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A non-polynomial four variable refined plate theory for free vibration of functionally graded thick rectangular plates on elastic foundation

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Abstract. This paper presents a free vibration analysis of plates made of functionally graded materials and resting on two-layer elastic foundations by proposing a non-polynomial four variable refined plate theory. Undetermined integral terms are introduced in the proposed displacement field and unlike the conventional higher shear deformation theory (HSDT), the present one contains only four unknowns. Equations of motion are derived via the Hamilton's principles and solved using Navier's procedure. Accuracy of the present theory is demonstrated by comparing the results of numerical examples with the ones available in literature.

Keywords: vibration; functionally graded plate; elastic foundation; plate theory

1. Introduction

Functionally graded materials (FGMs) are microscopically inhomogeneous composites often manufactured from a mixture of metals and ceramics. Material characteristics of FGM vary along the material size depending on a function. The concept of FGM was first proposed in Japan in 1984 during a space plane project. Since its developments in the 1980s, FGMs are alternative materials widely employed in aerospace, nuclear reactor, energy sources, biomechanical, optical, civil, automotive, electronic, chemical, mechanical, and shipbuilding industries (Koizumi 1993, Tounsi *et al*. 2013, Hamidi *et al*. 2015, Al-Basyouni *et al*. 2015, Tagrara *et al*. 2015, Bennai *et al*. 2015, Larbi Chaht *et al*. 2015, Bouderba *et al*. 2016, Bousahla *et al*. 2016, Boukhari *et al*. 2016, Turan *et al*. 2016, El-Hassar *et al*. 2016).

Several works have been performed to investigate the vibration behavior of functionally graded (FG) plates. Vel and Batra (2004) proposed a three dimensional exact solution for free and forced vibrations of simply supported FG rectangular plates. Ferreira *et al*. (2006) investigated the vibrations of FG plates by employing a global collocation technique, the first and the third-order shear deformation plate models. Qian *et al*. (2004) discussed bending, and free and forced vibrations of a thick rectangular FG plate by employing a higher order shear and normal deformable plate theory. Matsunaga (2008) studied natural frequencies and buckling stresses of FG plates by considering the effects of transverse shear and normal deformations and rotatory inertia. Lu *et al*. (2009) presented a free vibration analysis of FG thick plates resting on elastic foundation based on three-dimensional elasticity. Zhao *et al*. (2009) examined a free vibration of FG plates by employing the element-free kp-Ritz method. Chen *et al*. (2009) studied the dynamic and buckling of FG plates based on a higher-order deformation theory. Malekzadeh (2009) analyzed the free vibration response of thick FG plates on two-parameter elastic foundation based on the 3D elasticity theory. Ait Atmane *et al*. (2010) studied the free vibration response of FG plates resting on Winkler–Pasternak elastic foundations using a new shear deformation theory. Hosseini-Hashemi *et al*. (2010) investigated the free vibration of FG rectangular plates using first-order shear deformation plate theory. Neves *et al*. (2012a, b) proposed a sinusoidal shear deformation formulation and a hybrid quasi-3D hyperbolic shear deformation theory for static and dynamic analysis of FG plates. Akavci (2014) presented a free vibration analysis of FG plates resting on elastic foundation by using a hyperbolic shear deformation theory. Ait Amar Meziane *et al*. (2014) presented an efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions. Hebali *et al*. (2014) developed a new quasi-3D hyperbolic shear

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deformation theory for the static and free vibration analysis of FG plates. Belabed *et al*. (2014) developed an efficient and simple higher order shear and normal deformation theory for bending and free vibration of FG plates. Zidi *et al*. (2014) proposed a four variable refined plate theory for bending analysis of FG plates under hygro-thermo-mechanical loading. Ait Yahia *et al*. (2015) discussed the wave propagation in functionally graded plates with porosities by employing various higher-order shear deformation plate theories. Bakora and Tounsi (2015) investigated the thermo-mechanical post-buckling behavior of thick FG plates resting on elastic foundations. By developing a new simple shear and normal deformations theory, Bourada *et al*. (2015) studied the bending and free vibration of FG thick beam. Tebboune *et al*. (2015) investigated the thermal buckling analysis of FG plates resting on elastic foundation based on an efficient and simple trigonometric shear deformation theory. Belkorissat *et al*. (2015) discussed the vibration properties of FG nano-plate using a new nonlocal refined four variable model. Ait Atmane *et al*. (2015) proposed a computational shear displacement model for vibrational analysis of FG beams with porosities. Bousahla *et al*. (2014) presented a novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates. Attia *et al*. (2015) studied the free vibration analysis of FG plates with temperature-dependent properties using various four variable refined plate theories. Nguyen ~~*et al*.~~ (2015) developed a refined higher-order shear deformation theory for bending, vibration and buckling analysis of FG sandwich plates. Mahi *et al*. (2015) developed a new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates. The vibrational behavior of FG plates has been studied by Mantari and Granados (2015) using a novel first shear deformation theory. Bourada *et al*. (2016) discussed the buckling behavior of isotropic and orthotropic plates using a novel four variable refined plate theory. Bounouara *et al*. (2016) proposed a nonlocal zeroth-order shear deformation theory for free vibration of FG nanoscale plates resting on elastic foundation. Recently, Bennoun *et al*. (2016) proposed a novel five variable refined plate theory for vibration analysis of FG sandwich plates. Draiche *et al*. (2016) presented a refined theory with stretching effect for the flexure analysis of laminated composite plates. Tounsi *et al*. (2016) developed a new 3-unknowns non-polynomial plate theory for buckling and vibration of FG sandwich plate. In the same way, Houari *et al*. (2016) presented also a novel simple three–unknown sinusoidal shear deformation theory for FG plates.

In the present research, a new displacement field is proposed by considering a hyperbolic variation of in-plane displacements through the plate thickness and the obtained displacement field is applied to study the free vibration behavior of FG plates resting on two-parameter elastic foundations. The addition of the integral term in the displacement field leads to a reduction in the number of unknowns and equations of motion. The governing equations of the plates are obtained by considering the Hamilton's principle. These equations are then solved via Navier method. Comparison studies are performed to check the validity of the present results.

2. Fundamental formulations

In the present work, a FG simply supported rectangular plate having a uniform thickness h, the length a, and the width b is examined. The geometry of the plate and coordinate system are indicated in Fig. 1. The material characteristics of FG plate are considered to vary continuously across the thickness of the plate in according to the power law distribution as follows (Bouderba *et al*. 2013, Meksi *et al*. 2015, Meradjah *et al*. 2015)

$$P(z) = P_m + \left(P_c - P_m\right)\left(\frac{1}{2} + \frac{z}{h}\right)^p \tag{1}$$

where P is the effective material properties like Young's modulus E and mass density ρ, P_m and P_c represent the property of the top and the bottom faces of the plate, respectively; and p is the volume fraction exponent. The Poisson's ratio ν is considered to be constant (Sallai *et al*. 2015, Bellifa *et al*. 2016).

The displacement field of the novel theory is given as follows (Hebali *et al*. 2016, Merdaci *et al*. 2016)

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x} + k_1\, f(z)\int \theta(x,y,t)\,dx \tag{2a}$$

$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y} + k_2\, f(z)\int \theta(x,y,t)\,dy \tag{2b}$$

$$w(x,y,z,t) = w_0(x,y,t) \tag{2c}$$

where $u_0(x, y)$, $v_0(x, y)$, $w_0(x, y)$, and $\theta(x, y)$ are the four unknown displacement functions of middle surface of the plate. The last unknown is a mathematical term that allows obtaining the rotations of the normal to the midplate about the x and y axes (as in the ordinary HSDT). Note that the integrals do not have limits. In the present paper is considered only four unknown displacement functions instead of five terms in ordinary HSDT (Akavci 2014). The constants k_1 and k_2 depends on the geometry. $f(z)$ represents the shape function for determining the distributions of the

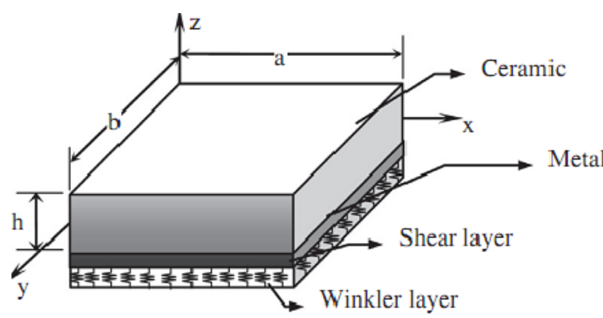

Fig. 1 Geometry and coordinates of the considered FG plate which is resting on elastic foundation

transverse shear strains and stresses along the thickness and given as (Nguyen *et al*. 2015)

$$f(z)=\sinh^{-1}\left(\frac{3z}{h}\right)-z\frac{6}{h\sqrt{13}} \tag{3}$$

Using the displacement field in Eq. (2) within the application of the linear, small-strain elasticity theory, normal and shear strains are obtained as

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}=\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix}+z\begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix}+f(z)\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix},\quad \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix}=g(z)\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix}, \tag{4}$$

where

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix}=\begin{Bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{\partial v_0}{\partial x}\\ \dfrac{\partial u_0}{\partial y}+\dfrac{\partial v_0}{\partial x}\end{Bmatrix},\quad \begin{Bmatrix}k_x^b\\ k_y^b\\ k_{xy}^b\end{Bmatrix}=\begin{Bmatrix}-\dfrac{\partial^2 w_0}{\partial x^2}\\ -\dfrac{\partial^2 w_0}{\partial y^2}\\ -2\dfrac{\partial^2 w_0}{\partial x\partial y}\end{Bmatrix},$$

$$\begin{Bmatrix}k_x^s\\ k_y^s\\ k_{xy}^s\end{Bmatrix}=\begin{Bmatrix}k_1\theta\\ k_2\theta\\ k_1\dfrac{\partial}{\partial y}\int\theta\,dx+k_2\dfrac{\partial}{\partial x}\int\theta\,dy\end{Bmatrix}, \tag{5a}$$

$$\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix}=\begin{Bmatrix}k_1\int\theta\,dy\\ k_2\int\theta\,dx\end{Bmatrix},$$

and

$$g(z)=\frac{df(z)}{dz} \tag{5b}$$

The integrals appearing in the above expressions shall be resolved by a Navier type solution and can be expressed as follows

$$\frac{\partial}{\partial y}\int\theta\,dx=A'\frac{\partial^2\theta}{\partial x\partial y},\quad \frac{\partial}{\partial x}\int\theta\,dy=B'\frac{\partial^2\theta}{\partial x\partial y},$$
$$\int\theta\,dx=A'\frac{\partial\theta}{\partial x},\quad \int\theta\,dy=B'\frac{\partial\theta}{\partial y} \tag{6}$$

where the coefficients A' and B' are defined according to the type of solution adopted, in this case via Navier. Therefore, A' and B' are expressed as follows

$$A'=-\frac{1}{\alpha^2},\quad B'=-\frac{1}{\beta^2},\quad k_1=\alpha^2,\quad k_2=\beta^2 \tag{7}$$

where α and β are defined in expression (24).

For the FG plates, the stress–strain relationships for plane-stress state can be expressed as

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&0&0&0\\ C_{12}&C_{22}&0&0&0\\ 0&0&C_{66}&0&0\\ 0&0&0&C_{44}&0\\ 0&0&0&0&C_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} \tag{8}$$

where (σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{xz}) and (ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{xz}) are the stress and strain components, respectively. Using the material properties defined in Eq. (1), stiffness coefficients, C_{ij}, can be given as

$$C_{11}=C_{22}=\frac{E(z)}{1-\nu^2},\quad C_{12}=\frac{\nu E(z)}{1-\nu^2},$$
$$C_{44}=C_{55}=C_{66}=\frac{E(z)}{2(1+\nu)}, \tag{9}$$

3. Equations of motion

Hamilton's principle is employed herein to obtain the equations of motion appropriate to the displacement field and the constitutive equations. The principle can be stated in analytical form as

$$0=\int_0^t(\delta U+\delta V_e-\delta K)\,dt \tag{10}$$

where δU is the variation of strain energy; δV_e is the variation of the potential energy of elastic foundation; and δK is the variation of kinetic energy.

The variation of strain energy of the plate is given by

$$\begin{aligned}\delta U&=\int_V\left[\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\gamma_{xy}+\tau_{yz}\delta\gamma_{yz}+\tau_{xz}\delta\gamma_{xz}\right]dV\\ &=\int_A\left[N_x\delta\varepsilon_x^0+N_y\delta\varepsilon_y^0+N_{xy}\delta\gamma_{xy}^0+M_x^b\delta k_x^b+M_y^b\delta k_y^b+M_{xy}^b\delta k_{xy}^b\right.\\ &\quad\left.+M_x^s\delta k_x^s+M_y^s\delta k_y^s+M_{xy}^s\delta k_{xy}^s+S_{yz}^s\delta\gamma_{yz}^s+S_{xz}^s\delta\gamma_{xz}^0\right]dA=0\end{aligned} \tag{11}$$

where A is the top surface and the stress resultants N, M, and S are defined by

$$\left(N_i,M_i^b,M_i^s\right)=\int_{-h/2}^{h/2}(1,z,f)\sigma_i\,dz,\quad (i=x,y,xy)$$
$$\text{and}\quad \left(S_{xz}^s,S_{yz}^s\right)=\int_{-h/2}^{h/2}g\left(\tau_{xz},\tau_{yz}\right)dz \tag{12}$$

The variation of the potential energy of elastic foundation can be expressed by

$$\delta V_e=\int_A f_e\delta w_0\,dA \tag{13}$$

where f_e is the density of reaction force of foundation. For the Pasternak foundation model

$$f_e=K_w w-K_s\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) \tag{14}$$

in which K_w and K_s are the Winkler foundation stiffness and the shear stiffness of the elastic foundation.

The variation of kinetic energy of the plate can be written as

$$
\begin{aligned}
\delta K &= \int_V \left[\dot{u}\delta\dot{u} + \dot{v}\delta\dot{v} + \dot{w}\delta\dot{w}\right]\rho(z)\,dV \\
&= \int_A \Bigg\{ I_0\left[\dot{u}_0\delta\dot{u}_0 + \dot{v}_0\delta\dot{v}_0 + \dot{w}_0\delta\dot{w}_0\right] \\
&\quad - I_1\left(\dot{u}_0\frac{\partial\delta\dot{w}_0}{\partial x} + \frac{\partial\dot{w}_0}{\partial x}\delta\dot{u}_0 + \dot{v}_0\frac{\partial\delta\dot{w}_0}{\partial y} + \frac{\partial\dot{w}_0}{\partial y}\delta\dot{v}_0\right) \\
&\quad + J_1\left(\begin{aligned}&(k_1A')\left(\dot{u}_0\frac{\partial\delta\dot{\theta}}{\partial x} + \frac{\partial\dot{\theta}}{\partial x}\delta\dot{u}_0\right)\\ &+(k_2\,B')\left(\dot{v}_0\frac{\partial\delta\dot{\theta}}{\partial y} + \frac{\partial\dot{\theta}}{\partial y}\delta\dot{v}_0\right)\end{aligned}\right) \\
&\quad + J_1\left(\begin{aligned}&(k_1\,A')\left(\dot{u}_0\frac{\partial\delta\dot{\theta}}{\partial x} + \frac{\partial\dot{\theta}}{\partial x}\delta\dot{u}_0\right)\\ &+(k_2\,B')\left(\dot{v}_0\frac{\partial\delta\dot{\theta}}{\partial y} + \frac{\partial\dot{\theta}}{\partial y}\delta\dot{v}_0\right)\end{aligned}\right) \\
&\quad + I_2\left(\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\dot{w}_0}{\partial x} + \frac{\partial\dot{w}_0}{\partial y}\frac{\partial\delta\dot{w}_0}{\partial y}\right) \\
&\quad + K_2\left((k_1\,A')^2\left(\frac{\partial\dot{\theta}}{\partial x}\frac{\partial\delta\dot{\theta}}{\partial x}\right) + (k_2\,B')^2\left(\frac{\partial\dot{\theta}}{\partial y}\frac{\partial\delta\dot{\theta}}{\partial y}\right)\right) \\
&\quad - J_2\left(\begin{aligned}&(k_1\,A')\left(\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\dot{\theta}}{\partial x} + \frac{\partial\dot{\theta}}{\partial x}\frac{\partial\delta\dot{w}_0}{\partial x}\right)\\ &+(k_2\,B')\left(\frac{\partial\dot{w}_0}{\partial y}\frac{\partial\delta\dot{\theta}}{\partial y} + \frac{\partial\dot{\theta}}{\partial y}\frac{\partial\delta\dot{w}_0}{\partial y}\right)\end{aligned}\right)\Bigg\} dA
\end{aligned}
\tag{15}
$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density given by Eq. (1); and (I_i, J_i, K_i) are mass inertias expressed by

$$
(I_0, I_1, I_2) = \int_{-h/2}^{h/2} (1, z, z^2)\rho(z)dz \tag{16a}
$$

$$
(J_1, J_2, K_2) = \int_{-h/2}^{h/2} (f, z\,f, f^2)\rho(z)dz \tag{16b}
$$

Using the generalized displacement–strain relations (4) and stress–strain relations (8), and the fundamentals of calculus of variations and collecting the coefficients of δu_0, δv_0, δw_0 and $\delta\theta$ in Eq. (10), the equations of motion are obtained as

$$
\begin{aligned}
\delta u_0 &: \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0\ddot{u}_0 - I_1\frac{\partial\ddot{w}_0}{\partial x} + k_1A'J_1\frac{\partial\ddot{\theta}}{\partial x} \\
\delta v_0 &: \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0\ddot{v}_0 - I_1\frac{\partial\ddot{w}_0}{\partial y} + k_2B'J_1\frac{\partial\ddot{\theta}}{\partial y} \\
\delta w_0 &: \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x\partial y} + \frac{\partial^2 M_y^b}{\partial y^2} - f_e \\
&= I_0\ddot{w}_0 + I_1\left(\frac{\partial\ddot{u}_0}{\partial x} + \frac{\partial\ddot{v}_0}{\partial y}\right) - I_2\nabla^2\ddot{w}_0 \\
&\quad + J_2\left(k_1A'\frac{\partial^2\ddot{\theta}}{\partial x^2} + k_2B'\frac{\partial^2\ddot{\theta}}{\partial y^2}\right) \\
\delta\theta &: -k_1M_x^s - k_2M_y^s - (k_1A' + k_2B')\frac{\partial^2 M_{xy}^s}{\partial x\partial y} \\
&\quad + k_1A'\frac{\partial S_{xz}^s}{\partial x} + k\,B'\frac{\partial S_{yz}^s}{\partial y} = -J_1\left(k\,A'\frac{\partial\ddot{u}_0}{\partial x} + k_2B'\frac{\partial\ddot{v}_0}{\partial y}\right) \\
&\quad - K_2\left((k_1A')^2\frac{\partial^2\ddot{\theta}}{\partial x^2} + (k_2B')^2\frac{\partial^2\ddot{\theta}}{\partial y^2}\right) \\
&\quad + J_2\left(k_1A'\frac{\partial^2\ddot{w}_0}{\partial x^2} + k_2B'\frac{\partial^2\ddot{w}_0}{\partial y^2}\right)
\end{aligned}
\tag{17}
$$

where stress and moment resultants are defined as

$$
\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & B & B^s \\ B & D & D^s \\ B^s & D^s & H^s \end{bmatrix}\begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix}, \qquad S = A^s\gamma, \tag{18}
$$

in which

$$
N = \{N_x, N_y, N_{xy}\}^t, \quad M^b = \{M_x^b, M_y^b, M_{xy}^b\}^t, \quad M^s = \{M_x^s, M_y^s, M_{xy}^s\}^t, \tag{19a}
$$

$$
\varepsilon = \{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0\}^t, \quad k^b = \{k_x^b, k_y^b, k_{xy}^b\}^t, \quad k^s = \{k_x^s, k_y^s, k_{xy}^s\}^t, \tag{19b}
$$

$$
A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \quad D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \tag{19c}
$$

$$
B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & 0 \\ B_{12}^s & B_{22}^s & 0 \\ 0 & 0 & B_{66}^s \end{bmatrix}, \quad D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & 0 \\ D_{12}^s & D_{22}^s & 0 \\ 0 & 0 & D_{66}^s \end{bmatrix}, \quad H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & 0 \\ H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & H_{66}^s \end{bmatrix}, \tag{19d}
$$

$$
S = \{S_{xz}^s, S_{yz}^s\}^t, \quad \gamma = \{\gamma_{xz}^0, \gamma_{yz}^0\}^t, \quad A^s = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix}, \tag{19e}
$$

and stiffness components are given as

$$\begin{Bmatrix} A_{11} & B_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & B_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{66} & B_{66} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix} = \int_{-h/2}^{h/2} C_{11}\left(1, z, z^2, f(z), z\,f(z), f^2(z)\right)\begin{Bmatrix} 1 \\ \nu \\ \dfrac{1-\nu}{2} \end{Bmatrix} dz, \tag{20a}$$

$$\left(A_{22}, B_{22}, D_{22}, B_{22}^s, D_{22}^s, H_{22}^s\right) = \left(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s\right), \tag{20b}$$

$$A_{44}^s = A_{55}^s = \int_{-h/2}^{h/2} C_{44}\left[g(z)\right]^2 dz, \tag{20c}$$

Introducing Eq. (18) into Eq. (17), the equations of motion can be expressed in terms of displacements (u_0, v_0, w_0, θ) and the appropriate equations take the form

$$\begin{aligned} &A_{11}d_{11}u_0 + A_{66}d_{22}u_0 + \left(A_{12}+A_{66}\right)d_{12}v_0 - B_{11}d_{111}w_0 \\ &-\left(B_{12}+2B_{66}\right)d_{122}w_0 + \left(B_{66}^s\left(k_1A'+k_2\,B'\right)\right)d_{122}\theta \\ &+\left(B_{11}^s k_1 + B_{12}^s k_2\right)d_1\theta = I_0\ddot{u}_0 - I_1 d_1\ddot{w}_0 + J_1 A' k_1 d_1\ddot{\theta}, \end{aligned} \tag{21a}$$

$$\begin{aligned} &A_{22}d_{22}v_0 + A_{66}d_{11}v_0 + \left(A_{12}+A_{66}\right)d_{12}u_0 - B_{22}d_{222}w_0 \\ &-\left(B_{12}+2B_{66}\right)d_{112}w_0 + \left(B_{66}^s\left(k_1A'+k_2B'\right)\right)d_{112}\theta \\ &+\left(B_{22}^s k_2 + B_{12}^s k_1\right)d_2\theta = I_0\ddot{v}_0 - I_1 d_2\ddot{w}_0 + J_1 B' k_2 d_2\ddot{\theta}, \end{aligned} \tag{21b}$$

$$\begin{aligned} &B_{11}d_{111}u_0 + \left(B_{12}+2B_{66}\right)d_{122}u_0 + \left(B_{12}+2B_{66}\right)d_{112}v_0 \\ &+ B_{22}d_{222}v_0 - D_{11}d_{1111}w_0 - 2\left(D_{12}+2D_{66}\right)d_{1122}w_0 \\ &- D_{22}d_{2222}w_0 + \left(D_{11}^s k_1 + D_{12}^s k_2\right)d_{11}\theta \\ &+ 2\left(D_{66}^s\left(k_1A'+k_2B'\right)\right)d_{1122}\theta + \left(D_{12}^s k_1 + D_{22}^s k_2\right)d_{22}\theta \\ &- f_e = I_0\ddot{w}_0 + I_1\left(d_1\ddot{u}_0 + d_2\ddot{v}_0\right) - I_2\left(d_{11}\ddot{w}_0 + d_{22}\ddot{w}_0\right) \\ &+ J_2\left(k_1A' d_{11}\ddot{\theta} + k_2 B' d_{22}\ddot{\theta}\right) \end{aligned} \tag{21c}$$

$$\begin{aligned} &-\left(B_{11}^s k_1 + B_{12}^s k_2\right)d_1u_0 - \left(B_{66}^s\left(k_1A'+k_2B'\right)\right)d_{122}u_0 \\ &-\left(B_{66}^s\left(k_1A'+k_2B'\right)\right)d_{112}v_0 - \left(B_{12}^s k_1 + B_{22}^s k_2\right)d_2v_0 \\ &+\left(D_{11}^s k_1 + D_{12}^s k_2\right)d_{11}w_0 + 2\left(D_{66}^s\left(k_1A'+k_2B'\right)\right)d_{1122}w_0 \\ &+\left(D_{12}^s k_1 + D_{22}^s k_2\right)d_{22}w_0 - H_{11}^s k_1^2\theta - H_{22}^s k_2^2\theta \\ &-2H_{12}^s k_1k_2\theta - \left(\left(k_1A'+k_2B'\right)^2 H_{66}^s\right)d_{1122}\theta \\ &+ A_{44}^s\left(k_2B'\right)^2 d_{22}\theta + A_{55}^s\left(k_1A'\right)^2 d_{11}\theta \\ &= -J_1\left(k_1A' d_1\ddot{u}_0 + k_2B' d_2\ddot{v}_0\right) + J_2\left(k_1A' d_{11}\ddot{w}_0 + k_2B' d_{22}\ddot{w}_0\right) \\ &- K_2\left(\left(k_1A'\right)^2 d_{11}\ddot{\theta} + \left(k_2B'\right)^2 d_{22}\ddot{\theta}\right) \end{aligned} \tag{21d}$$

where d_{ij}, d_{ijl} and d_{ijlm} are the following differential operators

$$d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad d_{ijl} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \quad d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \quad d_i = \frac{\partial}{\partial x_i}, \quad (i, j, l, m = 1, 2). \tag{22}$$

4. Analytical solution for simply-supported FG plates

For the analytical solution of the partial differential Eq. (21), the Navier technique, based on double Fourier series, is employed under the specified boundary conditions. Using Navier's procedure, the solution of the displacement variables satisfying the above boundary conditions can be expressed in the following Fourier series

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ \theta \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn}e^{i\omega t}\cos(\alpha\,x)\sin(\beta\,y) \\ V_{mn}e^{i\omega t}\sin(\alpha\,x)\cos(\beta\,y) \\ W_{mn}e^{i\omega t}\sin(\alpha\,x)\sin(\beta\,y) \\ X_{mn}e^{i\omega t}\sin(\alpha\,x)\sin(\beta\,y) \end{Bmatrix} \tag{23}$$

where U_{mn}, V_{mn}, W_{mn}, X_{mn} are arbitrary parameters to be determined and ω is the natural frequency. α and β are expressed as

$$\alpha = m\pi / a \quad \text{and} \quad \beta = n\pi / b \tag{24}$$

Substituting Eq. (23) into equations of motion (21) we get below eigenvalue equation for any fixed value of m and n, for free vibration problem

$$\left(\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{22} & S_{23} & S_{24} \\ S_{13} & S_{23} & S_{33} & S_{34} \\ S_{14} & S_{24} & S_{34} & S_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ m_{13} & m_{23} & m_{33} & m_{34} \\ m_{14} & m_{24} & m_{34} & m_{44} \end{bmatrix}\right)\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{25}$$

where

$$\begin{aligned} &S_{11} = -\left(A_{11}\alpha^2 + A_{66}\beta^2\right), \quad S_{12} = -\alpha\beta\,\left(A_{12}+A_{66}\right), \\ &S_{13} = \alpha\left(B_{11}\alpha^2 + B_{12}\beta^2 + 2B_{66}\beta^2\right), \\ &S_{14} = \alpha\left(k_1B_{11}^s + k_2B_{12}^s - \left(k_1A'+k_2B'\right)B_{66}^s\beta^2\right), \\ &S_{22} = -\left(A_{66}\alpha^2 + A_{22}\beta^2\right), \\ &S_{23} = \beta\left(B_{22}\beta^2 + B_{12}\alpha^2 + 2B_{66}\alpha^2\right), \\ &S_{24} = \beta\left(k_2B_{22}^s + k_1B_{12}^s - \left(k_1A'+k_2B'\right)B_{66}^s\alpha^2\right), \\ &S_{33} = -\left(D_{11}\alpha^4 + 2(D_{12}+2D_{66})\alpha^2\beta^2 + D_{22}\beta^4\right) \\ &\qquad + K_w + K_s(\alpha^2+\beta^2), \\ &S_{34} = -k_1\left(D_{11}^s\alpha^2 + D_{12}^s\beta^2\right) \\ &\qquad + 2(k_1A'+k_2B')\,D_{66}^s\alpha^2\beta^2 - k_2\left(D_{22}^s\beta^2 + D_{12}^s\alpha^2\right), \\ &S_{44} = -k_1\left(H_{11}^s k_1 + H_{12}^s k_2\right) - (k_1A'+k_2B')^2 H_{66}^s\alpha^2\beta^2 \\ &\qquad - k_2\left(H_{12}^s k_1 + H_{22}^s k_2\right) - \left(k_1A'\right)^2 A_{55}^s\alpha^2 - \left(k_2B'\right)^2 A_{44}^s\beta^2, \end{aligned} \tag{26}$$

$$\begin{aligned} &m_{11} = -I_0, \quad m_{13} = \alpha\,I_1, \quad m_{14} = -J_1k_1A'\alpha, \\ &m_{22} = -I_0, \quad m_{23} = \beta\,I_1, \quad m_{24} = -k_2B'\,\beta\,J_1, \\ &m_{33} = -I_0 - I_2(\alpha^2+\beta^2), \; m_{34} = J_2\left(k_1\,A'\alpha^2 + k_2\,B'\beta^2\right), \\ &m_{44} = -K_2\left(\left(k_1\,A'\right)^2\alpha^2 + \left(k_2\,B'\right)^2\beta^2\right) \end{aligned}$$

The natural frequencies of FG plate can be found from the nontrivial solution of Eq. (25).

Table 1 Material properties used in the FG plates

Material	Properties		
	Young's modulus (GPa)	Poisson's ratio	Mass density kg/m^3
Aluminium (Al)	70	0.3	2702
Alumina (Al_2O_3)	380	0.3	3800
Zirconia (ZrO_2)	200	0.3	5700

5. Numerical examples and discussions

In this section various numerical examples are presented and discussed to verify the accuracy of the present model in predicting the dynamic responses of simply supported FG plates. Two types of FG plates of Al/Al_2O_3 and Al/ZrO_2 are employed in this work. The material properties of FG plates are listed in Table 1.

Unless otherwise has been stated, the following expressions have been employed for presentations of non-dimensional natural frequencies and non-dimensional coefficients of foundation:

Table 2 Non-dimensional natural frequencies $\hat{\omega} = \omega a^2 \sqrt{\rho h / D_0}$ for simply supported isotropic square plate

h/a	Theory	Mode							
		(1,1)	(1,2)	(2,1)	(2,2)	(3,1)	(1,3)	(3,2)	(2,3)
0.001	Leissa (1973)	19.7392	49.3480	49.3480	78.9568	98.6960	98.6960	128.3021	128.3021
	Zhou et al. (2002)	19.7115	49.3470	49.3470	78.9528	98.6911	98.6911	128.3048	128.3048
	Akavci (2014)	19.7391	49.3476	49.3476	78.9557	98.6943	98.6943	128.3019	128.3019
	Present	19.7391	49.3476	49.3476	78.9557	98.6943	98.6943	128.3019	128.3019
0.01	Leissa (1973)	19.7319	49.3027	49.3027	78.8410	98.5150	98.5150	127.9993	127.9993
	Zhou et al. (2002)	19.7320	49.3050	49.3050	78.8460	98.5250	98.5250	128.0100	128.0100
	Akavci (2014)	19.7322	49.3045	49.3045	78.8456	98.5223	98.5223	128.0115	128.0115
	Present	19.7321	49.3040	49.3040	78.8442	98.5202	98.5202	128.0080	128.0080
0.1	Leissa (1973)	19.0584	45.4478	45.4478	69.7167	84.9264	84.9264	106.5154	106.5154
	Zhou et al. (2002)	19.0653	45.4869	45.4869	69.8093	85.0646	85.0646	106.7350	106.7350
	Akavci (2014)	19.0850	45.5957	45.5957	70.0595	85.4315	85.4315	107.3037	107.3037
	Present	19.0775	45.5548	45.5548	69.9664	85.2958	85.2958	107.0953	107.0953
0.2	Leissa (1973)	17.4524	38.1884	55.2539	65.313	65.313	65.3130	78.9864	78.9864
	Zhou et al. (2002)	17.4523	38.1883	55.2543	65.3135	65.3135	65.3135	78.9865	78.9865
	Akavci (2014)	17.5149	38.4722	38.4722	55.8358	66.1207	66.1207	80.1637	80.1637
	Present	17.4916	38.3701	38.3701	55.6322	65.8425	65.8425	79.7662	79.7662

Table 3 Non-dimensional natural frequencies $\hat{\omega} = \omega a^2 \sqrt{\rho h / D_0}$ for simply supported square plate resting on elastic foundation (*h/b* = 0.2)

$k_w^{(a)}$	$k_s^{(a)}$	$\hat{w}_{11}$			$\hat{w}_{12}$			$\hat{w}_{13}$		
		Matsunaga (2000)	Akavci (2014)	Present	Matsunaga (2000)	Akavci (2014)	Present	Matsunaga (2000)	Akavci (2014)	Present
0	0	17.5260	17.5149	17.4916	38.4827	38.4722	38.3701	65.9961	66.1207	65.8425
10		17.7847	17.7859	17.7630	38.5929	38.5929	38.4912	66.0569	66.1899	65.9120
10^2		19.9528	20.0603	20.0405	39.5669	39.6620	39.5640	66.5995	66.8087	66.5346
10^3		34.3395	35.5261	35.5178	47.8667	49.0757	49.0040	71.5577	72.6997	72.4588
10^4		45.5260	45.5260	45.5260	71.9829	71.9829	71.9829	97.4964	101.799	101.7992
10^5		45.5260	45.5260	45.5260	71.9829	71.9829	71.9829	101.7992	101.799	101.7992
	10	22.0429	22.2607	22.2435	43.4816	44.0294	43.9447	71.4914	72.6178	72.3765
		22.2453	22.4745	22.4574	43.5747	44.1347	44.0503	71.5423	72.6806	72.4396
		23.9830	24.3133	24.2980	44.3994	45.0711	44.9893	71.9964	73.2430	73.0050
		36.6276	38.0839	38.0767	51.6029	53.5296	53.4675	76.1848	78.6389	78.4273
		45.5260	45.5260	45.5260	71.9829	71.9829	71.9829	99.0187	101.799	101.799
		45.5260	45.5260	45.5260	71.9829	71.9829	71.9829	101.7992	101.799	101.799

* $k_w^{(a)} = K_w a^4 / D_0$, $k_s^{(a)} = K_s a^2 / D_0$

Table 4 Comparison on of non-dimensional fundamental frequencies $\bar{\omega} = \omega h\sqrt{\rho_m / E_m}$ of Al/ZrO_2 functionally graded square plates (a/h = 5)

Theory	$p = 2$	$p = 3$	$p = 5$
Vel and Batra (2004)	0.2197	0.2211	0.2225
Neves et al. (2012a) ($\varepsilon_z = 0$)	0.2189	0.2202	0.2215
Neves et al. (2012a) ($\varepsilon_z \neq 0$)	0.2198	0.2212	0.2225
Neves et al. (2012b) ($\varepsilon_z = 0$)	0.2191	0.2205	0.2220
Neves et al. (2012b) ($\varepsilon_z \neq 0$)	0.2201	0.2216	0.2230
Matsunaga (2008)	0.2264	0.2270	0.2280
Hosseini-Hashemi et al. (2011)	0.2264	0.2276	0.2291
Akavci (2014)	0.2264	0.2269	0.2278
Present	0.2261	0.2266	0.2275

- For isotropic plate

$$\hat{\omega} = \omega a^2 \sqrt{\rho h / D_0} \quad \text{with} \quad D_0 = \frac{Et^3}{12(1-\nu^2)} \tag{27a}$$

- For FG plate

$$\begin{aligned} \tilde{\omega} &= \omega a^2 / h\sqrt{\rho_m / E_m}, \quad \beta = \omega h\sqrt{\rho_c / E_c}, \\ \bar{\omega} &= \omega h\sqrt{\rho_m / E_m}, \quad \bar{\beta} = \omega a^2 / h\sqrt{\rho_c / E_c}, \end{aligned} \tag{27b}$$

- For foundations elastic parameters

$$k_w = K_w a^4 / A, \quad k_s = K_s a^2 / A \quad \text{with}$$

$$A = \frac{h^3}{12(1-\nu^2)} \frac{\left[p(8+3p+p^2)E_m + 3(2+p+p^2)E_c\right]}{\left[(1+p)(2+p)(3+p)\right]} \tag{27c}$$

5.1 Isotropic plates

As the first example, simply supported isotropic square plates are examined for the convergence study of the present non-polynomial four variable refined plate theory. The first eight non-dimensional natural frequencies for different thickness ratios are calculated and compared with other published results in Table 2. The results given by Leissa (1973) were based on 3D exact solution, Zhou *et al.* (2002) were based on a 3D Ritz method with Chebyshev polynomials, Liu and Liew (1999) were based on a differential quadrature element method, Nagino *et al.* (2008) were based on a 3D B-spline Ritz method, Hosseini-Hashemi *et al.* (2011) based on a exact closed form Levy-type solution, and Shufrin and Eisenberger (2005) and Akavci (2014) based on two dimensional HSDTs. It can be observed from this table that, not only for thin plates but also thick plates, the natural frequencies are predicted as accurately by the present theory.

The second example is carried out for an isotropic square plate resting on elastic foundation. The first three dimensionless natural frequencies of a square plate on elastic foundation are reported in Table 3 and compared with the results of different HSDTs of Matsunaga (2000) and Akavci (2014). The results in Table 3 demonstrate good agreement in all cases.

5.2 Functionally graded plates

In this section, to check the accuracy of present work for FG plates, natural frequencies of the plates with simply supported edges are examined.

The next two examples are established for Al/ZrO_2 FG square plates. In Table 4, dimensionless fundamental

Table 5 Comparison of non-dimensional fundamental frequencies $\bar{\omega} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/ZrO_2 functionally graded square plates ($m = n = 1$)

Mode no.	Theory	$p = 0$ [(a)]		$p = 1$			$a/h = 5$		
		$a/h = \sqrt{10}$	$a/h = 5$	$a/h = 5$	$a/h = 10$	$a/h = 20$	$p = 2$	$p = 3$	$p = 5$
1	Vel and Batra (2004)	4.6582	5.7769	5.4806	5.9609	6.1076	5.4923	5.5285	5.5632
	Matsunaga (2008)	4.6582	5.7769	5.7123	6.1932	6.3390	5.6599	5.6757	5.7020
	Akavci (2014)	4.6569	5.7754	5.7110	6.1924	6.3388	5.6593	5.6718	5.6941
	Present	4.6445	5.7731	5.7039	6.1901	6.3381	5.6522	5.6647	5.6866
2	Vel and Batra (2004)	8.7132	27.5540	14.5580	29.1230	58.2500	14.2780	14.1500	14.0260
	Matsunaga (2008)	8.7132	27.5540	15.3390	30.6850	61.3740	14.9700	14.7420	14.4760
	Akavci (2014)	8.7132	27.5536	15.3408	30.6861	61.3744	14.9718	14.7436	14.4772
	Present	8.7132	27.5536	15.3438	30.6876	61.3751	14.9776	14.7502	14.4830
3	Vel and Batra (2004)	14.4630	46.5030	24.3810	49.0130	98.1450	23.9090	23.6960	23.4940
	Matsunaga (2008)	14.4630	46.5030	25.7760	51.7950	103.7100	25.1400	24.7410	24.2780
	Akavci (2014)	14.7280	46.5741	25.9255	51.8664	103.7404	25.2966	24.9091	24.4606
	Present	14.7280	46.5741	25.9253	51.8662	103.7404	25.2962	24.9087	24.4601
4	Vel and Batra (2004)	24.8300	201.3400	57.6200	212.2200	828.7800	54.6850	53.1790	52.0680
	Matsunaga (2008)	24.8300	201.3400	61.5090	227.2900	888.6000	57.5760	55.2370	53.2880
	Akavci (2014)	25.4268	203.9805	62.8857	231.5235	904.2521	58.9929	56.3726	54.0672
	Present	25.3381	202.9566	62.6150	230.3785	899.5930	58.7874	56.2176	53.9320

(a) $\bar{\omega} = \omega a^2 / h\sqrt{\rho_c / E_c}$

Table 6 The first eight non-dimensional natural frequencies $\beta = \omega h \sqrt{\rho_c / E_c}$ for simply supported square Al/Al_2O_3 plate ($h/b = 0.1$)

p	Theory	Mode no. 2	4	6	8
	Mode	(1, 1, 1)	(1, 2, 1)	(2, 2, 1)	(1, 3, 1)
0	Matsunaga (2008)	0.0577	0.1381	0.2121	0.2587
	Akavci (2014)	0.0578	0.1380	0.2120	0.2585
	Present	0.0573	0.1378	0.2117	0.2581
0.5	Matsunaga (2008)	0.0491	0.1180	0.1819	0.2222
	Akavci (2014)	0.0491	0.1176	0.1813	0.2214
	Present	0.0490	0.1175	0.1811	0.2211
1	Matsunaga (2008)	0.0442	0.1063	0.1640	0.2004
	Akavci (2014)	0.0442	0.1061	0.1636	0.1999
	Present	0.0442	0.1060	0.1634	0.1996
4	Matsunaga (2008)	0.0381	0.0904	0.1383	0.1681
	Akavci (2014)	0.0381	0.0903	0.1379	0.1677
	Present	0.0381	0.0902	0.1378	0.1674
10	Matsunaga (2008)	0.0364	0.0858	0.1306	0.1583
	Akavci (2014)	0.0364	0.0858	0.1305	0.1582
	Present	0.0364	0.0857	0.1303	0.1579
∞	Matsunaga (2008)	0.0283	0.0701	0.1077	0.1313
	Akavci (2014)	0.0294	0.0702	0.1079	0.1316
	Present	0.0294	0.0702	0.1077	0.1313

frequencies of plate are calculated for three different volume fraction exponent and compared with 3D exact solution of Vel and Batra (2004), quasi three-dimensional sinusoidal and hyperbolic shear deformation theories of Neves *et al*. (2012a, b) and two dimensional higher order shear deformation theories of Matsunaga (2008), Hosseini-Hashemi *et al*. (2011) and Akavci (2014). It can be confirmed from the table that the results of the present model agree with the results of other two and three dimensional deformation theories. In Table 5, dimensionless natural frequencies, computed using the present model, are compared with the 3D theory of Vel and Batra (2004) and 2D HSDTs of Matsunaga (2008) and Akavci (2014). It can be observed from this example that a good agreement is achieved between the results of present method and those of other theory.

To check the higher order modes, the first eight and four frequencies of the Al/Al_2O_3 FG square and rectangular plates are calculated and illustrated in Tables 6 and 7. Table 6 shows a comparison between the first eight dimensionless natural frequencies of FG square plates computed by the present theory and those given by Matsunaga (2008) and Akavci (2014) using another 2D higher order deformation theories. A good agreement between the results is confirmed from this comparison. In Table 7, the first four dimensionless natural frequencies of FG rectangular plates

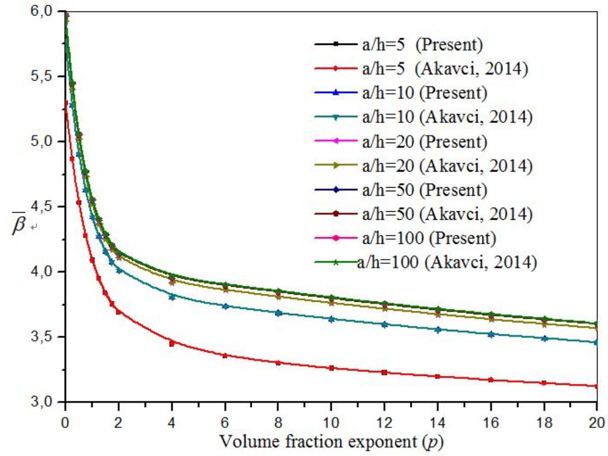

Fig. 2 Variation of dimensionless fundamental frequency $\bar{\beta} = \omega a^2 / h \sqrt{\rho_c / E_c}$ of Al/Al_2O_3 FG square plates with volume fraction exponent

for different thickness ratios are presented as compared with 2D higher order shear deformation theory of Akavci (2014) and exact closed-form Mindlin theory of Hosseini-Hashemi *et al*. (2010). It can be concluded from this example that there is an excellent agreement with the results of present method and those of the HSDT of Akavci (2014).

In Fig. 2, the variations of dimensionless natural frequencies of simply supported Al/Al_2O_3 FG square plates with respect to volume fraction exponent are plotted and the results are also compared to those obtained using the HSDT proposed by Akavci (2014). It is seen from the figure that, increasing value of volume fraction exponent leads to the decrease in the natural frequency.

5.3 Functionally graded plates on elastic foundation

In this section, to check the accuracy of present method for FG plates on two-layer elastic foundations, natural frequencies of the plates are compared with those found in literature.

Table 8 shows dimensionless fundamental frequencies of Al/ZrO_2 FG rectangular plates resting on two-layer elastic foundation. The results of proposed model are compared with the results of the first order shear deformation theory of Hosseini-Hashemi *et al*. (2010) and HSDT of Hasani Baferani *et al*. (2011) and Akavci (2014). It can be observed from the Table 8 that, the results of proposed model are in excellent agreement with the results of other theories.

The results presented in Tables 9 and 10 are performed for the Al/Al_2O_3 FG rectangular plates on elastic foundation. In Table 9, dimensionless fundamental frequencies for different aspect ratios are calculated and compared with other published solutions by employing HSDTs (Hasani Baferani *et al*. 2011, Akavci 2014) and the FSDT (Hosseini-Hashemi *et al*. 2010). It can be observed from the results that, the non-polynomial four variable refined plate model agrees well with the other shear deformation theories. Table 10 demonstrates the comparison of dimensionless fundamental frequencies of FG rectangular

Table 7 Comparison of first four non-dimensional natural frequencies $\beta = \omega a^2 / h\sqrt{\rho_c / E_c}$ of Al/Al_2O_3 FG graded rectangular plate (b/a = 2)

a / h	Mode no. (m, n)	Theory	p						
			0	0.5	1	2	5	8	10
5	1 (1,1)	Ref [(a)]	3.4409	2.9322	2.6473	2.4017	2.2528	2.1985	2.1677
		Ref [(b)]	3.4495	2.9408	2.6529	2.3989	2.2275	2.1724	2.1455
		Present	3.4463	2.9385	2.6509	2.3970	2.2260	2.1703	2.1431
	2 (1,2)	Ref [(a)]	5.2802	4.5122	4.0773	3.6953	3.4492	3.3587	3.3094
		Ref [(b)]	5.3002	4.5321	4.0906	3.6900	3.3952	3.3031	3.2626
		Present	5.2932	4.5269	4.0859	3.6859	3.3919	3.2985	3.2571
	3 (1,3)	Ref [(a)]	8.0710	6.9231	6.2636	5.6695	5.2579	5.1045	5.0253
		Ref [(b)]	8.1179	6.9690	6.2950	5.6613	5.1479	4.9921	4.9313
		Present	8.1021	6.9572	6.2845	5.6519	5.1404	4.9820	4.9193
	4 (2,1)	Ref [(a)]	9.7416	8.6926	7.8711	7.1189	6.5749	5.9062	5.7518
		Ref [(b)]	10.1828	8.7640	7.9209	7.1105	6.4181	6.2111	6.1355
		Present	10.1587	8.7459	7.9047	7.0958	6.4064	6.1957	6.1175
10	1 (1,1)	Ref [(a)]	3.6518	3.0983	2.7937	2.5386	2.3998	2.3504	2.3197
		Ref [(b)]	3.6542	3.1008	2.7952	2.5376	2.3915	2.3418	2.3124
		Present	3.6533	3.1001	2.7946	2.5370	2.3911	2.3412	2.3117
	2 (1,2)	Ref [(a)]	5.7693	4.8997	4.4192	4.0142	3.7881	3.7072	3.6580
		Ref [(b)]	5.7754	4.9059	4.4231	4.0118	3.7682	3.6864	3.6403
		Present	5.7731	4.9042	4.4216	4.0105	3.7671	3.6849	3.6385
	3 (1,3)	Ref [(a)]	9.1876	7.8145	7.0512	6.4015	6.0247	5.8887	5.7575
		Ref [(b)]	9.2029	7.8300	7.0612	6.3959	5.9766	5.8388	5.7662
		Present	9.1973	7.8258	7.0575	6.3925	5.9740	5.8351	5.7617
	4 (2,1)	Ref [(a)]	11.8310	10.0740	9.0928	8.2515	7.7505	7.5688	7.4639
		Ref [(b)]	11.8560	10.0992	9.1093	8.2428	7.6738	7.4892	7.3965
		Present	11.8467	10.0924	9.1033	8.2374	7.6695	7.4832	7.3892
20	1 (1,1)	Ref [(a)]	3.7123	3.1456	2.8352	2.5777	2.4425	2.3948	2.3642
		Ref [(b)]	3.7130	3.1462	2.8356	2.5774	2.4402	2.3924	2.3623
		Present	3.7127	3.1461	2.8355	2.5773	2.4401	2.3923	2.3621
	2 (1,2)	Ref [(a)]	5.9198	5.0175	4.5228	4.1115	3.8939	3.8170	3.7681
		Ref [(b)]	5.9215	5.0191	4.5238	4.1108	3.8883	3.8112	3.7632
		Present	5.9208	5.0189	4.5234	4.1104	3.8880	3.8108	3.7626
	3 (1,3)	Ref [(a)]	9.5668	8.1121	7.3132	6.6471	6.2903	6.1639	6.0843
		Ref [(b)]	9.5711	8.1164	7.3159	6.6453	6.2759	6.1488	6.0715
		Present	9.5695	8.1153	7.3149	6.6443	6.2752	6.1477	6.0702
	4 (2,1)	Ref [(a)]	12.4560	10.5660	9.5261	8.6572	8.1875	8.0207	7.9166
		Ref [(b)]	12.4633	10.5729	9.5307	8.6542	8.1634	7.9954	7.8950
		Present	12.4606	10.5710	9.5289	8.6527	8.1621	7.9936	7.8928

[(a)] Hosseini-Hashemi *et al.* (2010)
[(b)] Akavci (2014)

plates on elastic foundation with those given by Akavci (2014) via HSDT. It can be seen confirmed from the Table 10 that, the results of present model are in excellent agreement with the results of Akavci (2014).

Fig. 3 shows the variations of dimensionless fundamental frequencies of simply supported Al/Al_2O_3 FG square plates resting on elastic foundation with respect to volume fraction exponent. In the presence of elastic foundation, also, with the increase of volume fraction exponent, the fundamental frequencies decrease. It can be seen from the figure that, increasing value of Winkler and Pasternak coefficients causes the increase of the fundamental frequency. The figure demonstrates also that Pasternak modulus parameter of foundation has more significant influence than Winkler modulus parameter on the fundamental frequency of plate.

Table 8 Comparison of non-dimensional fundamental frequencies $\beta = \omega h \sqrt{\rho_c / E_c}$ of Al/ZrO_2 FG rectangular plates (a/b = 1.5)

(k_w, k_s)	h/a	p	Theory: Baferani *et al.* (2011)	Hosseini-Hashemi *et al.* (2010)	Akavci (2014)	Present
(0, 0)	0.05	0	–	0.02392	0.02393	0.02392
		0.25	–	0.02269	0.02309	0.02308
		1	–	0.02156	0.02202	0.02201
		5	–	0.02180	0.02244	0.02243
		∞	–	0.02046	0.02056	0.02055
	0.1	0	–	0.09188	0.09203	0.09197
		0.25	–	0.08603	0.08895	0.08889
		1	–	0.08155	0.08489	0.08484
		5	–	0.08171	0.08576	0.08570
		∞	–	0.07895	0.07908	0.07902
	0.2	0	–	0.32284	0.32472	0.32408
		0.25	–	0.31003	0.31531	0.31473
		1	–	0.29399	0.30152	0.30097
		5	–	0.29099	0.29762	0.29704
		∞	–	0.27788	0.27902	0.27842
(250, 25)	0.05	0	0.03421	0.03421	0.03422	0.03421
		0.25	0.03321	0.03285	0.03312	0.03312
		1	0.03249	0.03184	0.03214	0.03213
		5	0.03314	0.03235	0.03277	0.03276
		∞	–	0.02937	0.02940	0.02940
	0.1	0	0.13365	0.13365	0.13375	0.13371
		0.25	0.13004	0.12771	0.12959	0.12955
		1	0.12749	0.12381	0.12585	0.12581
		5	0.12950	0.12533	0.12778	0.12775
		∞	–	0.11484	0.11492	0.11490
	0.2	0	0.43246	0.49945	0.50044	0.50061
		0.25	0.42868	0.48327	0.48594	0.48581
		1	0.46406	0.46997	0.47298	0.47208
		5	0.44824	0.47400	0.47637	0.47561
		∞	–	0.43001	0.43001	0.42989

Table 9 Comparison of non-dimensional fundamental frequencies $\beta = \omega h \sqrt{\rho_c / E_c}$ of Al/Al_2O_3 FG rectangular plates (h/a = 0.15)

(k_w, k_s)	h/a	p	Theory: Baferani *et al.* (2011)	Hosseini-Hashemi *et al.* (2010)	Akavci (2014)	Present
(0, 0)	0.5	0	–	0.08006	0.08018	0.08014
		0.25	–	0.07320	0.07335	0.07331
		1	–	0.06335	0.06148	0.06145
		5	–	0.05379	0.05215	0.05213
		∞	–	0.04100	0.04081	0.04078
	1	0	–	0.12480	0.12508	0.12497
		0.25	–	0.11354	0.11457	0.11449
		1	–	0.09644	0.09613	0.09606
		5	–	0.08027	0.08089	0.08084
		∞	–	0.06335	0.06366	0.06360
	2	0	–	0.28513	0.28660	0.28610
		0.25	–	0.25555	0.26356	0.26314
		1	–	0.20592	0.22190	0.22157
		5	–	0.16315	0.18232	0.18208
		∞	–	0.14591	0.14587	0.14557
(1000, 10)	0.5	0	0.12869	0.12870	0.12876	0.12874
		0.25	0.11885	0.11842	0.11847	0.11845
		1	0.10498	0.10519	0.10388	0.10386
		5	0.09227	0.09223	0.09098	0.09096
		∞	–	0.06591	0.06554	0.06554
	1	0	0.17020	0.17020	0.17039	0.17032
		0.25	0.15734	0.15599	0.15665	0.15659
		1	0.13854	0.13652	0.13592	0.13587
		5	0.12077	0.11786	0.11774	0.11772
		∞	–	0.08663	0.08673	0.08671
	2	0	0.31449	0.32768	0.32890	0.32848
		0.25	0.30484	0.29612	0.30270	0.30235
		1	0.26966	0.24674	0.25901	0.25874
		5	0.22932	0.20359	0.21785	0.21766
		∞	–	0.16773	0.16740	0.16718

In Figs. 4(a) and (b), the variations of dimensionless fundamental frequency of simply supported Al/Al_2O_3 FG square plate resting on Winkler and Pasternak foundations with respect to Winkler modulus parameter of foundation are illustrated. It is seen from the results that, increasing the volume fraction exponent decreases the natural frequency for both of the foundations. It is also observed from the figure that, increasing value of volume fraction exponent increases the influence of elastic foundation on natural frequency. The figure demonstrates also, although increasing value of volume fraction exponent causes to decrease in the fundamental frequency, if the value of volume fraction exponent is more than 5, the effect of it on the fundamental frequency is negligible.

The results reported in Tables 2 to 10 and Figs. 2 to 4 demonstrate that the same accuracy is achievable with the present model by employing a lower number of variables than other models, and clearly highlights how the present model is simpler and more easily used in predicting the free vibration response of FG plates.

6. Conclusions

In the present work, analytical solutions for free vibration investigations of FG plates are developed by making further simplifying assumptions to the existing HSDT, with the inclusion of an undetermined integral term.

Table 10 Comparison of non-dimensional fundamental frequencies $\tilde{\omega} = \omega a^2 / h\sqrt{\rho_m / E_m}$ of Al/Al_2O_3 FG rectangular plates

(k_w, k_s) [(a)]	a/b	a/h	p = 0 Akavci (2014)	0 Present	1 Akavci (2014)	1 Present	5 Akavci (2014)	5 Present	10 Akavci (2014)	10 Present
(0, 0)	0.5	5	6.7771	6.7711	5.2122	5.2082	4.3763	4.3735	4.2153	4.2105
		10	7.1794	7.1775	5.4918	5.4906	4.6986	4.6978	4.5432	4.5417
		20	7.2948	7.2943	5.5712	5.5708	4.7943	4.7941	4.6411	4.6407
	1	5	10.4133	10.3995	8.0368	8.0276	6.6705	6.6640	6.4099	6.3992
		10	11.3468	11.3424	8.6900	8.6871	7.4033	7.4012	7.1521	7.1485
		20	11.6338	11.6326	8.8879	8.8871	7.6393	7.6389	7.3934	7.3924
	2	5	22.8734	22.8126	17.8289	17.7879	14.3625	14.3325	13.7120	13.6669
		10	27.1085	27.0842	20.8487	20.8328	17.5051	17.4938	16.8613	16.8422
		20	28.7174	28.7102	21.9670	21.9623	18.7946	18.7912	18.1727	18.1668
(0, 100)	0.5	5	11.1237	11.1204	10.8489	10.8474	10.9925	10.9917	11.0818	11.0806
		10	11.4503	11.4492	11.0940	11.0934	11.2538	11.2535	11.3313	11.3307
		20	11.5474	11.5471	11.1660	11.1658	11.3343	11.3342	11.4093	11.4091
	1	5	15.2095	15.2010	14.3923	14.3884	14.3071	14.3050	14.3829	14.3798
		10	15.9813	15.9782	14.9443	14.9427	14.8693	14.8683	14.9193	14.9177
		20	16.2285	16.2277	15.1189	15.1184	15.0607	15.0604	15.1056	15.1051
	2	5	28.6623	28.6184	25.6912	25.6688	24.3625	24.3498	24.3109	24.2936
		10	32.3444	32.3246	28.2316	28.2206	26.7223	26.7155	26.5586	26.5475
		20	33.8076	33.8015	29.2272	29.2237	27.7770	27.7748	27.5919	27.5881
(100, 0)	0.5	5	7.2276	7.2219	5.8746	5.8711	5.2360	5.2341	5.1288	5.1255
		10	7.6153	7.6136	6.1393	6.1383	5.5276	5.5269	5.4199	5.4187
		20	7.7272	7.7267	6.2152	6.2149	5.6156	5.6154	5.5087	5.5083
	1	5	10.7082	10.6948	8.4748	8.4665	7.2560	7.2502	7.0373	7.0272
		10	11.6261	11.6218	9.1107	9.1079	7.9520	7.9501	7.7356	7.7323
		20	11.9093	11.9081	9.3044	9.3037	8.1789	8.1783	7.9658	7.9501
	2	5	23.0053	22.9454	18.0231	17.9827	14.6363	14.6069	14.0098	13.9658
		10	27.2246	27.2005	21.0241	21.0083	17.7396	17.7285	17.1126	17.0938
		20	28.8295	28.8223	22.1378	22.1332	19.0187	19.0155	18.4115	18.4057
(100, 100)	0.5	5	11.4036	11.4004	11.1817	11.1801	11.3598	11.3593	11.4581	11.4584
		10	11.7285	11.7274	11.4284	11.4278	11.6243	11.6240	11.7103	11.7098
		20	11.8253	11.8250	11.5008	11.5007	11.7054	11.7052	11.7888	11.7887
	1	5	15.4127	15.4043	14.6407	14.6368	14.5862	14.5835	14.6702	14.6658
		10	16.1808	16.1778	15.1927	15.1912	15.1498	15.1489	15.2075	15.2060
		20	16.4271	16.4263	15.3674	15.3670	15.3414	15.3411	15.3938	15.3933
	2	5	28.7674	28.7273	25.8251	25.8033	24.5206	24.5080	24.4759	24.4596
		10	32.4417	32.4220	28.3613	28.3504	26.8763	26.8695	26.7186	26.7077
		20	33.9029	33.8968	29.3557	29.3523	27.9292	27.9270	27.7497	27.7460

[(a)] $k_w = K_w a^4 / D_m$, $k_s = K_s a^2 / D_m$ where $D_m = E_m h^3 / 12(1-\nu^2)$

The number of unknowns and equations of motion of the present HSDT are reduced by one, and hence, make this theory simple and efficient to use. Verification studies demonstrate that the predictions by the present HSDT and existing HSDT (Akavci 2014) for FG plates are close to each other. The results obtained by the present theory can be summarized as follows:

- It has been observed that the present analytical model can accurately predict fundamental frequencies of FG plates resting on two-layer elastic foundations.
- The fundamental frequencies of FG plate decrease with the increase of volume fraction exponent.
- In the presence of elastic foundation, increasing

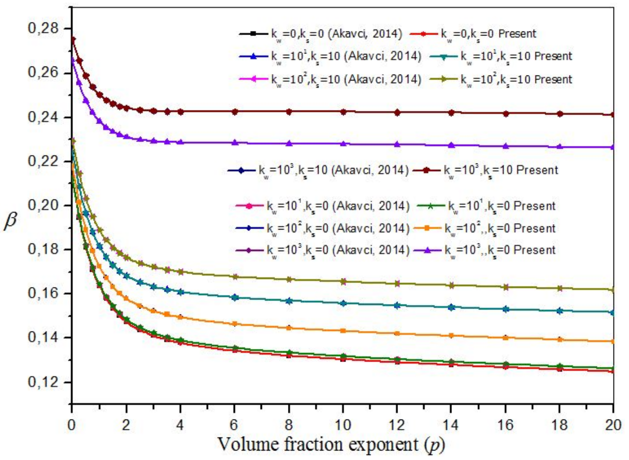

Fig. 3 Variation of non-dimensional fundamental frequency $\beta = \omega h\sqrt{\rho_c / E_c}$ of Al/Al_2O_3 FG square plates resting on elastic foundation with volume fraction exponent ($a / h = 5$)

value of Winkler and Pasternak coefficients causes to increase in the fundamental frequency of FG plate.

- The Pasternak modulus parameter of foundation has more significant influence on increasing natural frequency of FG plate than the Winkler modulus parameter.
- Increasing value of volume fraction exponent increases the influence of elastic foundation on natural frequency.

Finally it can be concluded that, the present model can improve the numerical computational cost due to their reduced degrees of freedom.

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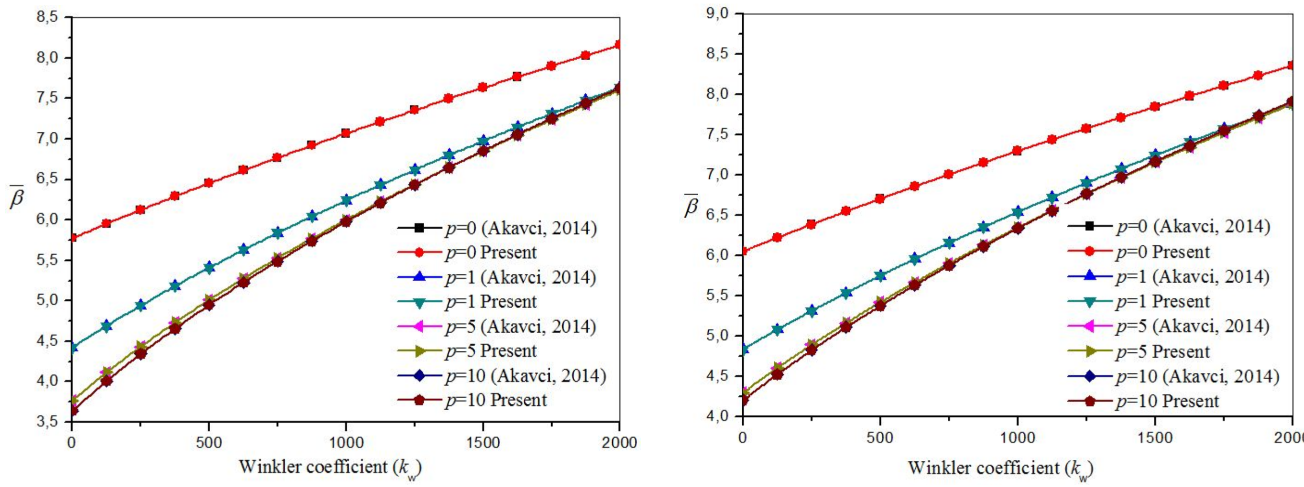

(a) $k_s = 0$ (b) $k_s = 10$

Fig 4 Variation of non-dimensional fundamental frequency $\overline{\beta} = \omega a^2 / h\sqrt{\rho_c / E_c}$ of Al/Al_2O_3 FG square plates resting on elastic foundation with Winkler coefficient ($a / h = 5$)

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