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Assessment of various nonlocal higher order theories for the bending and buckling behavior of functionally graded nanobeams

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Abstract. In this paper, various nonlocal higher-order shear deformation beam theories that consider the size dependent effects in Functionally Graded Material (FGM) beam are examined. The presented theories fulfill the zero traction boundary conditions on the top and bottom surface of the beam and a shear correction factor is not required. Hamilton's principle is used to derive equation of motion as well as related boundary condition. The Navier solution is applied to solve the simply supported boundary conditions and exact formulas are proposed for the bending and static buckling. A parametric study is also included to investigate the effect of gradient index, length scale parameter and length-to-thickness ratio (aspect ratio) on the bending and the static buckling characteristics of FG nanobeams.

Keywords: analytical solution; bending; buckling; functionally graded materials; higher order beam theory; nonlocal elasticity

1. Introduction

Nowadays,with the development in nano technology, beams, plates, rods, and shells in micro- or nano scale are frequently used as structure elements in micro- or nano electromechanical machines (Lavrik *et al*. 2004, Ekinci and Roukes 2005). In order to achieve the new advanced materials, the atomic or molecular characteristic length of material has a significant role. So, size effects are important in the mechanical performance of these structures in which sizes are small and analogous to molecular distances. Consequently, size dependent models of continuum mechanics have obtained growing consideration in recent years because of the need to model and study very small sized mechanical structures and mechanisms in the fast progresses of micro- or nanotechnologies (Wang *et al*. 2008, Sudak 2003, Wang and Hu 2005, Reddy 2007, Janghorban 2012, Saggam 2012). (McFarland and Colton 2005) noticed a significant difference between the stiffness values anticipated by the classical beam theory and the stiffness values achieved during a bending test of polypropylene micro-cantilever. These experiments expose that the size-dependent behavior is an inherent property of materials which appears for a beam when the characteristic size such as thickness or diameter is close to the internal material length scale parameter (Kong *et al*. 2008). The classical continuum mechanics models are not able to predict the precise responses of the size-dependent analyses, which happen in micron- and nano-scale structures. So, some higher order continuum models such as the nonlocal elasticity theory, the higher- order gradient theory, and the couple stress theory have been suggested to reasonably interpret the size-dependencies of nanostructures.

To clarify the effect of nonlocality on size-dependent analysis of micro/nano structures, some articles have been extended, for example an elastic buckling analysis of nanotubes on the basis of nonlocal Timoshenko beam theory by Wang *et al*. (2006). Also, a nonlocal Timoshenko beam model has been applied to derive the free vibration responses of micro/nanobeams by Wang *et al*. (2007). In addition, Wang *et al*. (2008) has investigated the bending solutions of nanobeams using the Timoshenko beam model and the nonlocal beam theory of Eringen for different boundary conditions. Based on nonlocal Timoshenko beam model, forced vibration of a SWCNT has been studied by Mesut (2011). Besides, the effect of the size-dependent, shear deformation, and shear stiffness of the free vibration analysis of microbeams with the simply-supported boundary conditions have been proposed by Duan *et al*. (2013). An analysis of microstructure beam buckling using the Timoshenko beam model and nonlocal elasticity theory has been studied by Zhang *et al*. (2013). Nguyen *et al*. (2014) has presented analytical solutions for bending of nanobeam based on the nonlocal Euler-Bernoulli beam model. Recently, Rahmani *et al*. has investigated on static and dynamic behavior of nanostructures based on nonlocal elasticity theory (Rahmani *et al*. 2015, 2016a, b, Hosseini *et al*. 2016).

In recent years, improvements in the field of materials engineering lead to a new type of materials with smooth and continuous variation of the material properties called Functionally Graded Materials (FGMs). The mechanical

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and thermal response of materials with spatial gradients in composition and microstructure is of notable interest in various technological areas such as optoelectronics, tribology, biomechanics, nanotechnology and high temperature technology. FGMs are made of two materials continuously which are often metal and ceramics. The properties of material in FGMs are assumed to vary smoothly from one surface of nanobeam to another. Regarding these benefits, numerous investigations, dealing with static, buckling and dynamic features of FG structures, were available in the scientific literature (Chakraborty *et al*. 2003, Aydogdu and Taskin 2007, Benatta *et al*. 2008, Li 2008, Ke *et al*. 2009, Sallai *et al*. 2009, Şimşek and Kocatürk 2009, Sina *et al*. 2009, Khalili *et al*. 2010, Şimşek 2010, Fallah and Aghdam 2011, Kocatürk *et al*. 2011, Şimşek and Cansız 2012, Hosseini and Rahmani 2016a, b, c, Jandaghian and Rahmani 2016).

With the fast growth of technology, FGbeams and plates have been used in Micro/Nano ElectroMechanicalSystems (MEMS/NEMS), such as the constituents in the form of shape memory alloy thin films with a global thickness in micro- or nano- scale (Fu *et al*. 2003, Witvrouw and Mehta 2005, Lü *et al*. 2009), electrically actuated MEMS mechanisms (Hasanyan *et al*. 2008, Mohammadi-Alasti *et al*. 2011, Zhang and Fu 2012), and Atomic Force Microscopes (AFMs) (Rahaeifard *et al*. 2009). As the dimension of these structural devices usually falls below micron- or nanoscale in at least one direction, a necessary feature triggered in these tools is that, their mechanical properties such as Young's modulus and flexural rigidity are size-dependent. Up to now, only a few works have been proposed for FG nanobeams based on the nonlocal elasticity theory. (Janghorban and Zare 2011) studied nonlocal free vibration axially FG nanobeams using differential quadrature method. Free vibration analysis of FG nanobeam according to the nonlocal Euler–Bernoulli beam theory has been investigated by Eltaher *et al*. (2012). Mohanty *et al*. (2012) have studied the evaluation of static and dynamic behavior of FG beams based on the Timoshenko beam theory. Free vibration of axially FG tapered nanorods has been studied based on the nonlocal elasticity theory by Şimşek (2012). Jha *et al*. (2013) has proposed a free vibration analysis of FG plates based on a higher order shear theory. In another paper, based on the nonlocal Euler–Bernoulli and Timoshenko beam theory, (Şimşek and Yurtcu 2013) have studied analytical solutions for static bending and buckling of the FG nanobeam. Eltaher *et al*. (2014) has examined the static bending and the buckling of FG nanobeam by applying the nonlocal Timoshenko model. Refaeinejad *et al*. (2016b) has developed the vibrational analysis of nanostructures based on FG nonlocal higher order theory. Also, Refaeinejad *et al*. (2016a) has presented static and dynamic behavior of FG nanobeam lying on elastic foundation based on various nonlocal higher order theories.

Lately, many theories have been developed to predict the mechanical behavior of beams. Among them Euler–Bernoulli beam model is suitableto be usedfor predicting the mechanical behavior of slender beams. For moderately deep beams, it underestimates deflection and overestimates buckling load because of ignoring the shear deformation influence. The Timoshenko beam model accounts for the shear deformation influence by supposing a constant shear strain through the thickness of the beam. In this model, a shear correction factor is necessary to compensate for the existing difference between the actual stress state and the constant stress state. To avoid the use of shear correction factor and obtain a better prediction of the deep beam responses, many higher-order shear deformation theories have been developed, among them are the third-order theory of Reddy (1984), the sinusoidal theory of Touratier (1991), the hyperbolic theory of Soldatos (1992), the exponential theory of Karama *et al*. (2003), and general exponential shear deformation beam theory (ABT) of (Aydogdu). There are a limited number of studies using HSDTs to obtain the responses of the static and dynamic behavior of nanostructures, for example, Akgöz and Civalek (2014a) have studied the buckling of homogeneous microbeams based on the trigonometric modified strain gradient theory. In another study (Akgöz and Civalek 2013) developed a size-dependent sinusoidal shear deformation beam model in conjunction with modified strain gradient elasticity theory (MSGT). Recently, they have used trigonometric beam model on the basis of modified strain gradient elasticity theory to study the nonhomogeneous microbeams (Akgöz and Civalek 2014b). A nonlocal beam theory for bending, buckling and free vibration of nanobeams has been used by Thai (2012), Also in another study, Thai and Vo (2012a) have investigated a nonlocal sinusoidal shear deformation beam theory with application to bending, buckling, and vibration of nanobeams. Şimşek and Yurtcu (2013) have investigated the bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory. A size-dependent higher-order beam theory was presented for the static bending and free vibration of the FG microbeam based on the modified couple stress theory by Şimşek and Reddy (2013). Recently, Ahouel *et al*. (2016) has developed a nonlocal trigonometric shear deformation beam model to predict a size-dependent mechanical behavior of FG nanobeam. Hence, based on the aforementioned discussions, there is a strong motivation to understand the mechanical behavior of FG nanobeams in the design of nano structures like nanobeams and nanoplates as the structural elements of MEMS/NEMS.

The present work is a comprehensive study on the bending and static buckling of FG nanobeams based on different nonlocal higher order shear deformation beam theories. This study considers the effect of mico/nanoscales, shear deformation, and material distribution parameter. The Hamilton's principle is used to derive equation of motion and the Navier's solution is applied to simply supported boundary conditions. Analytical solutions for the bending and buckling of FG nanobeams are presented and the effects of various parameters on the mechanical behavior of FG nanostructure are investigated, and also the obtained results are compared with those predicted by previous studies to verify the accuracy of the present solution.

Summarily, the present study shows that the bending deflection, and static buckling behavior of FG nanobeam are extremely dependent on the gradient index, aspect ratio, and nonlocal parameter. Also, it can be considered that using HSDTs theories can improve the responses of the static and stability behavior of FG nanobeam especially thick nanobeam. The obtained results of the present study might be useful to analyze the bending, and buckling behavior of FG nano-devices as a smart controller in reality.

2. Nonlocal theory

Based on the Eringen nonlocal elasticity theory (Eringen 1983, Eringen and Edelen 1972), stress at point x not only depends on the strain on that point, but also depends on all points of the continuum body. The 3-D structural nonlocal constitutive relation which has the integral form is written as follows

$$\begin{aligned} &\sigma_{ij,j} = 0 \\ &\sigma_{ij}(\mathrm{x}) = \int \alpha(|\mathrm{x}-\mathrm{x}'|,\tau)\mathrm{C}_{ijkl}\,\varepsilon_{kl}(\mathrm{x}')dV(\mathrm{x}') \qquad \forall x \in V \\ &\varepsilon_{ij} = \frac{1}{2}(\mathrm{u}_{i,j} + \mathrm{u}_{j,i}) \end{aligned} \tag{1}$$

C_{ijkl} denotes the elastic tensor for classical isotropic elasticity σ_{ij}, ε_{ij} and u_i are stress tensor, strain tensor and displacement vector, respectively. The kernel α ($|x - x'|$, τ) represents the nonlocal modulus or attenuation function and includes the nonlocal effect into the constitutive equation at reference point x which is produced by the local strain at the point x'. ($|x - x'|$ is the Euclidean metric. τ is equal to $e_0\alpha/l$ where α and l are internal and external characteristic length, respectively and e_0 denotes a constant which varies in accordance with each material and can be obtained using atomistic simulation or the dispersive curve of the Born-Karman model of lattice dynamics. Based on the Eringen nonlocal theory, when $\tau \rightarrow 0$, α is reverted to Dirac delta measure. Thus the internal characteristic length is omitted and consequently classical elasticity constitutive equation is governed. Besides, in small internal characteristic length when $\tau \rightarrow 1$, the atomic lattice dynamics should be determined by nonlocal elasticity theory. Therefore, by matching the dispersive curves of planes wave with molecular dynamics simulation or atomic lattice dynamics α is obtained. More convenient relations to find analytical solutionsfor static and dynamic problems in nanostructures mechanics are presented by (Eringen 1983) which is called the differential constitutive relation.

$$(1-\tau^2 l^2 \nabla^2)\sigma = T\,. \tag{2}$$

Where T is the classical stress tensor at point x ($T_{ij} = C_{ijkl}\varepsilon_{kl}$), and ∇^2 is the Laplacian operator. The following equations represent the nonlocal constitutive equations in differential form, for one dimensional elastic body

$$\begin{aligned} \sigma_{xx} - \mu\frac{\partial^2\sigma_{xx}}{\partial x^2} &= E\,\varepsilon_{xx} \\ \sigma_{xz} - \mu\frac{\partial^2\sigma_{xz}}{\partial x^2} &= G\,\gamma_{xz} \end{aligned} \tag{3}$$

E and $G = E$ / [2 (1 + v)] are elastic and shear modulus of the beam, respectively (where v is the Poisson's ratio). The nonlocal parameter is denoted by $\mu = (e_0\alpha)^2(\mathrm{nm}^2)$ and σ_{xx} and σ_{xz} represent the normal and shear nonlocal stresses, respectively. Combining nonlocal elasticity theory and higher order beam theories results in obtaining the constitutive relation for nonlocal higher order nanobeam. For a specific material, the corresponding nonlocal parameter can be estimated by fitting the results of atomic lattice dynamic and experiment. When the nonlocal parameter is taken as ($\mu = 0$), the constitutive relation of the local theory is obtained.

3. The FG material properties

The FG nanobeam is considered to have length L, thickness h and width b which consists of two different materials at the top and bottom surfaces. The FG nanobeam is subjected to the transverse distributed load $q(x)$ and axial compressive force $\overline{N}$, as shown in Fig. 1.

Poisson's ratio, bulk elastic modulus, and mass density change in the thickness direction (in the z direction) are based on the power-law distribution. According to the rule of mixture, the effective material properties Π can be expressed as

$$\Pi = \Pi_m\, v_m + \Pi_c\, v_c \tag{4}$$

Where Π_m, Π_c are the effective material properties, v_m, v_c are the volume fractions of the first (metal) and second (ceramics) material related by

$$v_m + v_c = 1 \tag{5}$$

The subscript m and c represent the metal and ceramics constituent. In this paper, the effective material properties of the FG nanobeam are defined by the power-law form. The volume fraction of the second material is supposed by

$$v_m = \left(\frac{z}{h} + \frac{1}{2}\right)^p \tag{6}$$

The variation profile of material properties through the FG nanobeam thickness is denoted by gradient index p. Gradient index p is dimensionless and has a range between $p = 0$ to $p = \infty$. When $p = 0$, the FG nanobeam becomes a homogeneous nanobeam made of pure ceramic and when $p = \infty$, it becomes homogeneous nanobeam made of pure metal. Using Eqs. (4)-(6), the effective material properties of the FG nanobeam can be given as

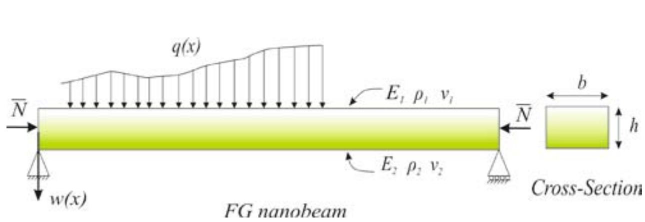


Fig. 1 Geometry and boundary condition of a functionally graded nanobeam

$$\Pi(z) = (\Pi_m - \Pi_c)(\frac{z}{h} + \frac{1}{2})^p + \Pi_c \tag{7}$$

Eq. (7) can be expressed in the terms of Young's modulus, density, shear modulus and Poisson's ratio.

4. Governing equations

Imagine a nanobeam consisting of FG material with length L, width b and height h. The rectangular Cartesian coordinate system x, y, z is taken along the length, width, and height of the nanobeam, respectively, as shown in Fig. 1. The displacement fields according to higher-order shear deformation beam theories state as

$$\begin{aligned} u_x(x,z,t) &= u(x,t) - z\frac{dw_b}{dx} - f(z)\frac{dw_s}{dx} \\ u_y(x,z,t) &= 0 \\ u_z(x,z,t) &= w_b(x,t) + w_s(x,t) \end{aligned} \tag{8}$$

Where u, w_b, and w_s are the axial displacement, bending, and shear components of transverse displacement on the neutral axis of the nanobeam; and $f(z)$ determines the transverse shear strain and shear stress distributions through the thickness of the nanobeam called a shape function. Because the shape function $f(z)$ fulfils the traction free conditions on the both surfaces of the nanobeam, therefore there is no need to shear correction factor. Different shape functions $f(z)$ which will be employed in the present study are given in Table 1.

The nonzero strain fields are obtained as

$$\begin{aligned} \varepsilon_x &= \frac{du}{dx} - z\frac{d^2w_b}{dx^2} - f(z)\frac{d^2w_s}{dx^2} \\ \gamma_{xz} &= (1 - \frac{df}{dz})\frac{dw_s}{dx} \equiv g(z)\frac{dw_s}{dx} \end{aligned} \tag{9}$$

Where $g(z) = 1 - df/dz$ is the shape function of transverse shear strains. The distribution of the transverse shear strain and hence the transverse shear stress, through the depth of the nanobeam are represented by these shape functions.

Table 1 Different shape functions are used in the present study

Model	$f(z)$	Abbreviation
Reddy (Reddy 1984)	$\frac{4z^3}{3h^2}$	TBT
Touratier (Touratier 1991)	$z - \frac{h}{\pi}\sin(\frac{\pi z}{h})$	SBT
Soldatos (Soldatos 1992)	$z - h\sinh(\frac{z}{\pi}) + z\cosh(\frac{1}{2})$	HBT
Karama (Karama *et al*. 2003)	$z - ze^{-2(z/h)^2}$	EBT
Aydogdu (Aydogdu 2009)	$z - z \times 3^{\frac{-2}{\ln(3)} \times (\frac{z}{h})^2}$	ABT
Euler-Bernoulli	z	CBT

5. Equations of motion

Hamilton's principle states as

$$0 = \int_{t_1}^{t_2} \delta(U + V - K)dt \tag{10}$$

Where the virtual strain energy δU, the virtual kinetic energy δK, and the virtual potential of external loading δV are

$$\begin{aligned} \delta U &= \int_v (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz})dv \\ &= \int_0^L (N\frac{d\,\delta u}{dx} - M_b\frac{d^2\delta w_b}{dx^2} - M_s\frac{d^2\delta w_s}{dx^2} + Q\frac{d\,\delta w_s}{dx})dx \end{aligned} \tag{11}$$

Where N, M_b, M_s and Q are the stress resultants define as

$$\begin{aligned} [N, M_b, M_S] &= \int_A (1, z, f)\sigma_{xx}dA \\ Q &= \int_A g\,\sigma_{xz}dA \end{aligned} \tag{12}$$

The variation of the potential energy of the applied can be written as follows

$$\begin{aligned} \delta v &= -\int_0^L q\,\delta(w_b + w_s)dx \\ &- \int_0^L N_0\frac{d(w_b + w_s)}{dx}\frac{d\,\delta(w_b + w_s)}{dx}dx \end{aligned} \tag{13}$$

Where q and N_0 are the transverse and axial loads, respectively. δK are set to zero in order to derive bending and buckling responses. By putting expression for δU and δV from Eqs. (11) and (13) in Eq. (10) and integrating by parts, and gathering the coefficients of δu, δw_b and δw_s the following equations of motion of the functionally graded nanobeam are obtained as follows

$$\begin{aligned} \delta u &: \frac{dN}{dx} = 0 \\ \delta w_b &: \frac{d^2M_b}{dx^2} - N_0(\frac{d^2w_b}{dx^2} + \frac{d^2w_s}{dx^2}) + q = 0 \\ \delta w_s &: \frac{d^2M_s}{dx^2} - N_0(\frac{d^2w_b}{dx^2} + \frac{d^2w_s}{dx^2}) + q = 0 \end{aligned} \tag{14}$$

The boundary conditions are given as below

$$\begin{aligned} &Specify\ u\ or\ N \\ &w_b\ or\ Q_b \equiv \frac{dM_b}{dx} - N_0\frac{d}{dx}(w_b + w_s) \\ &w_s\ or\ Q_s \equiv \frac{dM_s}{dx} - N_0\frac{d}{dx}(w_b + w_s) + Q \\ &\frac{dw_b}{dx}\ or\ M_b \\ &\frac{dw_s}{dx}\ or\ M_s \end{aligned} \tag{15}$$

By substituting Eq. (9) in Eq. (3) and the subsequent results into Eq. (12) the stress resultants are achieved as

$$\begin{aligned}
&N-\mu\frac{d^2N}{dx^2}=A\frac{du}{dx}-B\frac{d^2w_b}{dx^2}-B_s\frac{d^2w_s}{dx^2}\\
&M_b-\mu\frac{d^2M_b}{dx^2}=B\frac{du}{dx}-D\frac{d^2w_b}{dx^2}-D_s\frac{d^2w_s}{dx^2}\\
&M_s-\mu\frac{d^2M_s}{dx^2}=B_s\frac{du}{dx}-D_s\frac{d^2w_b}{dx^2}-H_s\frac{d^2w_s}{dx^2}\\
&Q-\mu\frac{d^2Q}{dx^2}=A_s\frac{dw_s}{dx}
\end{aligned}\tag{16}$$

where

$$\begin{aligned}
&[\mathrm{A},\mathrm{B},\mathrm{B}_s,\mathrm{D},\mathrm{D}_s,\mathrm{H}_s]=\int_A(1,\mathrm{z},\mathrm{f},\mathrm{z}^2,\mathrm{zf},\mathrm{f}^2)E(\mathrm{z})\,\mathrm{d}\mathrm{A}\\
&A_s=\int_A g^2G(\mathrm{z})dA
\end{aligned}\tag{17}$$

By putting Eq. (16) in Eq. (14) the equations of motion can be written as follows

$$\begin{aligned}
&A\frac{d^2u}{dx^2}-B\frac{d^3w_b}{dx^3}-B_s\frac{d^3w_s}{dx^3}=0\\
&B\frac{d^3u}{dx^3}-D\frac{d^4w_b}{dx^4}-D_s\frac{d^4w_s}{dx^4}-N_0\left(\frac{d^2w_b}{dx^2}+\frac{d^2w_s}{dx^2}\right)\\
&+\mathrm{N}_0\,\mu\left(\frac{d^4w_b}{dx^4}+\frac{d^4w_s}{dx^4}\right)+\mathrm{q}-\mu\frac{d^2q}{dx^2}=0\\
&B_s\frac{d^3u}{dx^3}-D_s\frac{d^4w_b}{dx^4}-H_s\frac{d^4w_s}{dx^4}+A_s\frac{d^2w_s}{dx^2}\\
&-N_0\left(\frac{d^2w_b}{dx^2}+\frac{d^2w_s}{dx^2}\right)+\mathrm{N}_0\,\mu\left(\frac{d^4w_b}{dx^4}+\frac{d^4w_s}{dx^4}\right)\\
&+\mathrm{q}-\mu\frac{d^2q}{dx^2}=0
\end{aligned}\tag{18}$$

6. Analytical solutions

In this section, analytical solutions of FG nanobeam have been proposed to solve governing equations for the bending and the static buckling of a simply-supported FG nanobeam. The Navier's solution technic is used to figure out the closed-form solutions for the simply-supported boundary conditions.

The solution is supposed to be of the form

$$\begin{aligned}
&u(\mathrm{x},\mathrm{t})=\sum_{n=1}^{\infty}U_n\cos\alpha x\\
&w_b(\mathrm{x},\mathrm{t})=\sum_{n=1}^{\infty}W_{bn}\sin\alpha x\\
&w_s(\mathrm{x},\mathrm{t})=\sum_{n=1}^{\infty}W_{sn}\sin\alpha x
\end{aligned}\tag{19}$$

Where (U_n, W_{bn}, W_{sn}) are the unknown maximum displacement coefficients, $\alpha = n\pi/L$ and $i=\sqrt{-1}$. The applied transverse q is expanded in the Fourier series as follows

$$\begin{aligned}
&q(\mathrm{x})=\sum_{n=1}^{\infty}Q_n\sin\alpha x\\
&Q_n=\frac{2}{L}\int_0^L q(\mathrm{x})\sin\alpha\,\mathrm{xdx}
\end{aligned}\tag{20}$$

Where Q_n are the Fourier coefficients, and are given below for some typical loads:

Uniform loading

$$q(\mathrm{x})=\mathrm{q}_0 \qquad Q_n=\frac{4q_0}{n\pi} \qquad n=1,3,5,... \tag{21}$$

Point load

$$q(\mathrm{x})=\mathrm{p}_0\,\delta(\mathrm{x}-\mathrm{x}_k) \quad Q_n=\frac{2p_0}{L}\sin\alpha x_k \quad n=1,2,3 \tag{22}$$

Where δ is the Dirac delta function, p_0 is the magnitude of the point load, x_k is the application position of the point load.

Substituting the expansions of u, w_b, w_s, and q from Eqs. (20)-(21) in Eq. (19), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} s_{11} & s_{12} & s_{13}\\ s_{12} & s_{22} & s_{23}\\ s_{13} & s_{23} & s_{33}\end{bmatrix}+\right)\begin{Bmatrix}U_n\\ W_{bn}\\ W_{sn}\end{Bmatrix}=(1+\mu\alpha^2)\begin{Bmatrix}0\\ Q_n\\ Q_n\end{Bmatrix}\tag{23}$$

where

$$\begin{aligned}
&s_{11}=A\alpha^2,\\
&s_{13}=-B_s\alpha^3,\\
&s_{23}=D_s\alpha^4-N_0\alpha^2(1+\mu\alpha^2),\\
&s_{12}=-B\alpha^3;\\
&s_{22}=D\alpha^4-N_0\alpha^2(1+\mu\alpha^2);\\
&s_{33}=H_s\alpha^4+A_s\alpha^2-N_0\alpha^2(1+\mu\alpha^2);
\end{aligned}\tag{24}$$

7. Results and discussion

In this section, new numerical examples are presented and discussed to verify the accuracy of the nonlocal beam model in predicting the bending and buckling responses of FG nanobeam. For numerical results, an Al/Al_2O_3 nanobeam composed of aluminum (as metal; E_m = 70 GPa, ρ_m = 2700 kG.m^3) and alumina (as ceramic; E_c 380 GPa, ρ_c = 3800 Kg/m^3) is considered. For convenience, the following dimensionless quantities are defined as follows

$$\bar{\mathrm{w}}=100\frac{\mathrm{E_m h^3}}{\mathrm{q_0 L^4}}\mathrm{w}\left(\frac{\mathrm{L}}{2}\right)\tag{25}$$

$$\bar{N}=\frac{N_{cr}12L^2}{E_m bh^3}\tag{26}$$

In order to illustrate the efficiency and accuracy of the presented model the obtained results are compared with the existing databased on local TBT, SBT, EBT, ABT ($\mu = (e_0 a)^2 = 0$) (Thai and Vo 2012b) and TBT homogenous beam theory ($P = 0$) (Thai 2012) subjected to uniformly distributed load. In Table 2, results are given to various values of the aspect ratios (L/h = 5,20) and the gradient indexes (P = 0, 0.5, 1, 2, 5, 10). Also, In Table 3, results are given to various values of the aspect ratios (L/h = 5, 10, 50) and the nonlocal parameters (μ = 0, 1, 2, 3, 4 (nm^2)). Tables 2, and 3 clearly shows that the presented dimensionless transverse deflections ($\overline{w}$) of FG nanobeam for uniform load agree very well with the solution of (Thai 2012, Thai and Vo 2012b).

Figs. 2-3 demonstrate the effect of gradient index p and aspect ratio L/h on the dimensionless transverse deflection and dimensionless buckling load, respectively, for different nonlocal parameter as the gradient index varies from $p = 0$ to $p = 10$, the results are compared for the different aspect ratio (L/h = 5, 10, and 20). Since there is no difference among the results of shear deformation beam theories, TBT is used only in Figs. 2-5. According to Fig. 2, it can be observed that as the length-to-thickness ration L/h increases (by decreasing the thickness of nanobeam), the dimensionless transverse deflection increases. Also, as the gradient index increases, the dimensionless transverse deflection decreases. It is notable that the effect of the shear deformation decreases as the gradient index decreases (i.e., $p < 2$) especially in the lower value of the nonlocal parameter. In addition, it is seen that the influence of the shear deformation is more significant for the thick nanobeams (i.e., $L/h = 5$). On the other hand, all beam theories used in this study produce similar results with some small differences for ($L/h < 10$). It means that the shear deformation loses its effect on the transverse deflections for the slender nanobeams (i.e., $L/h > 50$). It should be noted that the transverse deflections of all higher order beam theories (HSDTs) are almost the same, and also the results of ABT and EBT are identical.

In Table 3 non-dimensional transverse deflection with respect to aspect ratios, the nonlocal parameter, and the gradient index, are investigated. It can be seen that for all higher order theories with the increase in the value of L/h, non-dimensional deflection decreases. It is seen from this table, as nonlocal parameter increases, the dimensionless deflections increases as well, however by increasing length-to-thickness ratio the divergence between results by classical theory ($\mu = 0$) and higher-order models ($\mu \neq 0$) decreases. It means that the nonlocal parameter loses its effect on the transverse deflections for the slender nanobeams (i.e., $L/h > 100$). Furthermore, It is clear from Table 3 that the transverse deflections predicted by the local beam models ($\mu = 0$) are always larger than those predicted by the nonlocal beam models ($\mu \neq 0$), i.e., that inclusion of nonlocal model stiffens the nanobeam. Also, the difference between the deflections of the classical and non-classical models is significant for thick nanobeams (i.e., $L/h < 20$).

Table 2 Comparison of the dimensionless transverse deflections of FG nanobeam for the uniform load ($\mu = 0$)

L/h	p	Method							
		TBT		SBT		EBT		ABT	
		Present	Thai	Present	Thai	Present	Thai	Present	Thai
5	0	3.16537	3.1654	3.16489	3.1649	3.16532	3.1635	3.16532	-
	0.5	4.82854	4.8285	4.82785	4.8278	4.82600	4.8260	4.82600	-
	1	6.25944	6.2594	6.25863	6.2586	6.25632	6.2563	6.25632	-
	2	8.0677	8.0677	8.06831	8.0683	8.06665	8.0667	8.06664	-
	5	9.82805	9.8281	9.83675	9.8367	9.84143	9.8414	9.84143	-
	10	10.9381	10.9381	10.942	10.9420	109404	10.9404	10.9404	-
20	0	2.89625	2.8962	2.89622	2.8962	2.89614	2.8961	2.89614	-
	0.5	4.46444	4.4644	4.46441	4.6444	4.4643	4.4643	4.46430	-
	1	5.80492	5.8049	5.80488	5.8049	5.80474	5.8047	5.80474	-
	2	7.44206	7.4421	7.44211	7.4421	7.44201	7.4420	7.44201	-
	5	8.8182	8.8182	8.81877	8.8188	8.81908	8.8191	8.81908	-
	10	9.69051	9.6905	9.69078	9.6908	9.69071	9.6907	9.69071	-

Table 3 Comparison of dimensionless transverse deflection of homogenous nanobeam (p = 0)

Method	$L/h = 5$					$L/h = 10$					$L/h = 50$				
	μ (nm^2)														
	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
Thai	1.4320	1.5673	1.7027	1.8381	1.9735	1.3346	1.4622	1.5898	1.7174	1.8450	1.3102	1.4359	1.5615	1.6872	1.8128
Present	1.4319	1.5673	1.7027	1.8381	1.9735	1.3345	1.4621	1.5897	1.7173	1.8449	1.3034	1.4285	1.5540	1.6787	1.8038

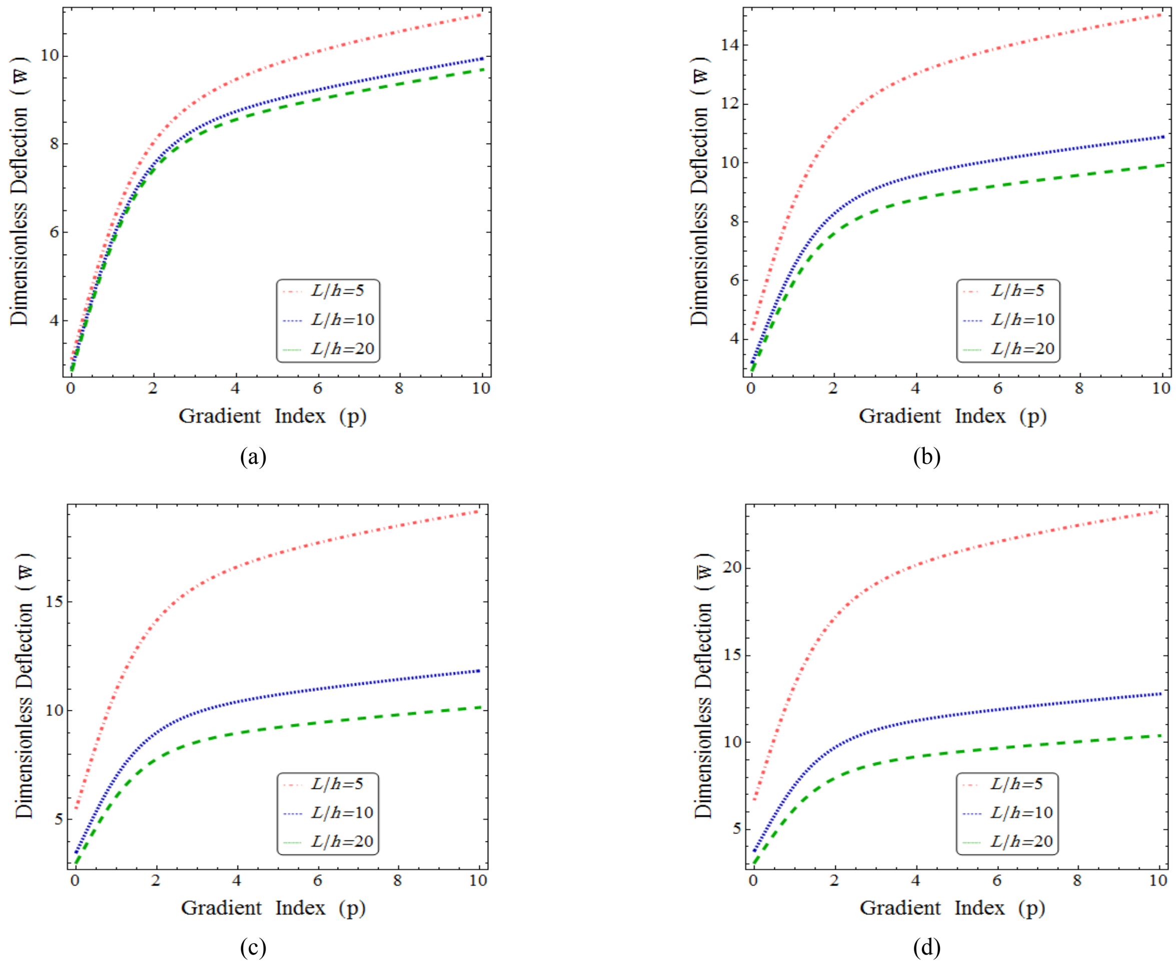


Fig. 2 Effect of gradient index p on dimensionless deflection for uniform load and different aspect ratios: (a) $\mu = 0$; (b) $\mu = 1$; (c) $\mu = 2$; (d) $\mu = 3$ (nm^2)

Table 4 The variations of non-dimensional deflection $\overline{w}$ of FG nanobeam for different aspect ratios, nonlocal parameters, and gradient indexes

L/h	μ (nm^2)	p = 0					1					2				
		CBT	TBT	SBT	ABT	EBT	CBT	TBT	SBT	ABT	EBT	CBT	TBT	SBT	ABT	EBT
5	0	3.11776	3.16537	3.16489	3.16532	3.16532	6.17903	6.25944	6.25863	6.25632	6.25632	7.92529	8.06769	8.06831	8.06665	8.06665
	1	4.29917	4.36255	4.36194	4.36016	4.36016	8.52507	8.63212	8.63109	8.62809	8.62809	10.9339	11.1231	11.124	11.1219	11.1219
	2	5.48058	5.55973	5.55899	5.55681	5.55681	10.8711	11.0048	11.0036	10.9998	10.9998	13.9426	14.1787	14.1797	14.1771	14.1771
	3	6.66198	6.7569	6.75605	6.75345	6.75345	13.21715	13.3775	13.376	13.3716	13.3716	16.9512	17.234	17.2354	17.2324	17.2324
	4	7.84339	7.95408	7.95309	7.95009	7.95009	15.56319	15.7501	15.7485	15.7434	15.7434	19.9599	20.2894	20.2911	20.2876	20.2876
50	0	2.88068	2.88116	2.88116	2.88115	2.88115	5.77863	5.77944	5.77944	5.77941	5.77941	7.40555	7.40698	7.40699	7.40697	7.40697
	1	2.89174	2.89222	2.89222	2.89221	2.89221	5.80082	5.80163	5.80163	5.8016	5.8016	7.43398	7.43542	7.43543	7.43541	7.43541
	2	2.9028	2.90329	2.90328	2.90327	2.90327	5.82301	5.82382	5.82382	5.82379	5.82379	7.46242	7.46386	7.46387	7.46385	7.46385
	3	2.91386	2.91435	2.91434	2.91433	2.91433	5.84519	5.84601	5.84601	5.84598	5.84598	7.49085	7.49229	7.4923	7.49229	7.49229
	4	2.92493	2.92541	2.92541	2.92539	2.92539	5.86728	5.8682	5.86819	5.86817	5.86817	7.51928	7.52073	7.52074	7.52073	7.52073
100	0	2.87889	2.87901	2.87901	2.879	2.879	5.7756	5.7758	5.7758	5.77579	5.77579	7.40161	7.40197	7.40197	7.40197	7.40197
	1	2.88165	2.88177	2.88177	2.88177	2.88177	5.78114	5.78135	5.78135	5.78134	5.78134	7.40872	7.40908	7.40908	7.40907	7.40907
	2	2.88442	2.88454	2.88453	2.88453	2.88453	5.78669	5.78689	5.78689	5.78688	5.78688	7.41582	7.41618	7.41618	7.41618	7.41618
	3	2.88718	2.88729	2.88729	2.88729	2.88729	5.79223	5.79244	5.79243	5.79243	5.79234	7.42293	7.42329	7.42329	7.42328	7.42328
	4	2.88994	2.89006	2.89006	2.89006	2.89006	5.79778	5.79798	5.79798	5.79797	5.79797	7.43003	7.43039	7.43039	7.43039	7.43039

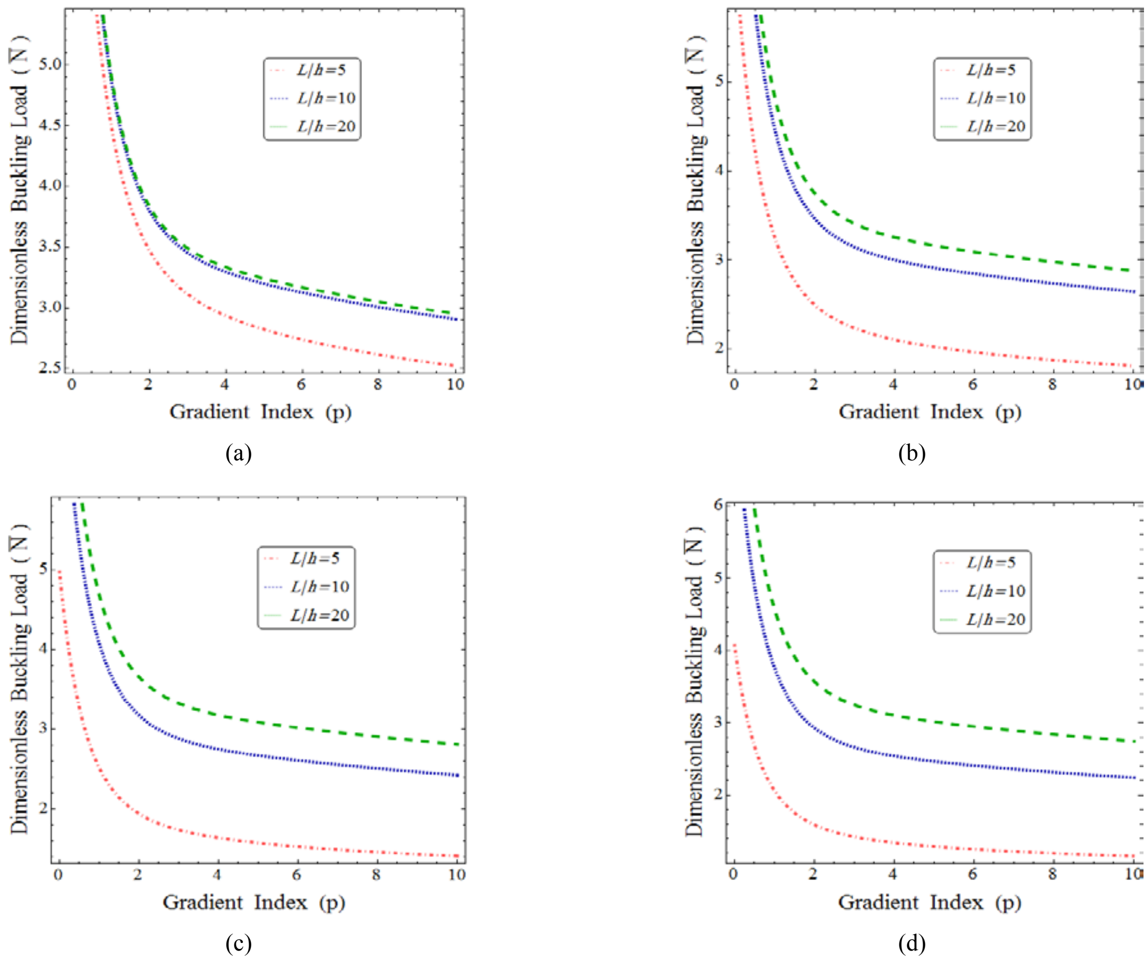


Fig. 3 Effect of gradient index p on dimensionless buckling load for uniform load and different aspect ratios: (a) $\mu = 0$; (b) $\mu = 1$; (c) $\mu = 2$; (d) $\mu = 3$ (nm^2)

Table 5 The variations of non-dimensional buckling load $\overline{N}$ of FG nanobeam for different aspect ratios, nonlocal parameters, and gradient indexes

L/h	μ (nm^2)	p = 0					1					2				
		CBT	TBT	SBT	ABT	EBT	CBT	TBT	SBT	ABT	EBT	CBT	TBT	SBT	ABT	EBT
5	0	9.54721	8.89229	8.77562	8.64566	8.64566	4.78343	4.50281	4.45219	4.39557	4.39557	3.78127	3.47268	3.42618	3.37423	3.37423
	1	6.84494	6.37539	6.29174	6.19856	6.19856	3.42952	3.22832	3.19203	3.15144	3.15144	2.67516	2.48976	2.45642	2.41917	2.41917
	2	5.33492	4.96896	4.90377	4.83114	4.83114	2.67295	2.51615	2.48786	2.45622	2.45622	2.08501	1.94051	1.91453	1.88549	1.88549
	3	4.37073	4.07091	4.01749	3.95799	3.95799	2.18986	2.06139	2.03822	2.0123	2.0123	1.70818	1.58979	1.56851	1.54473	1.54473
	4	3.70171	3.44778	3.40254	3.35215	3.35215	1.85466	1.74586	1.72623	1.70482	1.70482	1.44671	1.34645	1.32842	1.30828	1.30828
50	0	9.86957	9.86851	9.86819	9.86781	9.86781	4.91941	4.91895	4.91882	4.91866	4.91866	3.8387	3.83827	3.83815	3.83799	3.83799
	1	9.83076	9.8297	9.82939	9.82901	9.82901	4.90005	4.89961	4.89948	4.89932	4.89932	3.8236	3.82318	3.82305	3.82289	3.82289
	2	9.79225	9.7912	9.79089	9.79051	9.79051	4.88086	4.88042	4.88029	4.88013	4.88013	3.80862	3.80821	3.80808	3.80792	3.80792
	3	9.75405	9.753	9.75269	9.75231	9.75231	4.86182	4.86138	4.86125	4.86109	4.86109	3.79376	3.79335	3.79322	3.79307	3.79307
	4	9.71614	9.71509	9.71479	9.71441	9.71441	4.84292	4.84248	4.84235	4.84219	4.84219	3.77902	3.7786	3.77848	3.77832	3.77832
100	0	9.8696	9.86939	9.86932	9.86924	9.86924	4.91941	4.91932	4.91929	4.91926	4.91926	3.83871	3.83862	3.83859	3.83856	3.83856
	1	9.85987	9.85966	9.85959	9.85951	9.85951	4.91456	4.91447	4.91444	4.91441	4.91441	3.83492	3.83484	3.83481	3.83478	3.83478
	2	9.85016	9.84995	9.84988	9.84979	9.84979	4.90972	4.90963	4.9096	4.90957	4.90957	3.83114	3.83106	3.83103	3.83099	3.83099
	3	9.84047	9.84026	9.84019	9.84011	9.84011	4.90489	4.90479	4.90477	4.90474	4.90474	3.82737	3.82729	3.82726	3.82723	3.82723
	4	9.83079	9.83058	9.83052	9.83043	9.83043	4.90007	4.89998	4.89995	4.89991	4.89991	3.82361	3.82353	3.82349	3.82347	3.82347

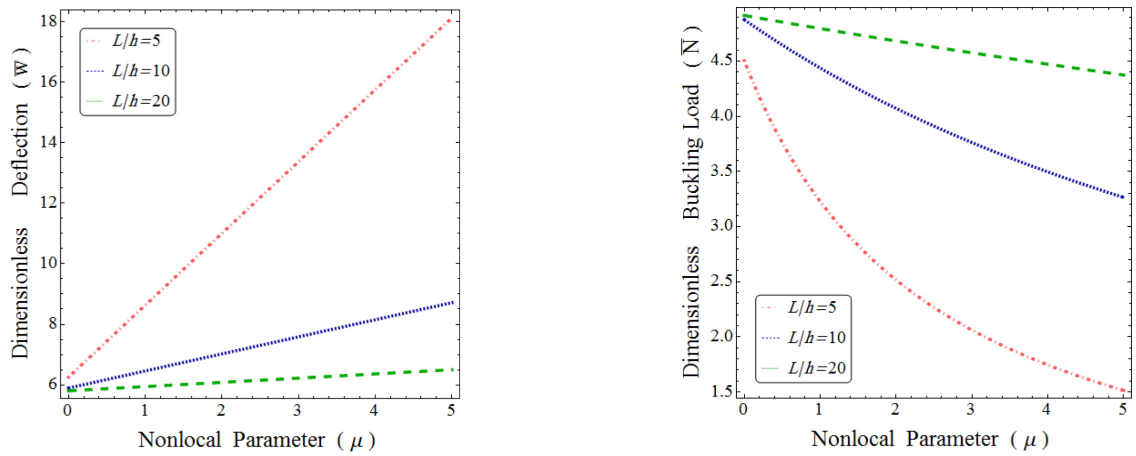
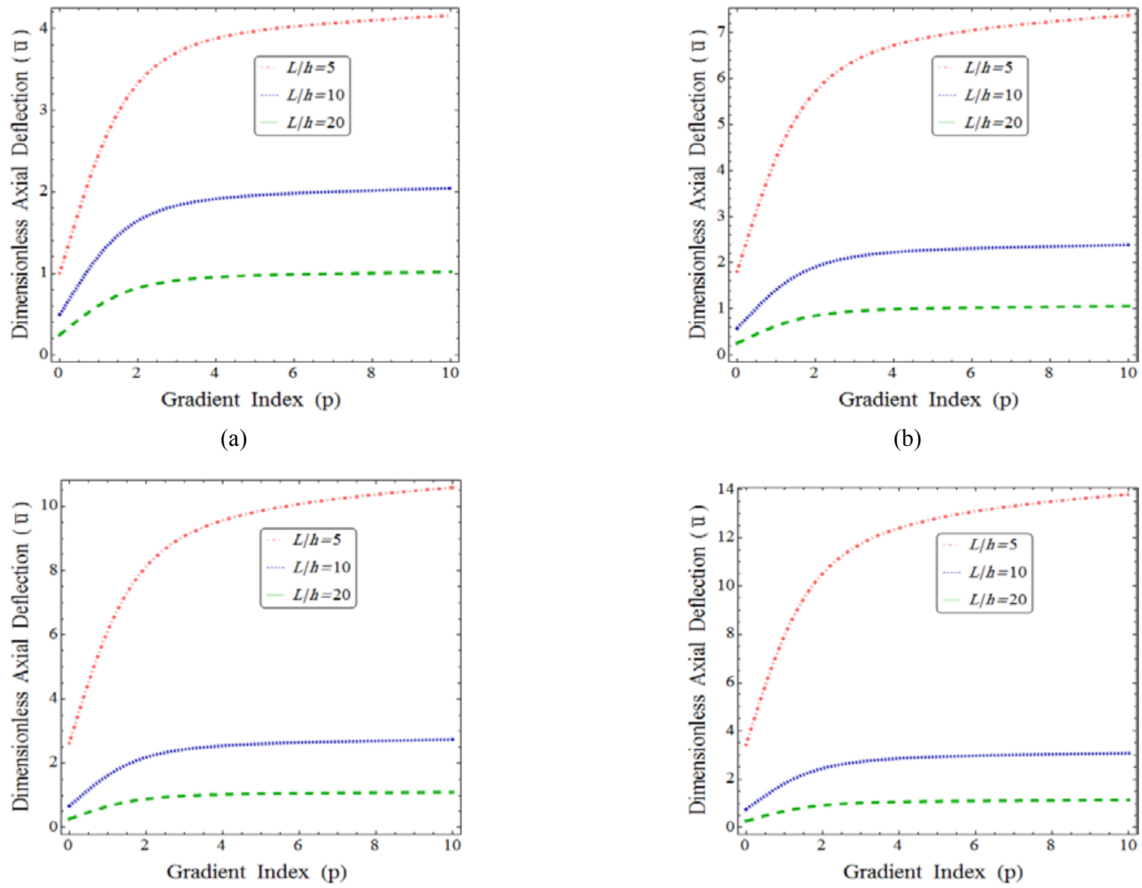


Fig. 4 Effect of nonlocal parameter μ at constant gradient index p = 1 on: (a) dimensionless transverse deflection for different aspect ratios; (b) dimensionless buckling load for different aspect ratios



(a) (b) (c) (d)

Fig. 5 Effect of gradient index p on the dimensionless axial deflection for the uniform load and different aspect ratios: (a) μ = 0; (b) μ = 1; (c) μ = 2; (d) μ = 3 (nm^2)

It is simple to understand from Fig. 3 that as the gradient index increases, the dimensionless buckling load decreases. Also, it decreasesas the aspect ratio increases. Besides, it can be observed that the non-dimensional buckling load decreases as gradient index (p) increases. As mentioned before, the FG nanobeam becomes a homogeneous nanobeam made of pure ceramic or pure metal for p = 0 or $p = \infty$, respectively. As a result, the volume fraction of the metal constituent becomes lower for smaller values of the gradient index. It is also notable that the FG nanobeam becomes stiffer for the smaller values of material property gradient index due to the elastic modulus of ceramic constituent.

In Table 4, buckling loads are proposed with respect to aspect ratios, nonlocal parameter, and gradient index. It can be seen that for all the higher order theories with increasing in value of *L*/*h*, non-dimensional buckling loads will increase, but by increasing the nonlocal parameter and gradient index, the non-dimensional buckling loads will decrease. Also, it can be concluded that the dimensionless buckling loads evaluated by HSDTs are almost equal to each other, but the difference between CBT and HSDTs is more considerable in the higher-order models for lower aspect ratios. Also, it can be said that the dimensionless buckling loads predicted by HBT and EBT are almost equal to each other for all cases. Moreover, It is notable when length -to-thickness ratio scale parameter increases (L/h > 20) the divergence between results predicted by classical (μ = 0) and non-classical models ($\mu \neq 0$) decreases, also the effects of shear deformation are more significant for thick nanobeam (i.e., L/h < 20). Summarily, it can be said that the divergence between results predicted by HSDTs and classic theory (CBT) decreases as the aspect ratio, nonlocal parameter, and gradient index decrease. Also, it can be concluded that nanostructure effects may be overlooked because of the increase in the characteristic sizes of the nanobeam. Though, variations in the length -to-thickness ratio, the gradient index and the beam theories have opposite effects on dimensionless buckling load compared with those on dimensionless transverse deflection.

Fig. 4 is plotted to show the dimensionless transverse deflection and the dimensionless buckling load respectively with respect to nonlocal parameter μ for constant gradient index p = 1 and different aspect ratios L/h = 5, 10, and 20. It is detected from Fig. 4 that an increase in slenderness ratio (by decreasing aspect ratio) leads to the reduction ofshear deformation effects. It is obviously seen that the differences between the dimensionless transverse deflection and the dimensionless buckling loads based on different aspect ratios are decreasing for lower value of the nonlocal parameter (i.e., μ < 1).

To clarify the effect of gradient index p and nonlocal parameter μ on the bending response of FG nanobeams under uniform load, the axial displacement u, is plotted in Fig. 5. It can be realized that increasing the gradient index p will reduce the stiffness of the FG nanobeams and consequently leads to an increase in the axial deflections. In addition, it can be understood that the inclusion of size effect leads to an increase in the deflections. Moreover, it is notable that increasing the dimensionless axial deflection is smoother for p = 2and more.

8. Conclusions

The present work is a comprehensive study on the bending and buckling of simply-supported FG nanobeams which includes both different higher order shear deformation beam theories and nonlocal beam theory of Eringen. The present study considers the influence of material length, shear deformation after loading, and the material distribution parameter by applying the different nonlocal higher order shear deformation beam theories on the FG nanobeams The proposed theories fulfill the zero traction boundary conditions on the top and bottom surfaces of the nanobeam, thus a shear correction factor is not necessary to use. Numerical results show that the nonlocal effects play an important role on the static and buckling behavior of the FG nanobeam. The discussed nonlocal beam models produce larger deflection and smaller buckling load in comparison with the classical (local) beam model. Therefore, the small scale effects should be considered in the evaluation of mechanical behavior of nanostructures. Further, it is found that the gradient index has a great influence on the responses of FG nanobeam, and the responses can be controlled by selecting appropriate values of the gradient index. The results of all proposed beam theories are nearly identical to each other, particularly for higher value of aspect ratio and agree well with the existing solution. The classical beam theory underestimates the bending responses and overestimates the buckling responses. Moreover, it is noticed that the inclusion of shear deformation effects leads to an increase in the deflections and a reduction of the buckling loads of nanobeams.

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