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A new four-unknown refined theory based on modified couple stress theory for size-dependent bending and vibration analysis of functionally graded micro-plate

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Abstract. This work investigates a novel plate formulation and a modified couple stress theory that introduces a variable length scale parameter is presented to discuss the static and dynamic of functionally graded (FG) micro-plates. A new type of third-order shear deformation theory of Reddy that use only 4 unknowns by including undetermined integral variables is proposed in this study. The equations of motion are derived from Hamilton's principle. Analytical solutions are obtained for a simply supported micro-plate. Numerical examples are presented to examine the effect of the length scale parameter on the responses of micro-plates. The obtained results are compared with the previously published results to demonstrate the correctness of the present formulation.

Keywords: micro-plate; modified couple stress theory; the length scale parameter; functionally graded material

1. Introduction

The first FGM was developed in Japan in 1984 as the result of a space plane project where the FGMS have gained wide application in variety branches of engineering such as mechanical, aerospace, chemical, electrical etc (El-Haina *et al*. 2017, Laoufi *et al*. 2016, Houari *et al*. 2016, Bousahla *et al*. 2016, Abdelbari *et al*. 2016, Abdelhak *et al*. 2016, Bounouara *et al*. 2016, Bouderba *et al*. 2016, Barati and Shahverdi, 2016, Barka *et al*. 2016, Beldjelili *et al*. 2016, Kar and Panda, 2015, Darılmaz, 2015, Belkorissat *et al*. 2015, Akbaş, 2015, Zidi *et al*. 2014, Tounsi *et al*. 2013). Functionally graded materials (FGMs) are the advanced materials in the family of engineering composites made from a mixture of ceramic and metal in which the ceramic component provides high-temperature resistance because of its low thermo conductivity, on the other hand, the ductile metal component prevents fracture due to thermal load. Compared with classical laminated composites, FGMS avoid the inter-laminar stress gaps that are induced by mismatches in the characteristics of two different materials. Such materials were introduced to gain benefits of the desired physical characteristics of each constituent material without interface problems. With the advance of technology, FGMS are started to be employed in micro/nano-electromechanical systems (MEMS/NEMS), such in the form of shape memory alloy thin films with a global thickness in micro-or nano-scale (Lü *et al.* 2009) electrically actuated MEMS devices (Zhang and Fu 2012) and atomic force microscopes (AFM) (Kahrobaiyan *et al.* 2010, Kahrobaiyan *et al.* 2011, Kahrobaiyan *et al.* 2012, Asghari *et al.* 2010).

In this context, the practical studies show as the thickness of the structures becomes on the magnitude of microns and sub-microns, the scale effect of material takes a considerable role in mechanical behaviors of such structures (Fleck and Hutchinson 1993, Lam *et al.* 2003, Mindlin 1963, Mindlin and Tiersten 1962, Toupin 1962). The classical continuum mechanics theory cannot be utilized to interpret the size-dependent effect as it does not constrain any material length scale parameter. Thus, size-dependent plate models such as the classical couple stress theory having internal material length scale parameter are necessary (Mindlin 1963, Mindlin and Tiersten 1962, Toupin 1962).

Based on the modified couple stress theory, several size-dependent plate models have been developed.

Park and Gao (2006), Ma *et al.* (2008) studied Euler-Bernoulli and Timoshenko beams via a modified couple stress theory. These models are used to analyze the behavior characteristics of microtubules (Ma *et al.* 2008, Kong *et al.* 2008, Xia *et al.* 2010, Ke and Wang 2011) and micro tubes conveying fluid (Ke *et al.* 2011, Ahangar *et al.* 2011, Wang 2010, Xia and Wang 2010).

Simsek and Reddy (2013) discussed the bending and vibration of FG micro-beam using a new higher order beam theory and the modified couple stress theory. Al-Basyouni *et al.* (2015) proposed a novel unified beam formulation with a modified couple stress theory that consider a variable length scale parameter to study bending and dynamic behavior of FG micro-beam.

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In this article, a new analytical formulation based on the modified couple stress theory is proposed to study the bending and vibration behaviors of FG micro-plate having a variable length scale parameter by employing a novel form of the third-order shear deformation theory of Reddy (TSDT). The addition of the integral term in the displacement field leads to a reduction in the number of variables and governing equations. The governing equations and related boundary conditions are deduced by employing the Hamilton's principle. The influences of the length scale parameter, the power law indices, shear deformation on the bending and dynamic behavior of FG micro-scale plates are examined in this work. The present results are also compared with previously published results to confirm the validity of the present approach.

2. Theoretical formulation

2.1 modified couple stress theory

Based on the modified couple stress theory (Yang *et al.* 2002), the strain energy, U, for a linear elastic material occupying region Ω is related to strain and curvature tensors and can written as

$$U = \frac{1}{2}\int \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right) dV, \quad (i, j = 1,2,3) \tag{1}$$

Where σ is the stress tensor, ε is the strain tensor, m is the deviatoric part of the couple stress tensor and χ is the symmetric curvature .these tensors are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{i,i}\right) \tag{2}$$

$$\chi_{ij} = \frac{1}{2}\left(\theta_{i,j} + \theta_{i,i}\right) \tag{3}$$

where is the displacement vector, and θ is the rotation vector that can be defined as

$$\theta = \frac{1}{2} e_{ijk} u_{k,j} \tag{4}$$

where e_{ijk} is the permutation symbol.

2.2 Kinematic relations and constitutive relations

The displacement field of the conventional TSDT of Reddy is given as follows (Boukhari *et al.* 2016)

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x} + f(z)\varphi_x(x,y) \tag{5a}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y} + f(z)\varphi_y(x,y) \tag{5b}$$

$$w(x,y,z) = w_0(x,y) \tag{5c}$$

where u_0, v_0, w_0, φ_x, φ_y, are five unknown displacements of the mid-plane of the plate, and $f(z)$ represents shape function defining the variation of the transverse shear strains and stresses across the thickness.

In this article, the conventional TSDTs of Reddy is modified by proposing some simplifying suppositions so that the number of unknowns is reduced as follows (Hebali *et al.* 2016, Merdaci *et al.* 2016, Besseghier *et al.* 2017, Chikh *et al.* 2017, Khetir *et al.* 2017, Fahsi *et al.* 2017)

$$u(x,y,z) = u_0(x,y) - z\frac{\partial w_0}{\partial x} + k_1\, f(z)\int \theta(x,y)\,dx \tag{6a}$$

$$v(x,y,z) = v_0(x,y) - z\frac{\partial w_0}{\partial y} + k_2\, f(z)\int \theta(x,y)\,dy \tag{6b}$$

$$w(x,y,z) = w_0(x,y) \tag{6c}$$

The coefficients k_1 and k_2 depend on the geometry. In this article, the shape function is considered given by Reddy (1984) as

$$f(z) = z\left[1 - \frac{4}{3}\left(\frac{z}{h}\right)^2\right] \quad \text{and} \quad g(z) = \frac{df(z)}{dz} \tag{7}$$

where (u_0, v_0, w_0, θ) are four unknown displacements of the mid-plane of the plate, and h is the plate thickness. The nonzero linear strains are

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix} + f(z)\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix}, \quad \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = g(z)\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix} \tag{8}$$

where

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{\partial v_0}{\partial x}\\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix} = \begin{Bmatrix}-\dfrac{\partial^2 w_0}{\partial x^2}\\ -\dfrac{\partial^2 w_0}{\partial y^2}\\ -2\dfrac{\partial^2 w_0}{\partial x \partial y}\end{Bmatrix}, \quad \begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix} = \begin{Bmatrix}k_1\theta\\ k_2\theta\\ k_1\dfrac{\partial}{\partial y}\int \theta\,dx + k_2\dfrac{\partial}{\partial x}\int \theta\,dy\end{Bmatrix}, \tag{9}$$

$$\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix} = \begin{Bmatrix}k_2\int \theta\,dy\\ k_1\int \theta\,dx\end{Bmatrix} \tag{10}$$

The integrals used in the above equations shall be resolved by a Navier type method and can be written as follows

$$\frac{\partial}{\partial y}\int \theta\,dx = A'\frac{\partial^2\theta}{\partial x \partial y}, \quad \frac{\partial}{\partial x}\int \theta\,dy = B'\frac{\partial^2\theta}{\partial x \partial y}, \quad \int \theta\,dx = A'\frac{\partial\theta}{\partial x}, \quad \int \theta\,dy = B'\frac{\partial\theta}{\partial y} \tag{11}$$

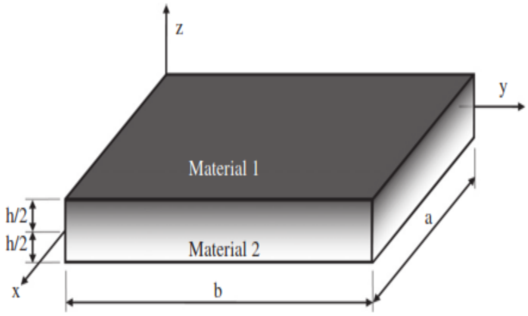

Fig. 1 Geometry of a FGM plate

where the coefficients A' and B' are expressed according to the type of solution employed, in this case by using Navier. Therefore, A' and B' are expressed as follows

$$A' = -\frac{1}{\alpha^2}, \quad B' = -\frac{1}{\beta^2}, \quad k_1 = \alpha^2, \quad k_2 = \beta^2 \tag{12}$$

where α and β are defined in expression (31).

In addition, using Eqs. (5) and (4), the components of the rotation vector are obtained as

$$\theta_x = \frac{\partial w_0}{\partial y} - \frac{1}{2} k_2 B' g(z) \frac{\partial \theta}{\partial y} \tag{13a}$$

$$\theta_y = -\frac{\partial w_0}{\partial x} + \frac{1}{2} k_1 A' g(z) \frac{\partial \theta}{\partial x} \tag{13b}$$

$$\theta_z = \frac{1}{2}\left(\frac{\partial v_0}{\partial x} - \frac{\partial u_0}{\partial y}\right) + \frac{1}{2} f(z)(k_2 B' - k_1 A') \frac{\partial^2 \theta}{\partial x \partial y} \tag{13c}$$

Substituting Eq. (13) into Eq. (3), the components of the curvature tensor take the form

$$\chi_x = \frac{\partial^2 w_0}{\partial x \partial y} - \frac{1}{2} k_2 B' g(z) \frac{\partial^2 \theta}{\partial x \partial y} \tag{14a}$$

$$\chi_y = -\frac{\partial^2 w_0}{\partial x \partial y} + \frac{1}{2} k_1 A' g(z) \frac{\partial^2 \theta}{\partial x \partial y} \tag{14b}$$

$$\chi_z = \frac{1}{2}(k_2 B' - k_1 A') g(z) \frac{\partial^2 \theta}{\partial x \partial y} \tag{14c}$$

$$\chi_{xy} = \frac{1}{2}\left(\frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 w_0}{\partial x^2}\right) - \frac{1}{4} g(z)\left(k_2 B' \frac{\partial^2 \theta}{\partial y^2} + k_1 A' \frac{\partial^2 \theta}{\partial x^2}\right) \tag{14d}$$

$$\chi_{xz} = -\frac{1}{4} k_2 B' g'(z) \frac{\partial \theta}{\partial y} + \frac{1}{4}\left(\frac{\partial^2 v_0}{\partial x^2} - \frac{\partial^2 u_0}{\partial x \partial y}\right) + \frac{1}{4} f(z)(k_2 B' - k_1 A') \frac{\partial^3 \theta}{\partial x^2 \partial y} \tag{14e}$$

$$\chi_{yz} = \frac{1}{4} k_1 A' g'(z) \frac{\partial \theta}{\partial x} + \frac{1}{4}\left(\frac{\partial^2 v_0}{\partial x \partial y} - \frac{\partial^2 u_0}{\partial y^2}\right) + \frac{1}{4} f(z)(k_2 B' - k_1 A') \frac{\partial^3 \theta}{\partial x \partial y^2} \tag{14f}$$

2.3 Constitutive relations

Consider a FG plate made of two constituent functionally graded materials as shown in Fig. 1. The material properties of the plate such as Young's modulus E

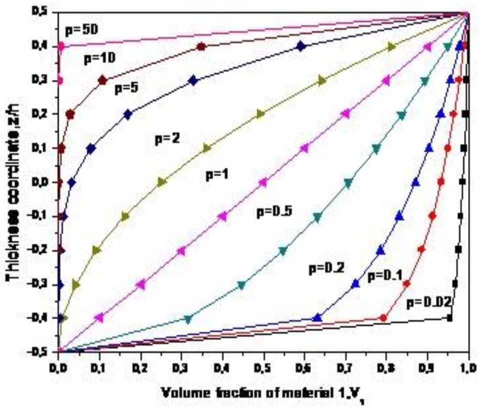

Fig. 2 Variation of volume fraction V_1 through the thickness of a FG plate for various gradient index p

and masse density ρ are considered to change continuously across the thickness by power law and the length scale parameter l are given by the rule of mixtures as (Hanifi Hachemi Amar *et al*. 2017, Bellifa *et al*. 2016, Bouderba *et al*. 2013).

$$E(z) = E_2 + (E_1 - E_2)V_1 \tag{15a}$$

$$\rho(z) = \rho_2 + (\rho_1 - \rho_2)V_1 \tag{15b}$$

$$l(z) = l_2 + (l_1 - l_2)V_1 \tag{15c}$$

Where $V_1=(0.5+z/h)^p$ is the volume fraction of material 1, the subscripts 1 and 2 indicate the two materials employed, and p is the gradient index indicating the volume fraction of material. The variation of the volume fraction V_1 across the thickness of the plate is plotted in Fig. 2 for various values of the power law index. The linear elastic constitutive relations are

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \frac{E(z)}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{16a}$$

$$m_{ij} = \frac{E(z)}{1+\nu}[l(z)^2]\chi_{ij} \tag{16b}$$

Where ν is the poison's ratio considered to be constant, l is the material length scale parameter which reflects the influence of couple stress.

2.4 Equations of motion

Hamilton's principle is employed in this work to determine the equations of motion. The principle can be expressed in analytical from as (Ait Amar Meziane *et al*. 2014, Attia *et al*. 2015, Ait Atmane *et al*. 2015, Mahi *et al*. 2015, Zemri *et al*. 2015, Taibi *et al*. 2015, Saidi *et al*. 2016, Ahouel *et al*. 2016, Klouche *et al*. 2017, Mouffoki *et al*.

2017, Meksi *et al*. 2017, Bellifa *et al*. 2017, Zidi *et al*. 2017).

$$\int_0^T (\delta U + \delta V - \delta K) dt = 0 \tag{17}$$

Where δU is the virtual strain energy, δV is the virtual work done by external loads, and δK is the virtual kinetic energy. The virtual strain energy is expressed by (see Eq. (1))

$$\begin{aligned}
\delta U &= \int_A \int_{-h/2}^{h/2} \left(\sigma_x \delta\varepsilon_x + m_y \delta\varepsilon_y + \sigma_{xy}\delta\gamma_{xy} + \sigma_{xz}\delta\gamma_{xz} + \sigma_{yz}\delta\gamma_{yz}\right) dAdz \\
&+ \int_A \int_{-h/2}^{h/2} (m_x \delta\chi_x + m_y \delta\chi_y + m_z \delta\chi_z + 2m_{xy}\delta\chi_{xy} \\
&+ 2m_{xz}\delta\chi_{xz} + 2m_{yz}\delta\chi_{yz}) dAdz \\
&= \int_A \Bigg[N_x \frac{\partial \delta u_0}{\partial x} + N_{xy}\left(\frac{\partial \delta u_0}{\partial y} + \frac{\partial \delta v_0}{\partial x}\right) \\
&+ \frac{1}{2} X_{yz}\left(\frac{\partial^2 \delta v_0}{\partial x \partial y} - \frac{\partial^2 \delta u_0}{\partial y^2}\right) \\
&+ \frac{1}{2} X_{xz}\left(\frac{\partial^2 \delta v_0}{\partial x^2} - \frac{\partial^2 \delta u_0}{\partial y \partial x}\right) + N_y \frac{\partial \delta v_0}{\partial y} \\
&- \frac{\partial^2 \delta w_0}{\partial x^2}\left(M_x + X_{xy}\right) + \frac{\partial^2 \delta w_0}{\partial y^2}\left(X_{xy} - M_y\right) \\
&+ \frac{\partial^2 \delta w_0}{\partial x \partial y}\left(X_x - 2M_{xy} - X_y\right) - S_x k_1 \theta \\
&+ \frac{1}{2} Y_{yz}\left(k_1 A' - k_2 B'\right)\frac{\partial^3 \delta w_0}{\partial x \partial y^2} \\
&+ \frac{1}{2} Y_{xz}\left(k_1 A' - k_2 B'\right)\frac{\partial^3 \delta w_0}{\partial x^2 \partial y} \\
&- S_y k_2 \theta - S_{xy}\left(k_1 A' \frac{\partial^2 \delta\theta}{\partial x \partial y} + k_2 B' \frac{\partial^2 \delta\theta}{\partial x \partial y}\right) \\
&- Q_{yz} k_2 B' \frac{\partial \delta\theta}{\partial y} - Q_{xz} k_1 A' \frac{\partial \delta\theta}{\partial x} \\
&+ \frac{1}{2} Z_x k_2 B' \frac{\partial^2 \delta\theta}{\partial x \partial y} - \frac{1}{2} Z_y k_1 A' \frac{\partial^2 \delta\theta}{\partial x \partial y} \\
&+ \frac{1}{2} Z_{xy} k_2 B' \frac{\partial^2 \delta\theta}{\partial y^2} - \frac{1}{2} Z_{xy} k_1 A' \frac{\partial^2 \delta\theta}{\partial x^2} \\
&- \frac{1}{2} W_{yz} k_1 A' \frac{\partial \delta\theta}{\partial x} + \frac{1}{2} W_{xz} k_2 B' \frac{\partial \delta\theta}{\partial y} \Bigg] dxdy
\end{aligned} \tag{18}$$

there N, M, S, Q, X, Z, and W are the stress resultants defined by

$$\left(N_i, M_i, S_i\right) = \int_{-h/2}^{h/2} (1, z, f) \sigma_i dz, i = x, y, xy \tag{19a}$$

$$Q_i = \int_{-h/2}^{h/2} g(z) \sigma_i dz, i = xy, yz \tag{19b}$$

$$\left(X_i, Y_i, Z_i, W_i\right) = \int_{-h/2}^{h/2} (1, f, g, g') m_i dz, i = x, y, xy, xz, yz \tag{19c}$$

The variation of the work done by the external applied forces can be expressed as

$$\delta V = -\int_A q \delta w dA = -\int_A q \delta w_0 dA \tag{20}$$

Where q is the transverse load.

The variation of kinetic energy is expressed as

$$\begin{aligned}
\delta K &= \int_A \int_{-h/2}^{h/2} \rho(z) \left[\dot{u}_0 \, \delta\dot{u}_0 + \dot{v}_0 \, \delta\dot{v}_0 + \dot{w}_0 \, \delta\dot{w}_0\right] dzdA \\
&= \int_A \Big\{ I_0 \left[\dot{u}_0 \, \delta\dot{u}_0 + \dot{v}_0 \, \delta\dot{v}_0 + \dot{w}_0 \, \delta\dot{w}_0\right] \\
&- I_1 \left[\dot{u}_0 \frac{\partial \delta\dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial x} \delta\dot{u}_0 + \dot{v}_0 \frac{\partial \delta\dot{w}_0}{\partial y} + \frac{\partial \dot{w}_0}{\partial y} \delta\dot{v}_0\right] \\
&+ I_1 \left[\frac{\partial \dot{w}_0}{\partial x} \frac{\partial \delta\dot{w}_0}{\partial x} + \frac{\partial \dot{w}_0}{\partial y} \frac{\partial \delta\dot{w}_0}{\partial y}\right] \\
&- J_1 \left[\begin{array}{l} k_1 A' \dot{u}_0 \dfrac{\partial \delta\dot{\theta}}{\partial x} + k_1 A' \delta\dot{u}_0 \dfrac{\partial \dot{\theta}}{\partial x} \\ + k_2 B' \dot{v}_0 \dfrac{\partial \delta\dot{\theta}}{\partial y} + k_2 B' \delta\dot{v}_0 \dfrac{\partial \dot{\theta}}{\partial y} \end{array}\right] \\
&+ J_2 \left[\begin{array}{l} k_1 A' \dfrac{\partial \dot{w}_0}{\partial x} \dfrac{\partial \delta\dot{\theta}}{\partial x} + k_1 A' \dfrac{\partial \delta\dot{w}_0}{\partial x} \dfrac{\partial \dot{\theta}}{\partial x} \\ + k_2 B' \dfrac{\partial \dot{w}_0}{\partial y} \dfrac{\partial \delta\dot{\theta}}{\partial y} + k_2 B' \dfrac{\partial \delta\dot{w}_0}{\partial y} \dfrac{\partial \dot{\theta}}{\partial y} \end{array}\right] \\
&+ K_2 \left[k_1^2 A'^2 \frac{\partial \dot{\theta}}{\partial x} \frac{\partial \delta\dot{\theta}}{\partial x} k_2^2 B'^2 \frac{\partial \dot{\theta}}{\partial y} \frac{\partial \delta\dot{\theta}}{\partial y}\right] \Big\} dA
\end{aligned} \tag{21}$$

where dot-superscript convention denotes the differentiation with respect to the time variable t, $\rho(z)$ is the masse density, and $(I_0, I_1, I_2, J_1, J_2, K_2)$ are the masse inertias defined as

$$\left(I_0, I_1, I_2, J_1, J_2, K_2\right) = \int_{-h/2}^{h/2} \left(1, z, z^2, f, zf, g\right) \rho(z) dz \tag{22}$$

Substituting Eqs. (18), (20) and (21) into Eq. (17) and integrating by parts, and collecting the coefficients of (δu_0, δv_0, δw_0, $\delta\theta$), the following equations of motion are obtained

$$\begin{aligned}
\delta u_0 &: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} + \frac{1}{2}\left(\frac{\partial^2 X_{xz}}{\partial x \partial y} + \frac{\partial^2 X_{yz}}{\partial y^2}\right) = \\
&\quad I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} - J_1 k_1 A' \frac{\partial \ddot{\theta}}{\partial x} \\
\delta v_0 &: \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} + \frac{1}{2}\left(\frac{\partial^2 X_{xz}}{\partial x^2} + \frac{\partial^2 X_{yz}}{\partial x \partial y}\right) = \\
&\quad I_0 \ddot{v}_0 - I_1 \frac{\partial \ddot{w}_0}{\partial x} - J_1 k_2 B' \frac{\partial \ddot{\theta}}{\partial y} \\
\delta w_0 &: \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \\
&+ \frac{1}{2}\left(k_1 A' - k_2 B'\right)\left(\frac{\partial^3 Y_{xz}}{\partial x^2 \partial y} + \frac{\partial^3 Y_{yz}}{\partial x \partial y^2}\right) \\
&+ \frac{\partial^2 X_{xy}}{\partial x^2} - \frac{\partial^2 X_{xy}}{\partial y^2} + \frac{\partial^2 X_y}{\partial x \partial y} - \frac{\partial^2 X_x}{\partial x \partial y} + q =
\end{aligned}$$

$$I_0\ddot{w}_0 + I_1\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}\right)$$
$$- J_2\left(k_1 A'\frac{\partial^2 \ddot{\theta}}{\partial x^2} + k_2 B'\frac{\partial^2 \ddot{\theta}}{\partial y^2}\right)$$

$$\begin{aligned}
\delta\theta :\ & k_1 S_x + k_2 S_y + (k_1 A' + k_2 B')\frac{\partial^2 S_{xy}}{\partial x \partial y} \\
& - k_1 A'\frac{\partial Q_{xz}}{\partial x} - k_2 B'\frac{\partial Q_{yz}}{\partial y} \\
& + \frac{1}{2}k_1 A'\left(\frac{\partial^2 Z_y}{\partial x \partial y} + \frac{\partial^2 Z_{xy}}{\partial x^2} - \frac{\partial W_{yz}}{\partial x}\right) \\
& - \frac{1}{2}k_2 B'\left(\frac{\partial^2 Z_x}{\partial x \partial y} + \frac{\partial^2 Z_{xy}}{\partial y^2} - \frac{\partial W_{xz}}{\partial y}\right) = \\
& J_1\left(k_1 A'\frac{\partial \ddot{u}_0}{\partial x} + k_2 B'\frac{\partial \ddot{v}_0}{\partial y}\right) \\
& - J_2\left(k_1 A'\frac{\partial^2 \ddot{w}_0}{\partial x^2} + k_2 B'\frac{\partial^2 \ddot{w}_0}{\partial y^2}\right) \\
& - J_3\left(k_1^2 A'^2\frac{\partial^2 \ddot{\theta}}{\partial x^2} + k_2^2 B'^2\frac{\partial^2 \ddot{\theta}}{\partial y^2}\right)
\end{aligned} \tag{23}$$

2.5 Equations of motion in terms of displacements

Substituting Eq. (19) into Eq. (23), the equations of motion can be expressed in terms of generalized displacements (δu_0, δv_0, δw_0, $\delta\theta$) as

$$\begin{aligned}
\delta u_0 :\ & A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + (A_{12} + A_{66})\frac{\partial^2 v_0}{\partial x \partial y} - B_{11}\frac{\partial^3 w_0}{\partial x^3} \\
& - (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x \partial y^2} - (k_1 B_{11}^s + k_2 B_{12}^s)\frac{\partial \theta}{\partial x} \\
& - B_{66}^s (k_1 A' + k_2 B')\frac{\partial^3 \theta}{\partial x \partial y^2} - \frac{1}{4}A_n \\
& \left[\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}\right)\right] \\
& + \frac{1}{4}(k_1 A' - k_2 B')\left[B_n\frac{\partial^3}{\partial x \partial y^2}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) - D_n\frac{\partial^3 \theta}{\partial x \partial y^2}\right] \\
& = I_0\ddot{u}_0 - I_1\frac{\partial \ddot{w}_0}{\partial x} - J_1 k_1 A'\frac{\partial \ddot{\theta}}{\partial x}
\end{aligned} \tag{24a}$$

$$\begin{aligned}
\delta v_0 :\ & A_{22}\frac{\partial^2 v_0}{\partial y^2} + A_{66}\frac{\partial^2 v_0}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2 u_0}{\partial x \partial y} - B_{22}\frac{\partial^3 w_0}{\partial y^3} \\
& - (B_{12} + 2B_{66})\frac{\partial^3 w_0}{\partial x^2 \partial y} - (k_1 B_{12}^s + k_2 B_{22}^s)\frac{\partial \theta}{\partial y} \\
& - B_{66}^s (k_1 A' + k_2 B')\frac{\partial^3 \theta}{\partial x^2 \partial y} - \frac{1}{4}A_n \\
& \left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 v_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x^2}\right) - \frac{\partial^2}{\partial x \partial y}\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right)\right] \\
& - \frac{1}{4}(k_1 A' - k_2 B')\left[B_n\frac{\partial^3}{\partial x^2 \partial y}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) - D_n\frac{\partial^3 \theta}{\partial x^2 \partial y}\right] \\
& = I_0\ddot{v}_0 - I_1\frac{\partial \ddot{w}_0}{\partial x} - J_1 k_2 B'\frac{\partial \ddot{\theta}}{\partial y}
\end{aligned} \tag{24b}$$

$$\begin{aligned}
\delta w_0 :\ & B_{11}\frac{\partial^3 u_0}{\partial x^3} + (B_{12} + 2B_{66})\frac{\partial^3 u_0}{\partial x \partial y^2} + (B_{12} + 2B_{66})\frac{\partial^3 v_0}{\partial x^2 \partial y} \\
& + B_{22}\frac{\partial^3 v_0}{\partial y^3} - D_{11}\frac{\partial^4 w_0}{\partial x^4} - 2(D_{12} + 2D_{66})\frac{\partial^4 w_0}{\partial x^2 \partial y^2} \\
& - \frac{B_n}{4}(k_1 A' - k_2 B')\nabla^2\left(\frac{\partial^3 u_0}{\partial x \partial y^2} - \frac{\partial^3 v_0}{\partial x^2 \partial y}\right) - D_{22}\frac{\partial^4 w_0}{\partial y^4} \\
& - A_n\nabla^4 w_0 - (k_1 D_{11}^s + k_2 D_{12}^s)\frac{\partial^2 \theta}{\partial x^2} - \frac{C_n}{2}\left(\frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial x^2}\right)^2 \\
& - 2(k_1 A' - 2k_2 B')D_{66}^s\frac{\partial^4 \theta}{\partial x^2 \partial y^2} - (k_1 D_{12}^s + k_2 D_{22}^s) \\
& \frac{\partial^2 \theta}{\partial y^2} - \frac{C_n}{2}\left(k_1 A'\frac{\partial^2}{\partial x^2} + k_2 B'\frac{\partial^2}{\partial y^2}\right)\nabla^2 \theta \\
& + \frac{F_n}{4}(k_1 A' - k_2 B')^2\frac{\partial^4}{\partial x^2 \partial y^2}\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) \\
& - \frac{E_n}{4}(k_1 A' - k_2 B')^2\frac{\partial^4 \theta}{\partial x^2 \partial y^2} + L_a\left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right) + q \\
& = I_0\ddot{w}_0 + I_1\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - I_2\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2} + \frac{\partial^2 \ddot{w}_0}{\partial y^2}\right) \\
& - J_2\left(k_1 A'\frac{\partial^2 \ddot{\theta}}{\partial x^2} + k_2 B'\frac{\partial^2 \ddot{\theta}}{\partial y^2}\right)
\end{aligned} \tag{24c}$$

$$\begin{aligned}
\delta\theta :\ & (k_1 B_{11}^s + k_2 B_{12}^s)\frac{\partial u_0}{\partial x} + (k_1 B_{12}^s + k_2 B_{22}^s)\frac{\partial v_0}{\partial y} \\
& + B_{66}^s (k_1 A' + k_2 B')\left(\frac{\partial^3 u_0}{\partial x \partial y^2} + \frac{\partial^3 v_0}{\partial x^2 \partial y}\right) \\
& + \frac{D_n}{4}(k_2 B' - k_1 A')\left(\frac{\partial^3 v_0}{\partial x^2 \partial y} - \frac{\partial^3 u_0}{\partial x \partial y^2}\right) \\
& - (k_1 D_{11}^s + k_2 D_{12}^s)\frac{\partial^2 w_0}{\partial x^2} - (k_1 D_{12}^s + k_2 D_{22}^s)\frac{\partial^2 w_0}{\partial y^2} \\
& - 2D_{66}^s (k_1 A' + k_2 B')\frac{\partial^4 w_0}{\partial x^2 \partial y^2} - \frac{C_n}{2}\left[k_1 A'\nabla^2\frac{\partial^2 w_0}{\partial x^2} + k_2 B'\nabla^2\frac{\partial^2 w_0}{\partial y^2}\right] \\
& - \left(H_{11}^s k_1^2 + 2H_{12}^s k_1 k_2 + H_{22}^s k_2^2\right)\theta \\
& - H_{66}^s (k_1 A' + k_2 B')^2\frac{\partial^4 \theta}{\partial x^2 \partial y^2}{}_z - k_1 A' A_{55}^s\left(-k_1 A'\frac{\partial^2 \theta}{\partial x^2}\right) \\
& - k_2 B' A_{44}^s\left(-k_2 B'\frac{\partial^2 \theta}{\partial y^2}\right) - \frac{E_n}{4}(k_1 A' - k_2 B')^2\frac{\partial^4 w_0}{\partial x^2 \partial y^2} \\
& + \frac{H_n}{4}k_1 A'\left(k_1 A'\frac{\partial^2 \theta}{\partial x^2}\right) + \frac{H_n}{4}k_2 B'\left(k_2 B'\frac{\partial^2 \theta}{\partial y^2}\right) \\
& - \frac{G_n}{4}\nabla^2\left[k_1 A'\left(k_1 A'\frac{\partial^2 \theta}{\partial x^2}\right) + k_2 B'\left(k_2 B'\frac{\partial^2 \theta}{\partial y^2}\right)\right] \\
& + \frac{G_n}{4}(k_1 A' - k_2 B')(3k_2 B' - 2k_1 A') \\
& \frac{\partial^4 \theta}{\partial x^2 \partial y^2} = J_1\left(k_1 A'\frac{\partial \ddot{u}_0}{\partial x} + k_2 B'\frac{\partial \ddot{v}_0}{\partial y}\right) \\
& - J_2\left(k_1 A'\frac{\partial^2 \ddot{w}_0}{\partial x^2} + k_2 B'\frac{\partial^2 \ddot{w}_0}{\partial y^2}\right) - J_3\left(k_1^2 A'^2\frac{\partial^2 \ddot{\theta}}{\partial x^2} + k_2^2 B'^2\frac{\partial^2 \ddot{\theta}}{\partial y^2}\right)
\end{aligned} \tag{24d}$$

Where A_{ij}, B_{ij}, D_{ij}, etc., are the plate stiffness, defined by

$$\begin{Bmatrix} A_{11} & B_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & B_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{66} & B_{66} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \lambda(z)\left[1, z, z^2, f, z\,f, f^2\right] \begin{Bmatrix} \frac{1-\nu}{\nu} \\ 1 \\ \frac{1-2\nu}{2\nu} \end{Bmatrix} dz \quad (25)$$

and

$$\left(A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[1, f, g, g', fg', f^2, g^2, g'^2\right] \mu(z) l^2 dz \quad (26)$$

Where

$$\lambda(z) = \frac{E(z)\nu(z)}{\left[1+\nu(z)\right]\left[1-2\nu(z)\right]} \quad \text{and} \quad \mu(z) = \frac{E(z)}{2\left[1+\nu(z)\right]} \quad (27)$$

2.6 Analytical solutions

In this section, analytical solutions for bending and free vibration are presented for a simply supported rectangular plate under transverse load q. Based on the Navier approach, the solutions are assumed as

$$\begin{Bmatrix} u_0(x,y,t) \\ v_0(x,y,t) \\ w_0(x,y,t) \\ \theta(x,y,t) \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}\cos(\alpha x)\sin(\beta y)e^{i\omega t} \\ V_{mn}\sin(\alpha x)\cos(\beta y)e^{i\omega t} \\ W_{mn}\sin(\alpha x)\sin(\beta y)e^{i\omega t} \\ \Theta_{mn}\sin(\alpha x)\sin(\beta y)e^{i\omega t} \end{Bmatrix} \quad (28)$$

where U_{mn}, V_{mn}, W_{mn}, Θ_{mn} are Fourier coefficients to be determined for each pair of m and n and $i = \sqrt{-1}$ with

$$\alpha = m\pi/a\,, \quad \beta = n\pi/b \quad (29)$$

The transverse load $q$is expanded in the double-Fourier sine series as

$$q(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn}\sin\alpha x \sin\beta y \quad (30)$$

Where

$$Q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\alpha x\sin\beta y\,dxdy = \begin{cases} q_0 & \text{for sinusoiddly distributed load,} \\ \frac{16q_0}{mn\pi^2} & \text{for uniformalydistributed load} \end{cases} \quad (31)$$

Substituting Eqs. (28) and (30) into Eq. (24), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31} & s_{32} & s_{33} & s_{34} \\ s_{41} & s_{42} & s_{43} & s_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}\right) \times \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ \Theta_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ Q_{mn} \\ 0 \end{Bmatrix} \quad (32)$$

Where

$$\begin{aligned}
s_{11} &= A_{11}\alpha^2 + A_{66}\beta^2 + \frac{A_n}{4}\beta^2\left(\alpha^2+\beta^2\right) \\
s_{12} &= \left(A_{12}+A_{66}\right)\alpha\beta - \frac{A_n}{4}\alpha\beta\left(\alpha^2+\beta^2\right) \\
s_{13} &= \frac{B_n}{4}\left(k_2B' - k_1A'\right)\alpha\beta^2\left(\alpha^2+\beta^2\right) - \left(B_{12}+2B_{66}\right)\alpha\beta^2 - B_{11}\alpha^3 \\
s_{14} &= \left(B_{11}^s k_1 + B_{12}^s k_2\right)\alpha - B_{66}^s\left(k_1A' + k_2B'\right)\alpha\beta^2 - \frac{D_n}{4}\left(k_1A' - k_2B'\right)\alpha\beta^2 \\
s_{22} &= A_{22}\beta^2 + A_{66}\alpha^2 + \frac{A_n}{4}\alpha^2\left(\alpha^2+\beta^2\right) \\
s_{23} &= -B_{22}\beta^3 - \left(2B_{66}+B_{12}\right)\alpha^2\beta + \frac{B_n}{4}\left(k_1A' - k_2B'\right)\alpha^2\beta\left(\alpha^2+\beta^2\right) \\
s_{24} &= \left(B_{12}^s k_1 + B_{22}^s k_2\right)\beta - B_{66}^s\left(k_1A' + k_2B'\right)\alpha^2\beta + \frac{D_n}{4}\left(k_1A' - k_2B'\right)\alpha^2\beta \\
s_{33} &= D_{11}\alpha^4 + D_{22}\beta^4 + 2\left(2D_{66}+D_{12}\right)\alpha^2\beta^2 + A_n\left(\alpha^2+\beta^2\right)^2 + \frac{F_n}{4}\left(k_1A' - k_2B'\right)^2\alpha^2\beta^2\left(\alpha^2+\beta^2\right) \\
s_{34} &= 2D_{66}^s\left(k_1A' + k_2B'\right)\alpha^2\beta^2 - k_2\left(D_{12}^s\alpha^2 + D_{22}^s\beta^2\right) - k_1\left(D_{11}^s\alpha^2 + D_{12}^s\beta^2\right) \\
&\quad + \frac{C_n}{2}\left(k_1A'\alpha^2 + k_2B'\beta^2\right)\left(\alpha^2+\beta^2\right) + \frac{E_n}{4}\left(k_1A' - k_2B'\right)^2\alpha^2\beta^2 \\
s_{44} &= H_{11}^s k_1^2 + H_{22}^s k_2^2 + 2H_{12}^s k_1k_2 + H_{66}^s\left(k_1A' - k_2B'\right)^2\alpha^2\beta^2 + A_{55}^s k_1^2A'^2\alpha^2 + A_{44}^s k_2^2B'^2\beta^2 \\
&\quad + G_n\left(k_1^2A'^2 + k_2^2B'^2 - \frac{3}{2}k_1A'k_2B'\right)\alpha^2\beta^2 + \frac{1}{4}G_n\left(k_1^2A'^2\alpha^4 + k_2^2B'^2\beta^4\right) - \frac{H_n}{4}k_2^2B'^2\beta^2
\end{aligned} \quad (33)$$

$$m_{11} = I_0\,,\ m_{12} = 0\,,\ m_{13} = -I_1\alpha\,,$$
$$m_{14} = -J_1k_1A'\alpha\,,\ m_{22} = I_0\,,$$
$$m_{23} = -I_1\beta\,,\ m_{24} = -J_1k_2B'\beta\,,$$
$$m_{33} = I_0 + I_2\left(\alpha^2+\beta^2\right),$$
$$m_{34} = J_2\left(k_1A'\alpha^2 + k_2B'\beta^2\right),$$
$$m_{44} = J_3\left(k_1^2A'^2\alpha^2 + k_2^2B'^2\beta^2\right)$$

Table 1 Comparison of non-dimensional $\overline{w}$ of a homogeneous square plate (l_1=l_2, h=88.h^{-6}m)

l_1/h	*a/h*=5			*a/h*=20			*a/h*=100		
	CPT[(a)]	Ref[(b)]	Present theory	CPT[(a)]	Ref[(b)]	Present theory	CPT[(a)]	Ref[(b)]	Present theory
0	0.2803	0.3433	0.3433	0.2803	0.2842	0.2842	0.2803	0.2804	0.2804
0.2	0.2399	0.2875	0.2875	0.2399	0.2430	0.2430	0.2399	0.2401	0.2401
0.4	0.1676	0.1934	0.1934	0.1676	0.1693	0.1693	0.1676	0.1677	0.1677
0.6	0.1116	0.1251	0.1251	0.1116	0.1124	0.1124	0.1116	0.1116	0.1116
0.8	0.0760	0.0838	0.0838	0.0760	0.0765	0.0765	0.0760	0.0760	0.0760
1	0.0539	0.0588	0.0588	0.0539	0.0542	0.0542	0.0539	0.0539	0.0539

(a) Tsiatas (2009)
(b) Thai and Thai *et al*. (2013)

Table 2 Comparison of non-dimensional fundamental frequency $\overline{\omega}$ of a homogeneous square plate (l_1=l_2, h=88.h^{-6}m)

l_1/h	*a/h*=5			*a/h*=20			*a/h*=100		
	CPT(a)	Ref(b)	Present theory	CPT(a)	Ref(b)	Present theory	CPT(a)	Ref(b)	Present theory
0	5.9734	5.2813	5.2813	5.9734	5.9199	5.9199	5.9734	5.9712	5.9712
0.2	6.4556	5.7699	5.7699	6.4556	6.4027	6.4027	6.4556	6.4535	6.4535
0.4	7.7239	7.0330	7.0330	7.7239	7.6708	7.6708	7.7239	7.7217	7.7217
0.6	9.4673	8.7389	8.7389	9.4673	9.4116	9.4116	9.4673	9.4651	9.4651
0.8	11.4713	10.6766	10.6766	11.4713	11.4108	11.4108	11.4713	11.4689	11.4689
1	13.6213	12.7408	12.7408	13.6213	13.5545	13.5545	13.6213	13.6186	13.6186

(a) Yin *et al*. (2010)
(b) Thai and Thai *et al*. (2013)

3. Numerical results and discussion

3.1 Verification studies

In this section, several numerical examples of bending and dynamic behaviour of FG micro-plate are presented based on modified couple stress theory. The present results are computed using the present theory type TSDT with only 4 unknowns. The results are compared with those reported by Thai *et al*. (2013), Yin *et al*. (2010) and Tsiatas *et al*. (2009). The constituents of the FG micro-plate used in this study include aluminum as material 2 and alumina as material 1 with the following properties:

$$E_1=380 \text{ GPa},\ E_2=70 \text{ GPa},\ \rho_1=3800 \text{ kg/m}^3 \text{ and } \rho_2=2702 \text{ kg/m}^3.$$

In this study, we take the length scale parameter of the aluminum component l_2 as 15μm, and in the other cases the ratio l_2/l_1 is varied so as to demonstrate the influence of the variation of the length scale parameter. The following dimensionless quantities can be defined for the convenience

$$\overline{w}=\frac{10wE_1}{q_0a^4},\quad \overline{\sigma}=\frac{\sigma h}{q_0a},\quad \overline{\omega}=\omega\frac{a^2}{h}\sqrt{\rho_1/E_1}$$

3.1 Parameter studies

The numerical results of simply supported square FG micro plate are presented. Examination of Tables 1-4 reveals that the present theory with only four variables provides similar results to those computed by the third-order shear deformation theory of Reddy (TSDT) used by Thai *et al*. (2013) and this for all examined values of the material length scale parameter (*l/h*) and with considering l_2=l_1=l.

Table 1 is performed for the dimensionless deflection $\overline{w}$ of a homogeneous micro plate subjected to a sinusoidal load q_0. Consider a simply supported micro plate made of epoxy with the following material properties (Reddy 2011):

$$E=1.44 \text{ GPa},\ \nu=0.3,\ \rho=1220 \text{ Kg/m}^3,\ h=88\times10^{-6} \text{ m}$$

The calculated deflections are compared with those predicted by Tai *et al*. (2013) based on the TSDT and by Tsiatas (2009) based on CPT. the analytical solutions of the CPT is given as $w=\frac{q_0}{(D+A_n)(\alpha^2+\beta^2)}$.

It can be seen that the computed results are found to be in excellent agreement with those of Thai *et al*. (2013). It can be seen clearly that the vertical deflection predicted by the CPT (Tsiatas 2009) are independent of the different values for the aspect ratio *a/h* because in CPT theory the shear effect is not introduced.

Table 2 presents the non-dimensional fundamental frequency $\overline{\omega}$ of a simply supported square plate. The obtained results are compared with those predicted by Tai *et al*. (2013) based on the TSDT of Reddy and Yin *et al*. (2010) based on CPT. The analytical solution of the CPT is given as $\omega=(\alpha^2+\beta^2)\sqrt{(D+A_n)/I_0}$. Again, the computed results are found to be excellent agreements with those Thai *et al*. (2013).

In Table 3 the non-dimensional deflections of the FG micro plate for the sinusoidal load based on the present formulation for values of the volume fraction exponent *p*,

Table 3 Non-dimensional deflection $\overline{w}$ of a simply supported square plate (l_1=l_2, h=88.h^{-6}m)

a/h	l_2/h	Plate theory	Gradient index (*p*)					
			0	0.5	1	2	5	10
5	0	Present theory	0. 3433	0.5177	0.6688	0.8671	1.0885	1.2276
		Thai *et al*. (2013)	0. 3433	0.5177	0.6688	0.8671	1.0885	1.2276
	0.2	Present theory	0.2875	0.4275	0.5468	0.7067	0.8981	1.0247
		Thai *et al*. (2013)	0.2875	0.4275	0.5468	0.7067	0.8981	1.0247
	0.4	Present theory	0.1934	0.2807	0.3535	0.4548	0.5925	0.6908
		Thai *et al*. (2013)	0.1934	0.2807	0.3535	0.4548	0.5925	0.6908
	0.6	Present theory	0.1251	0.1786	0.2224	0.2855	0.3802	0.4514
		Thai *et al*. (2013)	0.1251	0.1786	0.2224	0.2855	0.3802	0.4514
	0.8	Present theory	0.0838	0.1183	0.1464	0.1878	0.2539	0.3052
		Thai *et al*. (2013)	0.0838	0.1183	0.1464	0.1878	0.2539	0.3052
	1	Present theory	0.0588	0.0825	0.1017	0.1304	0.1782	0.2158
		Thai *et al*. (2013)	0.0588	0.0825	0.1017	0.1304	0.1782	0.2158
10	0	Present theory	0.2961	0.4537	0.5890	0.7573	0.9114	1.0087
		Thai *et al*. (2013)	0.2961	0.4537	0.5890	0.7573	0.9114	1.0087
	0.2	Present theory	0.2520	0.3798	0.4885	0.6284	0.7743	0.8697
		Thai *et al*. (2013)	0.2520	0.3798	0.4885	0.6284	0.7743	0.8697
	0.4	Present theory	0.1742	0.2551	0.3231	0.4161	0.5349	0.6175
		Thai *et al*. (2013)	0.1742	0.2551	0.3231	0.4161	0.5349	0.6175
	0.6	Present theory	0.1150	0.1649	0.2065	0.2664	0.3538	0.4177
		Thai *et al*. (2013)	0.1150	0.1649	0.2065	0.2664	0.3538	0.4177
	0.8	Present theory	0.0780	0.1103	0.1372	0.1772	0.2403	0.2879
		Thai *et al*. (2013)	0.0780	0.1103	0.1372	0.1772	0.2403	0.2879
	1	Present theory	0.0552	0.0774	0.0959	0.1238	0.1702	0.2058
		Thai *et al.* (2013)	0.0552	0.0774	0.0959	0.1238	0.1702	0.2058
20	0	Present theory	0.2842	0.4377	0.5689	0.7298	0.8669	0.9538
		Thai *et al*. (2013)	0.2842	0.4377	0.5689	0.7298	0.8669	0.9538
	0.2	Present theory	0.2430	0.3677	0.4737	0.6086	0.7429	0.8303
		Thai *et al*. (2013)	0.2430	0.3677	0.4737	0.6086	0.7429	0.8303
	0.4	Present theory	0.1693	0.2486	0.3153	0.4063	0.5201	0.5986
		Thai *et al.* (2013)	0.1693	0.2486	0.3153	0.4063	0.5201	0.5986
	0.6	Present theory	0.1124	0.1614	0.2025	0.2615	0.3470	0.4090
		Thai *et al*. (2013)	0.1124	0.1614	0.2025	0.2615	0.3470	0.4090
	0.8	Present theory	0.0765	0.1083	0.1349	0.1744	0.2368	0.2834
		Thai *et al*. (2013)	0.0765	0.1083	0.1349	0.1744	0.2368	0.2834
	1	Present theory	0.0542	0.0761	0.0944	0.1222	0.1681	0.2033
		Thai *et al*. (2013)	0.0542	0.0761	0.0944	0.1222	0.1681	0.2033
100	0	Present theory	0.2804	0.4326	0.5625	0.7209	0.8527	0.9362
		Thai *et al*. (2013)	0.2804	0.4326	0.5625	0.7209	0.8527	0.9362
	0.2	Present theory	0.2401	0.3639	0.4689	0.6022	0.7327	0.8176
		Thai *et al.* (2013)	0.2401	0.3639	0.4689	0.6022	0.7327	0.8176
	0.4	Present theory	0.1677	0.2465	0.3128	0.4031	0.5153	0.5925
		Thai *et al*. (2013)	0.1677	0.2465	0.3128	0.4031	0.5153	0.5925
	0.6	Present theory	0.1116	0.1603	0.2011	0.2599	0.3448	0.4061
		Thai *et al*. (2013)	0.1116	0.1603	0.2011	0.2599	0.3448	0.4061
	0.8	Present theory	0.0760	0.1076	0.1341	0.1736	0.2357	0.2820
		Thai *et al*. (2013)	0.0760	0.1076	0.1341	0.1736	0.2357	0.2820
	1	Present theory	0.0539	0.0756	0.0939	0.1216	0.1675	0.2024
		Thai *et al*. (2013)	0.0539	0.0756	0.0939	0.1216	0.1675	0.2024

the different values of thickness ratio *a/h*, and dimensionless material length scale parameter l_2/h. The obtained results are found to be excellent agreement with those of Thai *et al*. (2013).

It is also observed from Table 4 that the numerical results of the free vibration analysis of FG micro-plate are in good agreement with those of Tai *et al*. (2013).

Table 5 presents the nom-dimensionless deflections of

Table 4 Non-dimensional frequency $\overline{\omega}$ of a simply supported square plate (l_1=l_2, h=88.h^{-6}m)

a/h	l_2/h	Plate theory	Gradient index (*p*)					
			0	0.5	1	2	5	10
5	0	Present theory	5.2813	4.5180	4.0781	3.6805	3.3938	3.2514
		Thai *et al*. (2013)	5.2813	4.5180	4.0781	3.6805	3.3938	3.2514
	0.2	Present theory	5.7699	4.9715	4.5094	4.0755	3.7327	3 .5548
		Thai *et al*. (2013)	5.7699	4.9715	4.5094	4.0755	3.7327	3 .5548
	0.4	Present theory	7.0330	6.1339	5.6071	5.0763	4.5862	4.3200
		Thai *et al*. (2013)	7.0330	6.1339	5.6071	5.0763	4.5862	4.3200
	0.6	Present theory	8.7389	7.6895	7.0662	6.4011	5.7137	5.3335
		Thai *et al*. (2013)	8.7389	7.6895	7.0662	6.4011	5.7137	5.3335
	0.8	Present theory	10.6766	9.4456	8.7058	7.8861	6.9796	6.4759
		Thai *et al*. (2013)	10.6766	9.4456	8.7058	7.8861	6.9796	6.4759
	1	Present theory	12.7408	11.3086	10.4397	9.4536	8.3193	7.6895
		Thai *et al.* (2013)	12.7408	11.3086	10.4397	9.4536	8.3193	7.6895
10	0	Present theory	5.7694	4.9014	4 .4192	4.0090	3.7682	3.6368
		Thai *et al*. (2013)	5.7694	4.9014	4 .4192	4.0090	3.7682	3.6368
	0.2	Present theory	6.2537	5.3571	4.8526	4.4006	4.0876	3.9162
		Thai *et al*. (2013)	6.2537	5.3571	4.8526	4.4006	4.0876	3.9162
	0.4	Present theory	7.5210	6.5361	5.9664	5.4071	4.9169	4.6464
		Thai *et al*. (2013)	7.5210	6.5361	5.9664	5.4071	4.9169	4.6464
	0.6	Present theory	9.2543	8.1295	7.4619	6.7580	6.0447	5.6487
		Thai *et al*. (2013)	9.2543	8.1295	7.4619	6.7580	6.0447	5.6487
	0.8	Present theory	11.2396	9.9398	9.1537	8.2863	7.3338	6.8030
		Thai *et al*. (2013)	11.2396	9.9398	9.1537	8.2863	7.3338	6.8030
	1	Present theory	13.3651	11.8682	10.9511	9.9101	8.7135	8.0448
		Thai *et al.* (2013)	13.3651	11.8682	10.9511	9.9101	8.7135	8.0448
20	0	Present theory	5.9199	5.0180	4.5228	4.1100	3.8884	3.7622
		Thai *et al*. (2013)	5.9199	5.0180	4.5228	4.1100	3.8884	3.7622
	0.2	Present theory	6.4027	5.4744	4.9568	4.5006	4.2005	4.0323
		Thai *et al*. (2013)	6.4027	5.4744	4.9568	4.5006	4.2005	4.0323
	0.4	Present theory	7.6708	6.6585	6.0756	5.5082	5.0199	4.7488
		Thai *et al*. (2013)	7.6708	6.6585	6.0756	5.5082	5.0199	4.7488
	0.6	Present theory	9.4116	8.2630	7.5817	6.8661	6.1457	5.7453
		Thai *et al*. (2013)	9.4116	8.2630	7.5817	6.8661	6.1457	5.7453
	0.8	Present theory	11.4108	10.0895	9.2887	8.4062	7.4397	6.9013
		Thai *et al*. (2013)	11.4108	10.0895	9.2887	8.4062	7.4397	6.9013
	1	Present theory	13.5545	12.0372	11.1042	10.0450	8.8286	8.1494
		Thai *et al*. (2013)	13.5545	12.0372	11.1042	10.0450	8.8286	8.1494
100	0	Present theory	5.9712	5.0575	4.5579	4.1445	3.9299	3.8058
		Thai *et al.* (2013)	5.9712	5.0575	4.5579	4.1445	3.9299	3.8058
	0.2	Present theory	6.4535	5.5142	4.9922	4.5346	4.2394	4.0725
		Thai *et al*. (2013)	6.4535	5.5142	4.9922	4.5346	4.2394	4.0725
	0.4	Present theory	7.7217	6.7000	6.1126	5.5425	5.0552	4.7840
		Thai *et al*. (2013)	7.7217	6.7000	6.1126	5.5425	5.0552	4.7840
	0.6	Present theory	9.4651	8.3084	7.6224	6.9027	6.1800	5.7782
		Thai *et al*. (2013)	9.4651	8.3084	7.6224	6.9027	6.1800	5.7782
	0.8	Present theory	11.4689	10.1402	9.3344	8.4467	7.4755	6.9345
		Thai *et al*. (2013)	11.4689	10.1402	9.3344	8.4467	7.4755	6.9345
	1	Present theory	13.6186	12.0944	11.1560	10.0904	8.8673	8.1846
		Thai *et al*. (2013)	13.6186	12.0944	11.1560	10.0904	8.8673	8.1846

the FG micro plate based on the present theory for various values of the volume fraction exponent *p*, the different values of thickness ratio *a/h*, and variable length scale parameter l_1/l_2. Results are provided for the sinusoidal load. It is seen that the effect of the shear deformation becomes considerable for the thick micro plate (i.e., *a/h*=5). When l_1/l_2=1, the length scale parameter of the FG micro plate is a constant according to Eq. (15c).

The same equation also implies that, for the other remaining cases for which $l_1/l_2 \neq 1$, the length scale

Table 5 Non-dimensional deflection $\overline{w}$ of a simply supported square plate (l_2=15 μm, h/l_2=2)

l_1/l_2	Plate theory	*a/h*=5				*a/h*=100			
		p=0.3	*p*=1	*p*=3	*p*=10	*p*=0.3	*p*=1	*p*=3	*p*=10
1/3	CPT	0.29610	0.38165	0.47966	0.55014	0.29610	0.38165	0.47966	0.55014
	Present	0.34711	0.44033	0.55941	0.65065	0.29623	0.38180	0.47986	0.55040
1	CPT	0.17466	0.25019	0.36800	0.49092	0.17466	0.25019	0.36800	0.49092
	Present	0.19732	0.27940	0.41002	0.55714	0.17472	0.25027	0.36811	0.49109
3/2	CPT	0.11525	0.17827	0.29175	0.43895	0.11525	0.17827	0.29175	0.43895
	Present	0.12783	0.19502	0.31586	0.48362	0.11528	0.17832	0.29181	0.43907
2	CPT	0.07906	0.12955	0.23041	0.38684	0.07906	0.12955	0.23041	0.38684
	Present	0.08671	0.13968	0.24447	0.41610	0.07908	0.12958	0.23045	0.38692
Classical theory	CPT	0.37256	0.56227	0.79168	0.93546	0.37256	0.56227	0.79168	0.93546
	Present	0.44910	0.66876	0.97432	1.22755	0.37275	0.56254	0.79214	0.93619

Table 6 Non-dimensional frequency $\overline{\omega}$ of a simply supported square plate (l_2=15 μm, h/l_2=2)

l_1/l_2	Plate theory	*a/h*=5				*a/h*=100			
		p=0.3	*p*=1	*p*=3	*p*=10	*p*=0.3	*p*=1	*p*=3	*p*=10
1/3	CPT	5.82675	5.34605	4.95904	4.78176	6.01484	5.53362	5.15855	4.96468
	Present	5.43179	5.02603	4.65008	4.45554	6.01352	5.53253	5.15745	4.96351
1	CPT	7.58646	6.60186	5.66038	5.06159	7.83156	6.83441	5.88936	5.25562
	Present	7.20308	6.30583	5.42015	4.80523	7.83026	6.83339	5.88849	5.25470
3/2	CPT	9.33883	7.81959	6.35557	5.35235	9.64093	8.09646	6.61437	5.55801
	Present	8.94753	7.54243	6.16451	5.14885	9.63957	8.09548	6.61366	5.55727
2	CPT	11.27448	9.17047	7.14902	5.70080	11.63994	9.49764	7.44277	5.92053
	Present	10.86201	8.90560	8.90560	5.54215	11.63850	9.49668	7.44219	5.91995
Classical theory	CPT	5.19465	4.40481	3.86101	3.66791	5.36228	4.55900	4.01530	3.80729
	Present	4.77631	4.07809	3.52566	3.25135	5.36090	4.55792	4.01414	3.80580

parameter varies within the thickness. Thus, the ratio l_1/l_2 presents the degree of the length scale parameter variation within the plate. It is observed that increasing the length scale parameter ratio l_1/l_2 reduces the deflection and the results are significantly different to the case where the length scale parameter is considered to be constant (l_1/l_2=1). This observation is also validation of the premise of this work that the validation of the length scale parameter needs to be taken into consideration in the investigation of FG micro-plate. In addition, it is noted as the gradient index *p* increases, the increase of the deflection will be occur at the same conditions (length scale parameter ratio l_1/l_2 slenderness ratio *a/h*).

Table 6 presents the non-dimensional fundamental frequency $\overline{\omega}$ of FG micro plate for values of the gradient index *p*, for different values of the length scale parameter ratio (l_1/l_2) and for two different values of the aspect ratio (*a/h*=5,100). It can be observed that for each values of the gradient index, the non-dimensional frequency decreases with the reduction of the ration (l_1/l_2). However, the reduction of the gradient index leads to increase of the non-dimensional frequency. Again, from this Table it can be confirmed the need to consider the variation of the length scale parameter *l* within the micro-plate in dynamic analysis of FG micro-plate.

In Fig. 3, the variation of non-dimensional transverse deflections is presented versus the ratio (h/l_2) for different length scale parameter ratio (l_1/l_2) for square plate. It can be seen from Fig. 3 that the deflections given by the classical plate model are independent of the material length scale parameter (h/l_2) and they are always larger than those computed via the nom-classical plate model with the couple stress. This demonstrates that the incorporation of couple stress effect makes a plate stiffer, and hence, leads to a diminution of deflection. However, this influence can be ignored when the material length scale parameter (h/l_2) take high values as is shown in Fig. 3.

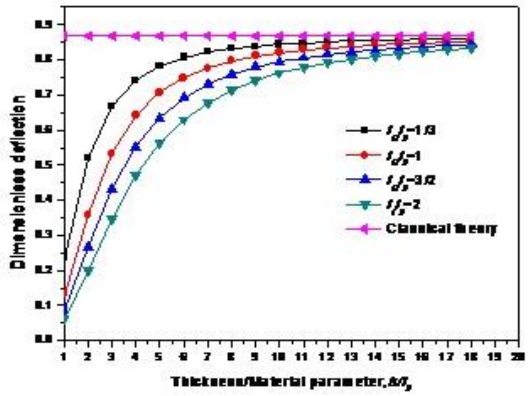

Fig. 3 Variation of the dimensionless transverse deflection of the FG micro-plate for different values of the length parameter ratio l_1/l_2 (*a/h*=5, l_2=15 μm, *a*=*b*, *p*=2)

Fig. 4 presents the variation of the non-dimensional deflection with the gradient index *p* and the length scale

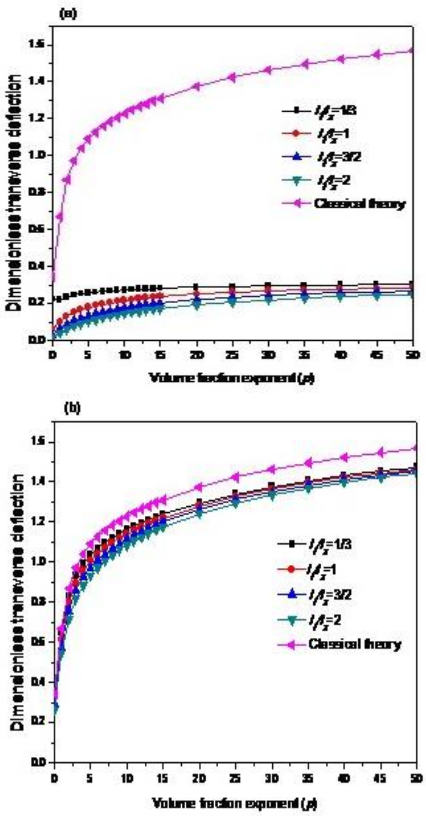

Fig. 4 Variation of the dimensionless transverse deflection of the FG micro-plate for different values of the volume fraction exponent for *a/h*=5, l_2=15 μm and *a*=*b*, (a) h/l_2=1, (b) h/l_2=8

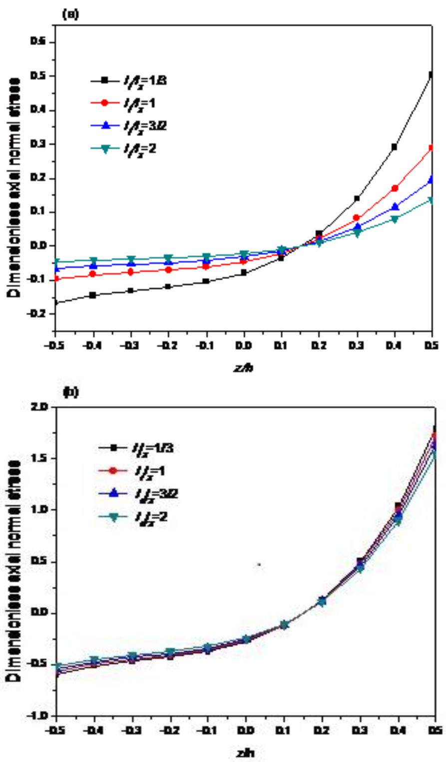

Fig. 5 Variation of the normal stress across the thickness of the FG micro-plate for different values of the length parameter ratio with (*a/h*=5, l_2=15 μm, *a*=*b*, *p*=2) (a) h/l_2=1, (b) h/l_2=8

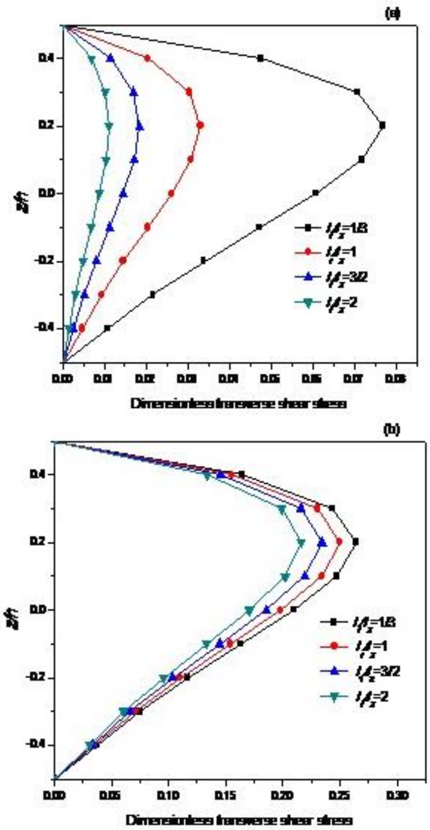

Fig. 6 Variation of the transverse stress across the thickness of the FG micro-plate for different values of the length parameter ratio with (*a/h*=5, l_2=15 μm, *a*=*b*, *p*=2) (a) h/l_2=1, (b) h/l_2=8

parameter ratio (l_1/l_2) for two different values of the non-dimensional material parameter (h/l_2) and for (*a/h*=5).

It can be observed that the increase of the gradient index leads to an increase in the deflection. However, the influence of the length scale parameter ratio (l_1/l_2) on the deflections is not obvious for h/l_2=8 comparatively to the case where h/l_2=1. Thus, the sensitivity of the non-dimensional deflection to the variations in (h/l_2) becomes rather remarked as this ratio takes small values.

In Fig. 5, the variation of the non-dimensional axial normal stress $\bar{\sigma}_x\,(a/2,b/2,z)$ of the FG micro plate with (*a/h*=5) within the thickness is presented for different values of the length scale parameter ratio (l_1/l_2).

Non-dimensional axial normal stress decreases when the ratio (h/l_2) is increased from 1/3 to 2. The reduction is much more significant when h/l_2=1, i.e., the ratio is relatively smaller.

Fig. 6 shows the variation of the dimensionless transverse shear stress $\bar{\tau}_{xz}\,(0,b/2,z)$ of the FG micro plate for different values of the length scale parameter ration (l_1/l_2). It can be observed that the transverse stress increases as the length scale parameter ration (l_1/l_2) decreases. This result demonstrates also the need to consider the variation of the length scale parameter *l* within the micro plate in the investigation of small-scale FG micro-plates.

In Fig. 7, the first and the third non-dimensional frequency are presented as a function of the ratio (h/l_2) for

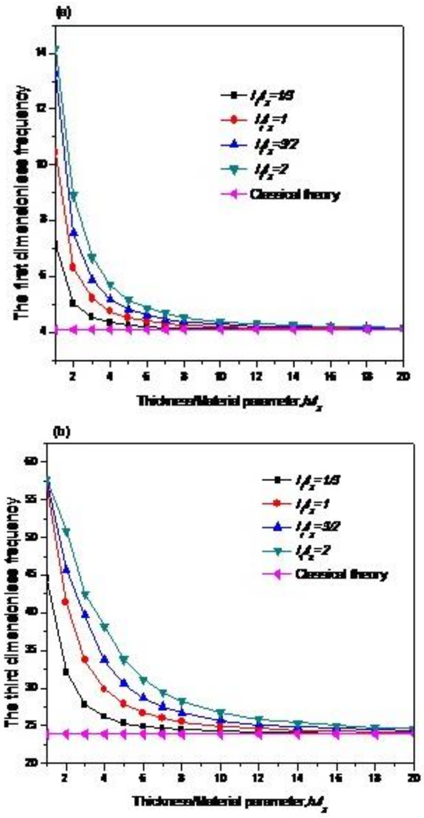

Fig. 7 Variation of the dimensionless frequencies of the FG micro-plate for different values of the length parameter ratio with (a/h=5, l_2=15 μm, a=b, p=1): (a) the first frequency, (b) the third frequency

different length scale parameter ratio (l_1/l_2) with considering a/h=5 and p=1. It can be seen that the frequency computed by the classical plate model are independent of the material length scale parameter (h/l_2) and they are always lower than those calculated by employing the non-classical plate model with the couple stress.

5. Conclusions

This work presents a novel size-dependent plate formulation based on the modified couple stress with only 4 unknowns. The theory considers a variable length scale parameter. A size-dependent model is developed for bending and vibration analysis of FG micro plates. The equations of motion are obtained using Hamilton's principle. Analytical solutions for bending and free vibration problems are obtained for a simply supported plate. This work justifies also the development of a general approach for the analysis of FG micro plate having a variable length scale parameter. It was confirmed that the parameter showing the degree of length scale parameter variation. An improvement of present formulation will be considered in the future work to consider the thickness stretching effect by using quasi-3D shear deformation models (Bessaim *et al*. 2013, Bousahla *et al*. 2014, Belabed *et al*. 2014, Fekrar *et al*. 2014, Hebali *et al*. 2014, Bennai *et al*. 2015, Meradjah *et al*. 2015, Larbi Chaht *et al*. 2015, Hamidi *et al*. 2015, Bourada *et al*. 2015, Bennoun *et al*. 2016, Draiche *et al*. 2016, Benbakhti *et al*. 2016, Benahmed *et al*. 2017, Ait Atmane *et al*. 2017, Benchohra *et al*. 2017, Bouafia *et al*. 2017) and the wave propagation problem (Mahmoud *et al*. 2015, Ait Yahia *et al*. 2015, Boukhari *et al*. 2016).

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