \tau_{111}\{\delta_1, \delta_1 z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}\mathrm{d}z,$$

$$\{N_{133}, M_{133}, S_{133}, Q_{133}, J_{133}\} = \int \tau_{133}\{1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}\mathrm{d}z, \tag{27}$$

$$\{N_{113}, I_{113}, L_{113}, G_{113}\} = \int \tau_{113}\{1, \delta_2\Phi_2, \delta_2(1 - 2\Phi_1'), \delta_2\Phi_2''\}\mathrm{d}z,$$

$$\{N_{333}, I_{333}, L_{333}, G_{333}\} = \int \tau_{333}\{1, \delta_2\Phi_2, \delta_2(1 - 2\Phi_1'), \delta_2\Phi_2''\}\mathrm{d}z,$$

$$\{M_{12}, I_{12}, O_{12}\} = \int m_{12}\{1, \delta_2\Phi_2, \delta_2(\Phi_1' + \Phi_2)\}\mathrm{d}z,$$

$$\{\bar{D}_1, \bar{B}_1\} = \int\{D_1, B_1\}\cos\left(\frac{\pi\tilde{x}_3}{h_p}\right)\mathrm{d}z,$$

$$\{\bar{D}_3, \bar{B}_3\} = \int\{D_1, B_1\}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{x}_3}{h_p}\right)\mathrm{d}z.$$

The substitution of strain components, electric and magnetic fields gives the resultant components in Appendix A.

3. Analytical solution and numerical results

Before presentation of full numerical results, the solution procedure must be described. The analytical solution is presented based on Navier's method and Fourier series for a simply-supported microbeam are expressed as

$$\begin{Bmatrix} u \\ w_2 \\ w_1 \\ w_3 \\ \psi \\ \phi \end{Bmatrix} = e^{i\omega t} \sum_{n=1}^{\infty} \begin{Bmatrix} U_n \cos(\lambda_n x) \\ V_n \sin(\lambda_n x) \\ W_n \sin(\lambda_n x) \\ X_n \sin(\lambda_n x) \\ Y_n \sin(\lambda_n x) \\ Z_n \sin(\lambda_n x) \end{Bmatrix}, \tag{28}$$

where $\lambda_n = n\pi/L$ and U_n, W_n, V_n, X_n, Y_n, and Z_n are unknown amplitudes of unknown variables used in our analysis. In addition, applied electric and magnetic potentials and transverse loads may be expressed as a single Fourier series as follows

$$\{q, \psi_0, \phi_0\} = \sum_{n=1}^{\infty} \{q^0, Y_0, Z_0\} \sin(\lambda_n x). \tag{29}$$

By substitution of proposed solutions and loadings into governing differential equations of the microbeam, we have

$$[K]\{\Sigma\} = \{F\} + \omega^2 [M]\{\Sigma\}, \tag{30}$$

where $\{\Sigma\} = \{U_n, V_n, W_n, X_n, Y_n, Z_n\}^T$, $[K]$ is the symmetric stiffness matrix and $\{F\}$ is force matrix. The elements of the stiffness matrix $K_{ij} = K_{ji}$, mass matrix $M_{ij} = M_{ji}$ and force matrix are given by

$$K_{11} = \lambda_n^4 \left(A_1 + \frac{3}{25} A_{94} + \frac{4}{25} A_{53} \right) + \lambda_n^2 A_{26},$$

$$K_{12} = -\lambda_n^5 \left(A_3 + \frac{4}{25} A_{55} + \frac{3}{25} A_{96} \right) - \lambda_n^3 \left(A_{28} + \frac{2}{25} A_{57} + \frac{4}{25} A_{98} \right),$$

$$K_{13} = -\lambda_n^5 \left(A_2 + \frac{4}{25} A_{54} + \frac{3}{25} A_{95} \right) - \lambda_n^3 A_{27},$$

$$K_{14} = -\lambda_n^3 \left(A_4 - \frac{4}{25} A_{56} - \frac{24}{75} A_{97} \right) - \lambda_n A_{29},$$

$$K_{15} = \lambda_n A_{30}, \quad K_{16} = \lambda_n A_{31},$$

$$M_{11} = B_1, \quad M_{12} = -\lambda_n B_3, \quad M_{13} = -\lambda_n B_2,$$

$$F_1 = -\frac{\mathrm{d}N_{11}^T}{\mathrm{d}x} - \frac{\mathrm{d}N_{13}^T}{\mathrm{d}x} + \frac{\mathrm{d}N_{11}^{\psi_0}}{\mathrm{d}x} + \frac{\mathrm{d}N_{11}^{\phi_0}}{\mathrm{d}x},$$

$$K_{22} = \lambda_n^6 \left(\frac{4}{25} A_{65} + \frac{3}{25} A_{106} + A_{11} \right) + \lambda_n^2 \left(\frac{16}{75} A_{161} + \frac{1}{25} A_{77} + A_{50} \right) + \delta_2^2 (K_1 + K_2 \lambda_n^2) + \lambda_n^4 \left(A_{40} + \frac{2}{25} A_{75} + A_{21} + \frac{2}{25} A_{67} + \frac{16}{75} A_{124} + \frac{1}{2} A_{134} + \frac{1}{25} A_{88} + \frac{12}{75} A_{108} + \frac{4}{25} A_{159} \right),$$

$$K_{23} = \lambda_n^6 \left(A_{10} + \frac{4}{25} A_{64} + \frac{3}{25} A_{105} \right) + \delta_2 (K_1 + K_2 \lambda_n^2) + \lambda_n^4 \left(A_{20} + A_{39} + \frac{2}{25} A_{74} - \frac{1}{25} A_{86} + \frac{1}{2} A_{133} + \frac{4}{25} A_{158} - \frac{16}{75} A_{122} \right),$$

$$K_{24} = \lambda_n^4 \left(A_{12} - \frac{4}{25} A_{66} + \frac{1}{25} A_{87} + \frac{1}{2} A_{135} - \frac{24}{75} A_{107} + \frac{16}{75} A_{123} \right) - \lambda_n^2 \left(\frac{32}{75} A_{160} - \frac{4}{25} A_{125} - \frac{2}{25} A_{89} - A_{22} - A_{41} - A_{50} + \frac{2}{25} A_{76} \right) + \delta_2^2 \Phi_2 (K_1 + K_2 \lambda_n^2),$$

$$K_{25} = \lambda_n^2 (A_{42} - A_{51}), \quad K_{26} = \lambda_n^2 (A_{43} - A_{52}),$$

$$M_{21} = -B_3 \lambda_n, \quad M_{22} = B_{10} + \lambda_n^2 B_6,$$

$$M_{23} = B_8 + \lambda_n^2 B_5, \quad M_{24} = B_{11}, \quad F_2 = \delta_2 q,$$

$$K_{33} = \lambda_n^6 \left(A_6 + \frac{4}{25} A_{59} + \frac{3}{25} A_{100} \right) + \lambda_n^4 \left(A_{17} + A_{33} + \frac{48}{225} A_{114} + \frac{1}{25} A_{78} + \frac{1}{2} A_{130} \right) + K_1 + K_2 \lambda_n^2,$$

$$K_{34} = \lambda_n^4 \left(\frac{1}{2} A_{132} - \frac{4}{25} A_{61} - \frac{1}{25} A_{79} - \frac{24}{75} A_{102} - \frac{48}{225} A_{115} + A_8 \right) + \lambda_n^2 \left(A_{19} + A_{35} - \frac{12}{75} A_{117} - \frac{2}{25} A_{81} \right) + \delta_2 \Phi_2 (K_1 + K_2 \lambda_n^2),$$

$$K_{35} = \lambda_n^2 A_{36}, \quad K_{36} = \lambda_n^2 A_{37},$$

$$M_{31} = -\lambda_n B_2, \quad M_{32} = B_8 + \lambda_n^2 B_5,$$

$$M_{33} = B_7 + \lambda_n^2 B_4, \quad M_{34} = B_9,$$

$$F_3 = -\frac{\mathrm{d}^2 M_{11}^T}{\mathrm{d}x^2} - \frac{\mathrm{d}^2 M_{13}^T}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 M_{11}^{\psi_0}}{\mathrm{d}x^2} + \frac{\mathrm{d}^2 M_{11}^{\phi_0}}{\mathrm{d}x^2} + q,$$

$$K_{44} = \lambda_n^4 \left(\frac{1}{25} A_{83} + \frac{1}{2} A_{138} + \frac{16}{75} A_{119} \right) + \frac{3}{25} A_{129} + \frac{2}{25} A_{93} + A_{47} + A_{25} + \delta_2 (K_1 + K_2 \lambda_n^2) \Phi_2 \big|_{z=-\frac{h}{2}} + \lambda_n^2 \left(+A_{50} + \frac{4}{25} A_{71} + \frac{2}{25} A_{85} + A_{16} + \frac{1}{25} A_{91} + \frac{8}{15} \frac{24}{15} A_{112} + \frac{4}{25} A_{121} + \frac{12}{75} A_{127} \right),$$

$$K_{45} = -\lambda_n^2 A_{51} + A_{48}, \quad K_{46} = -\lambda_n^2 A_{52} + A_{49},$$

$$M_{42} = B_{11}, \quad M_{34} = B_9, \quad M_{44} = B_{12},$$

$$F_4 = Q_{11}^T + Q_{13}^T - Q_{11}^{\psi_0} - Q_{11}^{\phi_0} + \delta_2 q \Phi_2 \big|_{z=\frac{h}{2}},$$

$$K_{55} = A_{140} \lambda_n^2 + A_{146}, \quad K_{56} = A_{141} \lambda_n^2 + A_{147},$$

$$F_5 = \bar{D}_3^T - \bar{D}_3^{\psi_0} - \bar{D}_3^{\phi_0},$$

$$K_{66} = A_{150} \lambda_n^2 + A_{156}, \quad F_6 = \bar{B}_3^T - \bar{B}_3^{\psi_0} - \bar{B}_3^{\phi_0}.$$

In this stage, we present material properties for calculation of stiffness matrix and numerical results. The material properties of core and face-sheets are expressed as (Pradhan and Phadikar 2009)

Core:

$$E = 1.02 \times 10^6 \text{ MPa}, \quad \nu = 0.3.$$

Piezomagnetic face-sheets (BiTiO3–CoFe2O4)

$c_{11} = 226$ GPa, $c_{13} = 124$ GPa,
$c_{33} = 216$ GPa, $c_{55} = 44.2$ GPa,
$e_{13} = -2.2$, $e_{23} = -2.2$, $e_{15} = 5.8$,
$q_{15} = 275$, $\epsilon_{11} = 5.64 \times 10^{-9}$,
$m_{11} = 5.367 \times 10^{-12}$, $\epsilon_{33} = 6.35 \times 10^{-9}$,
$m_{33} = 5.367 \times 10^{-12}$,
$\mu_{11} = -297 \times 10^{-6}$, $\mu_{33} = 83.5 \times 10^{-6}$.

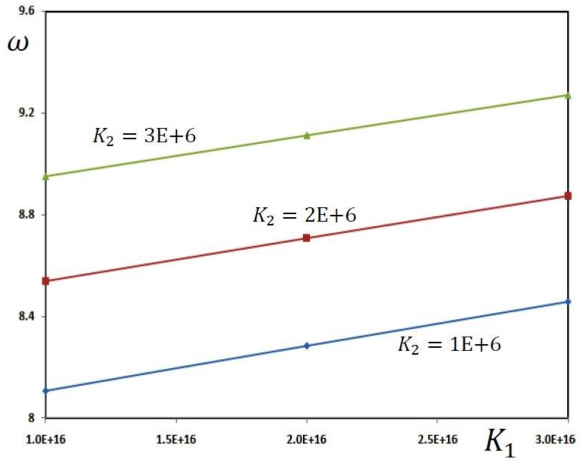

Fig. 2 Variation of fundamental natural frequencies of microbeam in terms of spring parameter of Pasternak's foundation for various shear parameters of foundation

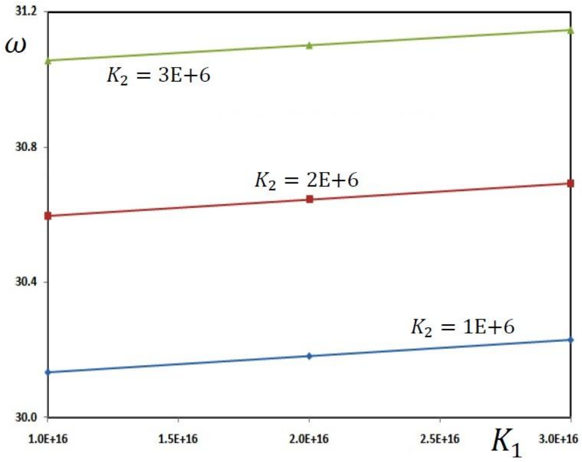

Fig. 3 Variation of second natural frequencies of microbeam in terms of spring parameter of Pasternak's foundation for various shear parameters of foundation

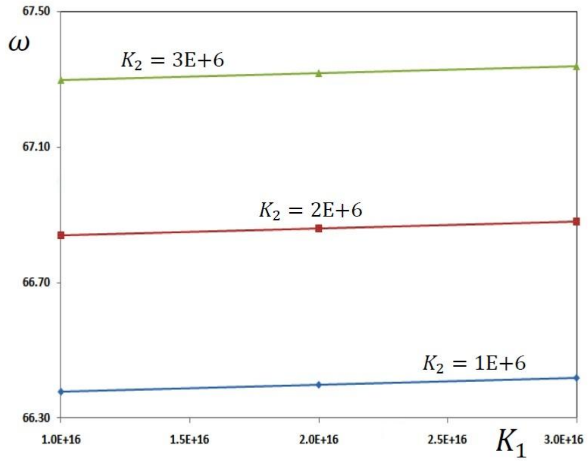

Fig. 4 Variation of third natural frequencies of microbeam in terms of spring parameter of Pasternak's foundation for various shear parameters of foundation

The natural frequencies (in GHz) of three-layered piezo-magnetic microbeam are studied in this section. Fig. 2 shows variation of fundamental natural frequencies of microbeam in terms of spring parameter of Pasternak's foundation for various shear parameters of foundation. It is observed that with increase of both parameters of foundation, the fundamental natural frequencies of micro-beam are increased significantly. Shown in Figs. 3, 4 are 2nd and 3rd natural frequencies of microbeam in terms of spring parameter of Pasternak's foundation for various shear parameters of foundation. It can be concluded that 2nd and 3rd natural frequencies of microbeam are increased with increase of two parameters of foundation due to increase of stiffness of foundation. In addition it can be concluded that the rate of increase of fundamental natural frequencies is more than second and third natural frequencies.

The effect of three micro-length-scale parameters l_0, l_1 and l_2 is studied in this section. for this study, three dimensionless parameters α, β and γ are employed as: $l_0 = \alpha\,(17.65\ \mu\text{m})$, $l_1 = \beta\,(17.65\ \mu\text{m})$ and $l_2 = \gamma\,(17.65\ \mu\text{m})$. Fig. 5 shows fundamental, second and third natural frequencies of microbeam in terms of first dimensionless micro-length-scale parameter α. It is observed that with increase of first dimensionless micro-length-scale parameter α, all natural frequencies of microbeam are increased significantly. One can conclude that increase of first dimensionless micro-length-scale parameter α leads to a stiffer beam and consequently increases natural frequencies. Figs. 6 and 7 shows variation of fundamental, second and third natural frequencies of microbeam in terms of second and third dimensionless micro-length-scale parameter β and γ. The results show that with increase of second and third dimensionless micro-length-scale parameter β and γ, stiffness of microbeam is increased and consequently all natural frequencies of microbeam are increased significantly.

The main conclusion of Figs. 5, 6 and 7 is that the effect

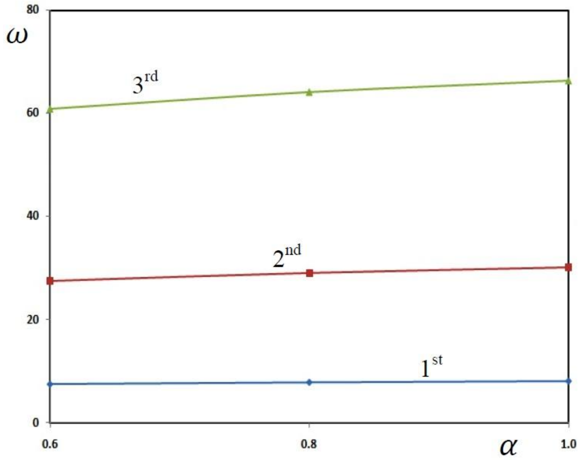

Fig. 5 Fundamental, second and third natural frequencies of microbeam in terms of first dimensionless micro-length-scale parameter α

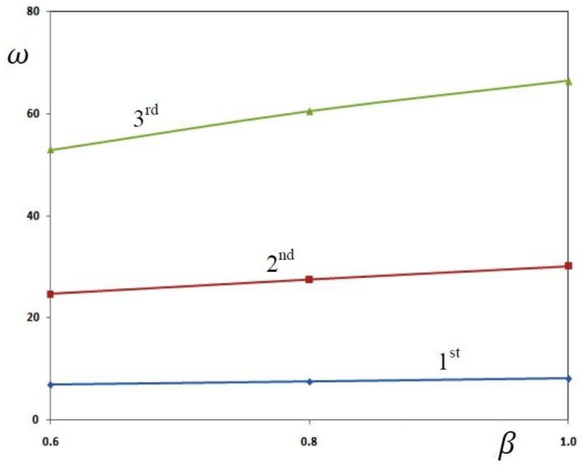

Fig. 6 Fundamental, second and third natural frequencies of microbeam in terms of second dimensionless micro-length-scale parameter β

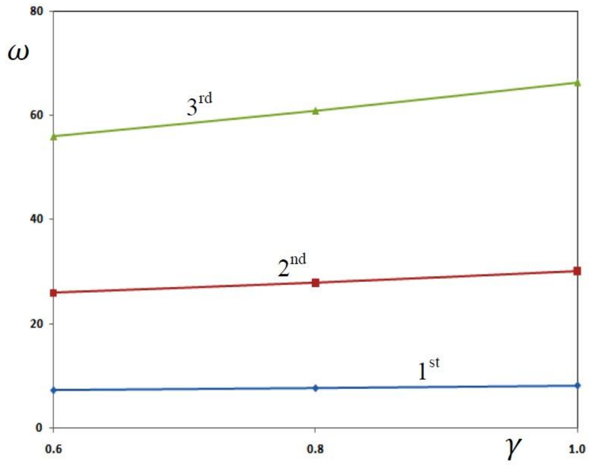

Fig. 7 Fundamental, second and third natural frequencies of microbeam in terms of third dimensionless micro-length-scale parameter γ

of second dimensionless micro-length-scale parameter β is more than effect of first and third dimensionless micro-length-scale parameter α and γ.

4. Conclusions

Free vibration analysis of a three-layered piezo-magnetic microbeam was studied in this paper. Strain gradient theory including three micro-length-scale parameters was employed for analysis of the problem. To increase accuracy of results and better modeling of the microbeam, three-unknown shear and normal deformations beam theory was used. The microbeam was rested on the Pasternak's foundation. A harmonic solution based on Navier's method was used to predict vibration characteristics of three-layered microbeam. The numerical results indicate that two parameters of foundation and three micro-length-scale parameters have significant influences on the vibration characteristics of microbeam. Some significant conclusions of our analysis are expressed as follows:

The foundation's characteristics including spring and shear parameters have significant effects on the first three natural frequencies of microbeam. The vibration analysis of microbeam shows that with increase of spring and shear parameters of foundation, fundamental, second and third natural frequencies are increased significantly.

The numerical results indicate that change of three micro-length-scale parameters α, β and γ leads to significant changes of natural frequencies of microbeam. One can conclude that increase of three parameters α, β and γ leads to significant increase of fundamental, second and third natural frequencies of microbeam. In addition, it can be concluded that influence of second dimensionless parameter β is more important among three dimensionless parameters.

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CC

Appendix

$$N_1 = A_1\frac{d^2u_0}{dx^2} - A_2\frac{d^3w_1}{dx^3} - A_3\frac{d^3w_2}{dx^3} + A_4\frac{dw_3}{dx},$$

$$M_1 = A_5\frac{d^2u_0}{dx^3} - A_6\frac{d^3w_1}{dx^3} - A_7\frac{d^3w_2}{dx^3} + A_8\frac{dw_3}{dx},$$

$$S_1 = A_9\frac{d^2u_0}{dx^2} - A_{10}\frac{d^3w_1}{dx^3} - A_{11}\frac{d^3w_2}{dx^3} + A_{12}\frac{dw_3}{dx},$$

$$Q_1 = A_{13}\frac{d^2u_0}{dx^2} - A_{14}\frac{d^3w_1}{dx^3} - A_{15}\frac{d^3w_2}{dx^3} + A_{16}\frac{dw_3}{dx},$$

$$N_3 = -A_{17}\frac{d^2w_1}{dx^2} - A_{18}\frac{d^2w_2}{dx^2} + A_{19}w_3,$$

$$F_3 = -A_{20}\frac{d^2w_1}{dx^2} - A_{21}\frac{d^2w_2}{dx^2} + A_{22}w_3,$$

$$G_3 = -A_{23}\frac{d^2w_1}{dx^2} - A_{24}\frac{d^2w_2}{dx^2} + A_{25}w_3,$$

$$N_{11} = A_{26}\frac{du_0}{dx} - A_{27}\frac{d^2w_1}{dx^2} - A_{28}\frac{d^2w_2}{dx^2} + A_{29}w_3 + A_{30}\psi + A_{31}\phi - N_{11}^T - N_{13}^T + N_{11}^{\psi_0} + N_{11}^{\phi_0},$$

$$M_{11} = A_{32}\frac{du_0}{dx} - A_{33}\frac{d^2w_1}{dx^2} - A_{34}\frac{d^2w_2}{dx^2} + A_{35}w_3 + A_{36}\psi + A_{37}\phi - M_{11}^T - M_{13}^T + M_{11}^{\psi_0} + M_{11}^{\phi_0},$$

$$S_{11} = A_{38}\frac{du_0}{dx} - A_{39}\frac{d^2w_1}{dx^2} - A_{40}\frac{d^2w_2}{dx^2} + A_{41}w_3 + A_{42}\psi + A_{43}\phi - S_{11}^T - S_{13}^T + S_{11}^{\psi_0} + S_{11}^{\phi_0},$$

$$Q_{33} = A_{44}\frac{du_0}{dx} - A_{45}\frac{d^2w_1}{dx^2} - A_{46}\frac{d^2w_2}{dx^2} + A_{47}w_3 + A_{48}\psi + A_{49}\phi - Q_{11}^T - Q_{13}^T + Q_{11}^{\psi_0} + Q_{11}^{\phi_0},$$

$$I_{13} = A_{50}\left(\frac{dw_2}{dx} + \frac{dw_3}{dx}\right) - A_{51}\frac{\partial\psi}{\partial x} - A_{52}\frac{\partial\phi}{\partial x},$$

$$N_{111} = \frac{2}{5}A_{53}\frac{d^2u_0}{dx^2} - \frac{2}{5}A_{54}\frac{d^3w_1}{dx^3} - \frac{2}{5}A_{55}\frac{d^3w_2}{dx^3} - \frac{2}{5}A_{56}\frac{dw_3}{dx} + \frac{1}{5}A_{57}\frac{dw_2}{dx},$$

$$M_{111} = \frac{2}{5}A_{58}\frac{d^2u_0}{dx^2} - \frac{2}{5}A_{59}\frac{d^3w_1}{dx^3} - \frac{2}{5}A_{60}\frac{d^3w_2}{dx^3} - \frac{2}{5}A_{61}\frac{dw_3}{dx} + \frac{1}{5}A_{62}\frac{dw_2}{dx},$$

$$S_{111} = \frac{2}{5}A_{63}\frac{d^2u_0}{dx^2} - \frac{2}{5}A_{64}\frac{d^3w_1}{dx^3} - \frac{2}{5}A_{65}\frac{d^3w_2}{dx^3} - \frac{2}{5}A_{66}\frac{dw_3}{dx} + \frac{1}{5}A_{67}\frac{dw_2}{dx},$$

$$Q_{111} = \frac{2}{5}A_{68}\frac{d^2u_0}{dx^2} - \frac{2}{5}A_{69}\frac{d^3w_1}{dx^3} - \frac{2}{5}A_{70}\frac{d^3w_2}{dx^3} - \frac{2}{5}A_{71}\frac{dw_3}{dx} + \frac{1}{5}A_{72}\frac{dw_2}{dx},$$

$$J_{111} = \frac{2}{5}A_{73}\frac{d^2u_0}{dx^2} - \frac{2}{5}A_{74}\frac{d^3w_1}{dx^3} - \frac{2}{5}A_{75}\frac{d^3w_2}{dx^3} - \frac{2}{5}A_{76}\frac{dw_3}{dx} + \frac{1}{5}A_{77}\frac{dw_2}{dx},$$

$$N_{333} = \frac{1}{5}A_{78}\frac{d^2w_1}{dx^2} - \frac{1}{5}A_{79}\frac{d^2w_3}{dx^2} - \frac{1}{5}A_{80}\frac{d^2w_2}{dx^2} + \frac{2}{5}A_{81}w_3,$$

$$I_{333} = \frac{1}{5}A_{82}\frac{d^2w_1}{dx^2} - \frac{1}{5}A_{83}\frac{d^2w_3}{dx^2} - \frac{1}{5}A_{84}\frac{d^2w_2}{dx^2} + \frac{2}{5}A_{85}w_3,$$

$$L_{333} = \frac{1}{5}A_{86}\frac{d^2w_1}{dx^2} - \frac{1}{5}A_{87}\frac{d^2w_3}{dx^2} - \frac{1}{5}A_{88}\frac{d^2w_2}{dx^2} + \frac{2}{5}A_{89}w_3$$

$$G_{333} = \frac{1}{5}A_{90}\frac{d^2w_1}{dx^2} - \frac{1}{5}A_{91}\frac{d^2w_3}{dx^2} - \frac{1}{5}A_{92}\frac{d^2w_2}{dx^2} + \frac{2}{5}A_{93}w_3,$$

$$N_{133} = N_{313} = N_{331} = -\frac{1}{5}A_{94}\frac{d^2u_0}{dx^2} + \frac{1}{5}A_{95}\frac{d^3w_1}{dx^3} + \frac{1}{5}A_{96}\frac{d^3w_2}{dx^3} + \frac{8}{15}A_{97}\frac{dw_3}{dx} - \frac{4}{15}A_{98}\frac{dw_2}{dx},$$

$$M_{133} = M_{313} = M_{331} = -\frac{1}{5}A_{99}\frac{d^2u_0}{dx^2} + \frac{1}{5}A_{100}\frac{d^3w_1}{dx^3} + \frac{1}{5}A_{101}\frac{d^3w_2}{dx^3} + \frac{8}{15}A_{102}\frac{dw_3}{dx} - \frac{4}{15}A_{103}\frac{dw_2}{dx},$$

$$S_{133} = S_{313} = S_{331} = -\frac{1}{5}A_{104}\frac{d^2u_0}{dx^2} + \frac{1}{5}A_{105}\frac{d^3w_1}{dx^3} + \frac{1}{5}A_{106}\frac{d^3w_2}{dx^3} + \frac{8}{15}A_{107}\frac{dw_3}{dx} - \frac{4}{15}A_{108}\frac{dw_2}{dx},$$

$$Q_{133}=Q_{313}=Q_{331}=-\frac{1}{5}A_{109}\frac{\mathrm{d}^2u_0}{\mathrm{d}x^2}+\frac{1}{5}A_{110}\frac{\mathrm{d}^3w_1}{\mathrm{d}x^3}+\frac{1}{5}A_{111}\frac{\mathrm{d}^3w_2}{\mathrm{d}x^3}+\frac{8}{15}A_{112}\frac{\mathrm{d}w_3}{\mathrm{d}x}-\frac{4}{15}A_{113}\frac{\mathrm{d}w_2}{\mathrm{d}x},$$

$$J_{133}=J_{313}=J_{331}=-\frac{1}{5}A_{157}\frac{\mathrm{d}^2u_0}{\mathrm{d}x^2}+\frac{1}{5}A_{158}\frac{\mathrm{d}^3w_1}{\mathrm{d}x^3}+\frac{1}{5}A_{159}\frac{\mathrm{d}^3w_2}{\mathrm{d}x^3}+\frac{8}{15}A_{160}\frac{\mathrm{d}w_3}{\mathrm{d}x}-\frac{4}{15}A_{161}\frac{\mathrm{d}w_2}{\mathrm{d}x},$$

$$N_{113}=N_{131}=N_{311}=-\frac{4}{15}A_{114}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}+\frac{4}{15}A_{115}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2}+\frac{4}{15}A_{116}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-\frac{1}{5}A_{117}w_3,$$

$$I_{113}=I_{131}=I_{311}=-\frac{4}{15}A_{118}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}+\frac{4}{15}A_{119}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2}+\frac{4}{15}A_{120}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-\frac{1}{5}A_{121}w_3,$$

$$L_{113}=L_{131}=L_{311}=-\frac{4}{15}A_{122}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}+\frac{4}{15}A_{123}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2}+\frac{4}{15}A_{124}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-\frac{1}{5}A_{125}w_3,$$

$$G_{113}=G_{131}=G_{311}=-\frac{4}{15}A_{126}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}+\frac{4}{15}A_{127}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2}+\frac{4}{15}A_{128}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-\frac{1}{5}A_{129}w_3,$$

$$M_{12}=-A_{130}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}-A_{131}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-A_{132}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2},$$

$$O_{12}=-A_{133}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}-A_{134}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-A_{135}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2},$$

$$I_{12}=-A_{136}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}-A_{137}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}-A_{138}\frac{\mathrm{d}^2w_3}{\mathrm{d}x^2},$$

$$\bar{D}_1=A_{139}\left(\frac{\mathrm{d}w_2}{\mathrm{d}x}+\frac{\mathrm{d}w_3}{\mathrm{d}x}\right)-A_{140}\frac{\partial\psi}{\partial x}-A_{141}\frac{\partial\phi}{\partial x},$$

$$\bar{D}_3=A_{142}\frac{\mathrm{d}u_0}{\mathrm{d}x}-A_{143}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}-A_{144}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}+A_{145}w_3-\bar{D}_3^T+\bar{D}_3^{\psi_0}+A_{146}\psi+\bar{D}_3^{\phi_0}+A_{147}\phi,$$

$$\bar{B}_1=A_{148}\left(\frac{\mathrm{d}w_2}{\mathrm{d}x}+\frac{\mathrm{d}w_3}{\mathrm{d}x}\right)-A_{149}\frac{\partial\psi}{\partial x}-A_{150}\frac{\partial\phi}{\partial x},$$

$$\bar{B}_3=A_{151}\frac{\mathrm{d}u_0}{\mathrm{d}x}-A_{152}\frac{\mathrm{d}^2w_1}{\mathrm{d}x^2}-A_{153}\frac{\mathrm{d}^2w_2}{\mathrm{d}x^2}+A_{154}w_3-\bar{B}_3^T+\bar{B}_3^{\psi_0}+A_{155}\psi+\bar{B}_3^{\phi_0}+A_{156}\phi,$$

in which the integration constants are expressed as

$$\{A_1,A_2,A_3,A_4\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z\right),$$

$$\{A_5,A_6,A_7,A_8\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}z\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}z\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}z\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z\right),$$

$$\{A_9,A_{10},A_{11},A_{12}\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z\right.$$
$$\left.+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z\right),$$

$$\{A_{13},A_{14},A_{15},A_{16}\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2'\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_2'\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z\right.$$
$$\left.+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2'\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}\mathrm{d}z\right),$$

$$\{A_{17},A_{18},A_{19}\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z\right),$$

$$\{A_{20},A_{21},A_{22}\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_1'\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_1'\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_1'\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z\right),$$

$$\{A_{23},A_{24},A_{25}\}=2\mu l_0^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2''\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_2''\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2''\{1,\delta_2\Phi_1',\delta_2\Phi_2''\}\mathrm{d}z\right),$$

$$\{A_{26},A_{27},A_{28}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{11}^{p}\delta_1\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{11}\delta_1\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{11}^{p}\delta_1\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z,$$

$$\{A_{29},A_{35},A_{41}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{13}^{p}\delta_2\Phi_1'\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{13}\delta_2\Phi_1'\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{13}^{p}\delta_2\Phi_1'\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z,$$

$$\{A_{30},A_{31}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\delta_1\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\delta_1\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z,$$

$$\{A_{32},A_{33},A_{34}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{11}^{p}z\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{11}z\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{11}^{p}z\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z,$$

$$\{A_{36},A_{37}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)z\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)z\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z,$$

$$\{A_{38},A_{39},A_{40}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{11}^{p}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{11}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{11}^{p}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z,$$

$$\{A_{42},A_{43}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\delta_2\Phi_1\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\delta_2\Phi_1\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z,$$

$$\{A_{44},A_{45},A_{46}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{13}^{p}\delta_2\Phi_2'\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{13}\delta_2\Phi_2'\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{13}^{p}\delta_2\Phi_2'\{\delta_1,z,\delta_2\Phi_1\}\mathrm{d}z,$$

$$A_{47}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{33}^{p}\delta_2\Phi_2'\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{33}\delta_2\Phi_2'\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{33}^{p}\delta_2\Phi_2'\mathrm{d}z,$$

$$\{A_{48},A_{49}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\delta_2\Phi_2'\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\delta_2\Phi_2'\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z,$$

$$A_{50}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} c_{55}^{p}(\delta_2\Phi_2)^2\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} c_{55}(\delta_2\Phi_2)^2\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} c_{55}^{p}(\delta_2\Phi_2)^2\mathrm{d}z,$$

$$\{A_{51},A_{52}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\cos\left(\frac{\pi\check{z}}{h_p}\right)\delta_2\Phi_2\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\cos\left(\frac{\pi\check{z}}{h_p}\right)\delta_2\Phi_2\{e_{13}^{p},q_{13}^{p}\}\mathrm{d}z,$$

$$\{A_{53},A_{54},A_{55},A_{56},A_{57}\}=2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z\right),$$

$$\{A_{58},A_{59},A_{60},A_{61},A_{62}\}=2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}z\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}z\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}z\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z\right),$$

$$\{A_{63},A_{64},A_{65},A_{66},A_{67}\}=2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_1\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2',\delta_2\Phi_1''\}\mathrm{d}z\right),$$

$$\{A_{68},A_{69},A_{70},A_{71},A_{72}\} = 2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2\Phi_2'\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2\Phi_2'\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2\Phi_2'\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz\right),$$

$$\{A_{73},A_{74},A_{75},A_{76},A_{77}\} = 2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2\Phi_1''\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2\Phi_1''\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2\Phi_1''\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz\right),$$

$$\{A_{78},A_{79},A_{80},A_{81}\} = 2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz\right),$$

$$\{A_{82},A_{83},A_{84},A_{85}\} = 2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2\Phi_2\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2\Phi_2\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2\Phi_2\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz\right),$$

$$\{A_{86},A_{87},A_{88},A_{89}\} = 2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2[1-2\Phi_1']\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2[1-2\Phi_1']\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2[1-2\Phi_1']\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz\right),$$

$$\{A_{90},A_{91},A_{92},A_{93}\} = 2\mu l_1^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2\Phi_2''\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2\Phi_2''\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2\Phi_2''\{1, \delta_2\Phi_2, \delta_2[1-2\Phi_1'], \delta_2\Phi_2''\}dz\right),$$

$$\{A_{94},A_{95},A_{96},A_{97},A_{98}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_1\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_1\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_1\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz\right),$$

$$\{A_{99},A_{100},A_{101},A_{102},A_{103}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} z\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} z\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} z\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz\right),$$

$$\{A_{104},A_{105},A_{106},A_{107},A_{108}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2\Phi_1\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2\Phi_1\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2\Phi_1\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz\right),$$

$$\{A_{109},A_{110},A_{111},A_{112},A_{113}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}} \delta_2\Phi_2'\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}} \delta_2\Phi_2'\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p} \delta_2\Phi_2'\{\delta_1, z, \delta_2\Phi_1, \delta_2\Phi_2', \delta_2\Phi_1''\}dz\right),$$

$$\{A_{114},A_{115},A_{116},A_{117}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz\right),$$

$$\{A_{118},A_{119},A_{120},A_{121}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_2\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz\right),$$

$$\{A_{122},A_{123},A_{124},A_{125}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2[1-2\Phi_1']\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2[1-2\Phi_1']\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2[1-2\Phi_1']\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz\right),$$

$$\{A_{126},A_{127},A_{128},A_{129}\} = 2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2''\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_2''\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2''\{1,\delta_2\Phi_2,\delta_2[1-2\Phi_1'],\delta_2\Phi_2''\}dz\right),$$

$$\{A_{130},A_{131},A_{132}\} = \frac{1}{2}\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz\right),$$

$$\{A_{133},A_{134},A_{135}\} = \frac{1}{2}\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2[\Phi_1'+\Phi_2]\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2[\Phi_1'+\Phi_2]\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2[\Phi_1'+\Phi_2]\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz\right),$$

$$\{A_{136},A_{137},A_{138}\} = \frac{1}{2}\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_2\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2\{1,\delta_2[\Phi_1'+\Phi_2],\delta_2\Phi_2\}dz\right),$$

$$\{A_{139},A_{140},A_{141}\} = \int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\cos\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{e_{15}^p\delta_2\Phi_2,\epsilon_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right),m_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right)\right\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\cos\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{e_{15}^p\delta_2\Phi_2,\epsilon_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right),m_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right)\right\}dz,$$

$$\{A_{142},A_{143},A_{144},A_{145}\} = \int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{x}_3}{h_p}\right)e_{13}^p\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}dz+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{x}_3}{h_p}\right)e_{13}^p\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2'\}dz,$$

$$\{A_{146},A_{147}\} = \int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\left[\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\right]^2\{\epsilon_{33}^p,m_{33}^p\}dz+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\left[\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\right]^2\{\epsilon_{33}^p,m_{33}^p\}dz,$$

$$\{A_{148},A_{149},A_{150}\} = \int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\cos\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{q_{15}^p\delta_2\Phi_2,m_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right),\mu_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right)\right\}dz + \int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\cos\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{q_{15}^p\delta_2\Phi_2,m_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right),\mu_{11}^p\cos\left(\frac{\pi\check{z}}{h_p}\right)\right\}dz,$$

$$\{A_{151},A_{152},A_{153},A_{154}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{x}_3}{h_p}\right)q_{13}^{p}\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2^{'}\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{x}_3}{h_p}\right)q_{13}^{p}\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2^{'}\}\mathrm{dz},$$

$$\{A_{155},A_{156}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\left[\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\right]^2\{m_{33}^{p},\mu_{13}^{p}\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\left[\frac{\pi}{h_p}\sin\left(\frac{\pi\check{z}}{h_p}\right)\right]^2\{m_{33}^{p},\mu_{13}^{p}\}\mathrm{dz},$$

$$\{A_{157},A_{158},A_{159},A_{160},A_{161}\}$$
$$=2\mu l_2^2\left(\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_1^{''}\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2^{'},\delta_2\Phi_1^{''}\}\mathrm{dz}+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_1^{''}\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2^{'},\delta_2\Phi_1^{''}\}\mathrm{dz}\right.$$
$$\left.+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_1^{''}\{\delta_1,z,\delta_2\Phi_1,\delta_2\Phi_2^{'},\delta_2\Phi_1^{''}\}\mathrm{dz}\right),$$

$$\{N_{11}^{T},N_{13}^{T}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_1\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz}+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_1\alpha T\{c_{11},c_{13}\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_1\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz},$$

$$\left\{N_{11}^{\psi_0},N_{11}^{\phi_0}\right\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_1\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_1\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz},$$

$$\{M_{11}^{T},M_{13}^{T}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}z\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz}+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}z\alpha T\{c_{11},c_{13}\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}z\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz},$$

$$\left\{M_{11}^{\psi_0},M_{11}^{\phi_0}\right\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}z\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}z\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz},$$

$$\{Q_{11}^{T},Q_{13}^{T}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2^{'}\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz}+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_2^{'}\alpha T\{c_{11},c_{13}\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2^{'}\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz},$$

$$\left\{Q_{11}^{\psi_0},Q_{11}^{\phi_0}\right\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_2^{'}\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_2^{'}\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz},$$

$$\{S_{11}^{T},S_{13}^{T}\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_1\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz}+\int_{-\frac{h_e}{2}}^{+\frac{h_e}{2}}\delta_2\Phi_1\alpha T\{c_{11},c_{13}\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_1\alpha T\{c_{11}^{p},c_{13}^{p}\}\mathrm{dz},$$

$$\left\{S_{11}^{\psi_0},S_{11}^{\phi_0}\right\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\delta_2\Phi_1\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\delta_2\Phi_1\left\{\frac{2\psi_0}{h_p}e_{13}^{p},\frac{2\phi_0}{h_p}q_{13}^{p}\right\}\mathrm{dz},$$

$$\{\bar{D}_3^{T},\bar{B}_3^{T}\}=2\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\{e_{13}^{p},q_{13}^{p}\}\alpha T\mathrm{dz}+2\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\{e_{13}^{p},q_{13}^{p}\}\alpha T\mathrm{dz},$$

$$\left\{\bar{D}_3^{\psi_0},\bar{D}_3^{\phi_0}\right\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{\frac{2\psi_0}{h_p}\epsilon_{33}^{p},\frac{2\phi_0}{h_p}m_{33}^{p}\right\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{\frac{2\psi_0}{h_p}\epsilon_{33}^{p},\frac{2\phi_0}{h_p}m_{33}^{p}\right\}\mathrm{dz},$$

$$\left\{\bar{B}_3^{\psi_0},\bar{B}_3^{\phi_0}\right\}=\int_{-\frac{h_e}{2}-h_p}^{-\frac{h_e}{2}}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{\frac{2\psi_0}{h_p}m_{33}^{p},\frac{2\phi_0}{h_p}\mu_{33}^{p}\right\}\mathrm{dz}+\int_{+\frac{h_e}{2}}^{+\frac{h_e}{2}+h_p}\frac{\pi}{h_p}\sin\left(\frac{\pi\tilde{z}}{h_p}\right)\left\{\frac{2\psi_0}{h_p}m_{33}^{p},\frac{2\phi_0}{h_p}\mu_{33}^{p}\right\}\mathrm{dz},$$

$$\{B_1,B_3,B_{10}\}=\int\rho\{\delta_1^2,\delta_1\delta_2\Phi_1,\delta_2^2\}\mathrm{dz},\quad\{\mathrm{B}_8,\mathrm{B}_9\}=\int\rho\delta_2\{1,\Phi_2\}\mathrm{dz},\quad\{\mathrm{B}_7,\mathrm{B}_4\}=\int\rho\{1,\mathrm{z}^2\}\mathrm{dz},$$

$$\{B_2,B_5\}=\int\rho\{\delta_1,\delta_2\Phi_1\}z\mathrm{dz},\quad\{B_6,B_{11},B_{12}\}=\int\rho\delta_2^2\{\Phi_1^2,\Phi_2,(\psi\Phi_2)^2\}\mathrm{dz}$$