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# Vibration of antisymmetric angle-ply laminated plates under higher order shear theory

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**Abstract.** This paper deals with the analysis of vibration of antisymmetric angle-ply plates using spline method for higher order shear theory. Free vibration of laminated plates is addressed to show the capability of the present method in the vicinity of higher order shear deformation theory and simply supported edges of plates. The coupled differential equations are obtained in terms displacement and rotational functions. These displacement and rotational functions are approximated using cubic and quantic spline. A generalized eigenvalue problem is obtained and solved numerically for an eigenfrequency parameter and an associated eigenvector of spline coefficients. The antisymmetric angle-ply fiber orientation are taken as design variables. Numerical results enable us to examine the frequencies for various geometric and material parameters and accuracy and effectiveness of the proposed method is also verified by comparative study.

**Keywords:** antisymmetric angle-ply; free vibration; shear deformation theory; spline approximation; eigenvalue

## 1. Introduction

Laminated composite plates are extensively been used in civil and aerospace engineering. The attractive features of using laminated composites is because they possess numerious favourable mechanical properties such as high stiffness to weight and low density, which is particulary useful for aerospace and submarine structures because they require high stiffness and light weight. To use laminated plates structures efficiently, it is necessary to develop appropriate theories (Pagano 1970, Ferreira 2005) to predict accurately their structural and dynamic behaviour. The 3D elasticity theory was proposed by Noor (1973) to improve the accuracy of transverse shear stresses. While calculating 3D continuum elements for describing the response of thin structures, the computational cost become a serious issue. Single layer plate theories were proposed to reduce the computational cost and to convert 3D problem into 2D problem. Moreover, several laminated

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plates theories were developed. Among them classical laminated plate theory (CLPT) capable of measuring the stresses of thin composite plates while ignoring the transverse shear deformation (Reissner and Stavsky 1961). The first order shear deformation theory (FSDT) (Reissner 1972) was developed to measure the stresses of thin and moderately thick plates. According to this theory the transverse shear stresses are constant through the thickness of the plate and a shear correction factor were introduce to correct the discrepancy in the shear forces of FSDT and 3D elasticity theory (Pai and Schulz 1999). Shear correction factor depends upon the laying angle, material constants, geometry and boundary conditions, thus making this factor difficult to determine (Mackerle 2002, Yang *et al*. 2000). To overcome the discrepancies of the FSDT, higher-order shear deformation theories (HSDT) were developed to accurately evaluate the transverse shear stresses which effectively exist in thick plates. In higher-order plate theories the displacements are expanded up to any desired degree in terms of thickness coordinates (Vinson 2001, Noor *et al*. 1996) which avoid the need for a shear correction factor. Further, HSDT yields more accurate inter-laminar stress distributions and satisfies the conditions of zero shear stress at the top and bottom surfaces of the plate. In third-order plate theory the displacement are expanded up to the cubic term in thickness coordinatesto have quadratic variation of transverse shear strains and transverse shear stresses through the plate thickness. This avoid the need for shear correction coefficient (Reddy 2006).

Higher-order shear deformation theory was used by Ferreira *et al*. (2003) to analyse the composite plates using multiquadric radial basis function. Neves *et al*. (2013) investigated the static, free and buckling analysis of plates using quasi 3-D higher-order shear deformation theory. Static inconsistencies of beams, plates and shells were investigated using higher-order shear deformation theory by Groh and Weaver (2015). Khandelwal *et al*. (2015) studied the vibration response of laminated composite plates with weakly boded layers. Free vibration of composite plates using higher-order shear deformation theory was investigated by Kant and Swaminathan (2001, 2002). Naserian-Nik and Tahani (2010) analysed free vibration of moderately thick rectangular laminated composite plates using semi-analytical method under different boundary conditions. Static and dynamic analysis of sandwich and laminated composite plates and shells were analysed using higher-order shear deformation theory by Mantari *et al*. (2011, 2012). Recently, Mantari and Granados (2015) examine the thermoelastic analysis of sandwich plates using quasi 3-D hybrid type HSDT. Further, Mantari and Soares (2015) used hybrid-type quasi-3D HSDT to analyse advanced composite plates. Pekovic *et al*. (2014) studied the bending analysis of composite plates under higher-order shear deformation theory. Phung-Van *et al*. (2015a) analyze cell-based three-node Mindline plate using $C^0$-type higher-order shear deformation for geometrically non linear analysis of composite plates. In another paper Phung-Van *et al*. (2015b) analyse composite plates having piezoelectric sensors and actuators using HSDT. Thai *et al*. (2015) studied the isogeometric analysis of composite plates using HSDT. $C^0$-type higher-order shear deformation theory was used for analysing laminated plates by Tran *et al*. (2015). Zhen and Wanji (2006) investigated the free vibration of laminated and sandwich plates using global-local higher-order shear deformation theory. Reddy's higher-order theory was used to examine the free vibration of composite sandwich plates by Nayak *et al*. (2002). Zuo *et al*. (2014, 2015) analysed the composite plates using wavelet finite element method and higher-order plate theory. A refined laminated plate theory was used by Wang and Shi (2015) to account the third order shear deformation and interlaminar transverse stress. Yang *et al*. (2013) conducted free vibration and buckling analysis of plates based on Reissner-Mindlin theory using a finite element method (FEM) of B-spline wavelet on the interval (BSWI).

This paper aims to investigate the free vibration of anti-symmetric angle-ply laminted platesusinghigher-order shear deformation theory and applying spline approximation technique. The plates kinematics is based on third order shear deformation theory (TSDT). The displacement and rotational functions are predicted using spline approximation of suitable order. Collocation with these splines yields a set of field equations which along with the equations of boundary conditions, reduce to system of homogeneous simultaneous algebraic equations on the assumed spline coefficients. Then the problem is solved using eigensolution technique to obtain the frequency parameter. The eigenvector are the spline coefficients from which the mode shapes can be constructed. The simply supported boundary condition is considered to show the parametric effects of plate's aspect ratio, side-to-thickness ratio, laying angle, number of lamina and different lamination materials on the frequency of the plate. The obtained results are presented in graphs and tables.

## 2. Formulation

The displacement field considered according to third order shear deformation theory (Reddy 2006).

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + z\phi_x(x,y,t) - \frac{4z^3}{3h^2}\left(\phi_x + \frac{\partial w_0}{\partial x}\right) \\ v(x,y,z,t) &= v_0(x,y,t) + z\phi_y(x,y,t) - \frac{4z^3}{3h^2}\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \qquad w(x,y,z,t) = w_0(x,y,t) \end{aligned} \tag{1}$$

were $u$, $v$ and $w$ are the displacement components in the $x$, $y$ and $z$ directions respectively, $u_0$ and $v_0$ and $w_0$ are the in-plane displacements of the middle plane and $\phi_x$ and $\phi_y$ are the shear rotations of any point on the middle surface of the plate.

### *2.1 Kinematics*

In-plane strains are defined as

$$\varepsilon = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}, \qquad \varepsilon = \varepsilon^0 + z\varepsilon^1 + z^3\varepsilon^3 \tag{2}$$

where

$$\varepsilon^0 = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial y} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \varepsilon^1 = \begin{Bmatrix} \varepsilon_x^1 \\ \varepsilon_y^1 \\ \gamma_{xy}^1 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \varepsilon^3 = \begin{Bmatrix} \varepsilon_x^3 \\ \varepsilon_y^3 \\ \gamma_{xy}^3 \end{Bmatrix} = \left(-\frac{4}{3h^2}\right) \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 w_0}{\partial y^2} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix}$$

and the shear strain components are defined as

$$\gamma = \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix}, \qquad \gamma = \gamma^0 + z^2 \gamma^2 \tag{3}$$

where

$$\gamma^0 = \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} \phi_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix}, \; \gamma^2 = \begin{Bmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{Bmatrix} = \left( -\frac{4}{h^2} \right) \begin{Bmatrix} \phi_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix}$$

### 2.2 Constitutive equations

The stress-strain relations for the $k$-th layer, after neglecting transverse normal strain and stress, are of the form

$$\begin{pmatrix} \sigma_x^{(k)} \\ \sigma_y^{(k)} \\ \tau_{xy}^{(k)} \\ \tau_{yz}^{(k)} \\ \tau_{xz}^{(k)} \end{pmatrix} = \begin{pmatrix} C_{11}^{(k)} & C_{12}^{(k)} & C_{16}^{(k)} & 0 & 0 \\ C_{12}^{(k)} & C_{22}^{(k)} & C_{26}^{(k)} & 0 & 0 \\ C_{16}^{(k)} & C_{26}^{(k)} & C_{66}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & C_{44}^{(k)} & C_{45}^{(k)} \\ 0 & 0 & 0 & C_{45}^{(k)} & C_{55}^{(k)} \end{pmatrix} \begin{pmatrix} \varepsilon_x^{(k)} \\ \varepsilon_y^{(k)} \\ \gamma_{xy}^{(k)} \\ \gamma_{yz}^{(k)} \\ \gamma_{xz}^{(k)} \end{pmatrix} \tag{4}$$

When the materials are oriented at an angle $\theta$ with the $x$-axis, the transformed stress-strain relations are

$$\begin{pmatrix} \sigma_x^{(k)} \\ \sigma_y^{(k)} \\ \tau_{xy}^{(k)} \\ \tau_{yz}^{(k)} \\ \tau_{xz}^{(k)} \end{pmatrix} = \begin{pmatrix} Q_{11}^{(k)} & Q_{12}^{(k)} & Q_{16}^{(k)} & 0 & 0 \\ Q_{12}^{(k)} & Q_{22}^{(k)} & Q_{26}^{(k)} & 0 & 0 \\ Q_{16}^{(k)} & Q_{26}^{(k)} & Q_{66}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & Q_{44}^{(k)} & Q_{45}^{(k)} \\ 0 & 0 & 0 & Q_{45}^{(k)} & Q_{55}^{(k)} \end{pmatrix} \begin{pmatrix} \varepsilon_x^{(k)} \\ \varepsilon_y^{(k)} \\ \gamma_{xy}^{(k)} \\ \gamma_{yz}^{(k)} \\ \gamma_{xz}^{(k)} \end{pmatrix} \tag{5}$$

where $Q_{ij}^{(k)}$, as functions of $C_{ij}^{(k)}$ are fully furnished in Viswanathan and Lee (2007). The stress resultants are defined as

$$\begin{Bmatrix} N_i \\ M_i \\ P_i \end{Bmatrix} = \int_{-h/2}^{h/2} \sigma_i \begin{Bmatrix} 1 \\ z \\ z^3 \end{Bmatrix} dz, \qquad \begin{Bmatrix} Q_i \\ R_i \end{Bmatrix} = \int_{-h/2}^{h/2} \tau_i \begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dz \tag{6}$$

where $N_i$, $M_i$ and $Q_i$ are stress, moment and shear resultants respectively. $P_i$ and $R_i$ denote higher-order stress resultants. The stress-strain relations are obtained as follows

$$\begin{pmatrix} N \\ M \\ P \\ Q \\ R \end{pmatrix} = \begin{pmatrix} A & B & E & 0 & 0 \\ B & D & F & 0 & 0 \\ E & F & H & 0 & 0 \\ 0 & 0 & 0 & A' & D' \\ 0 & 0 & 0 & D' & F' \end{pmatrix} \begin{pmatrix} \varepsilon^0 \\ \varepsilon^1 \\ \varepsilon^3 \\ \gamma^0 \\ \gamma^2 \end{pmatrix} \tag{7}$$

Stiffness coefficients are defined as

$$\begin{aligned} A_{ij} &= \sum_k \overline{Q}_{ij}^{(k)} (z_k - z_{k-1}), \qquad B_{ij} = \frac{1}{2}\sum_k \overline{Q}_{ij}^{(k)} (z^2{}_k - z^2{}_{k-1}), \qquad D_{ij} = \frac{1}{3}\sum_k \overline{Q}_{ij}^{(k)} (z^3{}_k - z^3{}_{k-1}) \\ E_{ij} &= \frac{1}{4}\sum_k \overline{Q}_{ij}^{(k)} (z^4{}_k - z^4{}_{k-1}), \quad F_{ij} = \frac{1}{5}\sum_k \overline{Q}_{ij}^{(k)} (z^5{}_k - z^5{}_{k-1}), \qquad H_{ij} = \frac{1}{7}\sum_k \overline{Q}_{ij}^{(k)} (z^7{}_k - z^7{}_{k-1}) \end{aligned} \tag{8}$$

for $i, j = 1, 2, 6$

$$A'_{ij} = \sum_k \overline{Q}_{ij}^{(k)} (z_k - z_{k-1}), \quad D'_{ij} = \frac{1}{3}\sum_k \overline{Q}_{ij}^{(k)} (z^3{}_k - z^3{}_{k-1}) \text{ and } F'_{ij} = \frac{1}{5}\sum_k \overline{Q}_{ij}^{(k)} (z^5{}_k - z^5{}_{k-1})$$

for $i, j = 4, 5$ where the elastic coefficients $A_{ij}$, $B_{ij}$ and $D_{ij}$ (extensional,bending-extensional coupling and bending stiffnesses) and $E_{ij}$, $F_{ij}$ and $H_{ij}$ are the higher-order stiffness coefficients. The equilibrium equations considered are as follows

$$\begin{aligned} &\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \frac{\partial^2 u}{\partial t^2}, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} + = I_0 \frac{\partial^2 v}{\partial t^2} \\ &\frac{\partial \bar{Q}_x}{\partial x} + \frac{\partial \bar{Q}_y}{\partial y} + c_1 \left( \frac{\partial^2 P_x}{\partial x^2} + 2\frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_y}{\partial y^2} \right) \\ &= I_0 \frac{\partial^2 w}{\partial t^2} - c_1^2 I_6 \left( \frac{\partial^4 w_0}{\partial x^2 \partial t^2} + \frac{\partial^4 w_0}{\partial y^2 \partial t^2} \right) + c_1 J_4 \left( \frac{\partial^3 \phi_x}{\partial x \partial t^2} + \frac{\partial^3 \phi_y}{\partial y \partial t^2} \right) \end{aligned} \tag{9}$$

$$\frac{\partial \bar{M}_{xx}}{\partial x} + \frac{\partial \bar{M}_{xy}}{\partial y} - \bar{Q}_x = K_2 \frac{\partial^2 \phi_x}{\partial t^2} - c_1 J_4 \frac{\partial^3 w}{\partial x \partial t^2}, \qquad \frac{\partial \bar{M}_{xy}}{\partial x} + \frac{\partial \bar{M}_{yy}}{\partial y} - \bar{Q}_y = K_2 \frac{\partial^2 \phi_y}{\partial t^2} - c_1 J_4 \frac{\partial^3 w}{\partial x \partial t^2}$$

Here

$$I_i = \int_z \rho^{(k)} (z)^i \, dz \quad (i = 0,1,2,3,.....,6)$$

and $\rho$ is the material density of the $k$-th layer and

$$\bar{M}_{\alpha\beta} = M_{\alpha\beta} - c_1 P_{\alpha\beta}, \quad \bar{Q}_{\alpha} = Q_{\alpha} - c_2 R_{\alpha}, \quad J_i = I_i - c_1 I_{i+2}, \quad K_2 = I_2 - 2c_1 I_4 + c_1^2 I_6 .$$

Substitute Eqs. (2) and (3) into Eq. (7) and then substituting into Eq. (9). Further, equating the following laminate stiffnesses equal to zero for anti-symmetric angle-ply laminates (Reddy 1997)

$$A_{16}, A_{26}, A_{45}, B_{11}, B_{12}, B_{22}, B_{66}, D_{16}, D_{26}, D_{45}, E_{11}, E_{12}, E_{22}, E_{66}, H_{16}, H_{26}, F_{16}, F_{26}, F_{45} = 0 .$$

The displacements and rotational functions are assumed in the separable form for antisymmetric angle-ply plates as

$$\begin{aligned} u(x,y) &= U(x)\cos\frac{n\pi y}{b}\, e^{i\omega t} \\ v(x,y) &= V(x)\sin\frac{n\pi y}{b} e^{i\omega t} \\ w(x,y) &= W(x)\sin\frac{n\pi y}{b} e^{i\omega t} \\ \phi_x(x,y) &= \Phi_x(x)\sin\frac{n\pi y}{b} e^{i\omega t} \\ \phi_y(x,y) &= \Phi_y(x)\cos\frac{n\pi y}{b} e^{i\omega t} \end{aligned} \tag{10}$$

The non-dimensional parameters introduced are as follows

$\lambda = \omega a\sqrt{\frac{I_0}{A_{11}}}$, a frequency parameter

$\delta_k = \frac{h_k}{h}$, the relative layer thickness of the k-th layer

$\phi = \frac{a}{b}$, aspect ratio

$X = \frac{x}{a}$, a distance coordinate

$H = \frac{a}{h}$, side-to-thickness ratio

Using Eq. (10) and introducing non-dimensional parameters, modified equations in the matrix form areobtained as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{21} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{31} & L_{32} & L_{33} & L_{34} & L_{35} \\ L_{41} & L_{42} & L_{43} & L_{44} & L_{45} \\ L_{51} & L_{52} & L_{53} & L_{54} & L_{55} \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \\ \Phi_X \\ \Phi_Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{11}$$

where $L_{ij}'s$ are shown in Appendix.

### *2.3 Method of solution*

The differential equations in Eq. (11) contain derivatives of second order in $U(X)$, third order in $V(X)$, fourth order in $W(X)$, third order in $\Phi_X(X)$ and second order in $\Phi_Y(X)$. These functions are approximated by using cubic and quantic spline functions , in the range of $X \in [0, 1]$, since splines are relatively simple and elegant and use series of lower order approximations rather than global higher order approximations, affording fast convergence and high accuracy.

The displacement functions $U(X)$, $V(X)$ and $W(X)$ and the rotational functions $\Phi_X(X)$ $\Phi_Y(X)$ are approximated respectively by the splines

$$
\begin{aligned}
U(X) &= \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j) \\
V(X) &= \sum_{i=0}^{4} c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^5 H(X - X_j) \\
W(X) &= \sum_{i=0}^{4} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^5 H(X - X_j) \\
\Phi_X(X) &= \sum_{i=0}^{4} g_i X^i + \sum_{j=0}^{N-1} p_j (X - X_j)^5 H(X - X_j) \\
\Phi_Y^*(X) &= \sum_{i=0}^{2} l_i X^i + \sum_{j=0}^{N-1} q_j (X - X_j)^3 H(X - X_j)
\end{aligned}
\tag{12}
$$

Here $H(X - X_j)$ is the Heaviside step function and $N$ is the number of intervals into which the range [0, 1] of $X$ is divided. The points $X = X_s = \frac{s}{N}$, ($s$ = 0, 1, 2, ....., $N$) are chosen as the knots of the splines, as well as the collocation points. Thus the splines are assumed to satisfy the differential equations given by Eq. (12), at all $X_s$. The resulting expressions contain ($5N$ + 5) homogeneous system of equations in the ($5N$ + 21) spline coefficients.

The boundary condition considered on the edges $x$ = 0 and $x = a$ are

(1) (S-S): both the ends simply supported

This boundary condition gives 13 more equations, thus making a total of ($5N$ + 18) equations, in the same number of unknowns. The resulting field and boundary condition equations may be written in the form

$$[M]\{q\} = \lambda^2 [P]\{q\} \tag{13}$$

where $[M]$ and $[P]$ are square matrices, $\{q\}$ is a column matrix. This is treated as a generalized eigenvalue problem in the eigenparameter $\lambda$ and the eigenvector $\{q\}$ whose elements are the spline coefficients.

## 3. Results and discussion

The higher order shear deformation theory is used to investigate the free vibration of anti-symmetric angle-ply plates for simply supported end condition. All numerical computations in this section, unless otherwise stated, two materials are considered: Kevlar-49/epoxy (KE) and Graphite/Epoxy (AS4/3501-6) (GE). Two, four, and six layered plates with anti-symmetric angle-ply orientations are considered.

### *3.1 Convergence study*

The frequency parameter with respect to different configurations are carried out to confirm the convergence of the spline method for antisymmetric angle-ply plates. The number of subintervals $N$ of the range $X \in [0, 1]$. The value of $N$ started from 4 and finally it is fixed for $N = 18$, since for the next value of $N$, the percent changes in the values of $\lambda$ are very low, the maximum being 3%.

### *3.2 Validation*

The accuracy and effectiveness of the proposed method is verified with the number of previous studies available in the literature. Table 1 shows the comparison of the fundamental frequency with respect to the side-to-thickness ratio and aspect ratio of the present results compared with Swaminathan and Patil (2008a), for two layered antisymmetric angle-ply square laminated plates under higher order shear deformation theory. It can be seen from the Table 1, some of the present values closer and few values are significantly lesser than the results obtained by Swaminathan and Patil (2008a). Table 2 depicts the comparison of fundamental frequencies of four layered anti

Table 1 Non-dimensioned fundamental frequencies for simply supported two layered antisymmetric angles-ply ($\theta/-\theta$) square laminated plate

$\bar{\omega} = (\omega b^2 / h)\sqrt{\rho / E_2}$, $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$

| $\theta$ | Source | | a/h | |
|---|---|---|---|---|
| | | | **2** | **4** |
| 15° | Present | | 4.8711 | 8.3495 |
| | Swaminathan and Patil (2008a) | Model 1 | 5.1600 | 8.5142 |
| | | Model 2 | 5.1390 | 8.4789 |
| | | Model 3 | 5.6280 | 8.8117 |
| | | Model 4 | 5.9909 | 9.4119 |
| | | | **a/b** | |
| | | | 1 | 1.5 |
| | Present | | 8.3495 | 3.8611 |
| | Swaminathan and Patil (2008a) | Model 1 | 8.5142 | 4.6111 |
| | | Model 2 | 8.4789 | 4.5911 |
| | | Model 3 | 8.8117 | 4.7931 |
| | | Model 4 | 9.4119 | 5.2226 |

Table 2 Non-dimensionalized fundamental frequencies for simply supported four layered antisymmetric angle-ply (45°/–45°/45°/–45°) square laminated plate

$\bar{\omega} = (\omega b^2 / h)\sqrt{\rho / E_2}$, $E_1/E_2 = 40$, $G_{12}/E_2 = G_{13}/E_2 = 0.6$, $G_{23}/E_2 = 0.5$, $\nu_{12} = \nu_{13} = \nu_{23} = 0.25$

| Lamination and number of layers | Source | a/h | |
|---|---|---|---|
| | | **2** | **4** |
| 45°/–45°/45°/–45° | Present | 5.7035 | 10.3677 |
| | Swaminathan and Patil (2008b) | 5.5674 | 10.0731 |
| | | 6.1067 | 10.6507 |

Table 3 Fundamental frequency $\lambda$ of four layered anti-symmetric plates (45°/–45°/45°/–45°) $a / h$ =20

$\lambda' = \omega a^2 \sqrt{\frac{\rho}{E_2 h^2}}$, $E_1/E_2 = 40$, $G_{12}/E_2 = 0.6$, $G_{13}/E_2 = G_{23}/E_2 = 0.5$, $\nu_{12} = 0.25$

| Sources | ***a*/*b*** | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 0.2 | 0.4 | 0.6 | 0.8 | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2 |
| Present | 8.8974 | 10.2512 | 12.8226 | 15.9767 | 19.6810 | 25.6715 | 29.6799 | 33.2168 | 38.7031 | 44.7735 |
| Ghosh and Dey (1994) | 9.52 | 11.70 | 14.72 | 18.26 | 22.19 | 26.45 | 31.02 | 35.89 | 41.07 | 46.54 |
| Reddy (1979) | 9.475 | 11.767 | 14.896 | 18.557 | 22.584 | 26.857 | 31.401 | 36.249 | 41.372 | 46.789 |
| Bert and Chen (1978) | 9.300 | 11.46 | 14.45 | 17.97 | 21.87 | 26.12 | 30.68 | 35.56 | - | 46.26 |

Table 4 Fundamental frequency variation of four layered plates with respect to side-to-thickness ratio $a / h = 0.4$

| ***a*/*h*** | 30°/–30°/30°/–30° (GE/GE/GE/GE) | 45°/–45°/45°/–45° (GE/GE/GE/GE) | 60°/–60°/60°/–60° (GE/GE/GE/GE) |
|---|---|---|---|
| 10 | 0.7485 | 0.8737 | 1.8039 |
| 20 | 0.5160 | 0.69328 | 1.4055 |
| 30 | 0.5043 | 0.6218 | 1.1095 |
| 40 | 0.4837 | 0.5912 | 0.9823 |
| 50 | 0.4043 | 0.5606 | 0.7622 |
| 60 | 0.3654 | 0.4959 | 0.6133 |

symmetric angle-ply plates for side-to-thickness ratio fixed as 2 and 4 showing that the present results are closer to the results of Swaminathan and Patil (2008b). Table 3 presents the comparison of current results with the FSDT with the fundamental frequency $\lambda$ for four layered anti-symmetric angle-ply plates (45°/–45°/45°/–45°). It is observed that thepresent results are significantly lesser than the results obtained using FSDT (Bert and Chen 1978, Ghosh and Dey 1994, Reddy 1979).

Table 4 depicts the effect of side-to-thickness ratio on the fundamental frequency of four layered plates consisting of GE material with different laying angles. The aspect ratio $a / b = 0.4$ is

fixed. It is seen that frequency increases with the increase of laying angle. Further, frequency decreases with the increase of side-to-thickness ratio.

### 3.3 Effect of differenr materials and geometric parameters on the frequency of antisymmetric angle-ply plates

#### 3.3.1 Effect of aspect ratio

Fig. 1 depicts the effect of aspect ratio $a$ / $b$, laying-angle and material sequence on the value of frequency parameter $\lambda_m$ of six layered plates. The laying angle 45°/–30°/60°/–60°/30°/–45° and

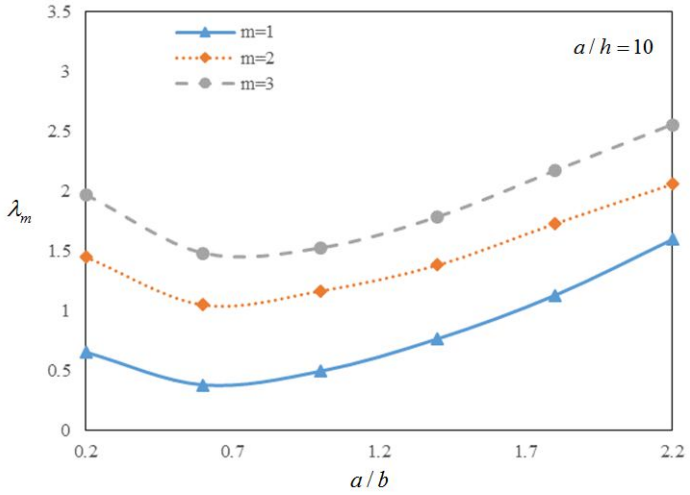


Fig. 1 The frequency variation of six layered anti-symmetric angle-ply plates with respect to aspect ratio. 45°/–30°/60°/–60°/30°/–45° (KE/GE/KE/KE/GE/KE)

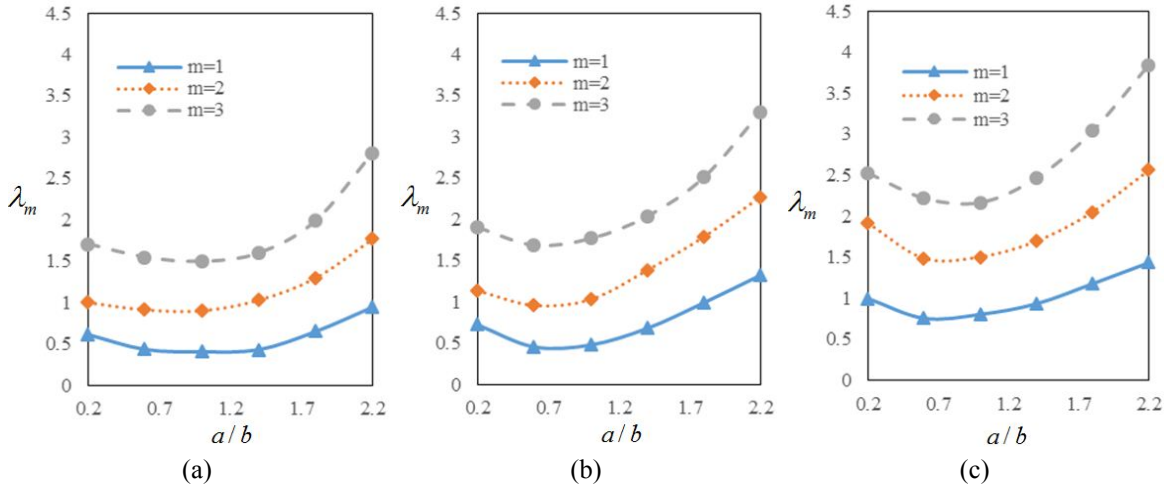


Fig. 2 The frequency variation of six layered anti-symmetric angle-ply plates with respect to aspect ratio: (a) 30°/–30°/30°/–30°/30°/–30°; (b) 45°/–45°/45°/–45°/45°/–45°; (c) 60°/–60°/60°/–60°/60°/–60° consisting of KE material

material combination (KE/GE/KE/KE/GE/KE) is considered and side-to-thickness ratio is fixed as $a / b = 10$. It can be seen that the frequency value decrease slowly till $a / b = 0.6$ and increases gradually afterwards.

The variation of frequency parameter $\lambda_m$ with respect to the aspect ratio is shown in Fig. 2. The material values of KE is used and six layered plates with different ply anglesare considered. The frequency valuesare decreasesin the range $0.2 < a / b < 0.6$ and slowly increases afterwards. Moreover, the frequency values are increases with increasing ply-angles.

Fig. 3 shows the variation of frequency value with respect tothe aspect ratio for four layered plates with the different ply-angles consisting of materials KE and GE arranged in the order of KE/GE/GE/KE. The maximum relative values increase for the first three modes, in the range $0.2 \leq$

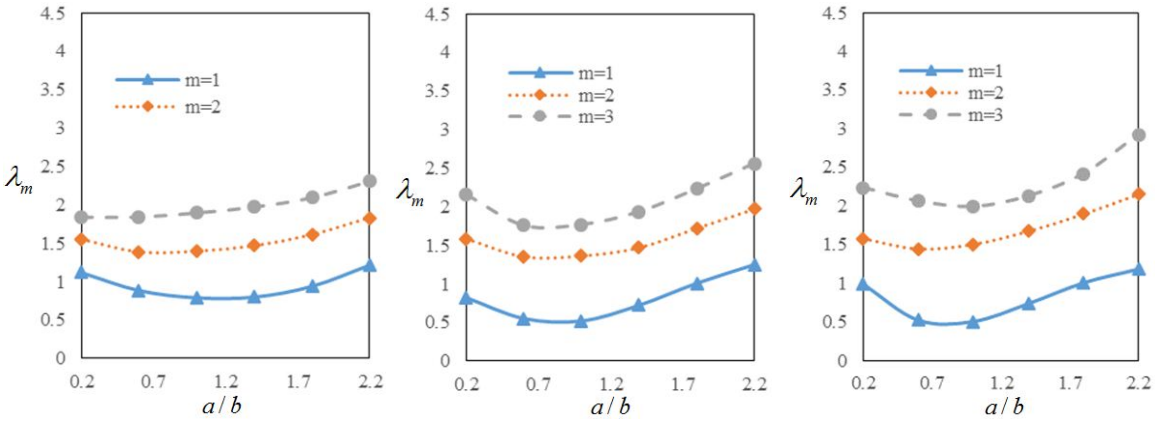


Fig. 3 The frequency variation of four layered anti-symmetric angle-ply plates with respect to aspect ratio: (a) 30°/–60°/60°/–30°; (b) 60°/–45°/45°/–60°; (c) 45°/–60°/60°/–45° consisting of KE/GE/GE/KE materials combination

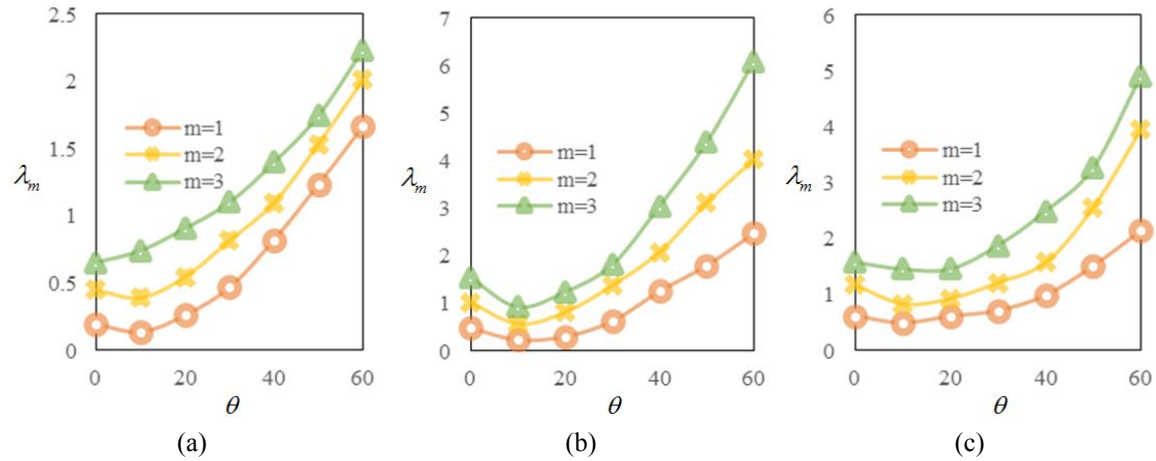


Fig. 4 The frequency variation of two layered anti-symmetric angle-ply plates with respect to laying-angle (a) $a / b = 0.4$; (b) $a / b = 1$; (c) $a / b = 1.6$ consisting of KE material

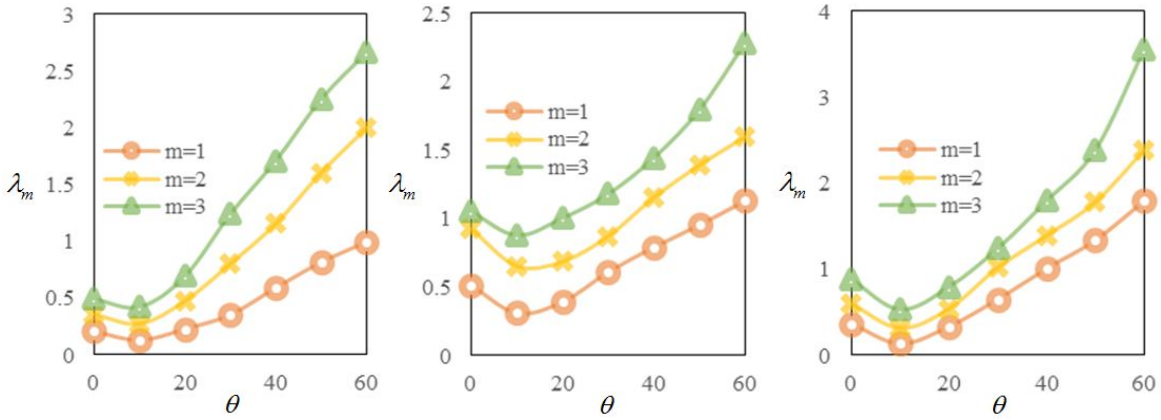


Fig. 5 The frequency variation of four layered anti-symmetric angle-ply plates with respect to laying-angle: (a) $a / b = 0.4$; (b) $a / b = 1$; (c) $a / b = 1.6$ consisting of (KE/GE/GE/KE) materials combination

$(a / b) \leq 2.2$, are 0.096119, 0.27516 and 0.467566 for Fig. 3(a), 0.426667, 0.384791 and 0.405663 for Fig. 3(b) and 0.193654, 0.578167 and 0.688356 for Fig. 3(c).

### *3.3.2 Effect of ply-angle*

The effect of ply-angle $\theta$ and aspect ratio shows that the frequency value decreases between $0 \leq \theta \leq 20$ and significantly increases afterwards for two layered antisymmetric plates using KE material. The aspect ratio is fixed as $a / b = 0.4$ for Fig. 4(a), $a / b = 1$ for Fig. 4(b), and $a / b = 1.6$ for Fig. 4(c).

Four layered anti-symmetric angle-ply plates with material combination (KE/GE/GE/KE) are considered to examine the effect of ply-angle on the value of the frequency parameter. Figure 5. shows that the frequency value iniallly decreases for certain value and increases afterwards.The aspect ratio is fixed as $a / b = 0.4$ for Fig. 5(a), $a / b = 1$ for Fig. 5(b), and $a / b = 1.6$ for Fig. 5 (c).

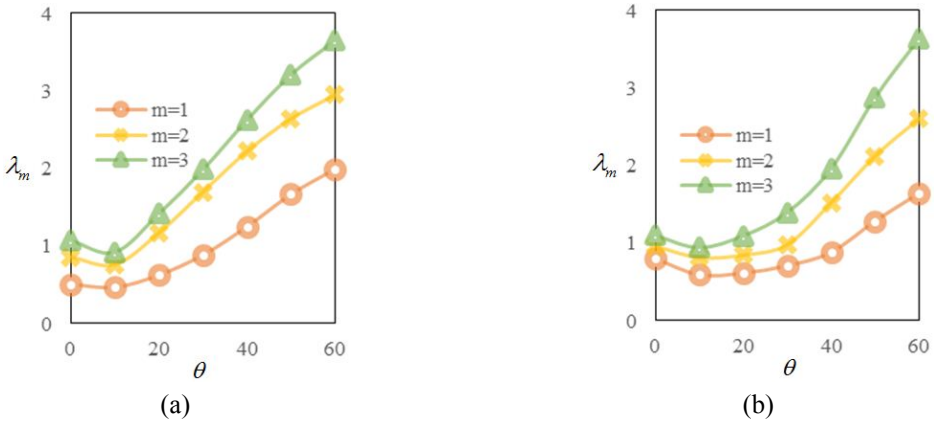


Fig. 6 The frequency variation of six layered anti-symmetric angle-ply plates with respect to laying-angle consisting of: (a) (KE/GE/KE/KE/GE/KE); (b) (GE/KE/GE/GE/KE/GE) materials combination $a / b = 1$

Fig. 6 shows six layered antisymmetric angle-ply plates with two different material combinations are considered (KE/GE/KE/KE/GE/KE), (GE/KE/GE/GE/KE/GE). The aspect ratio is fixed as $a/b = 1$. It is observed that there is a decrease and then increase in the frequency values with ply-angle increases.

### *3.3.3 Effect of side-to-thickness-ratio*

The variation of the frequency parameter between the range of side-to-thickness ratio $10 \leq (a/h) \leq 60$ is presented in Fig. 7 showing that frequency value decraeases strictly between $10 \leq (a/h) \leq 20$ and decraeases slowly afterwards. Six layered antisymmetric plates with laying-angle 60°/–45°/30°/–30°/45°/–60° (GE/KE/GE/GE/KE/GE) is considered. The aspect ratio is fixed as $a/b = 1$ and $a/b = 1.6$ for Figs. 7(a) and (b) repectively.

Fig. 8 illustrates the effect of different material combinations and side-to-thickness ratio with ply-angle arranged as 30°/–45°/60°/–60°/45°/–30° and material combinations for Fig. 8(a) as

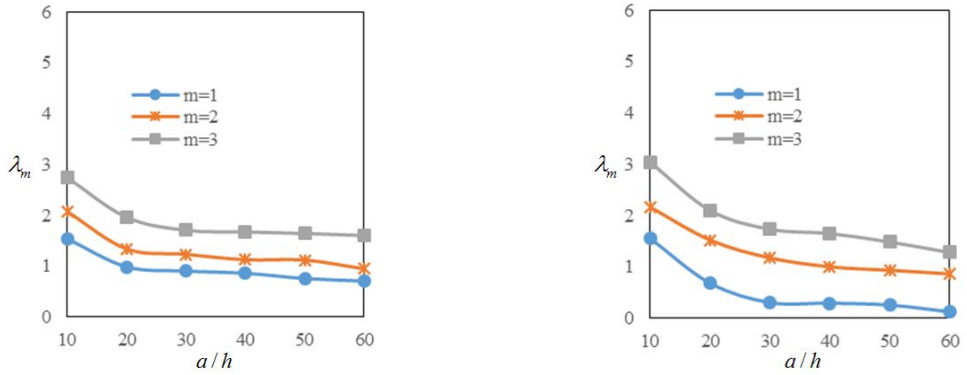


Fig. 7 The frequency variation of six layered anti-symmetric angle-ply plates with respect to side-to-thickness ratio consisting of (a) $a/b = 1$; (b) $a/b = 1.6$, laying-angle 60°/–45°/30°/–30°/45°/–60° materials combination (GE/KE/GE/GE/KE/GE)

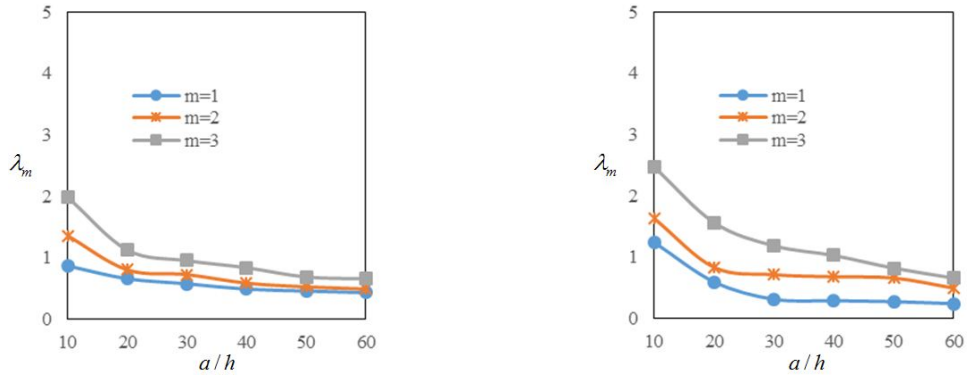


Fig. 8 The frequency variation of six layered anti-symmetric angle-ply plates with respect to side-to-thickness ratio consisting of laying-angle 30°/–45°/60°/–60°/45°/–30°; (a) (GE/KE/GE/GE/KE/GE); (b) (KE/GE/KE/KE/GE/KE) materials combination, $a/b = 0.4$

(GE/KE/GE/GE/KE/GE) and Fig. 8(b) as (KE/GE/KE/KE/GE/KE) on the frequency parameter value. The aspect ratio is fixed as $a/b = 0.4$. It can be seen that the frequency value decreases strictly between $10 \leq (a/h) \leq 20$ and slowly decreases afterwards.

The effect of different aspect ratio and side-to-thickness ratio on the frequency parameter value of six layered plates with laying-angle 30°/–45°/60°/–60°/45°/–30° (KE/GE/KE/KE/GE/KE) is presented in Fig. 9. The results shows that frequency value decreases with the increase of side to thickness ratio. Further, the frequency value is higher for aspect ratio $a/b = 1$ as compare to $a/b = 0.4$.

Four layered plates with anti-symmetric layers 30°/–30°/30°/–30°, 45°/–45°/45°/–45° and 60°/–60°/60°/–60° and material combination as (GE/KE/KE/GE) are considered in Fig. 10. The

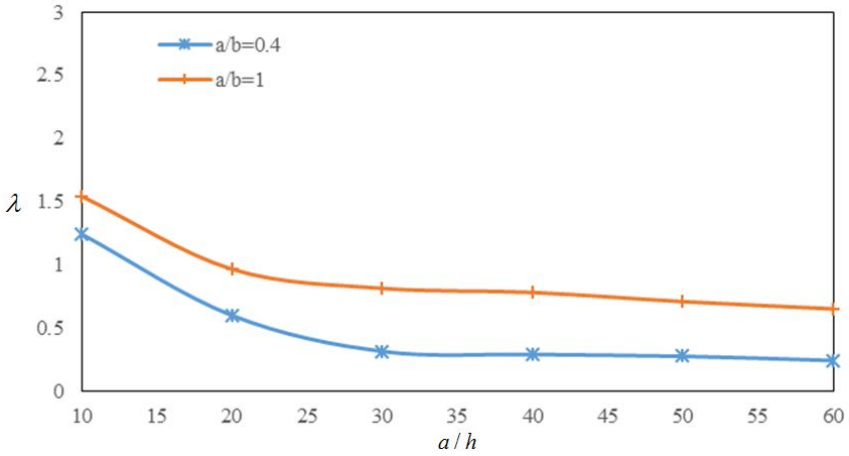


Fig. 9 The fundamental frequency variation of six layered anti-symmetric angle-ply plates with respect to side-to-thickness ratio consisting of laying-angle 30°/–45°/60°/–60°/45°/–30° (KE/GE/KE/KE/GE/KE) materials combination: (a) $a/b = 0.4$; (b) $a/b = 1$

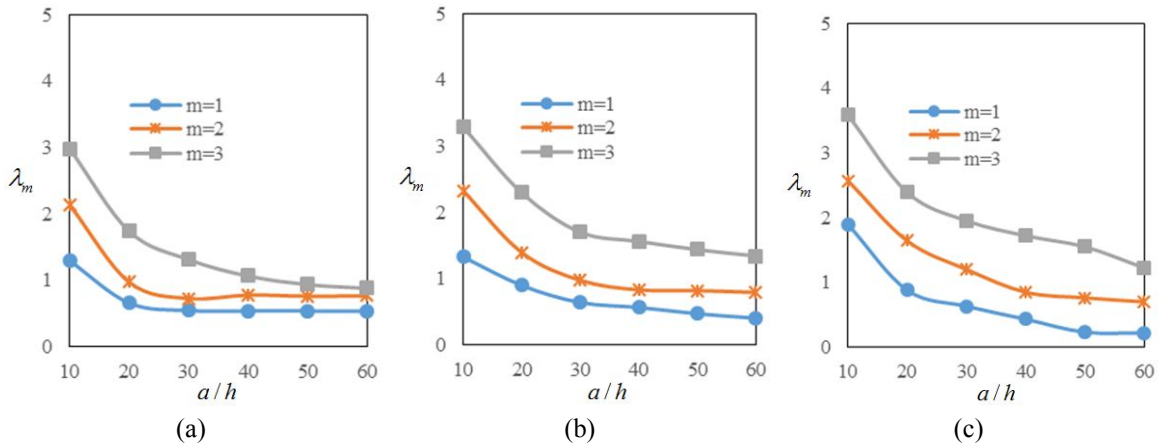


Fig. 10 The fundamental frequency variation of four layered anti-symmetric angle-ply plates with respect to side-to-thickness ratio consisting of laying-angle: (a) 30°/–30°/30°/–30°; (b) 45°/–45°/45°/–45°; (c) 60°/–60°/60°/–60° (GE/KE/KE/GE) materials combination, $a/b = 1$

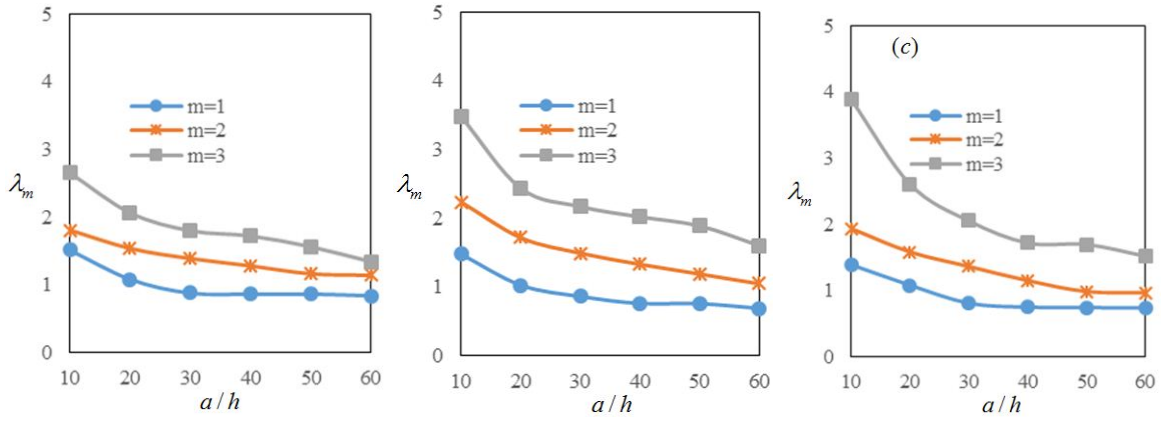


Fig. 11 The fundamental frequency variation of two layered anti-symmetric angle-ply plates with respect to side-to-thickness ratio consisting of laying-angle: (a) 30°/–30°; (b) 45°/–45°; (c) 60°/–60° (GE/GE) materials combination, $a / b = 1$

aspect ratio is fixed as $a / b = 1$. The effect of side-to-thickness ratio shows that frequency decreases significantly in the beginning and decreases slowly between $20 \leq a / b \leq 60$. Further, it is also seen that frequency value increases with the increase of laying-angle.

The frequency vatiation of two layered plates with laying-angles as 30°/–30°, 45°/–45°, 60°/–60° and material combination as (GE/GE) is presented in Fig. 11. The aspect ratio is fixed as $a / b = 1$. It is observed that frequency decreases with the increase of side-to-thickness ratio.

## 4. Conclusions

In this paper the free vibration of anti-symmetric angle-ply platesare analysed using spline approximationunder higher-order shear deformation theory. The vibrational behavior of laminatedplates is examined for simply suportedboundary conditions. The vibration characteristic of the plates is examined for aspect ratio, ply-angle, side-to-thickness ratio, differentnumber of layersand two different lamination materials. It is concluded that variation of the geometric parameters and materials effect the frequencies of plates, which may be beneficial for the designers of related fields.

## Acknowledgments

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## Appendix

$$L_{11}=\frac{d^2}{dX^2}-s_{10}\beta^2+\lambda^2 \qquad L_{12}=(s_2+s_{10})\beta\frac{d}{dX}$$
$$L_{13}=-3s_{33}c_1\beta\frac{d^2}{dX^2}+s_{34}c_1\beta^3 \qquad L_{14}=2\left(s_{15}-s_{33}c_1\right)\beta\frac{d}{dX} \tag{A1}$$
$$L_{15}=\left(s_{15}-s_{33}c_1\right)\frac{d^2}{dX^2}-\left(s_{16}-s_{34}c_1\right)\beta^2$$

$$L_{21}=-(s_{10}+s_2)\beta\frac{d}{dX}, \qquad L_{22}=s_{10}\frac{d^2}{dX^2}-s_3\beta^2+\lambda^2$$
$$L_{23}=-s_{33}c_1\frac{d^3}{dX^3}+3s_{34}c_1\beta^2\frac{d}{dX}, \qquad L_{24}=\left(s_{15}-s_{33}c_1\right)\frac{d^2}{dX^2}-\left(s_{16}-s_{34}c_1\right)\beta^2 \tag{A2}$$
$$L_{25}=-2\left(s_{16}-s_{34}c_1\right)\beta\frac{d}{dX}$$

$$L_{31}=-3s_{33}c_1\beta\frac{d^2}{dX^2}+s_{34}c_1\beta^3 \qquad L_{32}=s_{33}c_1\frac{d^3}{dX^3}-3s_{34}c_1\beta^2\frac{d}{dX}$$
$$L_{33}=-s_{23}c_1^2\frac{d^4}{dX^4}+\left(2s_{24}c_1^2\beta^2+4s_{28}c_1^2\beta^2+s_{14}-2s_{30}c_2+s_{32}c_2^2\right)\frac{d^2}{dX^2}$$
$$-\left(s_{25}c_1^2\beta^4+s_{13}\beta^2-2s_{29}c_2\beta^2+s_{31}c_2^2\beta^2\right)-c_1^2\lambda^2p_3\frac{d^2}{dX^2}+\left(\lambda^2+c_1^2\lambda^2p_3\beta^2\right)$$
$$L_{34}=\left(s_{20}c_1-s_{23}c_1^2\right)\frac{d^3}{dX^3}-\left(2s_{27}c_1\beta^2-2s_{28}c_1^2\beta^2+s_{21}c_1\beta^2\right. \tag{A3}$$
$$\left.-s_{24}c_1^2\beta^2-s_{14}+2s_{30}c_2-s_{32}c_2^2\right)\frac{d}{dX}+\left(c_1\lambda^2p_2-c_1^2\lambda^2p_3\right)\frac{d}{dX}$$
$$L_{35}=-\left(s_{21}c_1-s_{24}c_1^2+2s_{27}c_1-2s_{28}c_1^2\right)\beta\frac{d^2}{dX^2}$$
$$+\left(s_{22}c_1\beta^3-s_{25}c_1^2\beta^3-s_{13}\beta+2s_{29}c_2\beta-s_{31}c_2^2\beta\right)-\left(c_1\lambda^2p_2\beta-c_1^2\lambda^2p_3\beta\right)$$

$$L_{41}=-2\left(s_{15}-s_{33}c_1\right)\beta\frac{d}{dX}$$
$$L_{42}=\left(s_{15}-s_{33}c_1\right)\frac{d^2}{dX^2}-\left(s_{16}-s_{34}c_1\right)\beta^2$$
$$L_{43}=\left(s_{23}c_1^2-s_{20}c_1\right)\frac{d^3}{dX^3}-\left(s_{24}c_1^2\beta^2-s_{21}c_1\beta^2+2s_{28}c_1^2\beta^2-2s_{27}c_1\beta^2+s_{14}-2s_{30}c_2+s_{32}c_2^2\right)\frac{d}{dX} \tag{A4}$$
$$-\left(c_1\lambda^2p_2-c_1^2\lambda^2p_3\right)\frac{d}{dX}$$

$$
\begin{aligned}
L_{44} &= \left(s_7 - 2s_{20}c_1 + s_{23}c_1^2\right)\frac{d^2}{dX^2} - \left(s_{12}\beta^2 - 2s_{27}c_1\beta^2 + s_{28}c_1^2\beta^2 + s_{14} - 2s_{30}c_2 + s_{32}c_2^2\right) \\
&\quad + \left(\lambda^2 p_1 - 2c_1\lambda^2 p_2 + c_1^2\lambda^2 p_3\right) \\
L_{45} &= -\left(s_8 - 2s_{21}c_1 + s_{24}c_1^2 + s_{21} - 2s_{27}c_1 + s_{28}c_1^2\right)\beta\frac{d}{dX}
\end{aligned} \tag{A4}
$$

$$
L_{51} = \left(s_{15} - s_{33}c_1\right)\frac{d^2}{dX^2} - \left(s_{16} - s_{34}c_1\right)\beta^2, \qquad L_{52} = 2\left(s_{16} - s_{34}c_1\right)\beta\frac{d}{dX}
$$

$$
\begin{aligned}
L_{53} &= \left(2s_{28}c_1^2 - 2s_{27}c_1 + s_{24}c_1^2 - s_{21}c_1\right)\beta\frac{d^2}{dX^2} - \left(s_{25}c_1^2\beta^3 - s_{22}c_1\beta^3 + s_{13}\beta - 2s_{29}c_2\beta + s_{31}c_2^2\beta\right) \\
&\quad - \left(c_1\beta\lambda^2 p_2 - c_1^2\beta\lambda^2 p_3\right) \\
L_{54} &= \left(s_{12} - 2s_{27}c_1 + s_{28}c_1^2 + s_8 - 2s_{21}c_1 + s_{24}c_1^2\right)\beta\frac{d}{dX} \\
L_{55} &= \left(s_{12} - 2s_{27}c_1 + s_{28}c_1^2\right)\frac{d^2}{dX^2} - \left(s_9\beta^2 - 2s_{22}c_1\beta^2 + s_{25}c_1^2\beta^2 + s_{13} - 2s_{29}c_2 + s_{31}c_2^2\right) \\
&\quad + \left(\lambda^2 p_1 - 2c_1\lambda^2 p_2 + c_1^2\lambda^2 p_3\right)
\end{aligned} \tag{A5}
$$

where

$$
\lambda^2 = \frac{I_0\omega^2 a^2}{A_{11}}, \quad p_1 = \frac{I_2}{I_0 a^2},
$$

$$
p_2 = \frac{I_4}{I_0 a^2}, p_3 = \frac{I_6}{I_0 a^2}
$$

$$
\beta = n\pi\phi
$$

$$
s_2 = \frac{A_{12}}{A_{11}}, \; s_3 = \frac{A_{22}}{A_{11}}, s_7 = \frac{D_{11}}{a^2 A_{11}}, \; s_8 = \frac{D_{12}}{a^2 A_{11}}, \; s_9 = \frac{D_{22}}{a^2 A_{11}}, s_{10} = \frac{A_{66}}{A_{11}}, s_{12} = \frac{D_{66}}{a^2 A_{11}}, s_{13} = \frac{A_{44}}{A_{11}},
$$

$$
s_{14} = \frac{A_{55}}{A_{11}}, s_{15} = \frac{B_{16}}{aA_{11}}, s_{16} = \frac{B_{26}}{aA_{11}}, s_{20} = \frac{F_{11}}{a^4 A_{11}}, s_{21} = \frac{F_{12}}{a^4 A_{11}}, s_{22} = \frac{F_{22}}{a^4 A_{11}}, s_{23} = \frac{H_{11}}{a^6 A_{11}}, s_{24} = \frac{H_{12}}{a^6 A_{11}}
$$

$$
s_{25} = \frac{H_{22}}{a^6 A_{11}}, s_{27} = \frac{F_{66}}{a^4 A_{11}}, s_{28} = \frac{H_{66}}{a^6 A_{11}}, s_{29} = \frac{D_{44}}{a^2 A_{11}}, s_{30} = \frac{D_{55}}{a^2 A_{11}},
$$

$$
s_{31} = \frac{F_{44}}{a^4 A_{11}}, s_{32} = \frac{F_{55}}{a^4 A_{11}}, s_{33} = \frac{E_{16}}{a^3 A_{11}}, \; s_{34} = \frac{E_{26}}{a^3 A_{11}}
$$