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A functionally graded magneto-thermoelastic half space with memory-dependent derivatives heat transfer

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Abstract. In this work, the model of magneto-thermoelasticity based on memory-dependent derivative (MDD) is applied to a one-dimensional thermal shock problem for a functionally graded half-space whose surface is assumed to be traction free and subjected to an arbitrary thermal loading. The Lamé's modulii are taken as functions of the vertical distance from the surface of thermoelastic perfect conducting medium in the presence of a uniform magnetic field. Laplace transform and the perturbation techniques are used to derive the solution in the Laplace transform domain. A numerical method is employed for the inversion of the Laplace transforms. The effects of the time-delay on the temperature, stress and displacement distribution for different linear forms of Kernel functions are discussed. Numerical results are represented graphically and discussed.

Keywords: magneto-thermoelasticity; FGMs; variable Lamé's modulii; memory-dependent derivatives; perturbation method; numerical results

1. Introduction

Heat transfer continues to be a field of major interest to engineering and scientific researchers as well as designers, developers, and manufacturers. Considerable effort has been devoted to research in traditional applications such as chemical processing, general manufacturing, and energy devices, including general power systems, heat exchangers, and high performance gas turbines.

Physical observations and results of the conventional coupled dynamic thermoelasticity theories involving infinite speed of thermal signals, which were based on the mixed parabolic–hyperbolic governing equations, are mismatched (Biot 1955). To remove this paradox, the conventional theories of thermoelasticity have been generalized by Lord-Shulman (LS) (1967). Among the contributions to this theory are (Chandrasekharaiah 1998, Hetnarski and Ignaczak 1999, El-Karamany and Ezzat 2002, Lata *et al*. 2016, Abbas and Kumar 2016, Kumar *et al*. 2016 and Ezzat *et al*. 2001).

Fractional order differential equations have been the forefront of research due to their applications in many real life problems of fluid mechanics, viscoelasticity, biology, physics and engineering. It is a well-known fact that the integer order differential operator is a local operator but the fractional order differential operator is non-local. Hence, the next state of a system depends not only upon its current state but also upon all of its historical states. This is much more realistic and due to this reason, fractional derivative is also known as memory-dependent derivative. In recent times, various types of definition and approaches of fractional order derivatives have become popular amongst many researchers. The reason behind introduction of the fractional theory is that it predicts retarded response to physical stimuli, as is found in nature, as opposed to instantaneous response predicted by the generalized theory of thermoelasticity. The first application of fractional derivatives was given by Abel, who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. Caputo and Mainardi (1971) used fractional derivatives and their results found good agreement with the empirical evidences for description of viscoelastic materials. Recently, some efforts have been done to modify the classical Fourier law of heat conduction by using the fractional calculus (Povstenko 2011, Sherief *et al*. 2010, Ezzat 2011, 2012, Yu *et al*. 2013, Ezzat and El-Bary 2017 and Kumar and Sharma 2017). One can refer to Podlubny (1999) for a survey of applications of fractional calculus.

The memory-dependent derivative is defined in an integral form of a common derivative with a kernel function on a slipping interval. So this kind of definition is better than the fractional one for reflecting the memory effect (instantaneous change rate depends on the past state). Its definition is more intuitionistic for understanding the physical meaning and the corresponding memory-dependent differential equation has more expressive force (Wang and Li 2011). Recently, many works were devoted to investigate various theoretical and practical aspects in continuum mechanics with memory-dependent heat transfer. Yu *et al*. (2014) introduced memory-dependent derivative (MDD), instead of fractional calculus, into the Lord and Shulman generalized thermoelasticity. Ezzat *et al*. (2014, 2015 and 2017) constructed a new generalized

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thermo-viscoelasticity theory with memory–dependent derivatives.

Functionally graded materials (FGMs) are composite materials which are fabricated with combination of two or more materials with continuously varying material composition to obtain material properties suitable for specific application (Naebe and Shirvanimoghaddam 2016).

The increasing technological demand regarding variability, durability and other properties of materials led to the development of novel categories of high-performance composites, such as FGMs. Within a FGM the mechanical properties continuously vary in one or more directions. This is preferable over layered composites as it eliminates the possibility of delamination failure. The material properties of a FGM can be subtly adjusted and optimized for its application (Willert and Popov 2017).

Recently, various problems in solid mechanics are being studied where the elastic coefficients are no longer constant but they are functions of position. The investigations result from the fact that idea of non-homogeneity in elastic coefficients is not at all hypothetical, but more realistic. Elastic properties in soil may vary considerably with positions. The earth crust itself is non-homogeneous. Beside these, some structural materials such as functionally graded materials (FGMs) have distinct non-homogeneous character. For example, the functionally graded metal-ceramic composites have been used as thermal barriers or thermal shields in various applications (Lee *et al*. 1996 and Tsukamoto 2010). Especially, in severe temperature environments, such as extremely high temperature and thermal shock, widely potential applications are opening for FGMs. Containing various advantageous properties, FGMs are appropriate for various engineering applications and gained intense interest by several researchers (Javaheri and Eslami 2002, Chakraborty *et al*. 2003, Zhang 2013, Zenkour and Abouelregal 2015 and Shirvanimoghaddam 2016, 2017).

In this work we solve a thermal shock problem for a functionally graded perfect electrically conducting half space with memory-dependent derivative heat transfer and variable Lamé's moduli. Laplace transforms and perturbation techniques are used to solve the problem. The inversion of the Laplace transforms is carried out using a numerical approach proposed by Honig and Hirdes (1984). Some comparisons have been shown in figures to estimate the effects of time-delay, different forms of kernel function and Alfven velocity on all the studied fields.

2. Mathematical model

We shall consider a thermoelastic medium of perfect conductivity permeated by an initial magnetic field H_o. This produces an induced magnetic field h and induced electric field E, which satisfy the linearized equations of electro-magnetism and are valid for slowly moving media. The first set of equations constitutes the equations of electro-dynamics of slowly moving bodies

$$\text{Curl}\,\boldsymbol{h} = \boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t} \tag{1}$$

$$\text{Curl}\,\boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{h}}{\partial t} \tag{2}$$

$$\boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{u}}{\partial t} \wedge \boldsymbol{H}_o \tag{3}$$

$$\text{div}\,\boldsymbol{h} = 0. \tag{4}$$

Here the $\boldsymbol{J}$ is the electric current density vector, $\boldsymbol{u}$ is the displacement vector, t is the time, μ_o, ε_o are magnetic and electric permeability, respectively.

The second group of equations is the equations of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + \mu_o \left(\boldsymbol{J} \wedge \boldsymbol{H}_o\right)_i \tag{5}$$

where ρ is the density and σ_{ij} is the stress tensor represents the constitutive equation

$$\sigma_{ij} = \lambda e_{kk}\,\delta_{ij} + 2\mu e_{ij} - (3\lambda + 2\mu)\alpha_T\,\theta \tag{6}$$

where λ, μ are Lamé's modulii, e_{ij} are the components of strain tensor, δ_{ij} is the Kronecker delta, α_T is the coefficient of thermal expansion and $\theta = T - T_o$ where T is the absolute temperature, T_o is the temperature of the medium in its normal state such that $|\,\theta\,| \ll 1$.

The above equations should be supplemented by the relations between strain and displacements

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{7}$$

and heat conduction equation with memory-dependent derivative

$$\begin{aligned} kT_{,ii} &= \rho C_E \frac{\partial T}{\partial t} + T_o(3\lambda + 2\mu)\alpha_T \frac{\partial^2 u_{i,i}}{\partial t} \\ &\quad + \int_{t-\omega}^{t} K(t-\xi)\left(\rho C_E \frac{\partial^2 T}{\partial \xi^2} + T_o(3\lambda + 2\mu)\alpha_T \frac{\partial^2 u_{i,i}}{\partial \xi^2}\right) d\xi \end{aligned} \tag{8}$$

where k is the thermal conductivity and C_E is the specific heat at constant strain.

The previous Eqs. (1)-(8) constitute a complete system of generalized magneto-thermoelasticity for a perfectly conducting medium with memory-dependent derivative heat transfer.

3. One dimensional physical problem

In this work, we consider a homogeneous isotropic electrically conducting thermoelastic solid with perfect conductivity occupying the region $x \geq 0$ composed of a FGM material whose Lamé's parameters depend on the vertical distance "x" from the surface. The surface of the half-space is taken to be traction free and is subjected to a thermal shock that is a function of time. The initial conditions are assumed to be homogenous and the initial magnetic field H_o acts in the direction of the z-axis and has the components $(0, 0, H_o)$. The displacement vector will thus have the components (u, 0, 0). We assume also that there are no body forces or heat sources inside the medium. From the physics of the problem, it is clear that all the variables depend on x and t only.

Given the above assumptions, we have

(1) The Maxwell equations

$$J = -\left(\frac{\partial h}{\partial x} + \varepsilon_o \mu_o H_o \frac{\partial^2 u}{\partial t^2}\right) \tag{9}$$

$$h = -H_o \frac{\partial u}{\partial x} \tag{10}$$

$$E = \mu_o H_o \frac{\partial u}{\partial t}. \tag{11}$$

(2) The constitutive relation

$$\sigma = \sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - (3\lambda + 2\mu)\alpha_T \theta \tag{12}$$

(3) The equation of motion

$$\begin{aligned}&\left(\rho + \varepsilon_o \mu_o^2 H_o^2\right)\frac{\partial^2 u}{\partial t^2} \\ &= \left(\lambda + 2\mu + \mu_o H_o^2\right)\frac{\partial^2 u}{\partial x^2} - (3\lambda + 2\mu)\alpha_T \frac{\partial \theta}{\partial x} \\ &\quad + \left(\frac{\partial \lambda}{\partial x} + 2\frac{\partial \mu}{\partial x}\right)\frac{\partial u}{\partial x} - \alpha_T \left(3\frac{\partial \lambda}{\partial x} + 2\frac{\partial \mu}{\partial x}\right)\theta\end{aligned} \tag{13}$$

The generalized energy equation with memory-dependent derivative

$$\begin{aligned}k\frac{\partial^2 \theta}{\partial x^2} &= \rho C_E \frac{\partial \theta(x,t)}{\partial t} + T_o(3\lambda + 2\mu)\alpha_T \frac{\partial^2 u(x,t)}{\partial t \partial x} \\ &\quad + \int_{t-\omega}^{t} K(t-\xi)\left(\rho C_E \frac{\partial^2 \theta(x,\xi)}{\partial \xi^2}\right. \\ &\quad \left. + T_o(3\lambda + 2\mu)\alpha_T \frac{\partial^3 u(x,\xi)}{\partial x \partial \xi^2}\right) d\xi\end{aligned} \tag{14}$$

From now on, we shall take λ and μ in the form

$$\lambda = \lambda^o e^{-ax}, \quad \mu = \mu^o e^{-ax} \tag{15}$$

where λ^o, μ^o and "a" are constants. Thus, the Eqs. (12)-(14) yield

$$\sigma = \mathrm{e}^{-ax}\left[(\lambda^o + 2\mu^o)\frac{\partial u}{\partial x} - (3\lambda^o + 2\lambda^o)\alpha_T \theta\right] \tag{16}$$

$$\begin{aligned}\left(\rho + \varepsilon_o \mu_o^2 H_o^2\right)\frac{\partial^2 u}{\partial t^2} &= (\lambda^o + 2\mu^o)\frac{\partial}{\partial x}\left(e^{-ax}\frac{\partial u}{\partial x}\right) \\ &\quad - (3\lambda^o + 2\mu^o)\alpha_T \frac{\partial}{\partial x}\left(e^{-ax}\theta\right) + \mu_o H_o^2 \frac{\partial^2 u}{\partial x^2},\end{aligned} \tag{17}$$

$$k\frac{\partial^2 \theta}{\partial x^2} = \rho C_E \frac{\partial \theta(x,t)}{\partial t} + T_o(3\lambda_o + 2\mu_o)\alpha_T e^{-ax}\frac{\partial^2 u(x,t)}{\partial t \partial x}$$

$$\begin{aligned}&+T_o(3\lambda_o + 2\mu_o)\alpha_T e^{-ax}\frac{\partial^3 u(x,\xi)}{\partial x \partial \xi^2}\Bigg) d\xi \\ &+ \int_{t-\omega}^{t} K(t-\xi)\left(\rho C_E \frac{\partial^2 \theta(x,\xi)}{\partial \xi^2}\right.\end{aligned} \tag{18}$$

Let us introduce the following non-dimensional variables

$$x' = c_o \eta x, u' = c_o \eta u, t' = c_o^2 \eta t,$$

$$\sigma' = \frac{\sigma}{\lambda^o + 2\mu^o}, a' = \frac{a}{c_o \eta}, \eta = \frac{\rho C_E}{k}$$

$$\theta' = \frac{(3\lambda^o + 2\mu^o)\alpha_T \theta}{\lambda^o + 2\mu^o}, \; J' = \frac{J}{H_o c_o \eta}, E' = \frac{E}{\mu_o H_o c_o}, h' = \frac{h}{H_o}$$

The Eqs. (9)-(11) and (16)-(18) in non-dimensional form become

$$J = -\left(\frac{\partial h}{\partial x} + V^2 \frac{\partial^2 u}{\partial t^2}\right) \tag{19}$$

$$h = -\frac{\partial u}{\partial x} \tag{20}$$

$$E = \frac{\partial u}{\partial t} \tag{21}$$

$$\sigma = \mathrm{e}^{-ax}\left(\frac{\partial u}{\partial x} - \theta\right) \tag{22}$$

$$\alpha \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} e^{-ax}\left(\frac{\partial u}{\partial x} - \theta\right) + \beta \frac{\partial^2 u}{\partial x^2} \tag{23}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \left(1 + \omega D_\omega\right)\left(\frac{\partial \theta(x,t)}{\partial t} + \varepsilon e^{-ax}\frac{\partial^2 u(x,t)}{\partial x \, \partial t}\right) \tag{24}$$

where

$$D_\omega f(t) = \frac{1}{\omega}\int_{t-\omega}^{t} K(t-\xi)\, f'(\xi)\, \mathrm{d}\xi$$

where ω is the time delay and $K(t - \omega)$ is the kernel function, $\alpha_o = \sqrt{\dfrac{\mu_o H_o^2}{\rho}}$ is Alfven velocity, c is the speed of light given by $c = (1/\mu_o \varepsilon_o)^{1/2}$ $\alpha = 1 + \dfrac{\alpha_o^2}{c^2}$, $\beta = \dfrac{\alpha_o^2}{c^2}$, $V = \dfrac{c_o}{c}$, $c_o = \sqrt{\dfrac{\lambda^o + 2\mu^o}{\rho}}$ is speed of propagation of longitudinal waves and $\varepsilon = T_o(3\lambda_o + 2\mu_o)^2 \alpha_{T0}^2 / (\lambda_o + 2\mu_o)k\eta$.

We assume that the boundary conditions consist of:

(a) A thermal shock is applied to the boundary plane $x = 0$ in the form

$$\theta(0,t) = h(t), \qquad \theta(\infty,t) = 0, \qquad t > 0 \tag{25}$$

(2) Mechanical boundary condition

The bounding plane $x = 0$ is taken to be traction-free

$$\sigma(0,t)=0, \qquad \sigma(t,\infty)=0, \qquad t>0. \tag{26}$$

where $h(t)$ is a known function of t.

From now on, the kernel function form $K(t-\xi)$ can be chosen freely as

$$K(t-\xi)=1-\frac{2n}{\omega}(t-\xi)+\frac{m^2(t-\xi)^2}{\omega^2}$$
$$=\begin{cases}1 \quad if \quad m=n=0\\ 1-\dfrac{(t-\xi)}{\omega} \quad if \quad ,m=0, n=\dfrac{1}{2}\\ 1-(t-\xi) \quad if \quad m=0 \quad n=\omega/2\\ (1-\dfrac{t-\xi}{\omega})^2 \quad if \quad m=n=1,\end{cases} \tag{27}$$

where m and n are constants.

4. The solutions in the Laplace-transform domain

Performing the Laplace transform with parameter s defined by the relation

$$\mathrm{L}\{\mathrm{g}(x,t)\}=\bar{\mathrm{g}}(x,s)=\int_0^\infty \mathrm{e}^{-st}\mathrm{g}(x,t)\,\mathrm{d}t$$

of both sides Eqs. (19)-(24), with the homogeneous initial conditions

$$u(x,0)=\dot{u}(x,0)=\sigma(x,0)=\dot{\sigma}(x,0)$$
$$=\theta(x,0)=\dot{\theta}(x,0)=0,$$

we get a coupled system of the following equations:

$$\bar{J}=-\left(\mathscr{D}\bar{h}+V^2 s^2\bar{u}\right) \tag{28}$$

$$\bar{h}=-\mathscr{D}\bar{u} \tag{29}$$

$$\bar{E}=s\,\bar{u} \tag{30}$$

$$\alpha s^2\bar{u}=\mathscr{D}\left[e^{-ax}\left(\mathscr{D}\bar{u}-\theta\right)\right]+\beta\,\mathscr{D}^2\bar{u}, \tag{31}$$

$$\mathscr{D}^2\bar{\theta}=s(1+G)\left(\bar{\theta}+\varepsilon e^{-ax}\mathscr{D}\bar{u}\right), \tag{32}$$

$$\bar{\sigma}=\mathrm{e}^{-ax}\left(\mathscr{D}\bar{u}-\bar{\theta}\right) \tag{33}$$

where $\mathscr{D}=\dfrac{\partial}{\partial x}$

$$G(s)=(1-e^{-s\omega})(1-\frac{2n}{\omega s}+\frac{2m^2}{\omega^2 s^2})-(m^2-2n+\frac{2m^2}{\omega s})e^{-s\omega}$$

and m and n are constants such that

$$L\{\omega D_\omega f(t)\}$$
$$=F(s)\begin{cases}[(1-e^{-s\omega})], \quad m=n=0\\ [1-\dfrac{1}{\omega s}(1-e^{-s\omega})], \quad m=0, n=\dfrac{1}{2}\\ [(1-e^{-s\omega})-\dfrac{1}{s}(1-e^{-s\omega})+\omega e^{-s\omega}], m=0, n=\dfrac{\omega}{2}\\ [(1-\dfrac{2}{\omega s})+\dfrac{2}{\omega^2 s^2}(1-e^{-s\omega})], \quad m=n=1\end{cases}$$

$$F(s)=L\{(\frac{\partial\theta}{\partial t}+\varepsilon e^{-ax}\frac{\partial^2 u}{\partial x\,\partial t})\}=s(\bar{\theta}+\varepsilon e^{-ax}D\bar{u})$$

and the boundary conditions (26) and (27) become

$$\bar{\theta}(0,s)=\bar{f}(s), \quad \bar{\sigma}(0,s)=0 \tag{34}$$

We shall use the perturbation method to solve the above equations. By expanding the temperature, displacement and stress functions as follows

$$\bar{\theta}=\bar{\theta}^{(0)}+a\,\bar{\theta}^{(1)}+a^2\,\bar{\theta}^{(2)}+\cdots\cdots$$
$$\bar{u}=\bar{u}^{(0)}+a\bar{u}^{(1)}+a^2\bar{u}^{(2)}+\cdots\cdots$$
$$\bar{\sigma}=\bar{\sigma}^{(0)}+a\bar{\sigma}^{(1)}+a^2\bar{\sigma}^{(2)}+\cdots\cdots$$
$$\bar{E}=\bar{E}^{(0)}+a\bar{E}^{(1)}+a^2\bar{E}^{(2)}+\cdots\cdots$$

where $\bar{\theta}^{(i)}$ and $\bar{u}^{(i)}$, $i = 1, 2$ are functions to be determined.

Eqs. (30-(33) gives, upon equating the coefficients of "a" in both sides up to order 1

$$(1+\beta)\mathscr{D}^2\bar{u}^{(0)}-\mathscr{D}\bar{\theta}^{(0)}=\alpha s^2\bar{u}^{(0)} \tag{35}$$

$$\begin{aligned}&(1+\beta)\mathscr{D}^2\bar{u}^{(1)}-\mathscr{D}\bar{\theta}^{(1)}-\alpha s^2\bar{u}^{(1)}\\&=\alpha x\,s^2\bar{u}^{(0)}+\mathscr{D}\bar{u}^{(0)}-\bar{\theta}^{(0)}\end{aligned} \tag{36}$$

$$\mathscr{D}^2\bar{\theta}^{(0)}=s(1+G)\left(\bar{\theta}^{(0)}+\varepsilon\,\mathscr{D}\bar{u}^{(0)}\right) \tag{37}$$

$$\begin{aligned}&\mathscr{D}^2\bar{\theta}^{(1)}-s(1+G)\left(\bar{\theta}^{(1)}+\varepsilon\,\mathscr{D}\bar{u}^{(1)}\right)\\&=-x\,\varepsilon s(1+G)\mathscr{D}\bar{u}^{(0)}\end{aligned} \tag{38}$$

$$\bar{\sigma}^{(0)}=\mathscr{D}\bar{u}^{(0)}-\bar{\theta}^{(0)} \tag{39}$$

$$\bar{\sigma}^{(1)}-\mathscr{D}\bar{u}^{(1)}+\bar{\theta}^{(1)}=-x\,\bar{\sigma}^{(0)} \tag{40}$$

$$\bar{E}^{(0)}=s\,\bar{u}^{(0)} \tag{41a}$$

$$\bar{E}^{(1)}=s\,\bar{u}^{(1)} \tag{41b}$$

$$\bar{h}^{(0)}=-\mathscr{D}\bar{u}^{(0)} \tag{42a}$$

$$\bar{h}^{(1)}=-\mathscr{D}\bar{u}^{(1)} \tag{42b}$$

Eliminating $\bar{\theta}^{(0)}$ between Eqs. (35) and (37), we obtain

$$\left\{\mathscr{D}^4-\left[\alpha\varsigma s^2+s\,(1+G)\,(\varsigma\,\varepsilon+1)\right]\mathscr{D}^2+\alpha\varsigma s^3(1+G)\right\}\bar{u}^{(0)}=0 \tag{43}$$

where $\varsigma = 1/1+\beta$.

The general solution of Eq. (43) which is bounded for $x \geq 0$ has the form

$$\bar{u}^{(0)}(x,s)=-\sum_{i=1}^{2}A_i\,k_i e^{-k_i x} \tag{44}$$

where k_i, $i = 1, 2$ are the roots of the characteristic equation with positive real parts of

$$k^4-\left[\alpha\varsigma s^2+s\,(1+G)\,(\varsigma\,\varepsilon+1)\right]k^2+\alpha\varsigma s^3(1+G)=0,$$

satisfying the relations

$$\begin{aligned}&k_1^2+k_2^2=\alpha\varsigma s^2+s\,(1+G)(\varsigma\,\varepsilon+1)\\&k_1^2k_2^2=\alpha\varsigma s^3(1+G)\end{aligned} \tag{45}$$

and A_i, $i = 1, 2$ are parameters depending on s to be determined from the boundary conditions of the problem.

Substitution from Eq. (44) into Eq. (35), we get

$$\bar{\theta}^{(0)}(x,s)=\sum_{i=1}^{2}A_i\,[(1+\beta)k_i^2-\alpha s^2]\,e^{-k_i x} \tag{46}$$

The boundary conditions (34) become

$$\bar{\theta}^{(0)}(0,s)=\bar{f}\,(s), \tag{47a}$$

$$\bar{\theta}^{(1)}(0,s)=0 \tag{47b}$$

$$\bar{\sigma}^{(0)}(0,s)=0 \tag{47c}$$

$$\bar{\sigma}^{(1)}(0,s)=0 \tag{47d}$$

In order to determine A_i, $i = 1, 2$ we shall use the boundary conditions (34) to obtain

$$A_1=\frac{\alpha s^2-\beta k_2^2}{s^2(k_1^2-k_2^2)}\bar{f}\,(s),\quad A_2=-\frac{\alpha s^2-\beta k_1^2}{s^2(k_1^2-k_2^2)}\bar{f}\,(s)$$

Eliminating $\bar{u}^{(1)}$ between Eqs. (36) and (38), and using Eqs. (44) and (46) we obtain

$$\begin{aligned}&\left(\mathscr{D}^2-k_1^2\right)\left(\mathscr{D}^2-k_2^2\right)\bar{\theta}^{(1)}\\&=\left(\ell_1 x+\ell_2\right)e^{-k_1x}+\left(\ell_3 x+\ell_4\right)e^{-k_2x}\end{aligned} \tag{48}$$

which has a general solution in the form

$$\bar{\theta}^{(1)}=\left(B_1+L_1x^2+L_2x\right)e^{-k_1x}+\left(B_2+L_3x^2+L_4x\right)e^{-k_2x} \tag{49}$$

where

$$\begin{aligned}
\ell_1&=\varsigma\varepsilon k_1^2 s\,(1+G)(2\alpha\varsigma s^2-k_1^2)A_1,\\
\ell_2&=2\varsigma\varepsilon k_1 s\,(1+G)(k_1^2-\alpha\varsigma s^2)A_1,\\
\ell_3&=\varsigma\varepsilon k_2^2 s\,(1+G)(2\alpha\varsigma s^2-k_2^2)A_2,\\
\ell_4&=2\varsigma\varepsilon k_2 s\,(1+G)(k_2^2-\alpha\varsigma s^2)A_2,\\
L_1&=\frac{-\ell_1}{4k_1(k_1^2-k_2^2)},\\
L_2&=\frac{-1}{2k_1(k_1^2-k_2^2)}\left[\ell_2+\frac{5k_1^2-k_2^2}{2k_1(k_1^2-k_2^2)}\ell_1\right],\\
L_3&=\frac{\ell_3}{4k_2(k_1^2-k_2^2)},\\
L_4&=\frac{1}{2k_1(k_1^2-k_2^2)}\left[\ell_4-\frac{5k_1^2-k_2^2}{2k_1(k_1^2-k_2^2)}\ell_3\right],
\end{aligned}$$

In the same manner the displacement distribution $\bar{u}^{(1)}$ satisfies the differential equation

$$\begin{aligned}&\left(\mathscr{D}^2-k_1^2\right)\left(\mathscr{D}^2-k_2^2\right)\bar{\theta}^{(1)}\\&=\left(m_1x+m_2\right)e^{-k_1x}+\left(m_3x+m_4\right)e^{-k_2x}\end{aligned} \tag{50}$$

and has the general solution in the form

$$\begin{aligned}\bar{u}^{(1)}&=\left(B_3+M_1x^2+M_2x\right)e^{-k_1x}\\&+\left(B_4+M_3x^2+M_4x\right)e^{-k_2x}\end{aligned} \tag{51}$$

where

$$\begin{aligned}
m_1&=\varsigma k_1\left[s\,(1+G)(\alpha s^2+\varepsilon k_1^2)-\alpha s^2\right]A_1,\\
m_2&=\varsigma\left[\alpha(3k_1^2-1)s^2-\varepsilon s\,(1+G)\right]A_1,\\
m_3&=\varsigma k_2\left[s\,(1+G)(\alpha s^2+\varepsilon k_1^2)-\alpha s^2\right]A_2,\\
m_2&=\varsigma\left[(3k_1^2-1)\alpha s^2-\varepsilon s\,(1+G)\right]A_2,\\
M_1&=\frac{-m_1}{4k_1(k_1^2-k_2^2)},\\
M_2&=\frac{-1}{2k_1(k_1^2-k_2^2)}\left[m_2+\frac{5k_1^2-k_2^2}{2k_1(k_1^2-k_2^2)}m_1\right],\\
M_3&=\frac{m_3}{4k_2(k_1^2-k_2^2)},\\
M_4&=\frac{1}{2k_1(k_1^2-k_2^2)}\left[m_4-\frac{5k_1^2-k_2^2}{2k_1(k_1^2-k_2^2)}m_3\right].
\end{aligned}$$

From Eqs. (44), (49), (51) and (36) one can get

$$\begin{aligned}B_3=\frac{1}{\varepsilon\varsigma k_1 s\,(1+G)}&\left[\varsigma\varepsilon s\,(1+G)M_2\right.\\&\left.+s\,(1+G-k_1^2)B_1-\ell_2\right]\end{aligned} \tag{52a}$$

$$\begin{aligned}B_4=\frac{1}{\varepsilon\varsigma k_2 s\,(1+G)}&\left[\varsigma\varepsilon s\,(1+G)M_4\right.\\&\left.+s\,(1+G-k_2^2)B_2-\ell_4\right]\end{aligned} \tag{52b}$$

Table 1 Values of the constants

ρ = 8954 kg/m^3	k = 0.55 J/(m.sec.K)	E = 525×10^7 N/m^2
C_E = 381 J/(kg.K)	λ^o = 7.76×(10)^{10} kg/(ms^2)	α_T = 1.78(10)$^{-5}$ K^{-1}
γ = 7.76(10)^{10} N/(m^{2}K)	η = 8886.73 sec/m^2	c = 2200 m/sec
T_o = 293 K	μ^o = 3.86×(10)^{10} kg/(ms^2)	H_o = 1.0 C/m.sec
ε = 0.0168	$\omega \approx 10^{-4}$ sec	a = 0.03 m^{-1}

By using the boundary conditions (47b) and (47d) into Eqs. (49) and (51), we have

$$B_1 = -B_2 \tag{52c}$$

$$k_1 L_3 + k_2 L_4 = M_2 + M_4 \tag{52d}$$

Solving the above system, we have

$$B_1 = -\frac{\ell_2 + \ell_4}{k_1^2 - k_2^2}$$

The other constants can be easily obtained from Eqs. (52a)-(52c).

The stress and induced magnetic and electric fields can be obtained from Eqs. (39)-(42).

This completes the solution in the Laplace transform domain.

5. Numerical inversion of the Laplace transforms

We shall now outline the method used to invert the Laplace transforms in the above equations. Let $\bar{f}(s)$ be the Laplace transform of a function $f(t)$. The inversion formula for Laplace transforms can be written as Honig and Hirdes (1984)

$$f(t) = \frac{e^{dt}}{2\pi} \int_{-\infty}^{\infty} e^{ity}\, \bar{f}(d+iy)\, \mathrm{d}y\,,$$

where d is an arbitrary real number greater than all the real parts of the singularities of $\bar{f}(s)$.

Expanding the function $h(t) = \exp(-dt) f(t)$ in a Fourier series in the interval $[0, 2L]$, we obtain the approximate formula

$$f(t) \approx f_N(t) = \frac{1}{2} c_0 + \sum_{k=1}^{N} c_k\,, \qquad \text{for} \qquad 0 \le t \le 2L \tag{53}$$

where

$$c_k = \frac{e^{dt}}{L} \mathrm{Re}\left[e^{ik\pi t/L}\, \bar{f}\left(d + ik\pi/L\right)\right]. \tag{54}$$

Two methods are used to reduce the total error. First, the ‘Korrektur’ method is used to reduce the discretization error. Next, the ε-algorithm is used to reduce the truncation error and therefore to accelerate convergence.

The Korrektur-method uses the following formula to evaluate the function $f(t)$

$$f(t) = f_{NK}(t) = f_N(t) - e^{-2dL}\, f_{N'}(2L+t)\,. \tag{55}$$

We shall now describe the ε-algorithm that is used to accelerate the convergence of the series in Eq. (55). Let N be an odd natural number and let $s_m = \sum_{k=1}^{m} c_k$, be the sequence of partial sums of (55).We define the ε-sequence by

$$\varepsilon_{0,m} = 0,\ \varepsilon_{1,m} = s_m, \quad m = 1, 2, 3, \ldots$$

and

$$\varepsilon_{n+1,m} = \varepsilon_{n-1,m+1} + 1/\left(\varepsilon_{n,m+1} - \varepsilon_{n,m}\right), \quad n, m = 1, 2, 3, \ldots.$$

It can be shown from Honig and Hirdes (1984), that the sequence $\varepsilon_{1,1}$, $\varepsilon_{3,1}$,..., $\varepsilon_{N,1}$,... converges to $f(t) - c_0/2$ faster than the sequence of partial sums.

6. Numerical results

The method based on a Fourier series expansion proposed by Honig and Hirdes (1984) and developed in detail in many works such as Ezzat and Youssef (2010) is adopted to invert the Laplace transform in the previous Equations. The analysis is conducted for a copper material, whose physical data is given below in Table 1 (Ezzat and Fayik 2011)

Let us consider that (t) is varying sinusoidal pulse function with time described mathematically as

$$f(t) = \begin{cases} \sin\left(\dfrac{\pi t}{a}\right) & 0 \le t \le a \\ 0 & \text{otherwise} \end{cases}$$

or

$$\bar{f}(s) = \frac{\pi a\left(1 + e^{-as}\right)}{a^2 s^2 + \pi^2}$$

The computations were performed for one value of time, namely t = 0.12. The numerical technique outlined above was used to obtain the temperature, displacement, stress and induced magnetic and electric fields. The results are displayed graphically at different positions of x as shown in Figs. 1-5.

Fig. 1 depicts the space variation of temperature with distance x for different values of time-delay ω = 0.000, 0.009, 0.09 when the kernel function has the form $K(t, \xi) = 1 - (t - \xi)/\omega$. We noticed that the solution corresponding to the coupled theory (Biot theory, ω = 0) that the thermal waves propagate with infinite speeds, so the value of the temperature is not identically zero (though it may be very small) for any large value of x. While, in the non-Fourier

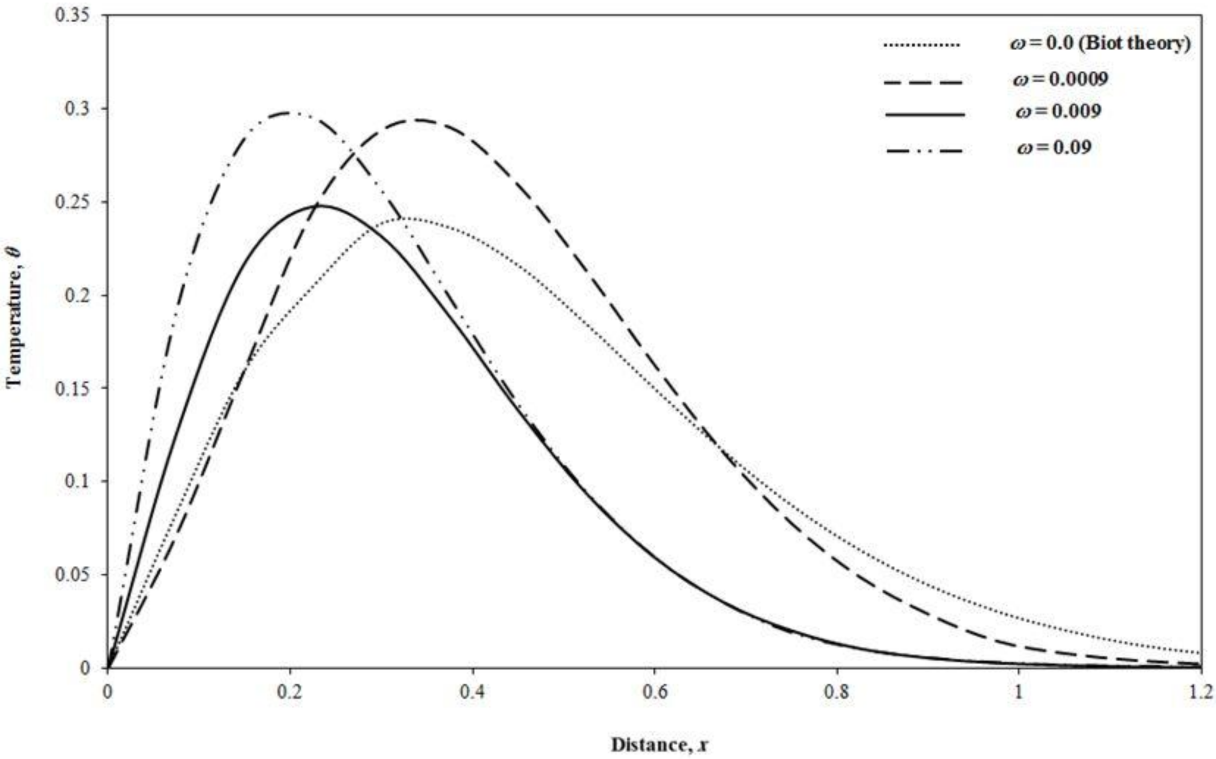

Fig. 1 The variation of temperature for different values of time-delay ω kernal fuction $K(t, \xi) = 1 - (t - \xi) - \omega$

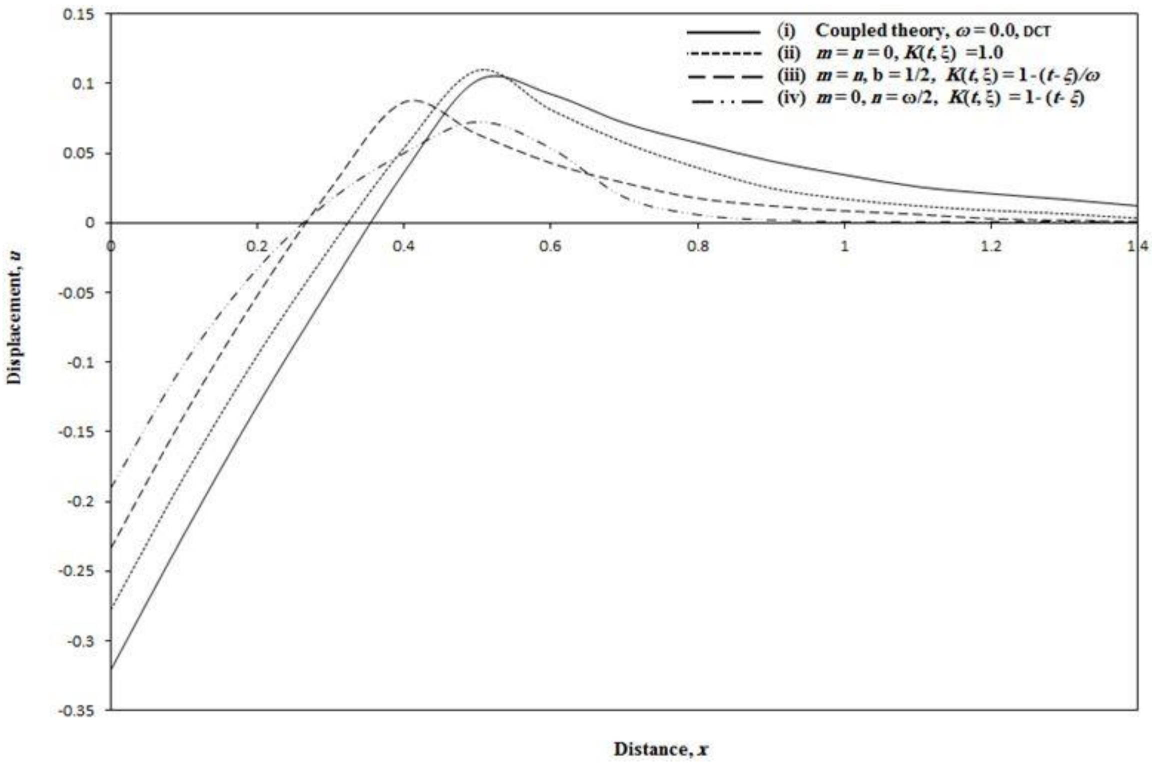

Fig. 2 The variation of displacement for different forms of kernal function $K(t, \xi)$ and time-delay $\omega = 0.009$

theory ($\omega > 0$), the response to the thermal effect in FGMs does not reach infinity instantaneously but remains in a bounded region. We observed also that the temperature field has been affected by the time-delay ω, where the decreasing of the value of the parameter ω causes increasing in the amplitude of the thermal waves which are continuous functions and smooth.

Figs. 2 and 3 depict the space variations of displacement

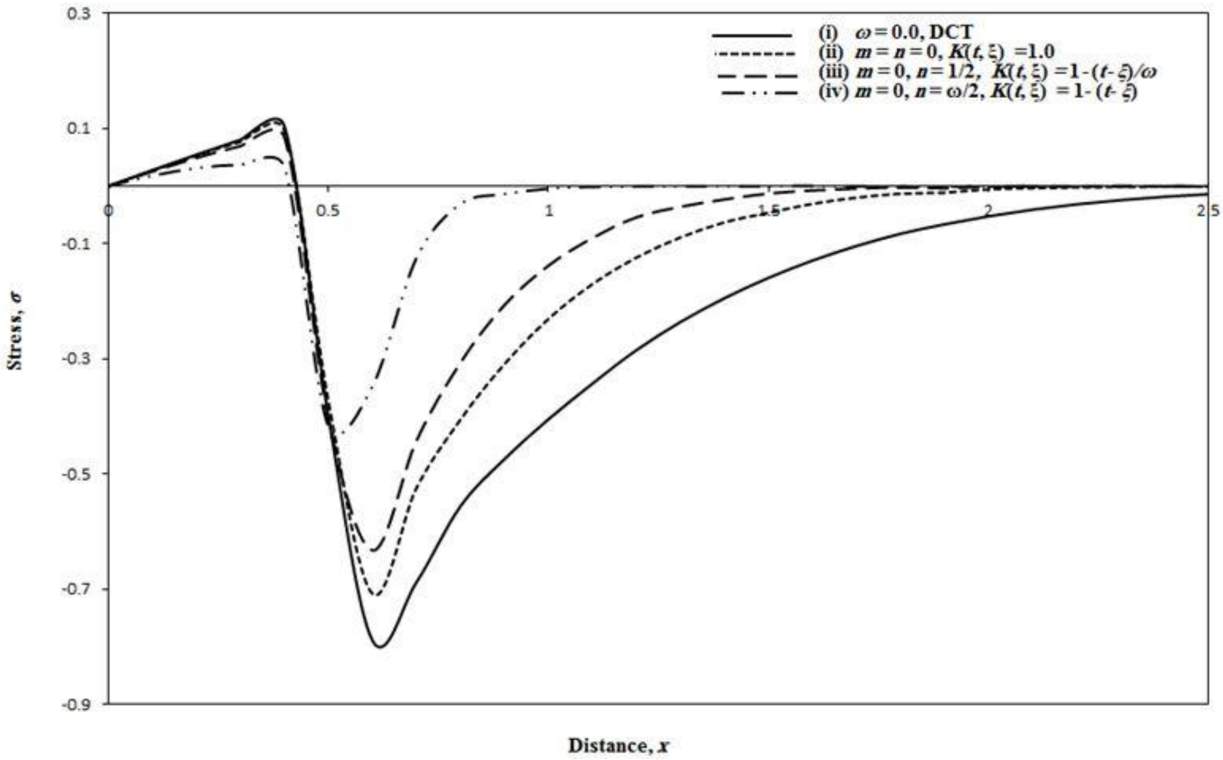

Fig. 3 The variation of stress for different forms of kernal function $K(t, \xi)$ and time-delay ω = 0.009

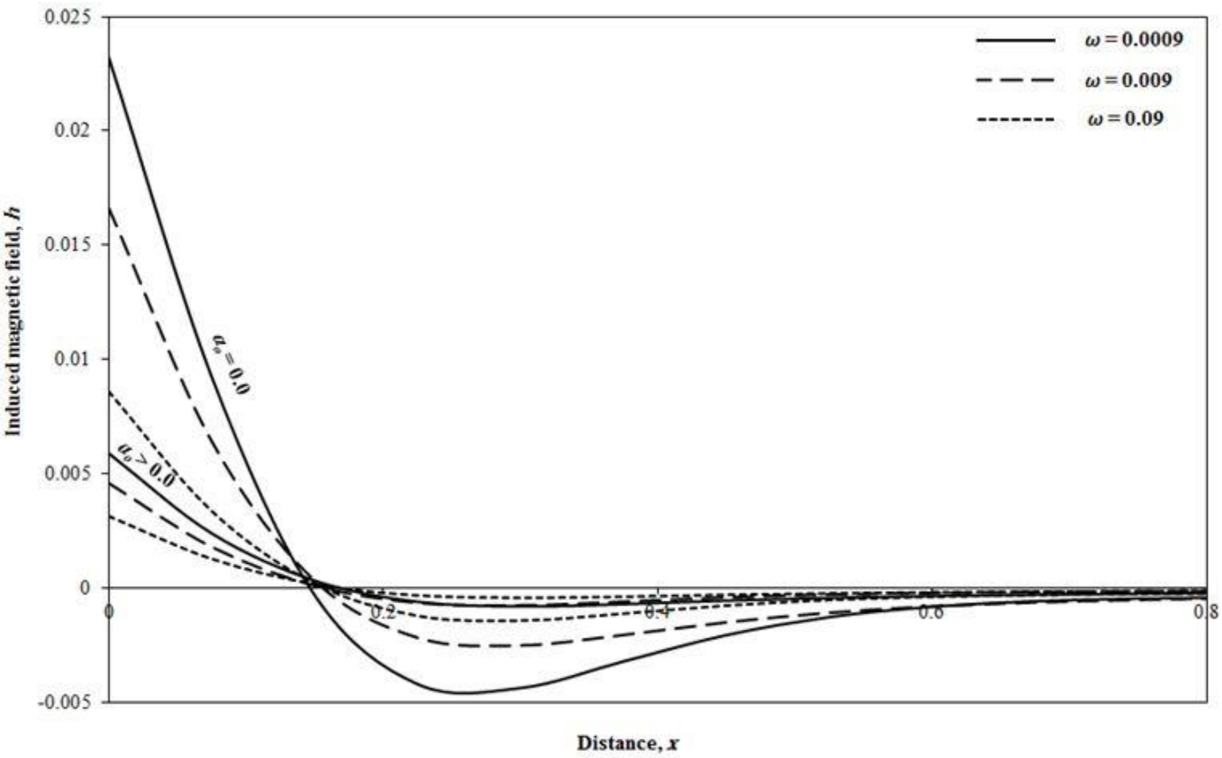

Fig. 4 The variation of induced magnetic field for different values of time-delay ω and Alfven velocity a_o

and stress distributions for the different forms of kernel function $K(t, \xi) = 1$, $1 - (t - \xi)$, $1 - (t - \xi)/\omega$ at one value of time-delay ω = 0.009. We learned from these figures that the displacement and stress fields in FGMs has been affected by the choosing of the form of kernel function $K(t, \xi)$. As usual in dealing with problems of the theory of generalized thermoelasticity of FGMs with memory-dependent derivative, the finite speed of wave propagation

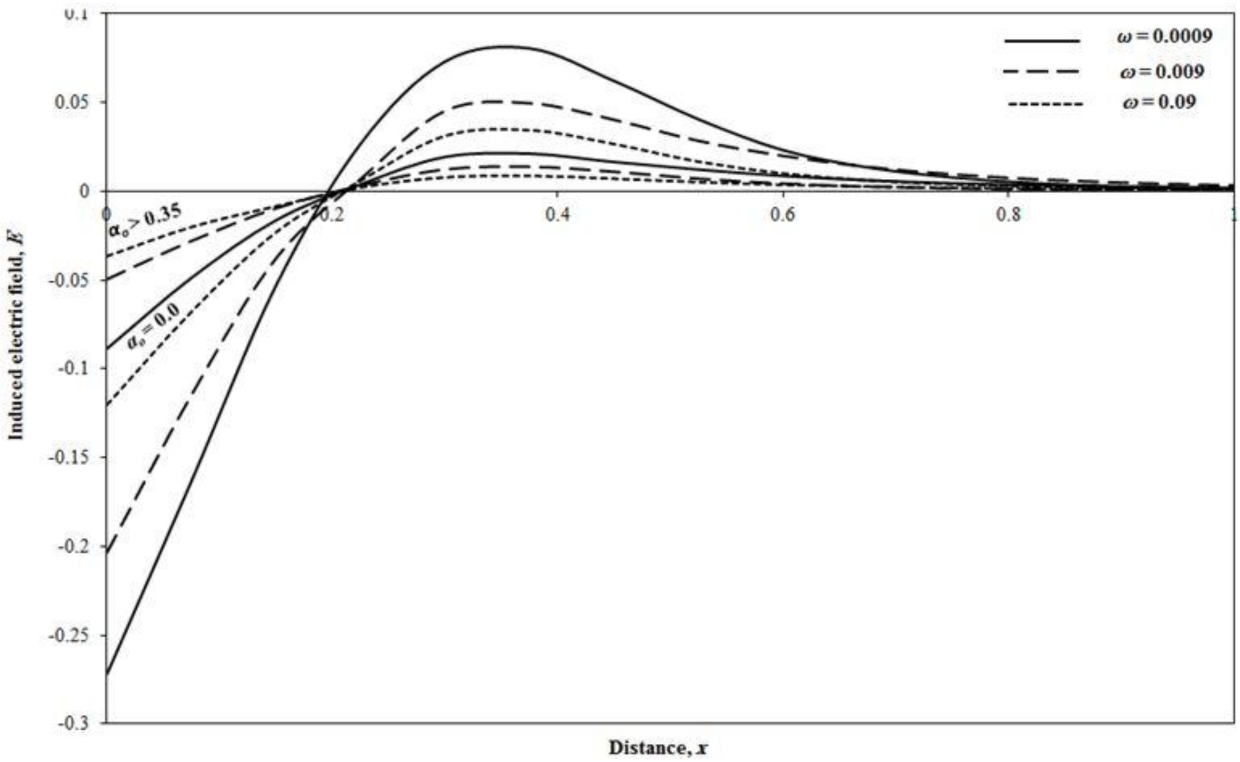

Fig. 5 The variation of induced electric field for different values of time-delay ω and Alfven velocity a_o

is apparent.

Figs. 5 and 6 show the variation of displacement and stress distributions for a kernel function namely, $K(t, \xi) = 1 - (t - \xi)$ at different values of time-delay. It is clear that see that time-delay ω has a significant effect on the induced magnetic and electric fields so that the magnitude value of stress and displacement distributions have the same behavior as the temperature and the absolute of the maximum value of these distributions decreases when time-delay increases. The Alfven velocity α_o acts to decrease the displacement and magnitude of stress distributions. This is mainly due to the fact that the magnetic field corresponds to a term signifying a positive force that tends to accelerate the charge carries.

7. Conclusions

The electro–magneto–thermoelastic analysis problem of a perfect conducting FGM half space based upon memory-dependent derivatives theory is presented. The main contribution in this article is to describe the effects of time-delay and kernel function on temperature, displacement, and stress distributions. According to this theory we have to construct a new classification for FGMs materials according to their, time-delay ω where this parameter becomes a new indicator of its ability to conduct heat in conducting medium. The advantage of Fourier law of heat conduction with time-delay and kernel function by using the definition for reflecting the memory effect (instantaneous change rate depends on the past state) where kernel function and time-delay can be arbitrarily chosen freely according to the necessity of applications.

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