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Evaluation of limit load analysis for pressure vessels – Part II: Robust methods

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Abstract. Determining limit load for a pressure bearing structure using elastic-plastic finite element analysis was computationally very expensive. A series of robust methods using elastic modulus adjustment techniques (EMAP) to identify the limit load directly were proposed. The numerical implementation of the robust method had the potential to be an attractive alternative to elastic-plastic finite element analysis since it was simple, and required less computational effort and computer storage space. Another attractive feature was that the method provided a go/no go criterion for the limit load, whereas the results of an elastic-plastic analysis were often difficult to interpret near the limit load since it came from human sources. To explore the performance of the method further, it was applied to a number of configurations that include two-dimensional and three-dimensional effects. In this study, limit load of cylinder with nozzle was determined by the robust methods.

Keywords: pressure vessel; limit load; robust method; ANSYS

1. Introduction

It was the objective of structural analysis to determine the load-carrying capacity. Pressurized components should be designed against excessive plastic deformation (or plastic collapse) under monotonic loading and against incremental collapse under cyclic loading. To design against excessive plastic deformation or plastic collapse under monotonic loadings, information on plastic limit loads would be useful. For this reason, extensive work had been reported in the literature on plastic limit loads for complex structures (such as elbows, branch junctions, nozzles and so on).

Li *et al*. (2008) reported the plastic limit load of cylindrical vessels with different lateral angles under increasing internal loadings by means of experimental testing. Moreover, a three-dimensional, nonlinear, finite element numerical simulation was also performed. The limit load of cylindrical vessels with different lateral angles was obtained using twice-elastic-slope criterion. It was found that the limit loads determined by experiment and numerical simulation methods were in good agreement. Patel and Kumat (2014) investigated limit load of pressure vessel with different inlet and outlet openings by means of experiment methods such as twice elastic slope method, tangent intersection method and nonlinear finite element method. Tangent intersection method which was used to estimate the lower value of limit pressure was more effective for higher elastic slope of limit pressure vs strain. Prakash *et al*. (2016) studied plastic limit load of cylindrical pressure vessels with combined inclination of nozzles (i.e., in longitudinal and radial plane). The plastic limit load was obtained with twice elastic slope method. The approximate closed-form plastic limit load solutions for branch junctions under out-of-plane bending and under combined pressure and out-of-plane bending were presented by Lee *et al*. (2012), based on three-dimensional finite element limit analyses for an elastic-perfectly plastic material. Likewise, the researcher observed that plastic limit load of cylinder with nozzle was determined by elastic - plastic finite element analysis. However, this method required considerable amount of computational effort and computer storage space, and came from human sources. Therefore, robust methods, such as elastic compensation method (ECM) (Mackenzie and Boyle 1993), Linear matching method (LMM) (Ponter and Carter 1997a, b), and so on, were applied for determining limit load, shakedown and ratcheting boundary of components or structures since limit analysis can be considered as a particular case of shakedown.

The elastic iterative methods for limit load and shakedown analysis determination had been reviewed in detail by Machenzie *et al*. (2000). Many scholars (Aman Ahmed Bolar 2001, Engelhardt 1999, Freeman 2000, Hossain 2009, Adibi-Asl 2008, Habibullah 2003, Fanous 2008, Xiao 2010) studied extensively one R-Node family method that was used to determine limit load or shakedown of some structures or components with and without defect. The results indicated the scientific, quick, accuracy and reliability of EMAP.

Plancq and Berton (1998) studied limit load of branch

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pipe tee connecting under different loading conditions. Hardy *et al*. (2001) studied limit and shakedown analysis of internal and external flanges. Yang *et al*. (2005) estimated limit load of nozzle-cylinder junctions. Based upon an upper bound approach using the re-parameterized exact Ilyushin yield surface and a nonlinear optimization procedure, Trân *et al*. (2008) determined limit load of 90° elbow pipe and vessel head. Abdalla *et al*. (2011a, b) determined the lower bound shakedown limit load of 90° elbow pipe with a simplified technique. Chen *et al*. (2011, 2012), Ure *et al*. (2013) and Li *et al*. (2011, 2012) estimated limit and shakedown analysis of defective pipe using linear matching method. Moreover, Chen *et al*. (Chen *et al*. 2015, Chen and Chen 2016) studied ratcheting strain and shakedown of pressurized straight pipe and 90° elbow pipe subjected to reversed bending by means of finite element analysis.

Robust methods were simple and required less computational effort and computer storage space. In this study, limit load of branch pipe tee was determined with the methods mentioned above.

2. Limit load determination methods

2.1 Lower and upper bound theorem

2.1.1 Classical lower bound theorem

The statement of the classical lower bound theorem (Calladine 2000) was as follows: "*If any stress distribution throughout the structure can be found, which was everywhere in equilibrium internally and balanced the external loads and at the same time did not violate the yield condition, those loads will be carried safely by the structure*".

$$\sigma_{ij} n_j = m_L T_i \tag{1}$$

$$f(\sigma) = \bar{\sigma}(\sigma) - \sigma_y \le 0 \tag{2}$$

Based on the lower bound limit theorem of limit analysis, the lower bound limit load multiplier m_L and limit load value P_L were expressed, respectively as follows

$$m_L = \frac{\sigma_y}{\sigma_{\max}} \tag{3}$$

$$P_L = P m_L \tag{4}$$

2.1.2 Classical upper bound theorem

The classical upper bound theorem (Calladine 2000) stated that "*If an estimate of the plastic collapse load of a body was made by equating the internal rate of dissipation of energy to the rate at which external forces do work in any postulated mechanism of deformation of the body, the estimate would be either high, or correct*".

$$m_U \int_{S_T} \bar{p}_i \dot{u}_i \, dS \le \int_V \sigma \dot{\varepsilon} \, dV \tag{5}$$

Therefore, the classical upper bound limit load multiplier was found as

$$m_U = \frac{\int_V \sigma \dot{\varepsilon} \, dV}{\int_{S_T} \bar{p}_i \dot{u}_i \, dS} \tag{6}$$

where, σ in the upper bound was the stress yield associated with the compatible strain rate $\dot{\varepsilon}$, $\bar{p}_i$ was the shape description of a unit load. $\dot{u}_i$ was the displacement rate field.

$$s_{ij} = \sqrt{\frac{2}{3}} \sigma_y \frac{\dot{\varepsilon}_{ij}}{\sqrt{\dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}}} \tag{7}$$

Then, substitution of Eq. (7) into Eq. (6) provided as follows

$$m_U = \frac{\int_V \sigma_y \dot{\bar{\varepsilon}} \, dV}{\int_S \bar{p}_i \dot{u} \, dS} \tag{8}$$

where, $\sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_y \dot{\bar{\varepsilon}}$ and $\dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3} \dot{\varepsilon}_{ij} \dot{\varepsilon}_{ij}}$.

In terms of finite element discretization scheme, Eq. (8) can be written as follows.

$$m_U = \frac{\int_V \sigma_y \varepsilon_{eq} dV}{\int_V \sigma_{ij} \varepsilon_{ij} dV} \Leftrightarrow \frac{\sigma_y \sum_{k=1}^{N} \left(\varepsilon_{eq} \Delta V \right)_k}{\sum_{k=1}^{N} \left(\sigma_{eq} \varepsilon_{eq} \Delta V \right)_k} \tag{9}$$

where, σ_{eq} and ε_{eq} were equivalent stress and strain, respectively.

The upper bound limit load value P_U were expressed as

$$P_U = P m_U \tag{10}$$

2.1.3 New lower and upper bound theorem

In limit analysis, the statically admissible stress field (equilibrium set) cannot lie outside the hypersurface of the yield criterion, and the stress field obtained from a kinematically admissible strain rate field (compatibility set) in calculating the plastic dissipation should be on the hypersurface. In order to eliminate such requirement, the concept of integral mean of yield based on a variational formulation was proposed by Mura and Lee (1965) who extended the classical lower and upper bound theorem.

2.1.3.1 New lower bound theorem

Mura and Lee (1965) proposed the new lower bound multiplier m', which was derived from Mura's extended variational principle and was expressed as follows

$$m' = \frac{2m^0 \sigma_y^2}{\sigma_y^2 + \left(m^0\right)^2 \left(\sigma_M^0\right)^2} \quad \text{or} \quad m' = \frac{2m^0}{1 + \left(\frac{m^0}{m_L}\right)^2} \tag{11}$$

where, m^0 was the following new upper bound multiplier, σ_y was yield stress, σ_M^0 was the maximum equivalent stress in a structure or component.

2.1.3.2 New upper bound theorem

Using the integral mean of yield criterion, Mura and Lee (1965) proposed the new upper bound multiplier m^0 which was expressed as follows

$$m^0 = \frac{\sigma_y \sqrt{V}}{\sqrt{\int_V (\sigma_{eq})^2 \, dV}} \tag{12}$$

$$m^{"} = \frac{m^0}{1+G} \tag{13}$$

where

$$G = \sqrt{\frac{\int_V \left[\left(m^0 \frac{\sigma_{eq}}{\sigma_y} \right)^2 - 1 \right]^2 dV}{4V}} \tag{14}$$

Mangalaramanan and Seshadri (1997) proposed the upper bound multiplier m_1^0 from Mura's formulation

$$m_1^0 = \frac{\sigma_y \sqrt{V}}{\sqrt{\int_V (\sigma_{eq})^2 \, dV}} \tag{15}$$

The upper bound multiplier m_1^0, was based on the total volume V. If plastic collapse occurred over a localized region of the structures, it would be significantly overestimated. Therefore, Pan and Seshadri (2001) proposed a new upper bound multiplier m_2^0 from Mura's formulation in order to address the above problem.

$$m_2^0 = \sigma_y \frac{\sqrt{\sum_{k=1}^{N} \varepsilon_{ek} V_k / \sigma_{sk}}}{\sqrt{\sum_{k=1}^{N} \varepsilon_{ek} \sigma_{ek} V_k}} \quad \text{or} \quad m_2^0 = \sigma_y \frac{\sqrt{\int_V \frac{\varepsilon_{eq}}{\sigma_{eq}} dV}}{\sqrt{\int_V \sigma_{eq} \varepsilon_{eq} dV}} \tag{16}$$

2.1.3.3 Reference volume approach

If plastic collapse occurred over a localized region of a component or structure, some regions may remain rigid or elastic, namely dead region V_D. The volumes of the remaining plastic regions were called reference volume V_R which carried the external loads at the limit state (Mangalaramanan and Seshadri 2001). The upper bound multiplier m^0, m_1^0 and m_2^0 would be largely overestimated if it was based on the total volume V_T. The multiplier m' would be underestimated. Therefore, the multiplier can be written as follows

$$m_1^0 = \frac{\sigma_y \sqrt{V_R}}{\sqrt{\int_{V_R} (\sigma_{eq})^2 \, dV}} \tag{17}$$

Moreover, the reference volume V_R ($V_R \le V$) was introduced by Reinhardt and Seshadri (2003) in order to identify the kinematically active portion of component or structure that participated in plastic action. Hence a new upper bound multiplier can be written as follows

$$m_1^0(V_R) = \frac{\sigma_y \sqrt{\sum_{k=1}^{\alpha} (\Delta V_k)}}{\sqrt{\sum_{k=1}^{\alpha} (\varepsilon_{ek}^0)^2 \Delta V_k}} \qquad (\alpha < N) \tag{18}$$

2.1.3.4 The m_α-method

Seshadri and Mangalaramanan (1997) developed an improved lower bound multiplier m_α-method based on Mura's variational theorem, which provided better lower bound limit load over Mura's lower bound estimate. The m_α -method adopted the elastic modulus adjustment procedure (EMAP) to estimate improved lower bound limit load. Further discussion on these methods was presented as follows

$$m_\alpha = 2m^0 \frac{2\left(\frac{m^0}{m_L}\right) + \sqrt{\frac{m^0}{m_L}\left(\frac{m^0}{m_L} - 1\right)^2 \left(1 + \sqrt{2} - \frac{m^0}{m_L}\right)\left(\frac{m^0}{m_L} - 1 + \sqrt{2}\right)}}{\left(\left(\frac{m^0}{m_L}\right)^2 + 2 - \sqrt{5}\right)\left(\left(\frac{m^0}{m_L}\right)^2 + 2 + \sqrt{5}\right)} \tag{19}$$

$$R_\alpha = 2R^0 \left[\frac{2\zeta^2 + \sqrt{\zeta(\zeta-1)^2 (1+\sqrt{2}-\zeta)(\zeta - 1 + \sqrt{2})}}{(\zeta^2 + 2 - \sqrt{5})(\zeta^2 + 2 + \sqrt{5})} \right] \tag{20}$$

where, $\zeta = m^0 / m_L$, $r_0 = m^0 / m$ and $R_a = m_\alpha / m$. Due to normalization, $R_a = 1$represented the boundary between the upper bound region $R_a > 1$ and lower bound region $R_a < 1$. The value of m_α became imaginary when $m^0 / m_L > 1 + \sqrt{2}$, as would be the case for components with notches and cracks.

2.1.3.5 The m_α^T-method

Once the $R_\alpha^T = 1$ line was identified, the multiplier m_α^T value (Seshadri and Hossain 2009) can be readily estimated by the relationship.

$$m_\alpha^T = \frac{m^0}{1 + \left(1 - \frac{1}{\sqrt{2}}\right)(\zeta - 1)} \tag{21}$$

where

$$\zeta = \frac{m^0}{m_L} \tag{22}$$

2.1.3.6 The m*-method

Based on Mura's lower bound multiplier and extended lower bound theorem, Seshadri and Indermohan (2004) introduced a multiplier m^*.

$$m^* = \frac{m_1^0}{1+G} \tag{23}$$

where the parameter G evaluated acted as a convergence parameter, and was indicative of any deviation of statically admissible stress distributions from limit state.

$$G=\sqrt{\frac{\int_V\left[\left(m^0\frac{\sigma_{eq}}{\sigma_y}\right)^2-1\right]^2dV}{4V}} \tag{24}$$

2.1.3.7 The m_β-method

The multiplier m^* need to be a lower bound. The parameter β was introduced in Eq. (25) by Seshadri and Indermohan (2004), the multiplier m_β was expressed as follows

$$m_\beta=\frac{m_1^0}{1+\beta G} \tag{25}$$

where

$$G=\sqrt{\frac{\int_V\left[\left(m^0\frac{\sigma_{eq}}{\sigma_y}\right)^2-1\right]^2dV}{4V}} \tag{26}$$

The multiplier m_β was sensitive to the value of β. When $\beta = \beta_R$, the exact multiplier m_β was achieved. β_R was the reference parameter and was as yet undefined. The reference parameter β_R would be the lowest possible value of β that would generate the multiplier m_β which satisfied the following four requirements. But the multiplier m_β was less than or equal to m.

$$\begin{cases}(1)\ if\ \zeta\geq 0,\ m^0\geq m\geq m_\beta\\ (2)\ if\ \zeta\geq 0,\ \dfrac{dm_\beta}{d\zeta}\geq 0\\ (3)\ if\ \zeta\geq 0,\ \dfrac{dm^0}{d\zeta}\leq 0\\ (4)\ if\ \zeta\to\zeta_L,\ m^0=m_\beta=m\end{cases} \tag{27}$$

where, ζ was iteration variable.

2.1.4 Reference stress method

In order to overcome some of complications of creep analysis, Sim (1968) proposed a useful simplified method, namely reference stress method. Reference stress was a function of stress components that must reach the value of yield stress in simple tension or compression for yielding to occur. The basic principle of reference stress method was that the deformation of structures subjected to multiaxial creep can be related to the results of a uniaxial creep test carried out at the reference stress through a scaling factor. Therefore, the deflection δ at a point in a structure at some time t was given by

$$\delta(t)=\xi\varepsilon_c(t) \tag{28}$$

where, ξ was the geometric scaling factor depending on configuration of structure and boundary conditions, $\varepsilon_c(t)$ was the creep strain at time t as obtained by uniaxial creep test performed at the reference stress σ_{ref}. Creep strain was expressed by the Norton equation.

$$\varepsilon_c=B\sigma^n \tag{29}$$

where, B and n were the material constants, σ was the stress.

Reference stress σ_{ref} at any other load was expressed as following since it was insensitive to exact creep exponent n in the strain rate to stress relationship (Anderson *et al.* 1963). If the creep exponent n approximated infinity, the limit solution to perfect plasticity would be approached, i.e., at limit load. The reference stress would equal to yield stress.

$$\sigma_{ref}=\left(\frac{P}{P_L}\right)\sigma_f \tag{30}$$

where, P was any other load, P_L was limit load, σ_f was the yield stress of the material.

2.1.5 Theorem of nesting surfaces

The reference stress can be expressed in another manner based on energy dissipation. The average dissipation rate at the reference stress equated to the dissipation rate in a structure of component under applied loading.

$$\sigma_R\varepsilon_R V=\int_V\sigma_{ij}\varepsilon_{ij}dV \tag{31}$$

Eq. (31) was written by equivalent stress and strain, and Eq. (29) was substituted in Eq. (31), provided

$$\sigma_R^{n+1}V=\int_V\sigma_{eq}^{n+1}dV \tag{32}$$

Thus, the reference stress can be obtained as follows

$$\sigma_R=\left[\int_V\sigma_{eq}^{n+1}dV\right]^{\frac{1}{n+1}} \tag{33}$$

Calladine and Drucker (1962) proposed the theorem of nesting surfaces for determining the reference stress which was given by the expression

$$\sigma_R=\left[\int_V\sigma_{eq}^{n+1}dV\right]^{\frac{1}{n+1}} \tag{34}$$

Eq. (34) indicated that the reference stress increased monotonically the increasing of the exponent n. It was bounded below by the result of $n = 1$ (elastic) and above by the limiting functional as $n \to \infty$ (perfect plasticity). The minimum and maximum values of the reference stress corresponded to $n = 1$ and $n \to \infty$, respectively. The stress distributions relating to the various values of n can be simulated by performing elastic analysis combining with elastic modulus adjustment procedures. Therefore, limit load corresponding to stress distributions can be identified when $n \to \infty$.

Assuming the external load P applied to be equal to the exact limit load and limit stress distribution, for a structure or component the completely became plastic at collapse, $\sigma_{ref} = \sigma_y$ and otherwise $\sigma_{ref} < \sigma_y$, namely

$$\sigma_{ref}\leq\sigma_y \tag{35}$$

The reference stress formula due to Sim (1970, 1971) given by Eq. (35) can be expressed as follows, i.e., limit load was written as follows

$$P_L = P\frac{\sigma_y}{\sigma_{ref}} \tag{36}$$

2.1.6 Brief summary

Clearly, the limit load multiplier m_1^0 and m_β was significantly overestimated if it was based on the total volume. It was shown in Table 6 that the limit load multiplier m_1^0 and m_2^0 were greater than the classical upper bound m_U and classical lower bound limit load multiplier m_L (Reinhardt and Seshadri 2003). If plastic collapse occurred over a localized region of the component or structure, the m_1^0 multiplier could be overestimated.

The lower bound limit load multiplier m' derived from Mura's extended variational principle was shown to be smaller than that obtained by applying classical lower bound theorem ($m' < m_L$), and less than the unknown actual collapse load multiplier m ($m' < m$). The multiplier m' was significantly underestimated if it was based on the total volume.

The m_α multiplier was an improved estimate of the analytical limit load multiplier compared to the bounds m_L and m^0. Although it was often found to be an improved lower bound, it could not be established as a lower bound in general. Reinhardt and Seshadri (2003) proposed that $1 \le m^0 / m_L \le 1 + \sqrt{2}$ and $1 \le m^0 / m \le 1 + \sqrt{2}$ was designated as the "m_α triangle". The m_α multiplier method was not applicable if a component falls outside the "m_α triangle". Therefore, the m_α^T method was applicable to a general class of mechanical components and structures containing significant amount of peak stresses. The estimates of m_α^T for all the worked out example problems were found to be lower bound to the corresponding analytical or inelastic finite element analysis results. Therefore, the reference volume had been presented to identify the "kinematically active" region of the component or structure that participated in plastic action.

2.2 Pseudo-elastic finite element method

In general, limit load was usually determined by the classical upper bound and lower bound theorems. However, this method which became highly tedious for structures with high degree of indeterminacy was impracticable for complex structures. Therefore, simplified methods which were developed to determine the limit load of structures including gloss R-Node method, elastic compensation method, modified elastic compensation method, m_α method, and so on.

2.2.1 Origin of simplified methods

Elastic modulus adjustment procedures (EMAP) or reduced modulus technique was used to determine the limit load of structures (Dhalla and Jones 1981, 1986, Dhalla 1984, 1987). A systematic adjustment procedure of elastic modulus resulted in inelastic response of the structures. The essence of EMAP was that local clamp stresses could be secondary owing to their redistribution on account of material or geometric non-linearity.

Marriott (1988) proposed an iterative procedure, reduced modulus technique for estimating lower bound limit load on the basis of elastic analysis by generating statically admissible stress fields and using them in conjunction with established theorems of limit analysis. With this method, a series of linear elastic finite element analysis (FEA) were performed. An arbitrary load P in the first FEA that guaranteed yielding in the component was applied on structures. Then, the method identified stress intensities of all the elements that exceeded the code allowable stresses and were selected in each linear elastic FEA. The elastic moduli of the selected elements were modified using the following equation

$$E_R = E_0\frac{S_m}{SI} \tag{37}$$

where, E_0 was the original value of elastic modulus, S_m was the code allowable stress, SI was the stress intensity.

The procedure was repeated until the maximum stress in the component did not change with further iteration or the equivalent stresses of all the elements were lower than the code allowable stress S_m or some other convergence criteria. Finally, limit load was determined using the following expression

$$P_L = P\frac{\sigma_y}{\sigma_{\max}} \tag{38}$$

where, σ_y was the yield strength, $\sigma_{\max}$ was the maximum equivalent stress.

2.2.2 Gloss R-Node method

Seshadri (Seshadri 1991, Seshadri and Fernando 1992) developed the generalized local stress strain (GLOSS) R-Node analysis which was a simple systematic method for inelastic evaluation of components and structures on basis of two linear elastic finite element analyses. The first linear elastic FEA was conducted for structures under consideration with an arbitrary proportional load factor. The elastic modulus of all elements was modified using the following expression

$$E_R = E_0\frac{\sigma_y}{\sigma_{ei}} \tag{39}$$

where, E_0 was the original value of elastic modulus, σ_y was the yield stress, σ_{ei} was the von Mises equivalent stress from the initial elastic analysis of the ith element.

The second linear elastic FEA which was carried out after making the above modification produced a stress distribution. The stresses of location points in the structures remained the same between the two FEA, because the stresses at these locations were insensitive to the material constitutive relations. These locations were called redistribution nodes (R-Nodes). R-Node location was determined by the follow up angle θ which was expressed in the following

$$\theta = \tan^{-1}\left(\frac{\sigma_A - \sigma_B}{\varepsilon_B - \varepsilon_A}\right) \tag{40}$$

When θ = 90°, the effective stresses $\sigma_{nj}(j)$ at R-Node locations were linearly proportional to externally applied loads for elastic-perfectly plastic material model. Limit load was the corresponding external load when R-Node locations reached the yield strength of the structures and mechanical components

$$P_L = P\left[\frac{\sigma_y}{\overline{\sigma}_n}\right] \tag{41}$$

where, $\overline{\sigma}_n$ was the average R-Node stress locations, and can be expressed as follows

$$\overline{\sigma}_n = \frac{\sum_{j=1}^{N}\sigma_{nj}}{N} \tag{42}$$

where, N was the number of R-Node locations in the structures and mechanical components.

2.2.3 Elastic compensation method

Based on the Gloss R-Node method, elastic compensation method (ECM) was proposed by Mackenzie and Boyle (1993) who used a sequence of linear elastic FEA with EMAP to estimate lower or upper bound limit load of structures and mechanical components. Similar to Gloss R-Node method, limit load determined by elastic compensation method was several repeated elastic iterations FEA. The elastic modulus of elements in each elastic iterations FEA was modified according to Eq. (43). An arbitrary load P was imposed on the structures under consideration. The elastic modulus adjustment of all elements in each iteration was carried out as follows

$$E_i = E_{i-1}\frac{\sigma_n}{\sigma_{(i-1)}} \tag{43}$$

where, i was the present iteration number, $\sigma_{(i-1)}$ was the maximum equivalent stress of element from previous iteration, σ_n was a nominal stress value which had a certain arbitrariness, $\sigma_n = [\max(\sigma_i) + \min(\sigma_i)]/2$ or $\sigma_n = \left[\frac{\int_V (\sigma_e^i)^2 dV}{V}\right]^{1/2}$.

Several iterations (5 or 10) were carried out until the maximum equivalent stress in the component at each subsequent iteration equals to or less than the yield strength. Therefore, the lower limit load P_{Li} or upper limit load P_{Ui} in each iteration were given by

$$P_{Li} = P\frac{\sigma_y}{\sigma_{\max,i}} \tag{44}$$

where, σ_y was the yield strength, $\sigma_{\max,j}$ was the maximum von Mises equivalent stress

$$P_{Ui} = P\frac{D_i}{U_i} \tag{45}$$

where, D_i was the energy dissipation, U_i was the strain energy.

The best estimation of lower or upper bound limit load for the structures under consideration was the iteration with minimum value of maximum stress. Thus, the lower bound limit load P_L can be expressed as

$$P_L = \max(P_{Li}) \tag{46}$$

The best estimation of upper bound limit load for the structures under consideration was the iteration with minimum value of maximum stress. Thus, the upper bound limit load P_U can be expressed as

$$P_U = \min(P_{Ui}) \tag{47}$$

2.2.4 Modified elastic compensation method

ECM can obtain better results for overall plastic damage of simple structures and uniform bearing structure. For local plastic damage of complex structures, the results of ECM often contained notable error. Because the essence of ECM was that the elastic modulus of all elements was modified, which resulted in the fact that von Mises equivalent stress of some elements decreased toward zero while the elastic modulus of these elements increased to infinity. This resulted in that excessively strengthened element stiffness, which would cause bigger numerical error for finite element analysis or numerical singularity of the procedure before the better limit load could be obtained. As a result, mistake occurred and the procedure aborted. Therefore, Yang *et al.* (2006) and Liu *et al.* (2009) proposed a modified ECM (MECM). Only when von Mises equivalent stress of the elements was larger than nominal stress did elastic modulus of these elements adjust as expressed in the following equation.

$$E_{i+1} = \begin{cases} E_i\dfrac{\sigma_n}{\sigma_i} & \sigma_i > \sigma_n \\ E_i & \sigma_i \le \sigma_n \end{cases} \tag{48}$$

where, i was the present iteration number, σ_n was a nominal stress value and σ_i was the maximum equivalent stress of element from previous iteration.

The nominal stress σ_n, which should satisfy

$$\min(\sigma_i) \le \sigma_n \le \max(\sigma_i) \tag{49}$$

The adjustment factor $\lambda \in [0, 1]$ was introduced in Eq. (49). Then the nominal stress σ_n was defined as follows

$$\sigma_n = \max(\sigma_i) - \lambda\left[\max(\sigma_i) - \min(\sigma_i)\right] \tag{50}$$

Limit load of MECM was determined by the above several equations.

2.2.5 General formulation of elastic modulus adjustment procedure

Elastic modulus adjustment procedure (EMAP) was adopted to modify the local elastic modulus, which was to generate statically admissible stress distribution in order to obtain the necessary stress redistribution. Numerous sets of statically admissible and kinematically admissible distributions can be generated in this manner, which enable calculation of both lower and upper bounds limit loads.

The elastic modulus of each element in the linear elastic finite element scheme was modified as follows

$$E^{i+1} = \left(\frac{\sigma^i_{ref}}{\sigma^i_{eq}} \right)^q E^i \tag{51}$$

where, q was the elastic modulus adjustment parameter, σ_{ref} was the reference stress, σ_{eq} was the equivalent stress and 'i' was the iteration index (i = 1 for the initial elastic analysis).

The reference stress σ^i_{ref} was given by the expression

$$\sigma^i_{ref} = \left[\frac{\int_{V_T} \sigma^2_{eq} dV}{V_T} \right]^{1/2}_i \tag{52}$$

In order to make use of Eq. (52) for the FEA solution, it can be written as follows

$$\sigma^i_{ref} = \left[\frac{\sum_{k=1}^{N} \left(\sigma^2_{eq} \Delta V \right)_k}{V_T} \right]^{1/2}_i \tag{53}$$

$$q = \frac{\ln\left(\frac{2\left(\sigma^i_{ref}\right)^2}{\left(\sigma^i_{ref}\right)^2 + \left(\sigma^i_{eq}\right)^2} \right)}{\ln\left(\frac{\sigma^i_{ref}}{\sigma^i_{eq}} \right)} \quad \text{or} \quad q = \frac{\ln\left(\frac{2\left(\sigma^i_{ref}\right)^2}{\left(\sigma^i_{ref}\right)^2 + \left(\sigma^i_{eq}\right)^2} \right)}{\ln\left(\frac{\sigma^i_{eq}}{\sigma^i_{ref}} \right)} \tag{54}$$

The above expressions described how the elastic modulus at a location with the equivalent stress σ_{eq} (e.g., the von Mises equivalent stress) was updated from the ith to the (i+1)th elastic iteration. This procedure was repeated until suitable convergence in a subsequent iteration was achieved.

The exponent q controlled the amount of redistribution within the element, as shown in Fig. 21. The detailed development of these formal bases for the elastic modulus adjustment and related procedures had been provided by Ponter and co-workers (Ponter and Engelhardt 2000, Ponter and Chen 2001). The generalized approach was similar with the elastic modulus adjustment procedures and can be better described as "linear matching methods" where a sequence of linear solutions was matched to the nonlinear problem. The elastic modulus adjustment methods relied on the convergence of the specific moduli adjustment procedure. This problem was also addressed by Ponter and Engelhardt (2000), who showed that the convergence of suitable matching methods was theoretically guaranteed for practically important yield functions.

2.2.6 Linear matching method

Linear matching method (LMM) (Ponter and Carter 1997a, b) was also a nonlinear programming technique, developed out of the ECM. LMM attempted to relate a series of incompressible linear solutions to the limit state described in Section 2.2.1. LMM was different with ECM. LMM was that the standard stress-strain relationship of

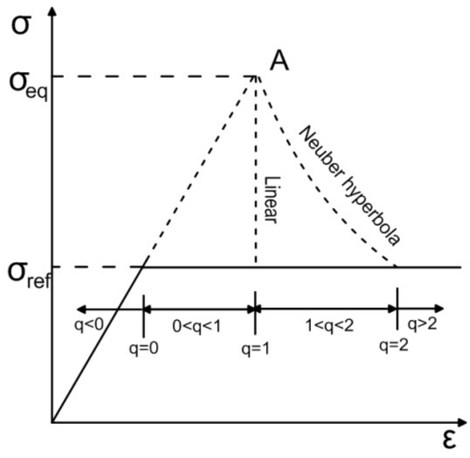

Fig. 1 Stress redistribution regions

isotropic material was divided into two separate linear relationships, namely the relationship of the deviatoric stress and hydrostatic stress versus strain rates components, respectively.

LMM in limit analysis which was also an iterative method with a linear elastic stress solution similar to ECM required varying spatially the shear modulus. A new distribution of shear modulus was evaluated by the following equation

$$\mu^{k+1} = \mu^k \frac{\sigma_y}{\overline{\sigma}\left(\sigma^k_{ij}\right)} \tag{55}$$

where, k was the iterative number, μ^{k+1} and μ^k was the shear modulus in the (k+1)th and kth iterative process. σ_y was the yield stress and $\overline{\sigma}(\sigma^k_{ij})$ was von Mises equivalent stress in the kth iterative number.

The primary application of these methods was the evaluation of limit load and shakedown limits for complex structural components. The lower and upper limit load multiplier may be evaluated by LMM at each iterative step, as the following Eqs. (60)~(63).

2.2.7 Equivalent strain energy density method

Based on total strain energy concept, Molski and Glinka (1981) found that the plastic stress and strain in notch root were equivalent to the elastic stress-strain field. The secant modulus E_s in the plastic zone was expressed as the ratio of the yield stress to the total strain, resulting in the following equation

$$E_s = \frac{2\sigma^2_y}{\sigma^2_y + \sigma^2} E \tag{56}$$

where, E was the actual elastic modulus, σ_y was the yield stress, σ was the stress in notch root.

Adibi-Asl *et al*. (2006) used Eq. (68) to modify the elastic modulus in the EMAP. The finite element implementation of the equation was expressed as follows

$$E^{i+1}_k = \frac{2\sigma^2_{arb}}{\sigma^2_{arb} + \left(\sigma^i_k\right)^2} E^i_k \tag{57}$$

where, E^{i+1}_k was the elastic modulus of the element k at increment i. σ_{arb} was an arbitrary value and similar to Eq.

(50).

2.2.8 Direct secant method

Direct secant method was proposed by Adluri (1999) based on R-Node family of techniques for limit load determination of structures or components. Direct secant method only applied to simple framed structures and plates. Similar to R-Node family method, two linear elastic finite element analyses (FEA) were performed.

A purely elastic analysis was first carried out, the maximum bending stress location was the first location of a plastic hinge. After the formation of the first plastic hinge, the stress will redistribute. The other peak moments in the first elastic analysis were thereby not at the locations of potential plastic hinges. And then the rigidity was modified based on elastic analysis results. The rigidity was modified in the following equation

$$I_{new}\left(x\right)=\left|\frac{M_{0-\max}}{M_0\left(x\right)}\right|^q I_{old}\left(x\right) \tag{58}$$

where, $M_0(x)$ was the bending moment from an initial analysis. The exponent q is usually taken between 1 and 2. $M_{0\text{-max}}$ was not less than $M(x)$, $I_{new}(x)$ was greater or equal to $I_{old}(x)$.

The second elastic analysis was executed on the basis of the modification Eq. (75). The modified structure was analyzed again with the same load, the same supported but the new rigidity. The peak moments and other relevant stress resultants were obtained. The same opportunities of all the elements led to all the potential locations of plastic hinges appearing at the same time. The selection of potential location of plastic hinges can be done in several ways. After selecting the locations of plastic hinges, limit load can be calculated by the following equation

$$\frac{P_{\lim}}{P}=\frac{M_P}{M_{peak-ave}} \tag{59}$$

where P was the externally applied vector load and can had any non-zero value. M_P was the plastic moment and was determined by the geometrical properties of the cross-section. $M_{Peak\text{-}ave}$ was the peak average bending moment in the second elastic analysis. At the location of these peak bending moments, plastic hinges form and they contributed to the plastic collapse.

2.2.9 Brief summary

In conclusion, a series of pseudo-elastic finite element methods based on R-Node method or similar to R-Node method, which were linear elastic stress solution and need to be several repeated elastic iterations, had a similar feature in their modified elastic modulus, shear modulus or rigidity. The flowchart of the R-Node family methods (i.e., EMAP) was given in Fig. 22.

3. Application and discussion

Taking a cylinder with nozzle (Chen 2005) as an example in this study, the limit load of a cylinder with

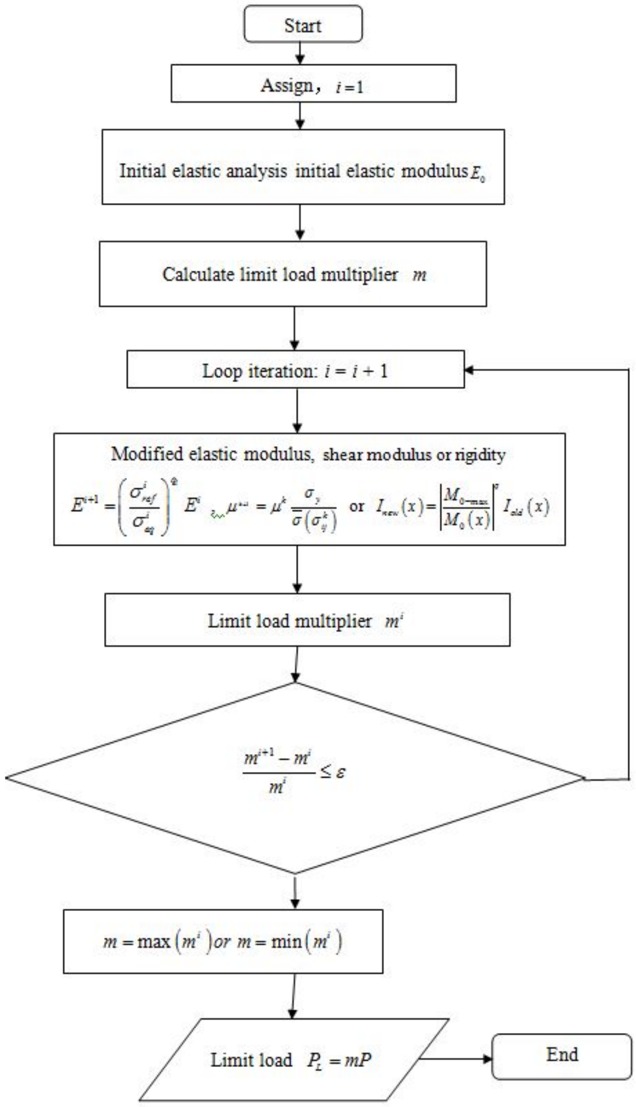

Fig. 2 Flowchart of the EMAP

nozzle was used by pseudo - elastic finite element method in conjunction with robust methods by mean of finite element software ANSYS.

3.1 Limit load determined by EMAP

Limit load of the cylinder with nozzle was determined by the general formulation of elastic modulus adjustment procedure combining with lower and upper bound theorem in Section 2.1, as listed in Table 6. The reference volume in the multiplier $m_2^0(\overline{V}_\eta)$ was sub-volume corresponding to last iteration. Fig 23 indicated the multipliers of robust methods.

3.2 Results and discussion

The cylinder with nozzle was modeled using commercial finite element program ANSYS. Limit load of the cylinder with nozzle was determined by linear elastic EMAP iteration. Theses evaluations were supported by a macro program, using the ANSYS Parametric Design Language

Table 1 Numerical solution of limit load

No.	Methods	Multiplier	Limit load/MPa
1	m_L	136.1	2.72
2	m_U	231.4	4.62
3	m'	108.2	2.16
4	m^0	275.2	5.5
5	m_1^0	275.2	5.5
6	m_2^0	231.4	4.63
7	m_α	168	3.38
8	m_α^T	211.8	4.24
9	m''	260.1	5.2
10	m_β	273.6	5.47
11	$m_2^0(\overline{V}_\eta)$	277	5.54

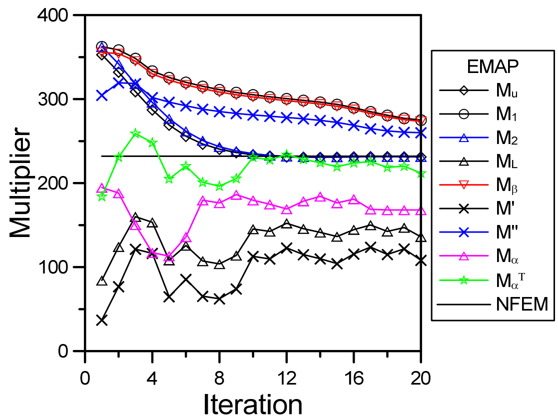

Fig. 3 The multipliers of robust methods

(APDL). This program can be conveniently used as part of a post-processing program. The main characteristics in this study can be described as follows:

(1) Reinhardt and Seshadri (2003) showed that the classical lower bound multiplier (m_L) was less than or equal to the exact limit load multiplier.

(2) Reinhardt and Seshadri (2003) showed that the classical upper bound multiplier (m_U) was greater than or equal to the exact limit load multiplier.

(3) If plastic collapse occurred over a localized region of the component or structure, the multiplier m_1^0 was significantly overestimated if it was based on the total volume. The m_1^0 multiplier were shown to be greater than the classical lower bound multiplier (m_L) and greater than the classical upper bound multiplier (m_U) (Reinhardt and Seshadri 2003).

(4) The m_2^0 multiplier were shown to be greater or equal than the classical lower bound multiplier (m_L) and greater or equal than the classical upper bound multiplier (m_U) (Pan and Seshadri 2001).

(5) The multiplier m_1^0 based on the total volume was significantly overestimated. Therefore, the reference volume had been presented to identify the "kinematically active" region of the component or structure that participated in plastic action.

(6) The lower bound limit load multiplier m' obtained from Mura's extended variational theorem was shown to be less than that obtained by applying classical lower bound theorem ($m' < m_L$). The multiplier m' was significantly underestimated if it was based on the total volume.

(7) Seshadri and Indermohan (2004) showed that the m'' multiplier based on m_1^0 need not be a lower bound.

(8) The m_α multiplier depended on the parameters m^0 and m_L. The m_α multiplier was an improved estimate of the analytical limit load multiplier compared to the bounds m^0 and m_L. Although it was often found to be an improved lower bound, it could not be established as a lower bound in general. The m_α multiplier method was applicable if a component fell inside the "m_α triangle". The region of "m_α triangle" was $1 \le m^0 / m_L \le 1 + \sqrt{2}$ and $1 \le m^0 / m \le 1 + \sqrt{2}$.

(9) In order to overcome these limitations, the m_α^T method is developed. The m_α^T method was developed as a viable tool for estimating the limit load of a general class of mechanical components and structures by using a single linear elastic analysis. The limit load multiplier m_L was evaluated by making use of the limiting tangent; upper bound multiplier m^0 and classical lower bound multiplier m_L. All necessary information can be extracted from the initial linear elastic analysis. The m_α^T method can take practically any value of m^0 / m_L, which extended the domain of application of the proposed m_α^T method beyond the "m_α triangle". The m_α^T method was applied to a number of mechanical components and structures, ranging from simple to relatively complex geometric configurations, and the results compared well with those obtained from the corresponding analytical and inelastic finite element analysis results.

(10) The m_β multiplier was a lower bound value and was determined that relied on the entire stress distribution rather than the maximum stress. For any given iteration, the (m_β, m_U) pair provided the minimum spread in terms of bounds on the limit load multipliers. The m_β multiplier was a better lower bound than the m_L multiplier and m_α multiplier.

3.3 Limit load determined

3.3.1 Numerical example

Gloss R-Node method, elastic compensation method and modified elastic compensation method were used to determined limit load of the connection of flat head and cylinder under internal pressure. Arbitrary load was 100 MPa, the iterations were ten. Elastic modulus of the material was E = 212 GPa, yield strength was S_m = 255 MPa. Element type PLANE82 with eight nodes was applied. Due to the symmetry of the structure, a quarter of the connection of flat head and cylinder was modeled, as shown in Fig. 24. Symmetry plane was applied symmetric load. The end of cylinder was applied axial constraint. Internal

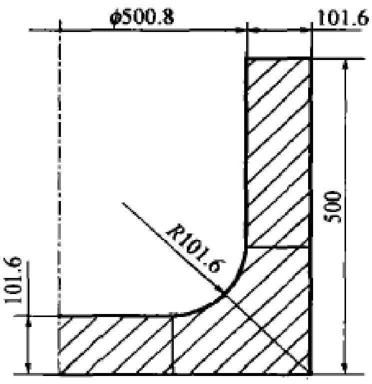

Fig. 4 Connection of flat head and cylinder under internal pressure

pressure was applied on the internal surface of flat head and cylinder. Limit load of the connection of flat head and cylinder under internal pressure, which was determined by Gloss R-Node method, elastic compensation method and modified elastic compensation method, was listed in Table 7.

3.3.2 Discussion

Zhou *et al*. (2006) observed that Gloss R-Node method, which was suitable for two dimensional problems, the mesh of finite element model need along the thickness direction was progressive refinement. Therefore, the application of Gloss R-Node method for three dimensional problems was restricted. Moreover, Mackenzie and Boyle (1993) investigated that Gloss R-Node method may appear to numerical singular problem in the process of the modified elastic modulus, thus elastic modulus of finite element may differ more than one order of magnitude. Yang *et al*. (2005) and Nadarajah *et al*. (1996) indicated that ECM, which was regarded as an evaluation method of ultimate bearing capacity of structures or component, was adopted by ASME code, and so on. ECM was widely applied in the design and safety assessment of pressure vessel. By mean of several elastic finite element iterations, limit load was determined by ECM. Elastic modulus of all elements was modified in each iteration, and then the stresses were redistributed. When Mises stress of the element tended to zero, and then element stiffness would be excessively strengthened, which would result in higher numerical error for finite element analysis. ECM can obtain high accuracy for simple structure or component. But ECM had high error for complex structure or component. When the structure appeared to lower stress or was applied uneven loading, the no convergence iteration results were usually caused. Therefore, Gloss R-Node method and ECM were not used to a cylinder with nozzle in this study.

According to the shortcoming of ECM, MECM was proposed by Liu *et al*. (2009) who thought that MECM was simple, efficient, convenient and significant for engineering application. For complex structures, limit load determined by MECM had high accuracy. Therefore, MECM can be applied to the designing of the structural engineering and safety assessment. Liu *et al*. (2009) studied limit load of the cylinder with nozzle which was determined by ECM, MECM

Table 2 The maximum allowable load

Determination method	GLOSS R-Node	ECM	MECM	Reference (Zhou et al. 2006)
Limit load	82.62	90.03	83.07	84.96

and elastic-plastic incremental method, it was found that the results of MECM were closer to those of elastic-plastic incremental method. But the computing time of MECM was longer than that of ECM.

4. Conclusions

The pseudo-elastic finite element method was a simple and robust methods for identifying the limit load for a component or structure. The method can be formulated to use only linear elastic tools available in all finite element software. It was easily automated in standard finite element packages with a moderate amount of user programming.

A computational implementation of the robust method was described in this study. The robust method was found to provide good accuracy approximations of the limit load obtained using other methods. Taking a cylinder with nozzle as an example, the results of limit loads of a cylinder with nozzle were calculated and compared. Through comparison of elastic-plastic finite element analysis, it was found that robust methods using elastic modulus adjustment techniques can save a lot of computation time. Limit load determined by some of robust methods was in well agreement with twice elastic slope criterion. In recent years, the development of robust methods has been beneficial for shakedown analysis and ratcheting boundary determination of structures or components.

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