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Vibration analysis of FG nanobeams based on third-order shear deformation theory under various boundary conditions

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Abstract. In this study, free vibration of functionally graded (FG) micro/nanobeams based on nonlocal third-order shear deformation theory and under different boundary conditions is investigated by applying the differential quadrature method. Third-order shear deformation theory can consider the both small-scale effects and quadratic variation of shear strain and hence shear stress along the FG nanobeam thickness. The governing equations are obtained by using the Hamilton's principle, based on third-order shear deformation beam theory. The differential quadrature (DQ) method is used to discretize the model and attain the natural frequencies and mode shapes. The properties of FG micro/nanobeam are assumed to be chanfged along the thickness direction based on the simple power law distribution. The effects of various parameters such as the nonlocal parameter, gradient index, boundary conditions and mode number on the vibration characteristics of FG micro/nanobeams are discussed in detail.

Keywords: nanobeam; free vibration; nonlocal elasticity; third-order shear deformation theory; functionally gradient materials

1. Introduction

Recent developments in material processing technology have resulted to the fabrication of a new kind of materials called functionally graded materials (FGMs) (Koizumi 1993, 1997). The most significant characteristic of this type of material is their continuous variation in composition and material properties. Hence, unwanted stress concentrations can be prevented in FGMs and they have excellent thermo-mechanical properties. Since FGMs have more superiority than laminated composites and homogeneous materials, they are widely used in numerous engineering fields such as, optics, biomedicine, aerospace, nuclear and mechanical. Between structures, beams have wide applications in engineering fields. Therefore, understanding the mechanical behaviour of these structures is very important in design of these structures. Sina *et al*. investigated free vibration of functionally graded beams by a new beam theory (Sina *et al*. 2009). The bending and free vibration of layered FGM beam using the third order zigzag theory studied by Kapuria *et al*. (2008). Alshorbagy *et al*. (2011) presented the dynamic characteristics of functionally graded beam with material graduation in axially or transversally through the thickness using finite element method. Mahi *et al*. (2010) presented an exact solutions for studying the free vibration of a FG beam with material properties vary continuously through the thickness according to a power law distribution (P-FGM), or an exponential law distribution (E-FGM) or a sigmoid law distribution (S-FGM).

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Recently, fast development in the use of micro/nano structures in engineering fields, led scientists to model nanostructures (Rahmani and Noroozi Moghaddam 2014, Rahmani *et al*. 2015, 2016a, b, 2017a, b). Using FG materials allow the designer to achieve a particular goal. For example, the static deflection does not exceed from a specified value, the critical buckling load is less than a pre-determined value, or the natural frequency of the system is less or more than a specified amount. Because of these advantages of functionally graded materials and by developing of the material technology, FGMs are widely used in micro and nano structures such as thin films, microswitches, micro piezoactuator, and micro/nano-electromechanical systems (MEMS and NEMS) (Lun *et al*. 2006, Batra *et al*. 2008, Carbonari *et al*. 2009, Jia *et al*. 2011, 2012). Due to the demands in applications and high sensibility of MEMS/NEMS to external excitations, obtain the mechanical properties of these nanoscale devices have attracted a lot of attention. Between the nanostructures, nanobeams due to their wide applications in engineering, such as nanowires, nanoprobes, atomic force microscope (AFM), and nanosensors have attracted a lot of consideration (Hung and Senturia 1999, Pei *et al*. 2004, Moser and Gijs 2007, Moghimi Zand and Ahmadian 2009, Hosseini and Rahmani 2016a, b, c, Rahmani *et al*. 2016a). Hence, many investigations have been carried out about buckling, vibration and bending of nanobeams (Jandaghian and Rahmani 2016a, Reddy 2007, Aydogdu 2009, Rahmani and Jandaghian 2015, Jandaghian and Rahmani 2016b). However, for micro/nanostructures, direct employ of classical continuum theory led to wrong results in predicting their mechanical behavior, the classical theory cannot capture the size effects, because structures at nanometer length scale display size-dependent behavior

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(Miller and Shenoy 2000). Since the experimental methods at the nanoscale are too time-consuming, so many researchers in nanotechnology use continuum mechanics for modeling the nanostructures. Such theories contain information about the forces between atoms, and the internal length scale is introduced into the constitutive equations as a material parameter. Among these theories, due to its simplicity and high accuracy, Eringen's theory has found wide application to study the behaviour of micro/nanostructures. Studies show that the results obtained from this theory are in good agreement with the results obtained by the method of molecular dynamics (Eringen and Edelen 1972, Eringen 1983, 2002, 2006). Therefore, in many studies about the nanostructures researchers use this theory (Murmu and Pradhan 2009, Pradhan and Phadikar 2009, Janghorban and Zare 2011, Hosseini-Hashemi *et al*. 2013, Bounouara *et al*. 2016). Recently, some researchers have studied FG micro/nanobeam based on Euler-Bernoulli and Timoshenko beam theory. Janghorban and Zare (2011) investigated free vibration analysis of functionally graded carbon nanotube with variable thickness based on the Timoshenko beam theory by differential quadrature method (DQM). Eltaher *et al*. (2012) have studied free vibration analyses of functionally graded (FG) size-dependent nanobeams using finite element method. Ke *et al*. (2012), Ke *et al*. (2012), and Asghari *et al*. (2010) investigated nonlinear vibrations of a micro-beam made of functionally graded materials using modified couple stress theory. Bending and buckling of analysis of FG nanobeams studied by Şimşek and Yurtcu (2013) based on Euler-Bernoulli and Timoshenko beams theory. Nguyen *et al*. (2014) present an analytical solutions for the size-dependent static analysis of the functionally graded (FG) nanobeams with various boundary conditions based on the nonlocal continuum model. Uymaz (2013) investigated forced vibration analysis of functionally graded (FG) nanobeams by using Navier method for various shear deformation theories. Rahmani and Pedram (2014) studied the vibration of FG nanobeams based on the nonlocal elasticity and Timoshenko beam model. Zemri *et al*. (2015) present a nonlocal shear deformation beam theory for bending, buckling and vibration of functionally graded nanobeams by using the nonlocal differential constitutive relations of Eringen. A new nonlocal hyperbolic refined plate model for free vibration properties of functionally graded (FG) plates was presented by Belkorissat *et al*. (2015). This nonlocal nano-plate model incorporates the length scale parameter, which can capture the small scale effect. To analyse FG micro/nanobeam with exact stress fields, more studies are needed. In the Euler-Bernoulli beam theory, the effect of the transverse shear strain and shear stress are neglected. In the Timoshenko beam theory, the transverse shear strain and consequently transverse shear stress are represented as constant through the beam thickness, which is a gross approximation of the true variation that vanishes on the top and bottom surfaces of the beam. Shear correction factor is introduced to validate this discrepancy between the true variation and the constant state of stress. The third-order shear deformation theory of Reddy is based on a displacement field that includes the cubic term in the thickness coordinate, thus the transverse shear strain and hence stress are expressed as quadratic across the beam thickness and vanish on the bottom and top surfaces of the beam. Therefore, the shear correction factor is not needed in this theory. Despite relatively further complex algebraic equations and computational effort compared to the Euler-Bernoulli and Timoshenko theories, the third-order shear deformation theory gives results that are close to 3-D elasticity solutions. Therefore, in this study we use the third-order shear deformation theory to investigate free vibration of FG micro/nanobeams based on the nonlocal elasticity. The governing equations of motion are derived by applying Hamilton's principle. Then these equations are solved by using Navier solution and the differential quadrature (DQ) method to determine the natural frequencies and mode shapes of FG micro/nanobeams under different boundary conditions. The effects of the nonlocal parameter, gradient index, length scale parameter and length-to-thickness ratio on the vibration characteristics of FGM micro/nanobeams with various boundary conditions are investigated.

2. Functionally graded materials

The material properties variation of FGMs are expressed in terms of a simple power law distribution (Praveen and Reddy 1998) as follow

$$P(z) = (P_2 - P_1)\left(\frac{1}{2} + \frac{z}{h}\right)^g + P_1, \tag{1}$$

where g is the power-law index, $0 \leq g \leq \infty$ and P_1 and P_2 are the corresponding material properties of the 1 and 2. Thus, for a functionally gradient material with two constituent materials, the Young's modulus and the mass density ρ can be stated as

$$E(z) = (E_2 - E_1)\left(\frac{1}{2} + \frac{z}{h}\right)^g + E_1, \tag{2}$$

$$\rho(z) = (\rho_2 - \rho_1)\left(\frac{1}{2} + \frac{z}{h}\right)^g + \rho_1, \tag{3}$$

3. Formulation

The displacement field for an FG beam of length l, width b and height h based on the Reddy beam theory is given as (Reddy 2002)

$$u_1 = u_1(x,t) + z\varphi(x,t) - c_1 z^3\left(\varphi + \frac{\partial w_0}{\partial x}\right), \quad u_2 = 0 \tag{4}$$

$$u_3 = w_0\left(x,t\right). \tag{5}$$

where u is the axial displacement, w_0 is the transverse displacement of any point of the beam, u_0 is the axial displacement, ϕ is the rotation of a point on the centroidal axis x of the beam and $c_1 = 4/(3h^2)$. The linear strains associated with the displacement field.

The linear strains associated with the displacement field

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)} \tag{6}$$

$$\gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \tag{7}$$

where

$$\varepsilon_{xx}^{(0)} = \frac{\partial u_0}{\partial x},\ \varepsilon_{xx}^{(1)} = \frac{\partial \phi}{\partial x},\ \varepsilon_{xx}^{(3)} = -c_1\left(\frac{\partial \phi}{\partial x} + \frac{\partial^2 w_0}{\partial x^2}\right) \tag{8}$$

$$\gamma_{xz}^{(0)} = \phi + \frac{\partial w_0}{\partial x},\ \gamma_{xz}^{(2)} = -c_2\left(\phi + \frac{\partial w_0}{\partial x}\right) \tag{9}$$

and $c_2 = 4/h^2$. It should be mentioned that $\sigma_{xz} = G\gamma_{xz}$ is quadratic in z direction and the transverse shear strain is zero on the top and bottom surfaces, $z = \pm h/2$, of the beam. Thus, there is no requiring applying shear correction factors in the Reddy beam theory.

By using Hamilton's principle, the governing equations and the corresponding boundary conditions of FG nanobeam are obtained. The strain energy variation δU of the FG nanobeam is written as

$$\begin{aligned}\delta U &= \int_V \sigma_{ij}\delta\varepsilon_{ij}\mathrm{dV} = \int_0^L\int_A \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz}\right)\mathrm{dxdA} \\ &= \int_0^L \left(N\delta\varepsilon_{xx}^{(0)} + M\delta\varepsilon_{xx}^{(1)} + P\delta\varepsilon_{xx}^{(3)} + Q\delta\gamma_{xz}^{(0)} + R\delta\gamma_{xz}^{(2)}\right)\mathrm{dx}\end{aligned} \tag{10}$$

where

$$\begin{Bmatrix} N \\ M \\ P \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z^2 \\ z^3 \end{Bmatrix} \sigma_{xx}\mathrm{dz},\quad \begin{Bmatrix} Q \\ R \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} \sigma_{xz}\mathrm{dz}. \tag{11}$$

The variation of the work done by applied forces can be written as

$$\begin{aligned}\delta V = \int_0^L &(N\delta\varepsilon_{xx}^0 + \hat{M}\frac{\partial\delta\phi}{\partial x} - c_1 P\frac{\partial^2\delta w_0}{\partial x^2} + \hat{Q}\delta\gamma_{xz}^0 \\ &- f\delta u_0 - q\delta w_0 - \bar{N}\frac{\partial w_0}{\partial x}\frac{\partial\delta w_0}{\partial x})\mathrm{dx}\end{aligned} \tag{12}$$

where

$$\hat{M} = M - c_1 P,\quad \hat{Q} = Q - c_2 R. \tag{13}$$

and the first variation of the kinetic energy is expressed as

$$\begin{aligned}\delta K = \int_0^L\int_A &\rho(z)\{[\dot{u}_0 + z\dot{\phi}_0 - c_1 z^3(\dot{\phi} + \frac{\partial\dot{w}_0}{\partial x})] \\ &[\delta\dot{u}_0 + z\delta\dot{\phi} - c_1 z^3(\delta\dot{\phi} + \frac{\partial\delta\dot{w}_0}{\partial x})] + \dot{w}_0\delta\dot{w}_0\}\mathrm{dAdx}\end{aligned} \tag{14}$$

f and q are the axial and transverse distributed loads and $\bar{N}$ is the applied axial compressive force. For obtaining the governing equations the Hamilton principle is written as

$$\int_0^t (\delta U + \delta V - \delta K)\mathrm{dt} = 0 \tag{15}$$

Substituting Eqs. (10), (12) and (14) into Eq. (15), the governing equations of motion of the Reddy beam theory are obtained as follow

$$\frac{\partial N}{\partial x} + f = I_0\frac{\partial^2 u_0}{\partial t^2}, \tag{16}$$

$$\begin{aligned}c_1\frac{\partial^2 P}{\partial x^2} + \frac{\partial\hat{Q}}{\partial x} - \frac{\partial}{\partial x}\left(\bar{N}\frac{\partial w_0}{\partial x}\right) + q = I_0\frac{\partial^2 w}{\partial t^2} + \\ c_1 I_4\frac{\partial^3\phi}{\partial x\partial t^2} - c_1^2 I_6\left(\frac{\partial^3\phi}{\partial x\partial t^2} + \frac{\partial^4 w_0}{\partial x^2\partial t^2}\right),\end{aligned} \tag{17}$$

$$\frac{\partial\hat{M}}{\partial x} - \hat{Q} = \hat{I}_2\frac{\partial^2\phi}{\partial t^2} - c_1\hat{I}_4\left(\frac{\partial^2\phi}{\partial t^2} + \frac{\partial^3 w_0}{\partial x\partial t^2}\right) \tag{18}$$

where

$$I_i = \int_{-h/2}^{h/2}\rho(z)z^i dz,\ \hat{I}_2 = I_2 - c_1 I_4, \hat{I}_4 = I_4 - c_1 I_6 \tag{19}$$

4. Nonlocal theory

In nonlocal theory, it is assumed that the stress at a point **x** not only depends on the strain at the same point but also on strains at all other points of the body. Therefore, the nonlocal stress tensor $\bar{\sigma}$ at point **x** is stated as

$$\bar{\sigma}_{ij} = \int_V K\left(|\mathbf{x}' - \mathbf{x}|, \tau\right)\sigma_{ij}(\mathbf{x}')\mathrm{d}\mathbf{x}' \tag{20}$$

K ($|\mathbf{x}' - \mathbf{x}|$, τ) is the nonlocal kernel function; τ is a material constant that depends on internal and external characteristic lengths (such as the lattice spacing and wavelength, respectively) in the body which is defined as $\tau = e_0 a/l$ where e_0 is a constant appropriate to each material, a is an internal characteristics length (e.g., lattice parameter, granular distance) and l is an external characteristics length (e.g., crack length, wavelength). σ is the classical, local stress tensor at point **x** and satisfies the constitutive relations

$$\sigma = \mathbf{C}(\mathbf{x}) : \varepsilon(\mathbf{x}) \tag{21}$$

where **C** is the fourth-order elasticity tensor, ε is the strain tensor and : indicates the 'double-dot product'. According to Eringen (1983), the nonlocal stress tensor $\bar{\sigma}$ at point **x** can be written as

$$\left(1 - \tau^2 l^2\nabla^2\right)t = \sigma,\ \tau = e_0 a/l \tag{22}$$

where ∇^2 is the Laplacian operator. For a beam type structure, the nonlocal behaviour can be neglected along the thickness. Hence, nonlocal constitutive equations for present FG nanobeam can be expressed as

$$\sigma_{xx}-\mu\frac{\partial^2\sigma_{xx}}{\partial x^2}=E(z)\varepsilon_{xx},\ \left(\mu=e_0^2a^2\right) \tag{23}$$

$$\sigma_{xz}-\mu\frac{\partial^2\sigma_{xz}}{\partial x^2}=G(z)\varepsilon_{xz} \tag{24}$$

where $E(z)$ and $G(z)$ are elastic and shear modulus of FG nanobeam, respectively. By using Eqs. (11) and (22)-(24), the stress resultants are attained as

$$N-\mu\frac{\partial^2 N}{\partial x^2}=A_{11}\varepsilon_{xx}^{(0)}+B_{11}\varepsilon_{xx}^{(1)}+E_{11}\varepsilon_{xx}^{(3)} \tag{25}$$

$$M-\mu\frac{\partial^2 M}{\partial x^2}=B_{11}\varepsilon_{xx}^{(0)}+D_{11}\varepsilon_{xx}^{(1)}+F_{11}\varepsilon_{xx}^{(3)} \tag{26}$$

$$P-\mu\frac{\partial^2 P}{\partial x^2}=E_{11}\varepsilon_{xx}^{(0)}+F_{11}\varepsilon_{xx}^{(1)}+H_{11}\varepsilon_{xx}^{(3)} \tag{27}$$

$$Q-\mu\frac{\partial^2 Q}{\partial x^2}=A_{55}\gamma_{xz}^{(0)}+D_{55}\gamma_{xz}^{(2)} \tag{28}$$

$$R-\mu\frac{\partial^2 R}{\partial x^2}=D_{55}\gamma_{xz}^{(0)}+F_{55}\gamma_{xz}^{(2)} \tag{29}$$

where

$$\left(A_{11},B_{11},D_{11},E_{11},F_{11},H_{11}\right)=\int_A E(z)(1,z,z^2,z^3,z^4,z^6)\,\mathrm{dA} \tag{30}$$

$$\left(A_{55},D_{55},F_{55}\right)=\int_A G(\mathrm{z})(1,z^2,z^4)\,\mathrm{dA} \tag{31}$$

$$G(z)=E(z)/2(1+\nu) \tag{32}$$

The governing equation of the axial displacement is derived for the nonlocal theory by substituting the second derivative of the axial force N from Eq. (16) into Eq. (25)

$$N=A_{11}\varepsilon_{xx}^{(0)}+B_{11}\varepsilon_{xx}^{(1)}+E_{11}\varepsilon_{xx}^{(3)}+\mu(I_0\frac{\partial^3 u_0}{\partial x\partial t^2}+\frac{\partial f}{\partial x}), \tag{33}$$

From Eqs. (26)-(28) the nonlocal constitutive equations for the stress resultants of the FG Reddy micro/nanobeam theory are

$$\widehat{M}-\mu\frac{\partial^2\widehat{M}}{\partial x^2}=\widehat{K}\varepsilon_{xx}^{(0)}+\widehat{L}\varepsilon_{xx}^{(1)}+\widehat{J}\varepsilon_{xx}^{(3)}, \tag{34}$$

$$\hat{Q}-\mu\frac{\partial^2\hat{Q}}{\partial x^2}=\bar{A}\gamma_{xz}^{(0)}+\bar{L}\gamma_{xz}^{(2)}, \tag{35}$$

$$P-\mu\frac{\partial^2 P}{\partial x^2}=E_{11}\varepsilon_{xx}^{(0)}+F_{11}\varepsilon_{xx}^{(1)}+H_{11}\varepsilon_{xx}^{(3)}. \tag{36}$$

where

$$\begin{gathered}\widehat{K}=B_{11}-c_1E_{11},\widehat{L}=D_{11}-c_1F_{11},\ \widehat{J}=F_{11}-c_1H_{11}\\ \widehat{I}_2=I_2-c_1I_4,\ \widehat{I}_4=I_4-c_1I_6,\\ \bar{A}=A_{55}-c_2D_{55},\bar{L}=D_{55}-c_2F_{55}\end{gathered} \tag{37}$$

Eliminating $\hat{Q}$ from Eqs. (17) and (18) results in the following equation

$$\begin{aligned}\frac{\partial^2\widehat{M}}{\partial x^2}&=-c_1\frac{\partial^2 P}{\partial x^2}-q+\frac{\partial}{\partial x}\left(\bar{N}\frac{\partial w_0}{\partial x}\right)+\\ &I_0\frac{\partial^2 w_0}{\partial t^2}+I_2\frac{\partial^3\phi}{\partial x\partial t^2}-c_1I_4\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^4 w_0}{\partial x^2\partial t^2}\right),\end{aligned} \tag{38}$$

Substituting the above result in the Eq. (34) the nonlocal bending moment is obtained as follow

$$\begin{aligned}\widehat{M}&=\widehat{K}\frac{\partial u_0}{\partial x}+\widehat{L}\frac{\partial\phi}{\partial x}-c_1\widehat{J}\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w_0}{\partial x^2}\right)\\ &\mu\left[-c_1\frac{\partial^2 P}{\partial x^2}-q+\frac{\partial}{\partial x}\left(\bar{N}\frac{\partial w_0}{\partial x}\right)+I_0\frac{\partial^2 w_0}{\partial t^2}+\right.\\ &\left.I_2\frac{\partial^3\phi}{\partial x\partial t^2}--c_1I_4\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^4 w_0}{\partial x^2\partial t^2}\right)\right]\end{aligned} \tag{39}$$

Substituting the second derivative of $\hat{Q}$ from Eq. (18) into the Eq. (39), yields to the following equation

$$\begin{aligned}\hat{Q}&=\tilde{A}\left(\phi+\frac{\partial w_0}{\partial x}\right)+\mu\left[-c_1\frac{\partial^3 P}{\partial x^3}+\frac{\partial^2}{\partial x^2}\left(\bar{N}\frac{\partial w_0}{\partial x}\right)-\frac{\partial q}{\partial x}\right]\\ &+\mu\frac{\partial}{\partial x}\left[I_0\frac{\partial^2 w_0}{\partial t^2}+c_1I_4\frac{\partial^3\phi}{\partial x\partial t^2}-c_1^2I_6\left(\frac{\partial^3\phi}{\partial x\partial t^2}+\frac{\partial^4 w_0}{\partial x^2\partial t^2}\right)\right]\end{aligned} \tag{40}$$

where

$$\tilde{A}=\bar{A}-c_2\bar{L} \tag{41}$$

Using Eq. (36), the following equation is obtained as

$$\begin{aligned}&c_1\frac{\partial^2}{\partial x^2}\left(P-\mu\frac{\partial^2 P}{\partial x^2}\right)=\\ &c_1\left[E_{11}\frac{\partial^3 u_0}{\partial x^3}+F_{11}\frac{\partial^2\phi}{\partial x^3}-H_{11}c_1(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^4 w}{\partial x^4})\right]\end{aligned} \tag{42}$$

By substituting Eqs. (33), (39), (40) and (42) into Eqs. (16)-(18) the nonlocal equations of motion in terms of the displacements can be written as

$$\begin{aligned}&A_{11}\frac{\partial^2 u_0}{\partial x^2}+B_{11}\frac{\partial^2\phi}{\partial x^2}-E_{11}c_1\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^3 w_0}{\partial x^3}\right)+f\\ &-\mu\frac{\partial^2 f}{\partial x^2}=I_0\left(\frac{\partial^2 u_0}{\partial t^2}-\mu\frac{\partial^4 u_0}{\partial x^2\partial t^2}\right),\end{aligned} \tag{43}$$

$$\hat{K}\frac{\partial^2 u_0}{\partial x^2}+\hat{L}\frac{\partial^2\phi}{\partial x^2}-c_1\hat{J}\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^3 w_0}{\partial x^3}\right)$$
$$-\hat{A}\left(\phi+\frac{\partial w_0}{\partial x}\right)=\hat{I}_2\frac{\partial^2\phi}{\partial t^2}-c_1\hat{I}_4\left(\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^3 w_0}{\partial x\partial t^3}\right) \quad (44)$$
$$-\mu\left[\hat{I}_2\frac{\partial^4\phi}{\partial x^2\partial t^2}-c_1\hat{I}_4\left(\frac{\partial^4\phi}{\partial x^2\partial t^2}+\frac{\partial^5 w_0}{\partial x^3\partial t^3}\right)\right],$$

$$c_1\left[E_{11}\frac{\partial^3 u_0}{\partial x^3}+F_{11}\frac{\partial^3\phi}{\partial x^3}-H_{11}c_1\left(\frac{\partial^3\phi}{\partial x^3}+\frac{\partial^4 w_0}{\partial x^4}\right)\right]$$
$$+\tilde{A}\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w_0}{\partial x^2}\right)-\frac{\partial^2 w_0}{\partial x^2}\bar{N}+q+\mu\left[\bar{N}\frac{\partial^4 w_0}{\partial x^4}-\frac{\partial^2 q}{\partial x^2}\right]$$
$$=I_0\frac{\partial^2 w_0}{\partial t^2}+c_1I_4\frac{\partial^3\phi}{\partial x\partial t^2}-c_1^2I_6\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial\phi}{\partial t}+\frac{\partial^2 w_0}{\partial x\partial t}\right) \quad (45)$$
$$-\mu\left[I_0\frac{\partial^4 w_0}{\partial x^2\partial t^2}+c_1I_4\frac{\partial^5\phi}{\partial x^3\partial t^2}-c_1^2I_6\frac{\partial^4}{\partial x^3\partial t}\left(\frac{\partial\phi}{\partial x}+\frac{\partial^2 w_0}{\partial x\partial t}\right)\right].$$

5. Navier solution

For free vibration, $\overline{N}$ and q are zero. Consider an FG simply supported micro/nanobeam. The Navier solution method is used for obtaining the analytical solution and natural frequency in this section. The boundary conditions related to the simply supported nanobeam are

$$w_0=M=0 \;\; at\; x=0\;,\; L. \quad (46)$$

According to the Navier solution, the axial displacement u_0, the transverse displacement w_0 and the rotation ϕ are defined as (Reddy 2007)

$$u_0(x,t)=\sum_{n=1}^{N}U_n\cos(\alpha_n x)e^{i\omega_n t},$$
$$w_0(x,t)=\sum_{n=1}^{N}W_n\sin(\alpha_n x)e^{i\omega_n t}, \quad (47)$$
$$\phi(x,t)=\sum_{n=1}^{N}\Phi_n\cos(\alpha_n x)e^{i\omega_n t}.$$

where $\alpha_n = n\pi/L$, ω_n is the natural frequency and (U_n, W_n, Φ_n) are coefficients. Substituting Eqs. (47) into Eqs. (43)-(45), the closed-form solutions can be obtained from the following system of equations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{Bmatrix}U_n\\ W_n\\ \Phi_n\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}. \quad (48)$$

where coefficients a_{11} through a_{33} are given in Appendix. By solving the characteristic equation that are obtained by setting the determinant of the coefficient matrix $[a]_{3\times 3}$ equal to zero, the natural frequencies of the FG nanoscale beam are obtained.

6. Differential quadrature method

The differential quadrature (DQ) method (Shu 2000) is used to solve Eqs. (43)-(45) and the associated boundary conditions to find the natural frequencies of FG micro/nano beam. The dimensionless quantities are introduced as follow

$$U=\frac{u_0}{L},\; W=\frac{w_0}{L},\; X=\frac{x}{L},\; \eta=(\frac{e_0a}{L})^2,\; \phi=\varphi \quad (49)$$

$$\frac{A_{11}}{L^2}\frac{\partial^2 U}{\partial X^2}+\frac{B_{11}}{L^2}\frac{\partial^2\varphi}{\partial X^2}-E_{11}c_1\left(\frac{1}{L^2}\frac{\partial^2\varphi}{\partial X^2}+\frac{1}{L^3}\frac{\partial^3 W}{\partial X^3}\right)$$
$$=I_0\left(1-\eta\frac{\partial^2}{\partial X^2}\right)\frac{\partial^2 U}{\partial t^2}, \quad (50)$$

$$\frac{\hat{K}}{L^2}\frac{\partial^2 U}{\partial X^2}+\frac{\hat{L}}{L}\frac{\partial^2\varphi}{\partial X^2}-c_1\hat{J}\left(\frac{1}{L^2}\frac{\partial^2\varphi}{\partial X^2}+\frac{1}{L^3}\frac{\partial^3 W}{\partial X^3}\right)$$
$$-\hat{A}\left(\varphi+\frac{1}{L}\frac{\partial W}{\partial X}\right)=\hat{I}_2\left(1-\eta\frac{\partial^2}{\partial X^2}\right)\frac{\partial^2\varphi}{\partial t^2} \quad (51)$$
$$-c_1\hat{I}_4\left(1-\eta\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial^2\varphi}{\partial t^2}+\frac{1}{L}\frac{\partial^3 W}{\partial X\partial t^2}\right)$$

$$c_1\left[\frac{E_{11}}{L^3}\frac{\partial^3 U}{\partial X^3}+\frac{F_{11}}{L^3}\frac{\partial^3\varphi}{\partial X^3}-H_{11}c_1\left(\frac{1}{L^3}\frac{\partial^3\varphi}{\partial X^3}+\frac{1}{L^4}\frac{\partial^4 W}{\partial X^4}\right)\right]$$
$$+\tilde{A}\left(\frac{1}{L}\frac{\partial\varphi}{\partial X}+\frac{1}{L^2}\frac{\partial^2 W}{\partial X^2}\right)=I_0\left(1-\eta\frac{\partial^2}{\partial X^2}\right)\frac{\partial^2 W}{\partial t^2}$$
$$+c_1\hat{I}_4\left(1-\eta\frac{\partial^2}{\partial X^2}\right)\frac{1}{L}\frac{\partial^3\varphi}{\partial X\partial t^2} \quad (52)$$
$$-c_1^2\hat{I}_6\left(1-\eta\frac{\partial^2}{\partial X^2}\right)\left(\frac{1}{L}\frac{\partial^3\varphi}{\partial X\partial t^2}+\frac{1}{L^2}\frac{\partial^4 W}{\partial X^2\partial t^2}\right)$$

The boundary conditions of the FG micro/nanobeam may be clamped (C), simply supported (S), or free (F). Thus, the end support conditions can be written in the dimensionless form

$$\text{Clamped (C):}\quad W=U=\varphi=0.\quad \frac{\partial W}{\partial X}=0 \quad (53)$$

Simply Supported (S):

$$W=U=0,$$
$$\hat{K}\frac{\partial\varphi}{\partial X}-\hat{I}\frac{\partial\varphi}{\partial X}-c_1\left(\frac{\partial\varphi}{\partial X}+\frac{1}{L}\frac{\partial^2 W}{\partial X^2}\right)=0, \quad (54)$$
$$E_{11}c_1\frac{\partial U}{\partial X}+F_{11}c_1\frac{\partial\varphi}{\partial X}-H_{11}c_1^2\left(\frac{\partial\varphi}{\partial X}+\frac{1}{L}\frac{\partial^2 W}{\partial X^2}\right)=0.$$

Free (F):

$$A_{11}\frac{\partial\varphi}{\partial X}+B_{11}\frac{\partial\varphi}{\partial X}-c_1E_{11}\left(\frac{\partial\varphi}{\partial X}+\frac{1}{L}\frac{\partial^2W}{\partial X^2}\right)=0,$$
$$\widehat{K}\frac{\partial U}{\partial X}+\hat{L}\frac{\partial\varphi}{\partial X}-c_1\hat{J}\left(\frac{\partial\varphi}{\partial X}+\frac{1}{L}\frac{\partial^2W}{\partial X^2}\right)=0,$$
$$E_{11}c_1\frac{\partial U}{\partial X}+F_{11}c_1\frac{\partial\varphi}{\partial X}-H_{11}c_1^2\left(\frac{\partial\varphi}{\partial X}+\frac{1}{L}\frac{\partial^2W}{\partial X^2}\right)=0. \quad (55)$$
$$\tilde{A}L^2\left(\varphi+\frac{1}{L}\frac{\partial W}{\partial X}\right)+c_1\left[E_{11}c_1\frac{\partial^2U}{\partial X^2}+F_{11}c_1\frac{\partial^2\varphi}{\partial X^2}-H_{11}c_1^2\left(\frac{\partial^2\varphi}{\partial X^2}+\frac{1}{L}\frac{\partial^3W}{\partial X^3}\right)\right]=0.$$

The functions U, W, and φ and their jth derivatives with respect to x can be approximated

$$\{U,W,\varphi\}=\sum_{k=1}^{N}l_k(X)\{U_k,W_k,\varphi_k\} \quad (56)$$

$$\left.\frac{\partial^j}{\partial X^j}(U,W,\varphi)\right|_{X=X_i}=\sum_{k=1}^{N}C_{ik}^{(j)}\{U_k,W_k,\varphi_k\} \quad (57)$$

where N is the total number of nodes distributed along the x-axis, $l_k(X)$ is the Lagrange interpolation polynomials, and $C_{ik}^{(j)}$ is the weighting coefficients of the jth-order differentiation. The recursive formula for which can be found in Shu (2000). In numerical computations, the cosine pattern is employed to generate the DQ point system because of its fast convergence characteristics

$$X_i=\frac{1}{2}\left[1-\cos\frac{\pi(i-1)}{N-1}\right],\quad i=1,2,\ldots,N \quad (58)$$

Applying the relationships between Eqs. (56) and (57) to Eqs. (43)-(45), result in a set of ordinary differential equations

$$\frac{A_{11}}{L^2}\sum_{k=1}^{N}B_{ik}U_k+\frac{B_{11}}{L^2}\sum_{k=1}^{N}B_{ik}\varphi_k-E_{11}c_1\left(\frac{1}{L^2}\sum_{k=1}^{N}B_{ik}\varphi_k+\frac{1}{L^3}\sum_{k=1}^{N}C_{ik}W_k\right)=m_0\left(\ddot{U}_i-\frac{\eta}{L^2}\sum_{k=1}^{N}B_{ik}U_k\right) \quad (59)$$

$$\frac{\widehat{K}}{L^2}\sum_{k=1}^{N}B_{ik}U_k+\frac{\hat{L}}{L^2}\sum_{k=1}^{N}B_{ik}\varphi_k-c_1\hat{J}\left(\frac{1}{L^2}\sum_{k=1}^{N}B_{ik}\varphi_k+\frac{1}{L^3}\sum_{k=1}^{N}C_{ik}W_k\right)-\hat{A}\left(\varphi_i+\frac{1}{L}\sum_{k=1}^{N}A_{ik}W_k\right)$$
$$=\hat{I}_2\left(\ddot{\varphi}_i-\frac{\eta}{L^2}\sum_{k=1}^{N}B_{ik}\ddot{\varphi}_k\right)-\hat{I}_4\left(\ddot{\varphi}_i+\frac{1}{L^2}\sum_{k=1}^{N}A_{ik}\ddot{W}_k\right)+c_1\hat{I}_4\eta\left(\ddot{\varphi}_i\frac{1}{L^2}\sum_{k=1}^{N}B_{ik}\ddot{\varphi}_k+\frac{1}{L}\sum_{k=1}^{N}C_{ik}\ddot{W}_k\right)0, \quad (60)$$

$$c_1\left[\frac{E_{11}}{L^3}\sum_{k=1}^{N}C_{ik}U_k+\frac{F_{11}}{L^3}\sum_{k=1}^{N}C_{ik}\varphi_k-H_{11}c_1\left(\frac{1}{L^3}\sum_{k=1}^{N}C_{ik}\varphi_k+\frac{1}{L^4}\sum_{k=1}^{N}D_{ik}W_k\right)\right.$$
$$+\tilde{A}\left(\frac{1}{L}\sum_{k=1}^{N}A_{ik}\varphi_k+\frac{1}{L^2}\sum_{k=1}^{N}B_{ik}W_k\right)=I_0\left(\ddot{W}_i-\frac{\eta}{L^2}\sum_{k=1}^{N}B_{ik}\ddot{W}_k\right)+ \quad (61)$$
$$c_1\hat{I}_4\left(\frac{1}{L}\sum_{k=1}^{N}A_{ik}\ddot{\varphi}_k-\frac{\eta}{L^3}\sum_{k=1}^{N}C_{ik}\ddot{\varphi}_k\right)-c_1^2\hat{I}_6\left(\frac{1}{L}\sum_{k=1}^{N}A_{ik}\ddot{\varphi}_k+\frac{1}{L^2}\sum_{k=1}^{N}B_{ik}\ddot{W}_k-\frac{\eta}{L^3}\sum_{k=1}^{N}C_{ik}\ddot{\varphi}_k-\frac{\eta}{L^4}\sum_{k=1}^{N}D_{ik}\ddot{W}_k\right)$$

where the over dots indicating the partial derivative with respect to the time. The related boundary conditions can be handled in the same method. For example, the boundary conditions of the most commonly used cantilever micro/nanobeam are approximated as

$$W_1=U_1=\varphi_1=0,\quad \sum_{k=1}^{N}A_{1k}W_k=0 \quad (62)$$

for the clamped end at X = 0, and

$$\widehat{K}\sum_{k=1}^{N}A_{Nk}U_k+I\sum_{k=1}^{N}A_{Nk}\varphi_k-c_1\hat{J}\left(\sum_{k=1}^{N}A_{Nk}\varphi_k+\frac{1}{L}\sum_{k=1}^{N}B_{Nk}W_k\right)=0$$
$$E_{11}c_1\sum_{k=1}^{N}A_{Nk}U_k+F_{11}c_1\sum_{k=1}^{N}A_{Nk}\varphi_k-H_{11}c_1^2\left(\sum_{k=1}^{N}A_{Nk}\varphi_k+\frac{1}{L}\sum_{k=1}^{N}B_{Nk}W_k\right)=0$$
$$A_{11}\sum_{k=1}^{N}A_{Nk}U_k+B_{11}\sum_{k=1}^{N}B_{Nk}\varphi_k-c_1E_{11}\left(\sum_{k=1}^{N}A_{Nk}\varphi_k+\frac{1}{L}\sum_{k=1}^{N}B_{Nk}W_k\right)=0 \quad (63)$$
$$\tilde{A}L^2\left(\varphi_k+\frac{1}{L}\sum_{k=1}^{N}A_{Nk}W_k\right)+c_1\left[E_{11}c_1\sum_{k=1}^{N}B_{Nk}U_k+F_{11}c_1\sum_{k=1}^{N}B_{Nk}\varphi_k\right.$$

$$-H_{11}c_1^2\left(\sum_{k=1}^{N}B_{Nk}\varphi_k+\frac{1}{L}\sum_{k=1}^{N}C_{Nk}W_k\right)\Bigg]=0 \quad (63)$$

for the free end at X = 1.

Denoting the unknown vector

$$\begin{aligned}&\boldsymbol{\delta}=\left[\{\mathbf{U}\}^{\mathrm{T}},\{\mathbf{W}\}^{\mathrm{T}},\{\boldsymbol{\varphi}\}^{\mathrm{T}}\right]^{\mathrm{T}},\\&\{\mathbf{U}\}=\left[U_1,U_2,\ldots,U_N\right]^{\mathrm{T}},\\&\{\mathbf{W}\}=\left[W_1,W_2,\ldots,W_N\right]^{\mathrm{T}},\\&\{\boldsymbol{\varphi}\}=\left[\varphi_1,\varphi_2,\ldots,\varphi_N\right]^{\mathrm{T}}\end{aligned} \quad (64)$$

Eqs. (59)-(61) can be written in matrix form as

$$\mathbf{K}\boldsymbol{\delta}=\mathbf{M}\ddot{\boldsymbol{\delta}} \quad (65)$$

where **M** and **K** are the mass and the stiffness matrix, respectively. Expanding the dynamic displacement vector $\delta=\bar{\delta}e^{i\Omega t}$ where Ω is the natural frequency, $\Omega=\omega L^2\sqrt{12\rho_c/E_ch^2}$ is the dimensionless frequency, and $\bar{\delta}$ is the vibration mode shape vector. Substituting this expansion into Eq. (65) yields the eigenvalues equations as below

$$\left(\mathbf{K}-\mathbf{M}\Omega^2\right)\bar{\boldsymbol{\delta}}=0 \quad (66)$$

The natural frequencies and associated mode shapes of FG micro/nanobeam can be determined by solving Eq. (66).

Table 1 Material properties of FGM constituents

Properties	Steel	Alumina (Al_2O_3)
E	210 (GPa)	390 (GPa)
ρ	7800 (kg/m^3)	3960 (kg/m^3)
v	0.30	0.3

7. Numerical results and discussion

The analytical and the numerical results obtained in the previous sections are presented for an FG micro/nanobeam under different boundary conditions, including SS, CS, CF and CC. The FG micro/nanobeam is composed of steel and alumina (Al_2O_3) with the material properties listed in Table 1 and its properties grade smoothly in the thickness direction. The top and the bottom surfaces of the beam are pure alumina and pure steel, respectively. The FG micro/nanobeam has following parameters, which are used in computing the numerical values (Eltaher *et al.* 2012) L (length) = 10000 (nm), b (width) = 1000 (nm), h (thickness) = 100 (nm).

In order to verify the analysis, results for simply supported nanobeams are compared with Eltaher *et al.* (2012) and Rahmani and Pedram (2014), see Table 2. The comparisons illustrate that the present results are in good agreement with those in the literatures. All frequencies are reduced by increasing the nonlocal parameter. It also

Table 2 The variation of non-dimensional frequency for different material distributions and nonlocal parameter for SS beam

Power index (g)	Nonlocal parameter (μ)	Eltaher *et al.* 2012	Rahmani and Pedram 2014	Present	
				Navier	HDQ
0	0	9.8797	9.8296	9.8271	9.8291
	1	9.4238	9.3777	9.3763	9.3772
	2	9.0257	8.9829	8.9811	8.9825
	3	8.6741	8.6341	8.6325	8.6337
	4	8.3607	8.3230	8.3213	8.3226
0.5	0	7.8061	7.7149	7.7142	7.7158
	1	7.4458	7.3602	7.3595	7.362
	2	7.1312	7.0504	7.0498	7.0521
	3	6.8533	6.7766	6.7764	6.7722
	4	6.6057	6.5325	6.5322	6.5411
1	0	7.0904	6.9676	6.9676	6.9627
	1	6.7631	6.6473	6.6468	6.6474
	2	6.4774	6.3674	6.3677	6.3629
	3	6.2251	6.1202	6.1199	6.1153
	4	6.0001	5.8997	5.8991	5.9846
5	0	6.0025	5.9172	5.9167	5.9145
	1	5.7256	5.6452	5.6442	5.6421
	2	5.4837	5.4075	5.4067	5.4011
	3	5.2702	5.1975	5.1967	5.1952
	4	5.0797	5.0103	5.0091	5.0115

Table 3 The variation of non-dimensional frequency for different material distributions and nonlocal parameter for CC beam

μ	Mode number	Power index (g)						
		0	0.1	0.2	0.5	1	5	10
0	1	22.0009	20.5157	19.4095	17.3377	15.6991	13.2913	12.6665
	2	59.3934	55.3974	52.4199	46.8354	42.4005	35.8166	34.1359
	3	113.4050	105.8034	100.1370	89.4939	81.0018	68.2479	65.0513
	4	181.7155	169.5835	160.5368	143.5176	129.8686	109.1162	104.0161
	5	262.2208	204.2284	194.0232	174.0444	156.6479	126.9721	120.8853
1	1	20.7636	19.3617	18.3175	16.3621	14.8159	12.5447	11.9550
	2	49.1713	45.8602	43.3936	38.7692	35.0999	29.6613	28.2690
	3	80.6520	75.2370	71.2022	63.6295	57.5973	48.5658	46.2897
	4	110.8427	103.4252	97.8973	87.5094	79.1983	66.6150	63.4984
	5	138.4790	129.2454	122.3624	109.4107	99.0005	83.0535	79.1747
2	1	19.7089	18.3780	17.3867	15.5306	14.0631	11.9082	11.3484
	2	42.8431	39.9570	37.8072	33.7776	30.5814	25.8477	24.6342
	3	66.0245	61.5893	58.2851	52.0851	47.1487	39.7645	37.9006
	4	86.9567	81.1348	76.7967	68.6464	62.1286	52.2681	49.8224
	5	105.7218	98.6693	93.4129	83.5240	75.5789	63.4160	60.4539
3	1	17.5272	16.5818	16.5818	14.8114	13.4120	11.3575	10.8236
	2	35.8559	33.9264	33.9264	30.3101	27.4424	23.1970	22.1079
	3	53.4530	50.5847	50.5847	45.2034	40.9198	34.5146	32.8967
	4	68.9922	65.3027	65.3027	58.3717	52.8302	44.4492	42.3692
	5	82.9557	78.5358	78.5358	70.2213	63.5423	53.3202	50.8296
4	1	17.9981	16.7824	15.8771	14.1819	12.8420	10.8754	10.3641
	2	35.1678	32.7979	31.0327	27.7247	25.1018	21.2198	20.2235
	3	51.3530	47.9021	45.3314	40.5087	36.6703	30.9321	29.4820
	4	65.4555	61.0718	57.8056	51.6701	46.7651	39.3481	37.5067
	5	78.2029	72.9847	69.0958	61.7805	55.9047	46.9131	44.7216

Table 4 The variation of non-dimensional frequency for different material distributions and nonlocal parameter for CS beam

μ	Mode number	Power index (g)						
		0	0.1	0.2	0.5	1	5	10
0	1	15.2483	14.2185	13.4525	12.0207	10.8910	9.2238	8.7860
	2	48.5579	45.2897	42.8525	38.2860	34.6675	29.3129	27.9332
	3	98.9298	92.2790	87.3279	78.0361	70.6397	59.5901	56.7936
	4	164.2756	153.2949	145.0971	129.6829	117.3498	98.7350	94.1205
	5	242.4770	204.4477	194.2158	174.1574	156.6867	127.0576	121.0019
1	1	14.4425	13.4669	12.7413	11.3852	10.3154	8.7368	8.3221
	2	40.6759	37.9364	35.8939	32.0683	29.0389	24.5611	23.4048
	3	71.2520	66.4550	62.8861	56.1928	50.8723	42.9415	40.9250
	4	101.4663	94.6735	89.6046	80.0854	72.4860	61.0413	58.1821
	5	129.5423	120.8870	114.4375	102.3089	92.5814	77.7621	74.1268
2	1	13.7500	12.8211	12.1303	10.8392	9.8208	8.3183	7.9234
	2	35.6996	33.2943	31.5013	28.1436	25.4854	21.5589	20.5439
	3	58.5678	54.6236	51.6892	46.1871	41.8154	35.3036	33.6456
	4	79.8192	74.4722	70.4837	62.9953	57.0200	48.0270	45.7767
	5	99.0253	92.4064	87.4750	78.2037	70.7714	59.4546	56.6742

Table 4 Continued

μ	Mode number	Power index (g)						
		0	0.1	0.2	0.5	1	5	10
3	1	13.1470	12.2587	11.5981	10.3637	9.3900	7.9538	7.5762
	2	32.1981	30.0281	28.4108	25.3823	22.9852	19.4457	18.5301
	3	50.9132	47.4842	44.9329	40.1497	36.3501	30.6925	29.2508
	4	67.9504	63.3969	60.0010	53.6262	48.5405	40.8885	38.9725
	5	83.2657	77.6994	73.5523	65.7565	59.5082	49.9966	47.6583
4	1	12.6158	11.7634	11.1294	9.9449	9.0107	7.6327	7.2703
	2	29.5635	27.5707	26.0856	23.3048	21.1041	17.8554	17.0146
	3	45.6544	42.5794	40.2914	36.0022	32.5955	27.5238	26.2309
	4	60.1875	56.1533	53.1453	47.4987	42.9945	36.2187	34.5215
	5	73.2444	68.3476	64.6994	57.8418	52.3461	43.9814	41.9243

Table 5 The variation of non-dimensional frequency for different material distributions and nonlocal parameter for CF beam

μ	Mode number	Power index (g)						
		0	0.1	0.2	0.5	1	5	10
0	1	3.5252	3.2865	3.1090	2.7773	2.5160	2.1342	2.0334
	2	21.5678	20.1069	19.0193	16.9845	15.3801	13.0420	12.4311
	3	59.7784	55.7420	52.7380	47.1153	42.6668	36.1080	34.4116
	4	113.7455	106.0837	100.3799	89.6928	78.3165	63.4868	60.4431
	5	183.2862	102.1139	97.0103	87.0165	81.2056	68.5765	65.3661
1	1	3.5393	3.2997	3.1215	2.7885	2.5261	2.1428	2.0416
	2	20.2575	18.8857	17.8644	15.9536	14.4468	12.2495	11.6754
	3	49.4342	46.0967	43.6124	38.9618	35.2817	29.8558	28.4535
	4	80.7445	75.3081	71.2598	63.6703	57.6409	48.6710	46.3924
	5	111.2410	100.8770	95.8353	85.9629	77.3701	62.7185	59.7112
2	1	3.5539	3.3133	3.1343	2.7999	2.5364	2.1516	2.0500
	2	19.1227	17.8279	16.8640	15.0606	13.6382	11.5632	11.0210
	3	43.1163	40.2056	38.0389	33.9823	30.7720	26.0382	24.8153
	4	65.9962	61.5536	58.2454	52.0432	47.1151	39.7793	37.9161
	5	87.2245	81.3742	77.0168	68.8368	62.3074	52.4692	50.0146
3	1	3.5252	3.2865	3.1090	2.7773	2.5160	2.1342	2.0334
	2	21.5678	20.1069	19.0193	16.9845	15.3801	13.0420	12.4311
	3	59.7784	55.7420	52.7380	47.1153	42.6668	36.1080	34.4116
	4	113.7455	106.0837	100.3799	89.6928	78.3165	63.4868	60.4431
	5	183.2862	102.1139	97.0103	87.0165	81.2056	68.5765	65.3661
4	1	3.5393	3.2997	3.1215	2.7885	2.5261	2.1428	2.0416
	2	20.2575	18.8857	17.8644	15.9536	14.4468	12.2495	11.6754
	3	49.4342	46.0967	43.6124	38.9618	35.2817	29.8558	28.4535
	4	80.7445	75.3081	71.2598	63.6703	57.6409	48.6710	46.3924
	5	111.2410	100.8770	95.8353	85.9629	77.3701	62.7185	59.7112

observed from Table 2 that the local theory (μ = 0) overestimates the natural frequency of the FG nanobeams compared to the nonlocal one, and the difference between local and nonlocal theories is significant for high value of the nonlocal parameter. This is due to the fact that the local theory is unable to capture the small scale effect of the micro/nanobeams.

The effects of boundary conditions, material graduation, and nonlocal parameter on the first five dimensionless frequencies are presented in Tables 3-5.

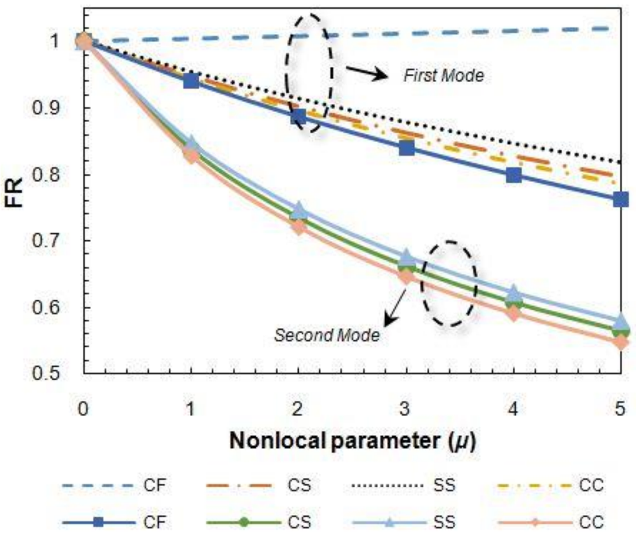

Fig. 2 The variation of the first and second frequencies with nonlocal parameter at power index (g = 0.5)

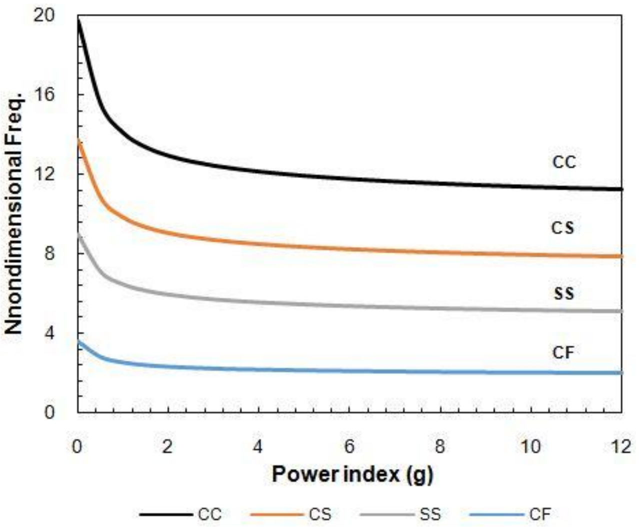

Fig. 3 The variation of the first and second frequencies with nonlocal parameter at power index (g = 0.5)

Table 3 illustrates the effect material graduation and nonlocal parameter on the vibration characteristics for Clamped FG micro/nanobeam. As shown in Table 3, by increasing the nonlocal parameter, natural frequency for all modes decreases and this reduction is more prominent at higher modes, and so the small-scale effect cannot be neglected. For instance, by increasing nonlocal parameter from 0 to 4 the first mode is reduced by 18.18%, but for the fifth mode for increasing the nonlocal parameterfrom0 to4natural frequency is reduced by 70.22%. The reduction may be described as follows. The nonlocal parameter effect makes the FG micro/nanobeam more flexible as the nonlocal model may be viewed as atoms connected by elastic springs whereas the local continuum model supposes the spring constant to take on an infinite value. In sum, the nonlocal beam theory should be employed if we need exact predictions of high frequencies of micro/nanobeams.

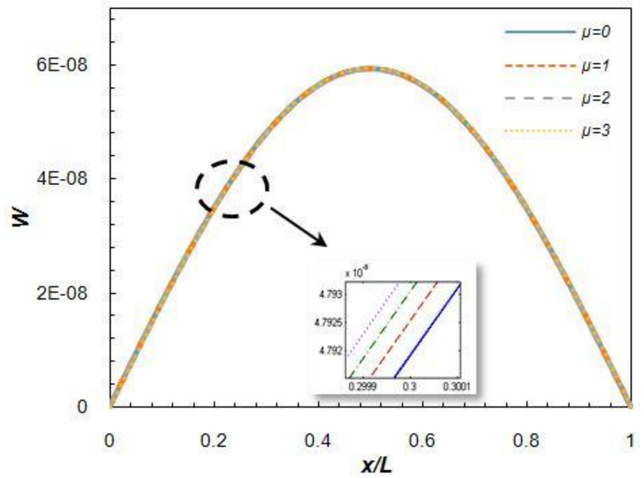

Fig. 4 The effect of the nonlocal parameter on the first mode shapes of the FG micro/nanobeams for SS ends

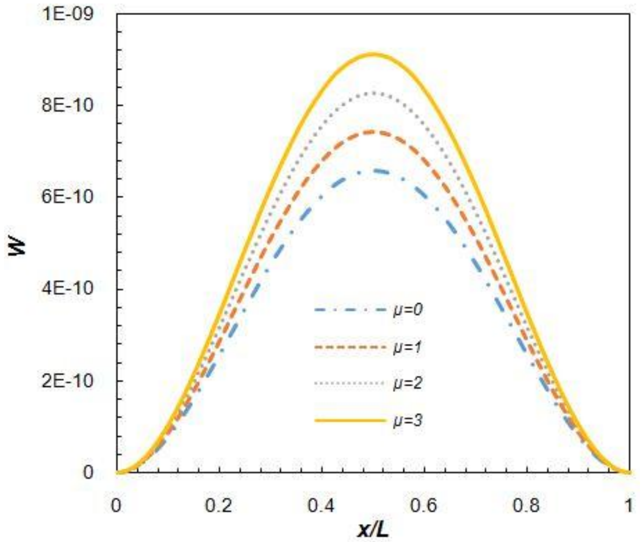

Fig. 5 The effect of the nonlocal parameter on the first mode shapes of the FG micro/nanobeams for CC ends

The effect of material graduation and nonlocal parameter for the frequencies in case clamped-simply supported beam is presented in Table 4. The behavior of this nanobeam is consistent with the simply (SS) and clamped (CC) micro/nanobeams. The frequencies of this case are higher than the simply supported micro/nanobeam and lower than the clamped micro/nanobeam.

Clamped-free beam shows unusual manner than the other type of boundary condition. While, the first frequency increased by increasing the nonlocal parameter and the nonlocal parameter has not a significant effect for the first frequency as presented in Table 5. Whereas, the other frequencies decrease as the nonlocality parameter increased for all power indexes.

In order to see better the effects of power index and nonlocal parameter variations on the natural frequencies of the FG micro/nanobeams with different boundary conditions, Figs. 2 and 3 also illustrate these variations. Fig. 2 shows the effect of nonlocal parameter on the first and second mode numbers of FG micro/nanobeams. For this purpose, natural frequency ratio is defined as follows

$$\text{Frequency Ratio} = \frac{\textit{Natural Frequency from nonlocal theory } (\omega_{nl})}{\textit{Natural Frequency from local theory } (\omega_{l})} \quad (67$$

It can be seen from Fig. 2 for the first mode Clamped-free beam shows a different behavior than the other type of boundary condition. Whereas, by increasing the nonlocal parameter the first frequency increased. Therefore, this observation must be considered in design of nanodevices. Similar to isotropic micro/nanobeams, the nonlocal parameters effects increase with increase in mode numbers. This is because of small wavelength effect for higher modes and thus the small-scale effect cannot be neglected. Also, it can be found that, as the rigidity of the beam increased the effect of nonlocality on the natural frequency will be increased.

Effect of power law index (g) on the natural frequency of the FG micro/nanobeam for various boundary conditions is illustrated in Fig. 3. As can be seen, by increasing the power law index (g) for all boundary conditions natural frequency decreases and this reduction is especially significant at lower power law index. This is due to the fact that higher values of power law index (g) correspond to high portion of material 1 in comparison with the material 2 part, thus makes such FG beams less flexible. The highest values for natural frequencies are obtained for full material 1 beams (g = 0), while the lowest natural frequencies are obtained for full material 2 beams ($g \to \infty$). This is because an increase in the value of the power law index results in a decrease in the value of elasticity modulus. In other words, the FG nanobeam becomes flexible as the power law index increases. As it is seen from Fig. 3, the behavior of the micro/nanobeam follows the same tendency in all boundary conditions, i.e., the natural frequencies of the micro/ nanobeam decrease by increasing of g and become stable for g values greater than 5. In fact, for $g >> 1$ the FG micro/nanobeam becomes a metal micro/nanobeam and its mechanical properties become almost steady.

The effect of the nonlocal parameter on the fundamental mode shapes (the dimensionless displacement W) of the CC and SS FG micro/nanobeams are plotted in Figs. 4 and 5. The nonlocal parameter nearly has no effect on the mode shapes W for the SS FG micro/nanobeam, but has a distinguished effect for the CC FG micro/nanobeam.

5. Conclusions

In the present work, vibration analysis of FG micro/ nanobeams based on Reddy beam theory is developed. Nonlocal constitutive equations of Eringen are being employed in the formulations to obtain the vibration behavior of FG micro/nanobeam. Equations of motion are obtained from the Hamilton's principal. Navier solution and DQ method are employed to find the natural frequencies and mode shapes of the FG micro/nanobeams with different boundary conditions. Effects of nonlocal parameter, the power law index, boundary condition, and mode number on the natural frequencies of FG micro/nanobeams are studied. From the present study, the following results are derived:

- It is seen that increasing the power law index g will decrease the stiffness of the FG micro/nanobeams, and consequently, leads to a decrease in the natural frequencies for all different nonlocal parameter μ and mode numbers. Therefore, by manipulating the material-distribution profile of FG micro/nanobeam, we can select a specific design frequency.
- Fundamental frequencies decreased as the nonlocal parameter increased for all boundary conditions except for the first mode in the case of CF FG micro/nanobeam where it increased.
- Similar to isotropic micro/nanobeams, the nonlocal parameters effects increase with increasing in mode numbers. This is because of small wavelength effect for higher modes and thus the small-scale effect cannot be neglected.

The analytical and numerical nonlocal Reddy beam solutions presented in this work can be helpful for engineers who are designing micro and nanoelectromechanical devices.

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