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Analytical solution for buckling of embedded laminated plates based on higher order shear deformation plate theory

Vahid Baseri, Gholamreza Soleimani Jafari* and Reza Kolahchi

Department of Mechanical Engineering, *Kashan Branch*, *Islamic Azad University*, *Kashan*, *Iran*

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Abstract. In this research, buckling analysis of an embedded laminated composite plate is investigated. The elastic medium is simulated with spring constant of Winkler medium and shear layer. With considering higher order shear deformation theory (Reddy), the total potential energy of structure is calculated. Using Principle of Virtual Work, the constitutive equations are obtained. The analytical solution is performed in order to obtain the buckling loads. A detailed parametric study is conducted to elucidate the influences of the layer numbers, orientation angle of layers, geometrical parameters, elastic medium and type of load on the buckling load of the system. Results depict that the highest buckling load is related to the structure with angle-ply orientation type and with increasing the angle up to 45 degrees, the buckling load increases.

Keywords: buckling; laminated composite; Reddy theory; analytical method; elastic medium

1. Introduction

Laminated composites have superior properties compared to the conventional materials like metal, wood and so on. These properties include high strength to weight ratio, excellent fatigue characteristics, high abrasion and bending strength, low weight to volume ratio, good thermal insulation and so forth. So in recent years, the study and analysis of the dynamics behavior of these structures are increased among the researchers. Dawe and Yuan (2001) analyzed the overall and local buckling of laminated composites plates. They used the high order shear deformation theory (HSDT) for mathematical modeling of the structure and applied the finite strip method (FSM) for solving the problem. They examined the effects of the geometrical parameters and also the orientation angle of the layers on the buckling behavior of the system. Chakrabarti and Sheikh (2006) studied the dynamic instability of laminated sandwich plates subjected to in-plane edge loading using finite element method (FEM). The plate model is based on refined HSDT. They solved a number of problems including various boundary conditions, plate geometry, thickness ratio and other aspects. Pandita *et al*. (2008a, b, 2009) discussed the vibration and buckling of the laminated sandwich plates. They presented an improved higher order zigzag theory and solved the problem by utilizing the FEM. The three-dimensional solution for static analysis of cross-ply rectangular plate embedded in piezoelectric layers was presented by Alibeigloo and Madoliat (2009). They applied differential quadrature method (DQM) and Fourier series approach for

*Corresponding author, Ph.D., E-mail: soleimani.kashan@gmail.com

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solving the problem. They introduced the upper layer and lower one as an actuator and sensor, respectively. Both the direct and inverse piezoelectric effects and the influence of piezoelectric layers on the mechanical behavior of the structure were investigated by the authors. Chen *et al.* (2012) developed a model for the composite laminated Reddy plate based on a new modified couple stress theory. They examined the deflections and stresses of the plate and showed that this model of plate can capture the scale effects of microstructures. Reddy (2012) investigated the bending of the beams and plates. He reformulated the classical and shear deformation beam and plate theories using the nonlocal differential constitutive relations of Eringen and von Karman nonlinear strains. The effect of the geometric nonlinearity and nonlocal parameters was studied by the author. Mantari *et al.* (2012a) proposed a new trigonometric shear deformation theory for laminated composite plates. They developed the finite element formulation to obtain the stresses. They compared the mentioned theory with other available theories and demonstrated that the accuracy of the results is higher than similar ones. Mantari *et al.* (2012b) also employed the Navier solution method for static bending analysis based on the trigonometric shear deformation theory. Sahoo and Singh (2013a, b, 2014) investigated the static analysis of the laminated composite plates in macro-scale. They modeled the structure using a trigonometric zigzag theory. The numerical FEM is used to calculate the bending of the laminated composite plate. A cell-based smoothed discrete shear gap method (CS-FEM-DSG3) using triangular elements was recently proposed by Nguyen-Thoi *et al.* (2013) to improve the performance of the discrete shear gap method (DSG3) for static and dynamics analyses of Mindlin plates. An isogeometric finite element approach (IGA) in combination with the third-order deformation plate theory (TSDT) was used by Tran *et al.* (2013) for thermal buckling analysis of functionally graded material (FGM) plates. Vidal and Polit (2013) probed the buckling analysis of laminated composite plates using a refined shear sinusoidal plate theory. A simple and effective formulation based on a fifth-order shear deformation theory (FSDT) in combination with IGA was presented by Nguyen-Xuan *et al.* (2013) for composite sandwich plates. Based on a CS-FEM-DSG3 and FSDT, Phung-Van *et al.* (2014a) investigated static and dynamics analyses of Mindlin plates resting on viscoelastic foundation. Sayyad and Ghugal (2014) developed the analytical solution for the biaxial bending analysis of isotropic, transversely isotropic and laminated composite plates based on a sinusoidal shear and normal deformation theory which taking into account effects of transverse shear and transverse normal. Luong-Van *et al.* (2014) used a cell-based smoothed finite element method using three-node shear-locking free Mindlin plate element (CS-FEM-MIN3) for dynamic response of laminated composite plates on viscoelastic foundation. The CS-FEM-DSG3 was extended to the C0-type HSDT by Phung-Van *et al.* (2014b) and was incorporated with damping–spring systems for dynamic responses of Mindlin plates on viscoelastic foundations subjected to a moving sprung vehicle. An edge-based smoothed stabilized discrete shear gap method (ES-DSG3) based on FSDT was recently proposed by Phung-Van *et al.* (2014c) for static and dynamic analyses of Mindlin plates. Thai *et al.* (2014) presented a generalized shear deformation theory for static, dynamic and buckling analysis of functionally graded material (FGM) made of isotropic and sandwich plates. A cell-based smoothed three-node Mindlin plate element (CS-MIN3) was extended by Phung-Van *et al.* (2014d) to geometrically nonlinear analysis of functionally graded plates (FGPs) subjected to thermo-mechanical loadings. The CS-FEM-DSG3 was extended and incorporated by Phung-Van *et al.* (2014e) with a layerwise theory for static and free vibration analyses of composite and sandwich plates. A simple and effective approach that incorporates IGA with a refined plate theory (RPT) was addressed by Nguyen-Xuan *et al.* (2014) for static, free vibration and buckling analysis of FGM plates. An efficient computational approach based on refined plate theory (RPT) including the thickness stretching effect, namely quasi-3D theory, in conjunction with IGA was proposed by Nguyen *et al*. (2015) for the size-dependent bending, free vibration and buckling analysis of functionally graded nanoplate structures. An efficient computational approach based on a generalized unconstrained approach in conjunction with IGA were proposed by Phung-Van *et al*. (2015a) for dynamic control of smart piezoelectric composite plates. Phung-Van *et al*. (2015b), the CS-FEM-MIN3 was extended to geometrically nonlinear analysis of laminated composite plates. A simple and effective formulation based on IGA and HSDT was applied by Phung-Van *et al*. (2015c) to investigate the static and dynamic behavior of functionally graded carbon nano-reinforced composite plates. Nguyen *et al*. (2016) introduced a unified framework on HSDTs, modelling and analysis of laminated composite plates.

In the present work, the buckling behavior of the laminated composite plate embedded in elastic medium is studied. The mathematical model of the structure is afforded based on higher order shear deformation theory (Reddy). By applying Navier solution method, the buckling load of the system is obtained and the effects of various parameters such as elastic medium, angle orientation of layers, geometric parameters and number of layers on the buckling behavior of the system are probed.

2. Mathematical formulation

Fig. 1 shows a laminated composite plate embedded in elastic medium which is modeled by Winkler springs and Pasternak shear layer. The length, width and thickness of the plate are *a*, *b* and *h*, respectively.

2.1 Third order shear deformation theory (*TSDT*)

This section examines the displacement field of the third order shear deformation theory (TSDT) which is proposed by Reddy. This theory assumes that the thickness does not change. So the displacement field is defined as a cubic function of z and transverse shear stresses are the functions of second order. For this reason, no need to use the shear correction factor in the FSDT and displacement field is considered as follows

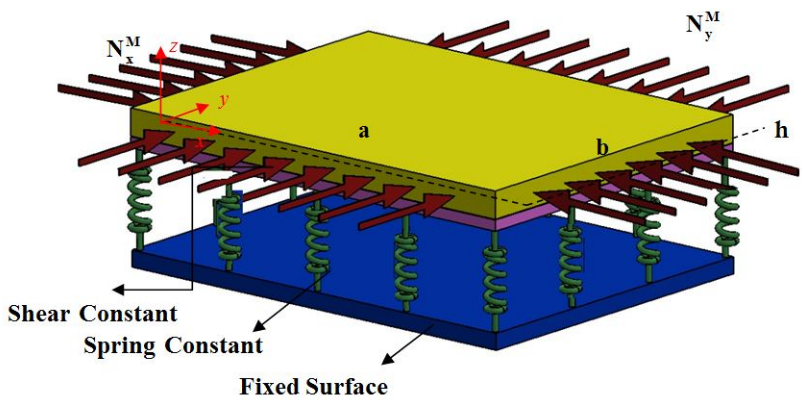

Fig. 1 Geometry of a laminated composite plate embedded in elastic medium

$$\begin{cases} u_1(x,y,z,t) = u(x,y,t) + z\,\phi_x(x,y,t) + c_1 z^3\left(\phi_x + \dfrac{\partial w}{\partial x}\right), \\ u_2(x,y,z,t) = v(x,y,t) + z\,\phi_y(x,y,t) + c_1 z^3\left(\phi_y + \dfrac{\partial w}{\partial y}\right), \\ u_3(x,y,z,t) = w(x,y,t), \end{cases} \tag{1}$$

where u, v and w are the displacement components of the mid-plane and ϕ_x, ϕ_y are the angle of rotation around the y and x axes of cross-section, respectively. Also $c_1 = -4 / 3h^2$ in which h is the thickness of the plate. So the kinematic relations are defined as follows

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{pmatrix} + z\begin{pmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{pmatrix} + z^3\begin{pmatrix} \varepsilon_{xx}^3 \\ \varepsilon_{yy}^3 \\ \gamma_{xy}^3 \end{pmatrix}, \tag{2a}$$

$$\begin{pmatrix} \gamma_{yz} \\ \gamma_{xz} \end{pmatrix} = \begin{pmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{pmatrix} + z^2\begin{pmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{pmatrix}, \tag{2b}$$

in which

$$\begin{pmatrix} \varepsilon_{xx}^0 \\ \varepsilon_{yy}^0 \\ \gamma_{xy}^0 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{xx}^1 \\ \varepsilon_{yy}^1 \\ \gamma_{xy}^1 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{xx}^3 \\ \varepsilon_{yy}^3 \\ \gamma_{xy}^3 \end{pmatrix} = c_1\begin{pmatrix} \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 w}{\partial y^2} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 w}{\partial x \partial y} \end{pmatrix}, \tag{2c}$$

$$\begin{pmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{pmatrix} = \begin{pmatrix} \phi_y + \dfrac{\partial w}{\partial y} \\ \phi_x + \dfrac{\partial w}{\partial x} \end{pmatrix}, \quad \begin{pmatrix} \gamma_{yz}^2 \\ \gamma_{xz}^2 \end{pmatrix} = c_2\begin{pmatrix} \phi_y + \dfrac{\partial w}{\partial y} \\ \phi_x + \dfrac{\partial w}{\partial x} \end{pmatrix}, \tag{2d}$$

where $c_2 = 3c_1$.

2.2 Stress-Strain relations

In this paper, the material of the layers obeys Hook's law and the constitutive equations are as follows

$$[\sigma]^{6\times1} = [C]^{6\times6}[\varepsilon]^{6\times1}, \tag{3}$$

in which $[\sigma]$, $[C]$ and $[\varepsilon]$ are stress, stiffness and strain matrices, respectively. Since in this research, the material of the layers is assumed to be orthotropic, Eq. (3) can be rewritten as follows

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{xz} \\ \sigma_{zy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{zy} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}. \tag{4}$$

To transform the stress-strain relations from local coordinate to reference one, we define

$$[\sigma] = [T][\sigma]_l, \tag{5a}$$

in which

$$[T] = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 0 & 0 & 0 & -\sin 2\theta \\ \sin^2\theta & \cos^2\theta & 0 & 0 & 0 & \sin 2\theta \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ \sin\theta\cos\theta & -\sin\theta\cos\theta & 0 & 0 & 0 & \cos^2\theta - \sin^2\theta \end{bmatrix}. \tag{5b}$$

By applying Eq. (5), the stress-strain relations in reference coordinate are obtained as follows

$$[\sigma] = [T][\sigma]_l = [T][C]_l[\varepsilon]_l = \underbrace{[T][C]_l[T]^T}_{[Q]}[\varepsilon], \tag{5c}$$

where Q_{ij} are the transformed material constants in the reference coordinate (Chow *et al.* 1992). So the constitutive equations can be expressed as follows

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix}, \tag{7}$$

where Q_{ij} are considered as follows

$$\begin{aligned} Q_{11} &= C_{11}\cos^4\theta - 4C_{16}\cos^3\theta\sin\theta + 2(C_{12} + 2C_{66})\cos^2\theta\sin^2\theta \\ &\quad - 4C_{26}\cos\theta\sin^3\theta + C_{22}\sin^4\theta, \end{aligned} \tag{8a}$$

$$\begin{aligned} Q_{12} &= C_{12}\cos^4\theta + 2(C_{16} - C_{26})\cos^3\theta\sin\theta + (C_{11} + C_{22} - 4C_{66})\cos^2\theta\sin^2\theta \\ &\quad + 2(C_{26} - C_{16})\cos\theta\sin^3\theta + C_{12}\sin^4\theta, \end{aligned} \tag{8b}$$

$$Q_{16}=C_{16}\cos^4\theta+(C_{11}-C_{12}-2C_{66})\cos^3\theta\sin\theta+3(C_{26}-C_{16})\cos^2\theta\sin^2\theta \\ +(2C_{66}+C_{12}-C_{22})\cos\theta\sin^3\theta-C_{26}\sin^4\theta, \tag{8c}$$

$$Q_{22}=C_{22}\cos^4\theta+4C_{26}\cos^3\theta\sin\theta+2(C_{12}+2C_{66})\cos^2\theta\sin^2\theta \\ +4C_{16}\cos\theta\sin^3\theta+C_{11}\sin^4\theta, \tag{8d}$$

$$Q_{26}=C_{26}\cos^4\theta+(C_{12}-C_{22}+2C_{66})\cos^3\theta\sin\theta+3(C_{16}-C_{26})\cos^2\theta\sin^2\theta \\ +(C_{11}-C_{12}-2C_{66})\cos\theta\sin^3\theta-C_{16}\sin^4\theta, \tag{8e}$$

$$Q_{66}=2(C_{16}-C_{26})\cos^3\theta\sin\theta+(C_{11}+C_{22}-2C_{12}-2C_{66})\cos^2\theta\sin^2\theta \\ +2(C_{26}-C_{26})\cos\theta\sin^3\theta+C_{66}(\cos^4\theta+\sin^4\theta), \tag{8f}$$

$$Q_{44}=C_{44}\cos^2\theta+2C_{45}\cos\theta\sin\theta+C_{55}\sin^2\theta, \tag{8g}$$

$$Q_{45}=(C_{55}-C_{44})\cos\theta\sin\theta+C_{44}(\cos^2\theta-\sin^2\theta), \tag{8h}$$

$$Q_{55}=C_{55}\cos^2\theta-2C_{45}\cos\theta\sin\theta+C_{44}\sin^2\theta. \tag{8i}$$

2.3 Derivation of governing equations

The strain energy stored in the structure can be considered as follows

$$U=\frac{1}{2}\int_{-h/2}^{h/2}\int_0^b\int_0^a\left(\sigma_{xx}\varepsilon_{xx}+\sigma_{yy}\varepsilon_{yy}+\sigma_{xy}\gamma_{xy}+\sigma_{xz}\gamma_{xz}+\sigma_{yz}\gamma_{yz}\right)dxdydz, \tag{9}$$

By applying Eq. (2) we have

$$U=\frac{1}{2}\int_{-h/2}^{h/2}\int_0^b\int_0^a\left[\begin{array}{l}\sigma_{xx}\left(\frac{\partial u}{\partial x}+z\frac{\partial\varphi_x}{\partial x}+c_1z^3\left(\frac{\partial\varphi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)\right)+\sigma_{yy}\left(\frac{\partial v}{\partial y}+z\frac{\partial\varphi_y}{\partial y}+c_1z^3\left(\frac{\partial\varphi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)\right)\\ +\sigma_{xy}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+z\left(\frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x}\right)+c_1z^3\left(\frac{\partial\varphi_x}{\partial y}+\frac{\partial\varphi_y}{\partial x}+2\frac{\partial^2 w}{\partial x\partial y}\right)+\right)\\ \sigma_{xz}\left(\varphi_x+\frac{\partial w}{\partial x}+c_2z^2\left(\varphi_x+\frac{\partial w}{\partial x}\right)\right)+\sigma_{yz}\left(\varphi_y+\frac{\partial w}{\partial y}+c_2z^2\left(\varphi_y+\frac{\partial w}{\partial y}\right)\right)\end{array}\right]dxdydz. \tag{10}$$

By defining the below relations

$$\begin{Bmatrix}N_{xx}\\ N_{yy}\\ N_{xy}\end{Bmatrix}=\int_{-h/2}^{h/2}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{bmatrix}dz, \tag{11a}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} z dz, \tag{11b}$$

$$\begin{Bmatrix} P_{xx} \\ P_{yy} \\ P_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} z^3 dz, \tag{11c}$$

$$\begin{bmatrix} Q_{xx} \\ Q_{yy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} dz, \tag{11d}$$

$$\begin{bmatrix} K_{xx} \\ K_{yy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} z^2 dz, \tag{11e}$$

the Eq. (10) can be rewritten as follows

$$\begin{aligned} U = \frac{1}{2}\int_0^b \int_0^a \Bigg(& N_{xx}\left(\frac{\partial u}{\partial x}\right) + N_{yy}\left(\frac{\partial v}{\partial y}\right) + Q_{yy}\left(\frac{\partial w}{\partial y} + \phi_y\right) + Q_{xx}\left(\frac{\partial w}{\partial x} + \phi_x\right) + N_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right). \\ & + M_{xx}\frac{\partial \phi_x}{\partial x} + M_{yy}\frac{\partial \phi_y}{\partial y} + M_{xy}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) + K_{yy}\left(c_2\left(\phi_y + \frac{\partial w}{\partial y}\right)\right) + K_{xx}\left(c_2\left(\phi_x + \frac{\partial w}{\partial x}\right)\right) \\ & + P_{xx}\left(c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)\right) + P_{yy}\left(c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right)\right) + P_{xy}\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right)\Bigg) dxdy \end{aligned} \tag{12}$$

The work due to the in-plane external loads and elastic medium can be expressed as (Ghorbanpour Arani *et al*. 2012)

$$W = -\frac{1}{2}\int_0^b \int_0^a \left[N_{xx}^M \left(\frac{\partial w}{\partial x}\right)^2 + N_{yy}^M \left(\frac{\partial w}{\partial y}\right)^2 \right] dxdy - \int_0^b \int_0^a \left(k_w w - k_g \nabla^2\right) dxdy, \tag{13}$$

where k_w and k_g are the Winkler and Pasternak stiffness coefficients, respectively. Also N_{xx}^M and $N_{yy}^M = \alpha N_{xx}^M$ are applied loads to the plate in x and y directions, respectively and α is a constant coefficient.

2.3.1 Principle of virtual work

To determine the equations of motion, Principle of Virtual Work is applied as follows

$$\int_0^t (-\delta U + \delta W) dt = 0. \tag{14}$$

Now by calculating the variation of the Eqs. (12) and (13), and substituting into Eq. (14), the equations of motion are obtained as follows

$$\delta u:\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0, \tag{15}$$

$$\delta v:\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_{yy}}{\partial y}=0, \tag{16}$$

$$\begin{aligned}\delta w:&\frac{\partial Q_{xx}}{\partial x}+\frac{\partial Q_{yy}}{\partial y}+c_2\left(\frac{\partial K_{xx}}{\partial x}+\frac{\partial K_{yy}}{\partial y}\right)+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}\\&-c_1\left(\frac{\partial^2 P_{xx}}{\partial x^2}+2\frac{\partial^2 P_{xy}}{\partial x\partial y}+\frac{\partial^2 P_{yy}}{\partial y^2}\right)+q=0,\end{aligned} \tag{17}$$

$$\delta\phi_x:\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{xy}}{\partial y}+c_1\left(\frac{\partial P_{xx}}{\partial x}+\frac{\partial P_{xy}}{\partial y}\right)-Q_{xx}-c_2K_{xx}=0, \tag{18}$$

$$\delta\phi_y:\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_{yy}}{\partial y}+c_1\left(\frac{\partial P_{xy}}{\partial x}+\frac{\partial P_{yy}}{\partial y}\right)-Q_{yy}-c_2K_{yy}=0. \tag{19}$$

Also by substituting the stress-strain relations (Eq. (7)) into Eqs. (11a)-(11e), we have

$$\begin{aligned}N_{xx}=&A_{11}\frac{\partial u}{\partial x}+A_{12}\frac{\partial v}{\partial y}+A_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{11}\frac{\partial \phi_x}{\partial x}+B_{12}\frac{\partial \phi_y}{\partial y}+B_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\\&+E_{11}c_1\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{12}c_1\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{16}c_1\left(\frac{\partial \phi_y}{\partial x}+\frac{\partial \phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\\N_{yy}=&A_{12}\frac{\partial u}{\partial x}+A_{22}\frac{\partial v}{\partial y}+A_{26}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{12}\frac{\partial \phi_x}{\partial x}+B_{22}\frac{\partial \phi_y}{\partial y}+B_{26}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\\&+E_{12}c_1\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{22}c_1\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{26}c_1\left(\frac{\partial \phi_y}{\partial x}+\frac{\partial \phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\\N_{xy}=&A_{16}\frac{\partial u}{\partial x}+A_{26}\frac{\partial v}{\partial y}+A_{66}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+B_{16}\frac{\partial \phi_x}{\partial x}+B_{26}\frac{\partial \phi_y}{\partial y}+B_{66}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\\&+E_{16}c_1\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+E_{26}c_1\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+E_{66}c_1\left(\frac{\partial \phi_y}{\partial x}+\frac{\partial \phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\end{aligned} \tag{20a}$$

$$\begin{aligned}M_{xx}=&B_{11}\frac{\partial u}{\partial x}+B_{12}\frac{\partial v}{\partial y}+B_{16}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+D_{11}\frac{\partial \phi_x}{\partial x}+D_{12}\frac{\partial \phi_y}{\partial y}+D_{16}\left(\frac{\partial \phi_x}{\partial y}+\frac{\partial \phi_y}{\partial x}\right)\\&+F_{11}c_1\left(\frac{\partial \phi_x}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+F_{12}c_1\left(\frac{\partial \phi_y}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)+F_{16}c_1\left(\frac{\partial \phi_y}{\partial x}+\frac{\partial \phi_x}{\partial y}+2\frac{\partial^2 w}{\partial x\partial y}\right),\end{aligned} \tag{20b}$$

$$M_{yy} = B_{12}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y} + B_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{12}\frac{\partial \phi_x}{\partial x} + D_{22}\frac{\partial \phi_y}{\partial y} + D_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$
$$+ F_{12}c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + F_{22}c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + F_{26}c_1\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right),$$
$$M_{xy} = B_{16}\frac{\partial u}{\partial x} + B_{26}\frac{\partial v}{\partial y} + B_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + D_{16}\frac{\partial \phi_x}{\partial x} + D_{26}\frac{\partial \phi_y}{\partial y} + D_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$
$$+ F_{16}c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + F_{26}c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + F_{66}c_1\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right), \tag{20b}$$

$$P_{xx} = E_{11}\frac{\partial u}{\partial x} + E_{12}\frac{\partial v}{\partial y} + E_{16}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + F_{11}\frac{\partial \phi_x}{\partial x} + F_{12}\frac{\partial \phi_y}{\partial y} + F_{16}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$
$$+ H_{11}c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + H_{12}c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + H_{16}c_1\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right),$$
$$P_{yy} = E_{12}\frac{\partial u}{\partial x} + E_{22}\frac{\partial v}{\partial y} + E_{26}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + F_{12}\frac{\partial \phi_x}{\partial x} + F_{22}\frac{\partial \phi_y}{\partial y} + F_{26}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$
$$+ H_{12}c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + H_{22}c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + H_{26}c_1\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right), \tag{20c}$$
$$P_{xy} = E_{16}\frac{\partial u}{\partial x} + E_{26}\frac{\partial v}{\partial y} + E_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + F_{16}\frac{\partial \phi_x}{\partial x} + F_{26}\frac{\partial \phi_y}{\partial y} + F_{66}\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right)$$
$$+ H_{16}c_1\left(\frac{\partial \phi_x}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) + H_{26}c_1\left(\frac{\partial \phi_y}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right) + H_{66}c_1\left(\frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_x}{\partial y} + 2\frac{\partial^2 w}{\partial x \partial y}\right),$$

$$Q_{xx} = A_{55}\left(\frac{\partial w}{\partial x} + \phi_x\right) + A_{45}\left(\frac{\partial w}{\partial y} + \phi_y\right) + D_{55}c_2\left(\phi_x + \frac{\partial w}{\partial x}\right) + D_{45}c_2\left(\frac{\partial w}{\partial y} + \phi_y\right),$$
$$Q_{yy} = A_{45}\left(\frac{\partial w}{\partial y} + \phi_y\right) + A_{44}\left(\frac{\partial w}{\partial y} + \phi_y\right) + D_{45}c_2\left(\phi_y + \frac{\partial w}{\partial y}\right) + D_{44}c_2\left(\frac{\partial w}{\partial y} + \phi_y\right), \tag{20d}$$

$$K_{xx} = D_{55}\left(\frac{\partial w}{\partial x} + \phi_x\right) + D_{45}\left(\frac{\partial w}{\partial y} + \phi_y\right) + F_{55}c_2\left(\phi_x + \frac{\partial w}{\partial x}\right) + F_{45}c_2\left(\frac{\partial w}{\partial y} + \phi_y\right),$$
$$K_{yy} = D_{45}\left(\frac{\partial w}{\partial y} + \phi_y\right) + D_{44}\left(\frac{\partial w}{\partial y} + \phi_y\right) + F_{45}c_2\left(\phi_y + \frac{\partial w}{\partial y}\right) + F_{44}c_2\left(\frac{\partial w}{\partial y} + \phi_y\right), \tag{20e}$$

where

$$A_{ij} = \sum_{k=1}^{N}\int_{z^{(k-1)}}^{z^{(k)}} Q_{ij}^{(k)}dz, \qquad (i, j = 1, 2, 6) \tag{21a}$$

$$B_{ij} = \sum_{k=1}^{N} \int_{z^{(k-1)}}^{z^{(k)}} Q_{ij}^{(k)} z dz, \tag{21b}$$

$$D_{ij} = \sum_{k=1}^{N} \int_{z^{(k-1)}}^{z^{(k)}} Q_{ij}^{(k)} z^2 dz, \tag{21c}$$

$$E_{ij} = \sum_{k=1}^{N} \int_{z^{(k-1)}}^{z^{(k)}} Q_{ij}^{(k)} z^3 dz, \tag{21d}$$

$$F_{ij} = \sum_{k=1}^{N} \int_{z^{(k-1)}}^{z^{(k)}} Q_{ij}^{(k)} z^4 dz, \tag{21e}$$

$$H_{ij} = \sum_{k=1}^{N} \int_{z^{(k-1)}}^{z^{(k)}} Q_{ij}^{(k)} z^6 dz. \tag{21f}$$

In which N is the number of composite layers. Finally the governing equations are obtained by substituting Eqs. (21a)-(21f) into governing equations (Eqs. (15)-(19)).

$$\begin{aligned}
&A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\frac{\partial^2 v}{\partial x \partial y} + A_{16}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + B_{11}\frac{\partial^2 \phi_x}{\partial x^2} + B_{12}\frac{\partial^2 \phi_y}{\partial x \partial y} \\
&+ B_{16}\left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2}\right) + E_{11}c_1\left(\frac{\partial^2 \phi_x}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) + E_{12}c_1\left(\frac{\partial^2 \phi_y}{\partial x \partial y} + \frac{\partial^3 w}{\partial x \partial y^2}\right) \\
&+ E_{16}c_1\left(\frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_x}{\partial x \partial y} + 2\frac{\partial^3 w}{\partial x^2 \partial y}\right) + A_{16}\frac{\partial^2 u}{\partial x \partial y} + A_{26}\frac{\partial^2 v}{\partial y^2} + A_{66}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) \\
&+ B_{16}\frac{\partial^2 \phi_x}{\partial x \partial y} + B_{26}\frac{\partial^2 \phi_y}{\partial y^2} + B_{66}\left(\frac{\partial^2 \phi_x}{\partial y^2} + \frac{\partial^2 \phi_y}{\partial x \partial y}\right) + E_{16}c_1\left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^3 w}{\partial y \partial x^2}\right) \\
&+ E_{26}c_1\left(\frac{\partial^2 \phi_y}{\partial y^2} + \frac{\partial^3 w}{\partial y^3}\right) + E_{66}c_1\left(\frac{\partial^2 \phi_y}{\partial x \partial y} + \frac{\partial^2 \phi_x}{\partial y^2} + 2\frac{\partial^3 w}{\partial x \partial y^2}\right) = 0,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&A_{16}\frac{\partial^2 u}{\partial x^2} + A_{26}\frac{\partial^2 v}{\partial x \partial y} + A_{66}\left(\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2}\right) + B_{16}\frac{\partial^2 \phi_x}{\partial x^2} + B_{26}\frac{\partial^2 \phi_y}{\partial x \partial y} \\
&+ B_{66}\left(\frac{\partial^2 \phi_x}{\partial x \partial y} + \frac{\partial^2 \phi_y}{\partial x^2}\right) + E_{16}c_1\left(\frac{\partial^2 \phi_x}{\partial x^2} + \frac{\partial^3 w}{\partial x^3}\right) + E_{26}c_1\left(\frac{\partial^2 \phi_y}{\partial x \partial y} + \frac{\partial^3 w}{\partial x \partial y^2}\right) \\
&+ E_{66}c_1\left(\frac{\partial^2 \phi_y}{\partial x^2} + \frac{\partial^2 \phi_x}{\partial x \partial y} + 2\frac{\partial^3 w}{\partial x^2 \partial y}\right) + A_{21}\frac{\partial^2 u}{\partial x \partial y} + A_{22}\frac{\partial^2 v}{\partial y^2} + A_{26}\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right)
\end{aligned} \tag{23}$$

$$
\begin{aligned}
&+B_{21}\frac{\partial^2\phi_x}{\partial x\partial y}+B_{22}\frac{\partial^2\phi_y}{\partial y^2}+B_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)+E_{21}c_1\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial y\partial x^2}\right)\\
&+E_{22}c_1\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+E_{26}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^2\phi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)=0,
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&A_{55}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+A_{45}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_y}{\partial x}\right)+D_{55}c_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+D_{45}c_2\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\
&+A_{45}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_x}{\partial y}\right)+A_{44}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)+D_{45}c_2\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_x}{\partial y}\right)+D_{44}c_2\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)\\
&+c_2\left(\begin{aligned}&D_{55}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+D_{45}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_y}{\partial x}\right)+F_{55}c_2\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\phi_x}{\partial x}\right)+F_{45}c_2\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_y}{\partial x}\right)\\&+D_{45}\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_x}{\partial y}\right)+D_{44}\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)+F_{45}c_2\left(\frac{\partial^2 w}{\partial x\partial y}+\frac{\partial\phi_x}{\partial y}\right)+F_{44}c_2\left(\frac{\partial^2 w}{\partial y^2}+\frac{\partial\phi_y}{\partial y}\right)\end{aligned}\right)\\
&+N_{xx}\frac{\partial^2 w}{\partial x^2}+N_{yy}\frac{\partial^2 w}{\partial y^2}-c_1\left(E_{11}\frac{\partial^3 u}{\partial x^3}+E_{12}\frac{\partial^3 v}{\partial x^2\partial y}+E_{16}\left(\frac{\partial^3 u}{\partial x^2\partial y}+\frac{\partial^3 v}{\partial x^3}\right)+F_{11}\frac{\partial^3\phi_x}{\partial x^3}\right.\\
&+F_{12}\frac{\partial^3\phi_y}{\partial x^2\partial y}+F_{16}\left(\frac{\partial^3\phi_x}{\partial x^2\partial y}+\frac{\partial^3\phi_y}{\partial x^3}\right)+H_{11}c_1\left(\frac{\partial^3\phi_x}{\partial x^3}+\frac{\partial^4 w}{\partial x^4}\right)+H_{12}c_1\left(\frac{\partial^3\phi_y}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\\
&+H_{16}c_1\left(\frac{\partial^3\phi_y}{\partial x^3}+\frac{\partial^3\phi_x}{\partial x^2\partial y}+2\frac{\partial^4 w}{\partial x^3\partial y}\right)+E_{12}\frac{\partial^3 u}{\partial y^2\partial x}+E_{22}\frac{\partial^3 v}{\partial y^3}+E_{26}\left(\frac{\partial^3 u}{\partial y^3}+\frac{\partial^3 v}{\partial x\partial y^2}\right)\\
&+F_{12}\frac{\partial^3\phi_x}{\partial x\partial y^2}+F_{22}\frac{\partial^3\phi_y}{\partial y^3}+F_{26}\left(\frac{\partial^3\phi_x}{\partial y^3}+\frac{\partial^3\phi_y}{\partial x\partial y^2}\right)+H_{12}c_1\left(\frac{\partial^3\phi_x}{\partial x\partial y^2}+\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\\
&+H_{22}c_1\left(\frac{\partial^3\phi_y}{\partial y^3}+\frac{\partial^4 w}{\partial y^4}\right)+H_{26}c_1\left(\frac{\partial^3\phi_y}{\partial x\partial y^2}+\frac{\partial^3\phi_x}{\partial y^3}+2\frac{\partial^4 w}{\partial x\partial y^3}\right)+2E_{16}\frac{\partial^3 u}{\partial y\partial x^2}+2E_{26}\frac{\partial^3 v}{\partial y^2\partial x}\\
&+2E_{66}\left(\frac{\partial^3 u}{\partial y^2\partial x}+\frac{\partial^3 v}{\partial x^2\partial y}\right)+2F_{16}\frac{\partial^3\phi_x}{\partial x^2\partial y}+2F_{26}\frac{\partial^3\phi_y}{\partial y^2\partial x}+2F_{66}\left(\frac{\partial^3\phi_x}{\partial y^2\partial x}+\frac{\partial^3\phi_y}{\partial x^2\partial y}\right)\\
&+2H_{16}c_1\left(\frac{\partial^3\phi_x}{\partial x^2\partial y}+\frac{\partial^4 w}{\partial x^3\partial y}\right)+2H_{26}c_1\left(\frac{\partial^3\phi_y}{\partial x\partial y^2}+\frac{\partial^4 w}{\partial y^3\partial x}\right)\\
&\left.+2H_{66}c_1\left(\frac{\partial^3\phi_y}{\partial x^2\partial y}+\frac{\partial^3\phi_x}{\partial y^2\partial x}+2\frac{\partial^4 w}{\partial x^2\partial y^2}\right)\right)-k_w w+k_g\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)=0,
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
& B_{11}\frac{\partial^2 u}{\partial x^2}+B_{12}\frac{\partial^2 v}{\partial x\partial y}+B_{16}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+D_{11}\frac{\partial^2\phi_x}{\partial x^2}+D_{12}\frac{\partial^2\phi_y}{\partial x\partial y}+D_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)\\
&+F_{11}c_1\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+F_{12}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+F_{16}c_1\left(\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_x}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)\\
&+B_{16}\frac{\partial^2 u}{\partial x\partial y}+B_{26}\frac{\partial^2 v}{\partial y^2}+B_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+D_{16}\frac{\partial^2\phi_x}{\partial x\partial y}+D_{26}\frac{\partial^2\phi_y}{\partial y^2}\\
&+D_{66}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)+F_{16}c_1\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial y\partial x^2}\right)+F_{26}c_1\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)\\
&+F_{66}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^2\phi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)+c_1\left(E_{11}\frac{\partial^2 u}{\partial x^2}+E_{12}\frac{\partial^2 v}{\partial x\partial y}+E_{16}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)\right.\\
&+F_{11}\frac{\partial^2\phi_x}{\partial x^2}+F_{12}\frac{\partial^2\phi_y}{\partial x\partial y}+F_{16}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+H_{11}c_1\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)\\
&+H_{12}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+H_{16}c_1\left(\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_x}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+E_{16}\frac{\partial^2 u}{\partial x\partial y}+E_{26}\frac{\partial^2 v}{\partial y^2}\\
&+E_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+F_{16}\frac{\partial^2\phi_x}{\partial x\partial y}+F_{26}\frac{\partial^2\phi_y}{\partial y^2}+F_{66}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)\\
&\left.+H_{16}c_1\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial y\partial x^2}\right)+H_{26}c_1\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+H_{66}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^2\phi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)\right)\\
&-A_{55}\left(\frac{\partial w}{\partial x}+\phi_x\right)-A_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)-D_{55}c_2\left(\phi_x+\frac{\partial w}{\partial x}\right)-D_{45}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right)\\
&+c_2\left(-D_{55}\left(\frac{\partial w}{\partial x}+\phi_x\right)-D_{45}\left(\frac{\partial w}{\partial y}+\phi_y\right)-F_{55}c_2\left(\phi_x+\frac{\partial w}{\partial x}\right)-F_{45}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right)\right)=0,
\end{aligned}
\tag{25}
$$

$$
\begin{aligned}
& B_{16}\frac{\partial^2 u}{\partial x^2}+B_{26}\frac{\partial^2 v}{\partial x\partial y}+B_{66}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)+D_{16}\frac{\partial^2\phi_x}{\partial x^2}+D_{26}\frac{\partial^2\phi_y}{\partial x\partial y}+D_{66}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)\\
&+F_{16}c_1\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)+F_{26}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+F_{66}c_1\left(\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_x}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)\\
&+B_{12}\frac{\partial^2 u}{\partial x\partial y}+B_{22}\frac{\partial^2 v}{\partial y^2}+B_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+D_{12}\frac{\partial^2\phi_x}{\partial x\partial y}+D_{22}\frac{\partial^2\phi_y}{\partial y^2}
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&+D_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)+F_{12}c_1\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial y\partial x^2}\right)+F_{22}c_1\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)\\
&+F_{26}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^2\phi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)+c_1\left(E_{16}\frac{\partial^2 u}{\partial x^2}+E_{26}\frac{\partial^2 v}{\partial x\partial y}+E_{66}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)\right.\\
&+F_{16}\frac{\partial^2\phi_x}{\partial x^2}+F_{26}\frac{\partial^2\phi_y}{\partial x\partial y}+F_{66}\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^2\phi_y}{\partial x^2}\right)+H_{16}c_1\left(\frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)\\
&+H_{26}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^3 w}{\partial x\partial y^2}\right)+H_{66}c_1\left(\frac{\partial^2\phi_y}{\partial x^2}+\frac{\partial^2\phi_x}{\partial x\partial y}+2\frac{\partial^3 w}{\partial x^2\partial y}\right)+E_{12}\frac{\partial^2 u}{\partial x\partial y}+E_{22}\frac{\partial^2 v}{\partial y^2}\\
&+E_{26}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)+F_{12}\frac{\partial^2\phi_x}{\partial x\partial y}+F_{22}\frac{\partial^2\phi_y}{\partial y^2}+F_{26}\left(\frac{\partial^2\phi_x}{\partial y^2}+\frac{\partial^2\phi_y}{\partial x\partial y}\right)\\
&+H_{12}c_1\left(\frac{\partial^2\phi_x}{\partial x\partial y}+\frac{\partial^3 w}{\partial y\partial x^2}\right)+H_{22}c_1\left(\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^3 w}{\partial y^3}\right)+H_{26}c_1\left(\frac{\partial^2\phi_y}{\partial x\partial y}+\frac{\partial^2\phi_x}{\partial y^2}+2\frac{\partial^3 w}{\partial x\partial y^2}\right)\\
&-A_{45}\left(\frac{\partial w}{\partial x}+\phi_x\right)-A_{44}\left(\frac{\partial w}{\partial y}+\phi_y\right)-D_{45}c_2\left(\phi_x+\frac{\partial w}{\partial x}\right)-D_{44}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right)\\
&+c_2\left(-D_{45}\left(\frac{\partial w}{\partial x}+\phi_x\right)-D_{44}\left(\frac{\partial w}{\partial y}+\phi_y\right)-F_{45}c_2\left(\phi_x+\frac{\partial w}{\partial x}\right)-F_{44}c_2\left(\frac{\partial w}{\partial y}+\phi_y\right)\right)=0.
\end{aligned}
\tag{26}
$$

3. Navier solution method

In this section, the Navier solution method is employed to obtain the buckling load of the structure so the all edges of the plate are assumed to be simply supported and the components of the displacement are considered as follows (Samaei *et al*. 2011)

$$u(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}u_{mn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{27a}$$

$$v(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}v_{mn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{27b}$$

$$w(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}w_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{27c}$$

$$\phi_x(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{xmn}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{27d}$$

$$\phi_y(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\phi_{ymn}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right), \tag{27e}$$

in which m and n are the wave numbers in x and y axes, respectively. By substituting the Eqs. (27a)-(27e), into the governing equations (Eqs. (22)-(26)), we have

$$\left\{\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix}\right\}\begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_{x0} \\ \phi_{y0} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{28}$$

where the components of matrix K_{mn} are mentioned in Appendix A. Setting the determinate of Eq. (28) to zero, the buckling load of the structure can be obtained.

4. Numerical results and discussion

This section examines the buckling load of the laminated composite plate resting on elastic medium. The main goal of this part is the study of various parameters such as number and orientation angle of layers, buckling modes and elastic medium on the buckling behavior of the system. The material of the layers is Graphite/Epoxy and the mechanical properties are listed in Table 1.

The buckling load and spring constant are defined as dimensionless parameters and considered as follows

$$P=N_x^M/(E_1a),\quad K_w=k_wa/E_1 \tag{29}$$

4.1 Buckling load versus circumferential mode number

The variation of buckling load of the system versus the circumferential mode number is plotted in Figs. 2-6. It can be observed that the buckling load decreases at first until reaches to the lowest

Table 1 Mechanical properties of Graphite/Epoxy (Phung-Van *et al*. 2015d)

Mechanical properties	Value
E_{11}	13238 GPa
$E_{22}=E_{33}$	10.76 GPa
G_{12}	3.61 GPa
$G_{13}=G_{23}$	5.65 GPa
$v_{11}=v_{23}$	0.24
v_{13}	0.49
ρ	1578 Kg/m^3

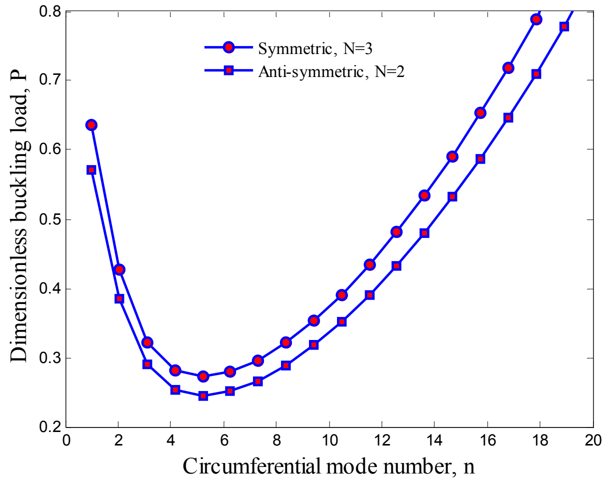

Fig. 2 Variation of dimensionless buckling load of the structures with various number of layers versus circumferential mode number

amount and after that increasing process begins. The critical buckling load appears in the point where the buckling load is minimal. Fig. 2 shows the effect of the number of layers on the dimensionless buckling load. It can be seen that in symmetric laminated composites (with three number of layers), the buckling phenomenon occurs later compared with the anti-symmetric ones (with two number of layers). The reason is that the symmetric laminated composite plates are more balance and stable.

The influence of the orientation type of layers is studied in Fig. 3. For this purpose, five various type of the orientation of layers are considered as follows

(0°, 0°, 0°): indicates a composite structure with zero angle of orientation in layers
(0°, 90°, 0°): indicates a composite structure with cross-ply orientation in layers
(15°, –15°, 15°), (30°, –30°, 30°) and (45°, –45°, 45°):
indicates a composite structure with angle-ply orientation in layers.

According to Fig. 3, it can be concluded that the composite structure with angle-ply orientation of layers has the highest buckling load and becomes stiffer by increasing the angle to 45 degrees since the buckling load increases and the system buckling occurs later. Also the composite structure with zero angle of orientation has the lowest buckling load and after that the structure with cross-ply orientation in layer.

In Fig. 4, the effect of the thickness of the structure on the buckling load is probed. From this figure it can be observed that with increasing the thickness of the structure, the dimensionless buckling load decreases and the critical buckling load is decreased. Thereby, with increasing the thickness, the stiffness of the structure decreases. According to Fig. 4, the dimensionless buckling load is about 0.37 for d = 0.05 m whilst for d = 0.2 m the dimensionless buckling load is about 0.25. Also it can be found that the effect of thickness changes, is negligible in lower mode numbers and this effect is prominent in higher mode numbers. In addition, the reduction rate of the buckling load decreases with increasing the thickness. So the excessive increase in the thickness caused the loss of the effect of this parameter on the buckling load.

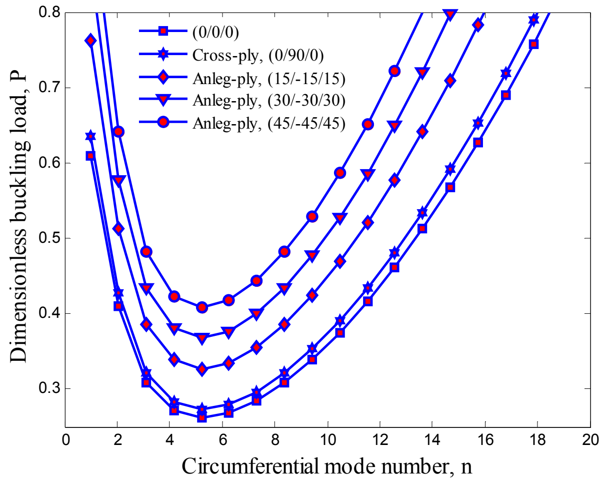

Fig. 3 Variation of dimensionless buckling load of the structures with various types of orientation angle of layers versus circumferential mode number

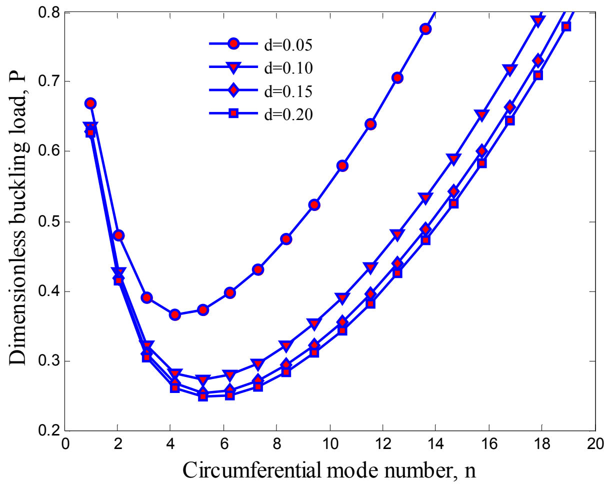

Fig. 4 Variation of dimensionless buckling load of the structures with different values of thickness versus circumferential mode number

The effect of the elastic medium which is modeled by the spring constant of Winkler medium and shear layer is studied in Fig. 5. It is noteworthy that the material of the elastic medium is chosen from reference (Ghorbanpour Arani *et al*. 2012) and the Winkler and Pasternak constants are 8.7×10^{17} N.m^3 and 2 N/m, respectively. Generally the existence of the elastic medium causes to increase the stiffness of the structure and thereby the buckling load increases. The Pasternak medium considers the vertical and shear loads however the Winkler medium considers only the vertical ones, therefore the effect of Pasternak medium is more than Winkler medium. According to Fig. 5, the effect of the elastic medium on the buckling load is significant and it can be a useful parameter to take away the system from buckling condition.

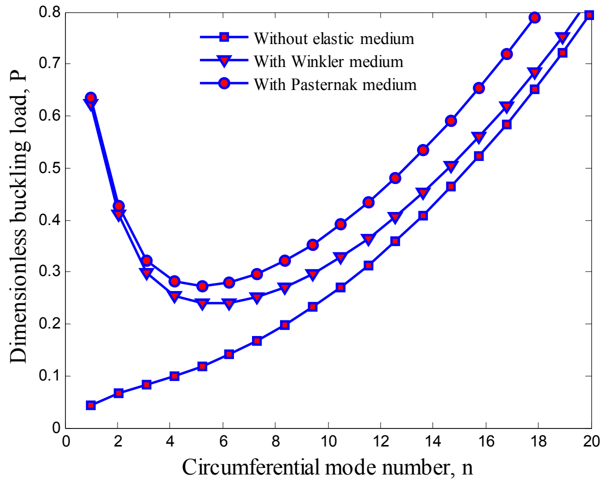

Fig. 5 The effect of the elastic medium on the dimensionless buckling load versus circumferential mode number

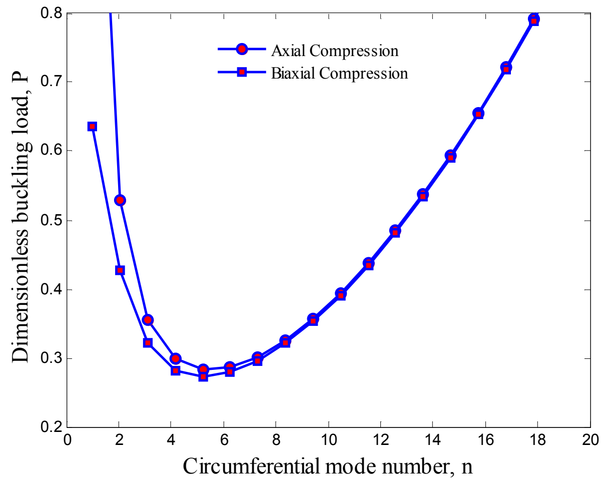

Fig. 6 The effect of the loading type on the dimensionless buckling load versus circumferential mode number

Fig. 6 examines the influence of the loading types on the dimensionless buckling load of the structure. Two types of loading include uniaxial (along the x axis) and biaxial (along the x and y axes) are considered. As it can be found, in biaxial loading type the dimensionless buckling load is lower than axial loading type. The reason is that, in biaxial loading type, the load which applied to the edges is higher than the axial loading type and therefore the buckling of the structure occurs sooner. Also the effect of loading type is apparent in lower mode numbers.

4.2 Buckling load versus spring constant of elastic medium

The effect of spring constant of elastic medium on the dimensionless buckling load is studied by plotting Figs. 7-11. In Fig. 7, the effect of the number of layers on the dimensionless buckling load versus spring constant of elastic medium is shown. It can be observed that the buckling load of the symmetric composite structures is higher than anti-symmetric ones. As it is noted before, the reason is that the stability of the symmetric structure is higher than the anti-symmetric ones. Also it can be found that with increasing the spring constant of the elastic medium, the dimensionless buckling load increases linearly.

The influence of the orientation type of the layers on the dimensionless buckling load of the structure is studied by Fig. 8. As it can be observed the highest buckling load belongs to angle-ply orientation type of the layers and after that cross-ply and zero orientation angle types. Also with increasing the angle of orientation up to 45 degrees, the stiffness of laminated composite structure and thereby the buckling load increases.

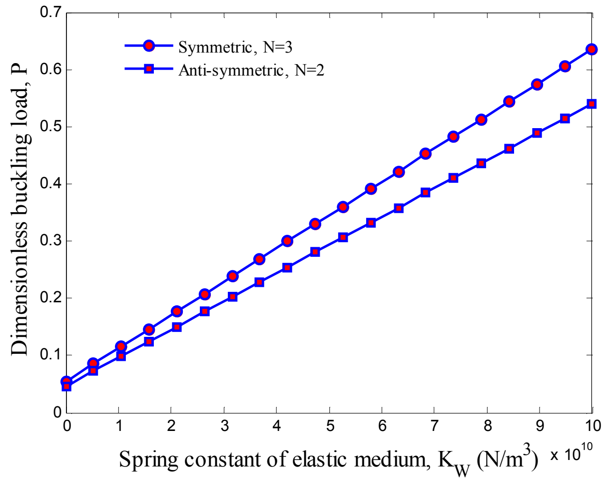

Fig. 7 The effect of the number of layers on the dimensionless buckling load versus the spring constant of elastic medium

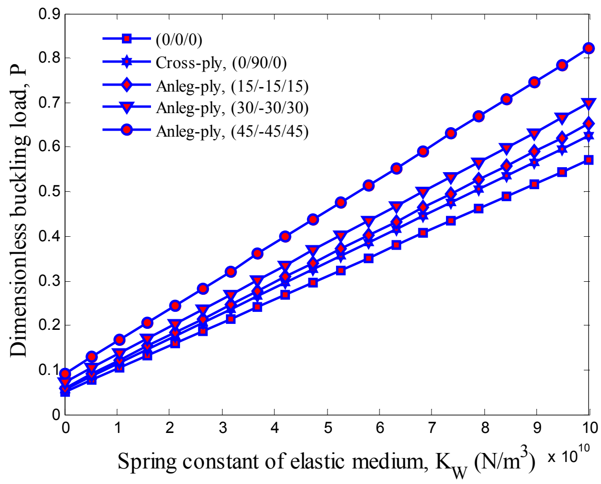

Fig. 8 Variation of dimensionless buckling load of the structures with different types of orientation angle versus spring constant of elastic medium

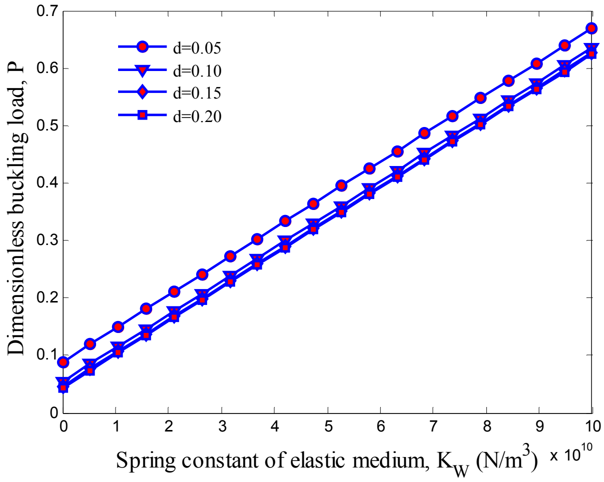

Fig. 9 Variation of dimensionless buckling load of the structures with different values of thickness versus spring constant of elastic medium

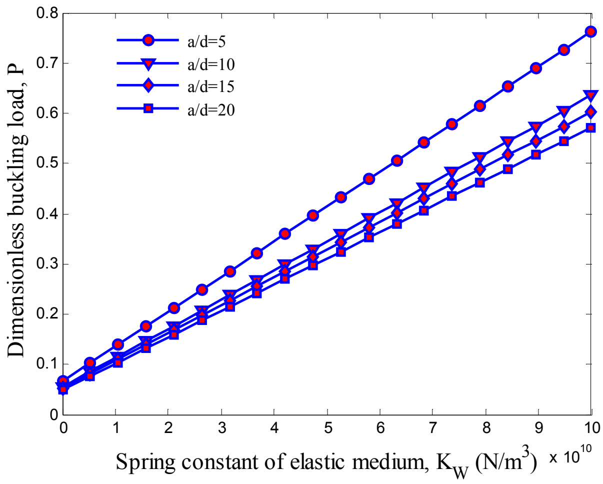

Fig. 10 Variation of dimensionless buckling load of the structures with different values of aspect ratio versus spring constant of elastic medium

In Fig. 9, the effect of the thickness of the structure on the dimensionless buckling parameter versus the spring constant of the elastic medium is probed. The plotted figure shows that with increasing the thickness of the structure, the dimensionless buckling load decreases and hence the structure with more thickness has less stiffness.

The influence of the length to thickness ratio (aspect ratio) of the structure on the dimensionless buckling load parameter is shown in Fig. 10. It is clear that with increasing the aspect ratio, the buckling load of the structure decreases because of the reduction of stiffness.

In Fig. 11, the buckling behavior of the structure is studied for two types of loading include axial and biaxial loadings. The results are the same as obtained from Fig. 6 and in axial loading type the buckling load of the structure is higher with respect to the biaxial loading type. Also it can

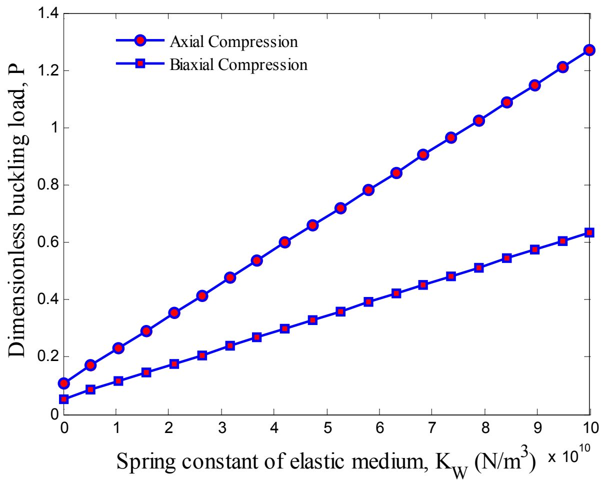

Fig. 11 Variation of dimensionless buckling load of the structures with different types of loading versus spring constant of elastic medium

be observed that the effect of the spring constant of the elastic medium is more significant in axial loading type.

4.3 Verification of the results

The buckling load of the structure in absence of elastic medium ($k_w = k_g = 0$) is obtained to verify the results. For this goal, all the mechanical properties and also the type of the loading are intended similar to reference (Putcha and Reddy 1986). The dimensionless buckling load is defined as $P = \dfrac{N_{xx} b^3}{E_2 h^3}$ in which E_2 is the Young's modulus, h is the thickness of the plate and b is the width of the structure. The results were compared with three different literatures. Putcha and Reddy (1986) used classical plate theory (CPT), first order (FSDT) and refined shear deformation theory for modeling the problem. Also Khedir and Liberscu (1988) and Matasunaga (2000) applied higher-order (HSDT) and global higher-order plate theory, respectively. As it can be concluded from Table 2, the results of the present work are in good agreement with the available literatures.

Table 2 The comparison of the results with other available literatures

Theory	E_1 / E_2				
	3	10	20	30	40
CPT (Putcha and Reddy 1986)	5.7538	11.4920	19.7120	27.9360	36.1600
FSDT (Putcha and Reddy 1986)	5.3991	9.9652	15.3610	19.7560	23.4530
Refined FSDT (Putcha and Reddy 1986)	5.3905	9.8336	14.8906	18.8778	22.1194
HSDT (Khedir and Liberscu 1988)	5.3920	9.8460	14.917	18.9120	22.1540
Global HSDT (Matasunaga 2000)	5.3208	9.7172	14.7290	168348.	21.8977
Present work	5.3918	9.8452	14.9167	18.8769	22.1531

Table 3 Normalized deflection and stresses of a three-layer square sandwich plate subjected to a uniform load

Method	$\overline{w}$				
	$\overline{w}$	$\overline{\sigma}_x$	$\overline{\sigma}_y$	$\overline{\tau}_{zx}$	τ_{xy}
HSDT (Pandya and Kant 1988)	256.1300	62.3800	30.3300	9.3820	3.0890
FSDT (Pandya and Kant 1988)	236.1000	61.8700	29.3200	9.8990	3.3130
Ferreira and Barbosa (1970)	258.7400	59.2100	29.5900	9.1220	3.5930
Exact (Srinivas 1973)	258.9700	60.3530	30.0970	9.3400	4.641
Third-order (Ferreira *et al*. 2003)	256.2387	60.1834	30.1642	9.3716	4.2768
Layerwise (Ferreira 2005)	255.9197	59.6503	29.8296	9.2073	3.9773
Layerwise (Ferreira *et al*. 2008)	258.1813	60.2973	30.1141	9.2928	4.0961
CS-FEM-DSG3 (Phung-Van *et al*. 2014a)	257.6447	59.8755	29.9477	9.2437	3.9133
Present work	256.3912	60.1944	30.2218	9.4473	4.1662

For another comparison of this paper, the results are validated with the work of Phung-Van *et al*. (2014a) on the bending of laminated composite plates. However, a three-layer square sandwich plate ((0/core/0)) subjected to a uniform load is considered. The material parameters are given by $E_1 = 25E_1$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.25E_2$ and $v_{12} = 0.25$. For this purpose, Eq. (28) becomes in the following form

$$\left\{\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} \end{bmatrix}\right\}\begin{bmatrix} u_0 \\ v_0 \\ w_0 \\ \phi_{x0} \\ \phi_{y0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q \\ 0 \\ 0 \end{bmatrix}, \tag{29}$$

Table 3 presents the normalized transverse deflection and stresses of the three-layer sandwich plate with the $a/t = 5$. It can be seen that the present results by Reddy plate theory agree well with other works.

5. Conclusions

In this paper, the buckling analysis of the laminated composite plate embedded in elastic medium is performed. The Navier solution method is applied to solve the equations. The effect of various parameters such as number and orientation angle of layers, elastic medium, geometric parameters and loading type on the buckling load of the system is studied. The remarkable results are listed as follows:

- The critical buckling load of the structure with $a/d = 10$ is in fifth circumferential mode number.
- In symmetric laminated composite plates, the buckling load is higher than the anti-symmetric ones and so the buckling of the system occurs lately.
- The buckling load of the structure with cross-ply orientation type in the layers is higher than the structure with zero orientation angles. Also the highest buckling load is belongs to the

structure with angle-ply orientation type and with increasing the angle up to 45 degrees, the buckling load increases.

- With increasing the thickness of the structure, the buckling load decreases and therefore the critical buckling load is appear in lowest values. Also the effect of thickness changes is more prominent in higher buckling modes.
- By considering the elastic medium, the stiffness of the system increases and hence the buckling load increases. Also the effect of Pasternak medium which includes the both vertical and shear loads is more than Winkler medium.
- In axial loading type, the buckling load of the structure is higher with respect to the biaxial loading type.
- The buckling load of the system decreases with increasing the aspect ratio. So it can be concluded that with increasing the aspect ratio, stiffness of the structure decreases and the buckling of the system occurs sooner.

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Appendix A

$$K_{11} = -\frac{2}{3}\frac{Q_{11\,m}\, m^2 \pi^2 d}{a^2} - \frac{2}{3}\frac{Q_{11\,t}\, m^2 \pi^2 d}{a^2} - \frac{2}{3}\frac{Q_{66\,m}\, n^2 \pi^2 d}{b^2} - \frac{2}{3}\frac{Q_{11\,b}\, m^2 \pi^2 d}{a^2} - \frac{2}{3}\frac{Q_{66\,t}\, n^2 \pi^2 d}{b^2} - \frac{2}{3}\frac{Q_{66\,b}\, n^2 \pi^2 d}{b^2} \tag{A1}$$

$$K_{12} = -\frac{2}{3}\frac{Q_{66\,m}\, m \pi^2 n\, d}{a\, b} - \frac{2}{3}\frac{Q_{66\,t}\, m \pi^2 n\, d}{a\, b} - \frac{2}{3}\frac{Q_{12\,t}\, m \pi^2 n\, d}{a\, b} - \frac{2}{3}\frac{Q_{66\,b}\, m \pi^2 n\, d}{a\, b} - \frac{2}{3}\frac{Q_{12\,m}\, m \pi^2 n\, d}{a\, b} - \frac{2}{3}\frac{Q_{12\,b}\, m \pi^2 n\, d}{a\, b} \tag{A2}$$

$$K_{13} = -\frac{20}{81}\left(\frac{Q_{12\,b}\, c1\, m \pi^3 n^2}{a\, b^2} + \frac{Q_{11\,b}\, c1\, m^3 \pi^3}{a^3}\right) d^4 + \frac{40}{81}\frac{Q_{66\,t}\, c1\, m \pi^3 n^2 d^4}{a\, b^2} + \frac{20}{81}\left(\frac{Q_{12\,t}\, c1\, m \pi^3 n^2}{a\, b^2} + \frac{Q_{11\,t}\, c1\, m^3 \pi^3}{a^3}\right) d^4 - \frac{40}{81}\frac{Q_{66\,b}\, c1\, m \pi^3 n^2 d^4}{a\, b^2} \tag{A3}$$

$$K_{14} = \frac{20}{81}\frac{Q_{11\,t}\, c1\, m^2 \pi^2 d^4}{a^2} - \frac{20}{81}\frac{Q_{66\,b}\, c1\, n^2 \pi^2 d^4}{b^2} - \frac{4}{9}\frac{Q_{66\,t}\, n^2 \pi^2 d^2}{b^2} + \frac{20}{81}\frac{Q_{66\,t}\, c1\, n^2 \pi^2 d^4}{b^2} + \frac{4}{9}\frac{Q_{66\,b}\, n^2 \pi^2 d^2}{b^2} - \frac{4}{9}\frac{Q_{11\,t}\, m^2 \pi^2 d^2}{a^2} - \frac{20}{81}\frac{Q_{11\,b}\, c1\, m^2 \pi^2 d^4}{a^2} + \frac{4}{9}\frac{Q_{11\,b}\, m^2 \pi^2 d^2}{a^2} \tag{A4}$$

$$K_{15} = -\frac{20}{81}\frac{Q_{12\,b}\, c1\, m \pi^2 n\, d^4}{a\, b} + \frac{4}{9}\frac{Q_{12\,b}\, m \pi^2 n\, d^2}{a\, b} + \frac{20}{81}\frac{Q_{12\,t}\, c1\, m \pi^2 n\, d^4}{a\, b} - \frac{20}{81}\frac{Q_{66\,b}\, c1\, m \pi^2 n\, d^4}{a\, b} - \frac{4}{9}\frac{Q_{66\,t}\, m \pi^2 n\, d^2}{a\, b} + \frac{20}{81}\frac{Q_{66\,t}\, c1\, m \pi^2 n\, d^4}{a\, b} \tag{A5}$$

$$+\frac{4}{9}\frac{Q_{66\,b}\,m\,\pi^2\,n\,d^2}{a\,b}-\frac{4}{9}\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^2}{a\,b} \tag{A5}$$

$$\begin{aligned}K_{21}=&-\frac{2}{3}\frac{Q_{66\,m}\,m\,\pi^2\,n\,d}{a\,b}-\frac{2}{3}\frac{Q_{66\,t}\,m\,\pi^2\,n\,d}{a\,b}\\&-\frac{2}{3}\frac{Q_{12\,t}\,m\,\pi^2\,n\,d}{a\,b}-\frac{2}{3}\frac{Q_{66\,b}\,m\,\pi^2\,n\,d}{a\,b}\\&-\frac{2}{3}\frac{Q_{12\,m}\,m\,\pi^2\,n\,d}{a\,b}-\frac{2}{3}\frac{Q_{12\,b}\,m\,\pi^2\,n\,d}{a\,b}\end{aligned} \tag{A6}$$

$$\begin{aligned}K_{22}=&-\frac{2}{3}\frac{Q_{22\,m}\,n^2\,\pi^2\,d}{b^2}-\frac{2}{3}\frac{Q_{66\,m}\,m^2\,\pi^2\,d}{a^2}-\frac{2}{3}\frac{Q_{22\,b}\,n^2\,\pi^2\,d}{b^2}\\&-\frac{2}{3}\frac{Q_{66\,t}\,m^2\,\pi^2\,d}{a^2}-\frac{2}{3}\frac{Q_{66\,b}\,m^2\,\pi^2\,d}{a^2}-\frac{2}{3}\frac{Q_{22\,t}\,n^2\,\pi^2\,d}{b^2}\end{aligned} \tag{A7}$$

$$\begin{aligned}K_{23}=&\frac{20}{81}\left(\frac{Q_{22\,t}\,c1\,n^3\,\pi^3}{b^3}+\frac{Q_{12\,t}\,c1\,m^2\,\pi^3\,n}{a^2\,b}\right)d^4\\&-\frac{20}{81}\left(\frac{Q_{22\,b}\,c1\,n^3\,\pi^3}{b^3}+\frac{Q_{12\,b}\,c1\,m^2\,\pi^3\,n}{a^2\,b}\right)d^4\\&-\frac{40}{81}\frac{Q_{66\,b}\,c1\,m^2\,\pi^3\,n\,d^4}{a^2\,b}+\frac{40}{81}\frac{Q_{66\,t}\,c1\,m^2\,\pi^3\,n\,d^4}{a^2\,b}\end{aligned} \tag{A8}$$

$$\begin{aligned}K_{24}=&-\frac{20}{81}\frac{Q_{12\,b}\,c1\,m\,\pi^2\,n\,d^4}{a\,b}+\frac{4}{9}\frac{Q_{12\,b}\,m\,\pi^2\,n\,d^2}{a\,b}\\&+\frac{20}{81}\frac{Q_{12\,t}\,c1\,m\,\pi^2\,n\,d^4}{a\,b}-\frac{20}{81}\frac{Q_{66\,b}\,c1\,m\,\pi^2\,n\,d^4}{a\,b}\\&-\frac{4}{9}\frac{Q_{66\,t}\,m\,\pi^2\,n\,d^2}{a\,b}+\frac{20}{81}\frac{Q_{66\,t}\,c1\,m\,\pi^2\,n\,d^4}{a\,b}\\&+\frac{4}{9}\frac{Q_{66\,b}\,m\,\pi^2\,n\,d^2}{a\,b}-\frac{4}{9}\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^2}{a\,b}\end{aligned} \tag{A9}$$

$$\begin{aligned}K_{25}=&-\frac{4}{9}\frac{Q_{22\,t}\,n^2\,\pi^2\,d^2}{b^2}+\frac{4}{9}\frac{Q_{22\,b}\,n^2\,\pi^2\,d^2}{b^2}\\&-\frac{20}{81}\frac{Q_{66\,b}\,c1\,m^2\,\pi^2\,d^4}{a^2}+\frac{20}{81}\frac{Q_{22\,t}\,c1\,n^2\,\pi^2\,d^4}{b^2}\end{aligned} \tag{A10}$$

$$
+\frac{4}{9}\,\frac{Q_{66\,b}\,m^2\,\pi^2\,d^2}{a^2}-\frac{20}{81}\,\frac{Q_{22\,b}\,c1\,n^2\,\pi^2\,d^4}{b^2}
-\frac{4}{9}\,\frac{Q_{66\,t}\,m^2\,\pi^2\,d^2}{a^2}+\frac{20}{81}\,\frac{Q_{66\,t}\,c1\,m^2\,\pi^2\,d^4}{a^2} \tag{A10}
$$

$$
\begin{aligned}
K_{31}=c1\Bigg(&\frac{20}{81}\,\frac{Q_{11\,t}\,m^3\,\pi^3\,d^4}{a^3}+\frac{40}{81}\,\frac{Q_{66\,t}\,m\,\pi^3\,n^2\,d^4}{a\,b^2}\\
&-\frac{40}{81}\,\frac{Q_{66\,b}\,m\,\pi^3\,n^2\,d^4}{a\,b^2}-\frac{20}{81}\,\frac{Q_{12\,b}\,m\,\pi^3\,n^2\,d^4}{a\,b^2}\\
&-\frac{20}{81}\,\frac{Q_{11\,b}\,m^3\,\pi^3\,d^4}{a^3}+\frac{20}{81}\,\frac{Q_{12\,t}\,m\,\pi^3\,n^2\,d^4}{a\,b^2}\Bigg)
\end{aligned} \tag{A11}
$$

$$
\begin{aligned}
K_{32}=c1\Bigg(&-\frac{20}{81}\,\frac{Q_{22\,b}\,n^3\,\pi^3\,d^4}{b^3}+\frac{40}{81}\,\frac{Q_{66\,t}\,m^2\,\pi^3\,n\,d^4}{a^2\,b}\\
&-\frac{40}{81}\,\frac{Q_{66\,b}\,m^2\,\pi^3\,n\,d^4}{a^2\,b}+\frac{20}{81}\,\frac{Q_{12\,t}\,m^2\,\pi^3\,n\,d^4}{a^2\,b}\\
&+\frac{20}{81}\,\frac{Q_{22\,t}\,n^3\,\pi^3\,d^4}{b^3}-\frac{20}{81}\,\frac{Q_{12\,b}\,m^2\,\pi^3\,n\,d^4}{a^2\,b}\Bigg)
\end{aligned} \tag{A12}
$$

$$
\begin{aligned}
K_{33}=\Bigg(&\frac{26}{27}\,\frac{Q_{55\,b}\,c1\,m^2\,\pi^2\,d^3}{a^2}+\frac{2}{27}\,\frac{Q_{55\,m}\,c1\,m^2\,\pi^2\,d^3}{a^2}\\
&+\frac{26}{27}\,\frac{Q_{55\,t}\,c1\,m^2\,\pi^2\,d^3}{a^2}-\frac{N_{ym}\,n^2\,\pi^2}{b^2}-\frac{N_{xm}\,m^2\,\pi^2}{a^2}\\
&+\frac{26}{27}\,\frac{Q_{44\,b}\,c1\,n^2\,\pi^2\,d^3}{b^2}-\frac{2}{3}\,\frac{Q_{55\,b}\,m^2\,\pi^2\,d}{a^2}\\
&+\frac{2}{27}\,\frac{Q_{44\,m}\,c1\,n^2\,\pi^2\,d^3}{b^2}-\frac{2}{3}\,\frac{Q_{44\,m}\,n^2\,\pi^2\,d}{b^2}\\
&+\frac{26}{27}\,\frac{Q_{44\,t}\,c1\,n^2\,\pi^2\,d^3}{b^2}-\frac{2}{3}\,\frac{Q_{55\,m}\,m^2\,\pi^2\,d}{a^2}\\
&+c1\left(\frac{2186}{15309}\left(-\frac{Q_{22\,t}\,c1\,n^4\,\pi^4}{b^4}-\frac{Q_{12\,t}\,c1\,m^2\,\pi^4\,n^2}{a^2\,b^2}\right)d^7\right.\\
&+\frac{2186}{15309}\left(-\frac{Q_{22\,b}\,c1\,n^4\,\pi^4}{b^4}-\frac{Q_{12\,b}\,c1\,m^2\,\pi^4\,n^2}{a^2\,b^2}\right)d^7
\end{aligned} \tag{A13}
$$

$$
\begin{aligned}
&- \frac{8744}{15309} \frac{Q_{66\,b}\, c1\, m^2 \pi^4 n^2 d^7}{a^2 b^2} + \frac{2186}{15309} \Bigg(\\
&- \frac{Q_{12\,b}\, c1\, m^2 \pi^4 n^2}{a^2 b^2} - \frac{Q_{11\,b}\, c1\, m^4 \pi^4}{a^4} \Bigg) d^7 \\
&- \frac{8744}{15309} \frac{Q_{66\,t}\, c1\, m^2 \pi^4 n^2 d^7}{a^2 b^2} + \frac{2186}{15309} \Bigg(\\
&- \frac{Q_{12\,t}\, c1\, m^2 \pi^4 n^2}{a^2 b^2} - \frac{Q_{11\,t}\, c1\, m^4 \pi^4}{a^4} \Bigg) d^7 + \frac{2}{15309} \Bigg(\\
&- \frac{Q_{12\,m}\, c1\, m^2 \pi^4 n^2}{a^2 b^2} - \frac{Q_{11\,m}\, c1\, m^4 \pi^4}{a^4} \Bigg) d^7 + \frac{2}{15309} \Bigg(\\
&- \frac{Q_{22\,m}\, c1\, n^4 \pi^4}{b^4} - \frac{Q_{12\,m}\, c1\, m^2 \pi^4 n^2}{a^2 b^2} \Bigg) d^7 \\
&- \frac{8}{15309} \frac{Q_{66\,m}\, c1\, m^2 \pi^4 n^2 d^7}{a^2 b^2} \Bigg) - \frac{2}{3} \frac{Q_{55\,t}\, m^2 \pi^2 d}{a^2} \\
&- \frac{2}{3} \frac{Q_{44\,b}\, n^2 \pi^2 d}{b^2} - c2 \Bigg(\frac{242}{405} \frac{Q_{55\,b}\, c1\, m^2 \pi^2 d^5}{a^2} \\
&- \frac{26}{81} \frac{Q_{55\,b}\, m^2 \pi^2 d^3}{a^2} + \frac{2}{405} \frac{Q_{55\,m}\, c1\, m^2 \pi^2 d^5}{a^2} \\
&- \frac{2}{81} \frac{Q_{55\,m}\, m^2 \pi^2 d^3}{a^2} + \frac{242}{405} \frac{Q_{55\,t}\, c1\, m^2 \pi^2 d^5}{a^2} \\
&- \frac{26}{81} \frac{Q_{55\,t}\, m^2 \pi^2 d^3}{a^2} \Bigg) - \frac{2}{3} \frac{Q_{44\,t}\, n^2 \pi^2 d}{b^2} \\
&- c2 \Bigg(\frac{242}{405} \frac{Q_{44\,b}\, c1\, n^2 \pi^2 d^5}{b^2} - \frac{26}{81} \frac{Q_{44\,b}\, n^2 \pi^2 d^3}{b^2} \\
&+ \frac{2}{405} \frac{Q_{44\,m}\, c1\, n^2 \pi^2 d^5}{b^2} - k_w - k_g \cdot \left(\frac{m^2 \pi^2}{a^2} + \frac{n^2 \pi^2}{b^2} \right) \\
&- \frac{2}{81} \frac{Q_{44\,m}\, n^2 \pi^2 d^3}{b^2} + \frac{242}{405} \frac{Q_{44\,t}\, c1\, n^2 \pi^2 d^5}{b^2} \\
&- \frac{26}{81} \frac{Q_{44\,t}\, n^2 \pi^2 d^3}{b^2} \Bigg) \Bigg)
\end{aligned}
\tag{A13}
$$

$$
K_{34} = \Bigg(-\frac{2}{3} \frac{Q_{55\,b}\, m \pi d}{a} + \frac{26}{27} \frac{Q_{55\,b}\, c1\, m \pi d^3}{a}
\tag{A14}
$$

$$
\begin{aligned}
&- \frac{2}{3}\,\frac{Q_{55\,t}\, m\,\pi\, d}{a} - \frac{2}{3}\,\frac{Q_{55\,m}\, m\,\pi\, d}{a} + \frac{26}{27}\,\frac{Q_{55\,t}\, c1\, m\,\pi\, d^3}{a} \\
&+ c1\left(-\frac{2186}{15309}\,\frac{Q_{11\,b}\, c1\, m^3\,\pi^3\, d^7}{a^3} \right. \\
&- \frac{2186}{15309}\,\frac{Q_{12\,b}\, c1\, m\,\pi^3\, n^2\, d^7}{a\, b^2} + \frac{242}{1215}\,\frac{Q_{11\,b}\, m^3\,\pi^3\, d^5}{a^3} \\
&+ \frac{4}{1215}\,\frac{Q_{66\,m}\, m\,\pi^3\, n^2\, d^5}{a\, b^2} - \frac{4372}{15309}\,\frac{Q_{66\,t}\, c1\, m\,\pi^3\, n^2\, d^7}{a\, b^2} \\
&+ \frac{242}{1215}\,\frac{Q_{11\,t}\, m^3\,\pi^3\, d^5}{a^3} - \frac{2186}{15309}\,\frac{Q_{11\,t}\, c1\, m^3\,\pi^3\, d^7}{a^3} \\
&+ \frac{242}{1215}\,\frac{Q_{12\,b}\, m\,\pi^3\, n^2\, d^5}{a\, b^2} - \frac{2}{15309}\,\frac{Q_{11\,m}\, c1\, m^3\,\pi^3\, d^7}{a^3} \\
&+ \frac{484}{1215}\,\frac{Q_{66\,b}\, m\,\pi^3\, n^2\, d^5}{a\, b^2} + \frac{484}{1215}\,\frac{Q_{66\,t}\, m\,\pi^3\, n^2\, d^5}{a\, b^2} \\
&+ \frac{2}{1215}\,\frac{Q_{12\,m}\, m\,\pi^3\, n^2\, d^5}{a\, b^2} + \frac{2}{1215}\,\frac{Q_{11\,m}\, m^3\,\pi^3\, d^5}{a^3} \\
&- \frac{4372}{15309}\,\frac{Q_{66\,b}\, c1\, m\,\pi^3\, n^2\, d^7}{a\, b^2} - \frac{2186}{15309}\,\frac{Q_{12\,t}\, c1\, m\,\pi^3\, n^2\, d^7}{a\, b^2} \\
&- \frac{2}{15309}\,\frac{Q_{12\,m}\, c1\, m\,\pi^3\, n^2\, d^7}{a\, b^2} + \frac{242}{1215}\,\frac{Q_{12\,t}\, m\,\pi^3\, n^2\, d^5}{a\, b^2} \\
&\left. - \frac{4}{15309}\,\frac{Q_{66\,m}\, c1\, m\,\pi^3\, n^2\, d^7}{a\, b^2} \right) + \frac{2}{27}\,\frac{Q_{55\,m}\, c1\, m\,\pi\, d^3}{a} \\
&- c2\left(\frac{242}{405}\,\frac{Q_{55\,b}\, c1\, m\,\pi\, d^5}{a} - \frac{26}{81}\,\frac{Q_{55\,b}\, m\,\pi\, d^3}{a} \right. \\
&+ \frac{2}{405}\,\frac{Q_{55\,m}\, c1\, m\,\pi\, d^5}{a} - \frac{2}{81}\,\frac{Q_{55\,m}\, m\,\pi\, d^3}{a} \\
&\left.\left. + \frac{242}{405}\,\frac{Q_{55\,t}\, c1\, m\,\pi\, d^5}{a} - \frac{26}{81}\,\frac{Q_{55\,t}\, m\,\pi\, d^3}{a} \right)\right)
\end{aligned}
\tag{A14}
$$

$$
\begin{aligned}
K_{35} = &\left(\frac{2}{27}\,\frac{Q_{44\,m}\, c1\, n\,\pi\, d^3}{b} - \frac{2}{3}\,\frac{Q_{44\,b}\, n\,\pi\, d}{b} \right. \\
&+ \frac{26}{27}\,\frac{Q_{44\,b}\, c1\, n\,\pi\, d^3}{b} - \frac{2}{3}\,\frac{Q_{44\,t}\, n\,\pi\, d}{b} + c1\,\Bigg(
\end{aligned}
\tag{A15}
$$

$$
\begin{aligned}
&-\frac{2186}{15309}\,\frac{Q_{12\,b}\,c1\,m^2\,\pi^3\,n\,d^7}{a^2\,b}+\frac{484}{1215}\,\frac{Q_{66\,t}\,m^2\,\pi^3\,n\,d^5}{a^2\,b}\\
&-\frac{2186}{15309}\,\frac{Q_{22\,b}\,c1\,n^3\,\pi^3\,d^7}{b^3}+\frac{2}{1215}\,\frac{Q_{12\,m}\,m^2\,\pi^3\,n\,d^5}{a^2\,b}\\
&+\frac{242}{1215}\,\frac{Q_{12\,b}\,m^2\,\pi^3\,n\,d^5}{a^2\,b}-\frac{2186}{15309}\,\frac{Q_{22\,t}\,c1\,n^3\,\pi^3\,d^7}{b^3}\\
&+\frac{4}{1215}\,\frac{Q_{66\,m}\,m^2\,\pi^3\,n\,d^5}{a^2\,b}+\frac{242}{1215}\,\frac{Q_{22\,t}\,n^3\,\pi^3\,d^5}{b^3}\\
&-\frac{4372}{15309}\,\frac{Q_{66\,t}\,c1\,m^2\,\pi^3\,n\,d^7}{a^2\,b}+\frac{242}{1215}\,\frac{Q_{12\,t}\,m^2\,\pi^3\,n\,d^5}{a^2\,b}\\
&-\frac{2186}{15309}\,\frac{Q_{12\,t}\,c1\,m^2\,\pi^3\,n\,d^7}{a^2\,b}+\frac{242}{1215}\,\frac{Q_{22\,b}\,n^3\,\pi^3\,d^5}{b^3}\\
&-\frac{2}{15309}\,\frac{Q_{12\,m}\,c1\,m^2\,\pi^3\,n\,d^7}{a^2\,b}+\frac{484}{1215}\,\frac{Q_{66\,b}\,m^2\,\pi^3\,n\,d^5}{a^2\,b}\\
&+\frac{2}{1215}\,\frac{Q_{22\,m}\,n^3\,\pi^3\,d^5}{b^3}-\frac{4372}{15309}\,\frac{Q_{66\,b}\,c1\,m^2\,\pi^3\,n\,d^7}{a^2\,b}\\
&-\frac{2}{15309}\,\frac{Q_{22\,m}\,c1\,n^3\,\pi^3\,d^7}{b^3}-\frac{4}{15309}\,\frac{Q_{66\,m}\,c1\,m^2\,\pi^3\,n\,d^7}{a^2\,b}\Bigg)\\
&-\frac{2}{3}\,\frac{Q_{44\,m}\,n\,\pi\,d}{b}+\frac{26}{27}\,\frac{Q_{44\,t}\,c1\,n\,\pi\,d^3}{b}\\
&-c2\left(\frac{242}{405}\,\frac{Q_{44\,b}\,c1\,n\,\pi\,d^5}{b}-\frac{26}{81}\,\frac{Q_{44\,b}\,n\,\pi\,d^3}{b}\right.\\
&+\frac{2}{405}\,\frac{Q_{44\,m}\,c1\,n\,\pi\,d^5}{b}-\frac{2}{81}\,\frac{Q_{44\,m}\,n\,\pi\,d^3}{b}\\
&\left.+\frac{242}{405}\,\frac{Q_{44\,t}\,c1\,n\,\pi\,d^5}{b}-\frac{26}{81}\,\frac{Q_{44\,t}\,n\,\pi\,d^3}{b}\right)\Bigg)
\end{aligned}
\tag{A15}
$$

$$
\begin{aligned}
K_{41}=&\frac{4}{9}\,\frac{Q_{66\,b}\,n^2\,\pi^2\,d^2}{b^2}-c2\left(\frac{20}{81}\,\frac{Q_{66\,b}\,n^2\,\pi^2\,d^4}{b^2}\right.\\
&\left.-\frac{20}{81}\,\frac{Q_{66\,t}\,n^2\,\pi^2\,d^4}{b^2}\right)-c1\left(\frac{20}{81}\,\frac{Q_{11\,b}\,m^2\,\pi^2\,d^4}{a^2}\right.\\
&\left.-\frac{20}{81}\,\frac{Q_{11\,t}\,m^2\,\pi^2\,d^4}{a^2}\right)+\frac{4}{9}\,\frac{Q_{11\,b}\,m^2\,\pi^2\,d^2}{a^2}
\end{aligned}
\tag{A16}
$$

$$-\frac{4}{9}\,\frac{Q_{66\,t}\,n^2\,\pi^2\,d^2}{b^2}-\frac{4}{9}\,\frac{Q_{11\,t}\,m^2\,\pi^2\,d^2}{a^2} \tag{A16}$$

$$\begin{aligned}K_{42}=&\frac{4}{9}\,\frac{Q_{12\,b}\,m\,\pi^2\,n\,d^2}{a\,b}-\frac{4}{9}\,\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^2}{a\,b}\\&-\frac{4}{9}\,\frac{Q_{66\,t}\,m\,\pi^2\,n\,d^2}{a\,b}-c2\left(\frac{20}{81}\,\frac{Q_{66\,b}\,m\,\pi^2\,n\,d^4}{a\,b}\right.\\&\left.-\frac{20}{81}\,\frac{Q_{66\,t}\,m\,\pi^2\,n\,d^4}{a\,b}\right)+\frac{4}{9}\,\frac{Q_{66\,b}\,m\,\pi^2\,n\,d^2}{a\,b}\\&-c1\left(\frac{20}{81}\,\frac{Q_{12\,b}\,m\,\pi^2\,n\,d^4}{a\,b}-\frac{20}{81}\,\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^4}{a\,b}\right)\end{aligned} \tag{A17}$$

$$\begin{aligned}K_{43}=&-c2\left(\frac{4372}{15309}\,\frac{Q_{66\,b}\,c1\,m\,\pi^3\,n^2\,d^7}{a\,b^2}\right.\\&+\frac{4}{15309}\,\frac{Q_{66\,m}\,c1\,m\,\pi^3\,n^2\,d^7}{a\,b^2}\\&\left.+\frac{4372}{15309}\,\frac{Q_{66\,t}\,c1\,m\,\pi^3\,n^2\,d^7}{a\,b^2}\right)-\frac{2}{3}\,\frac{Q_{55\,m}\,m\,\pi\,d}{a}\\&-\frac{2}{3}\,\frac{Q_{55\,b}\,m\,\pi\,d}{a}+\frac{4}{1215}\,\frac{Q_{66\,m}\,c1\,m\,\pi^3\,n^2\,d^5}{a\,b^2}\\&+\frac{26}{27}\,\frac{Q_{55\,b}\,c1\,m\,\pi\,d^3}{a}+\frac{484}{1215}\,\frac{Q_{66\,b}\,c1\,m\,\pi^3\,n^2\,d^5}{a\,b^2}\\&-c1\left(\frac{2186}{15309}\left(\frac{Q_{12\,b}\,c1\,m\,\pi^3\,n^2}{a\,b^2}+\frac{Q_{11\,b}\,c1\,m^3\,\pi^3}{a^3}\right)d^7\right.\\&+\frac{2}{15309}\left(\frac{Q_{12\,m}\,c1\,m\,\pi^3\,n^2}{a\,b^2}+\frac{Q_{11\,m}\,c1\,m^3\,\pi^3}{a^3}\right)d^7\\&\left.+\frac{2186}{15309}\left(\frac{Q_{12\,t}\,c1\,m\,\pi^3\,n^2}{a\,b^2}+\frac{Q_{11\,t}\,c1\,m^3\,\pi^3}{a^3}\right)d^7\right)+c2\left(\right.\\&-\frac{242}{405}\,\frac{Q_{55\,b}\,c1\,m\,\pi\,d^5}{a}+\frac{26}{81}\,\frac{Q_{55\,b}\,m\,\pi\,d^3}{a}\\&-\frac{2}{405}\,\frac{Q_{55\,m}\,c1\,m\,\pi\,d^5}{a}+\frac{2}{81}\,\frac{Q_{55\,m}\,m\,\pi\,d^3}{a}\\&\left.-\frac{242}{405}\,\frac{Q_{55\,t}\,c1\,m\,\pi\,d^5}{a}+\frac{26}{81}\,\frac{Q_{55\,t}\,m\,\pi\,d^3}{a}\right)\\&+\frac{2}{27}\,\frac{Q_{55\,m}\,c1\,m\,\pi\,d^3}{a}-\frac{2}{3}\,\frac{Q_{55\,t}\,m\,\pi\,d}{a}\end{aligned} \tag{A18}$$

$$
\begin{aligned}
&+\frac{484}{1215}\,\frac{Q_{66\,t}\,c1\;m\,\pi^3\,n^2\,d^5}{a\,b^2}+\frac{2}{1215}\left(\frac{Q_{12\,m}\,c1\;m\,\pi^3\,n^2}{a\,b^2}\right.\\
&\left.+\frac{Q_{11\,m}\,c1\;m^3\,\pi^3}{a^3}\right)d^5+\frac{26}{27}\,\frac{Q_{55\,t}\,c1\;m\,\pi\,d^3}{a}\\
&+\frac{242}{1215}\left(\frac{Q_{12\,b}\,c1\;m\,\pi^3\,n^2}{a\,b^2}+\frac{Q_{11\,b}\,c1\;m^3\,\pi^3}{a^3}\right)d^5\\
&+\frac{242}{1215}\left(\frac{Q_{12\,t}\,c1\;m\,\pi^3\,n^2}{a\,b^2}+\frac{Q_{11\,t}\,c1\;m^3\,\pi^3}{a^3}\right)d^5
\end{aligned}
\tag{A18}
$$

$$
\begin{aligned}
K_{44}=&-\frac{26}{81}\,\frac{Q_{66\,b}\,n^2\,\pi^2\,d^3}{b^2}-c2\left(\frac{2186}{15309}\,\frac{Q_{66\,b}\,c1\;n^2\,\pi^2\,d^7}{b^2}\right.\\
&-\frac{242}{1215}\,\frac{Q_{66\,t}\,n^2\,\pi^2\,d^5}{b^2}-\frac{242}{1215}\,\frac{Q_{66\,b}\,n^2\,\pi^2\,d^5}{b^2}\\
&+\frac{2186}{15309}\,\frac{Q_{66\,t}\,c1\;n^2\,\pi^2\,d^7}{b^2}-\frac{2}{1215}\,\frac{Q_{66\,m}\,n^2\,\pi^2\,d^5}{b^2}\\
&\left.+\frac{2}{15309}\,\frac{Q_{66\,m}\,c1\;n^2\,\pi^2\,d^7}{b^2}\right)-\frac{2}{81}\,\frac{Q_{11\,m}\,m^2\,\pi^2\,d^3}{a^2}\\
&+\frac{242}{1215}\,\frac{Q_{11\,b}\,c1\;m^2\,\pi^2\,d^5}{a^2}+\frac{2}{1215}\,\frac{Q_{66\,m}\,c1\;n^2\,\pi^2\,d^5}{b^2}\\
&+\frac{26}{27}\,Q_{55\,b}\,c1\;d^3+\frac{2}{1215}\,\frac{Q_{11\,m}\,c1\;m^2\,\pi^2\,d^5}{a^2}\\
&+\frac{242}{1215}\,\frac{Q_{66\,t}\,c1\;n^2\,\pi^2\,d^5}{b^2}+\frac{242}{1215}\,\frac{Q_{66\,b}\,c1\;n^2\,\pi^2\,d^5}{b^2}\\
&-\frac{2}{3}\,Q_{55\,b}\,d-\frac{26}{81}\,\frac{Q_{11\,b}\,m^2\,\pi^2\,d^3}{a^2}-\frac{2}{3}\,Q_{55\,m}\,d\\
&-\frac{2}{81}\,\frac{Q_{66\,m}\,n^2\,\pi^2\,d^3}{b^2}+\frac{2}{27}\,Q_{55\,m}\,c1\;d^3\\
&-c1\left(\frac{2186}{15309}\,\frac{Q_{11\,b}\,c1\;m^2\,\pi^2\,d^7}{a^2}-\frac{242}{1215}\,\frac{Q_{11\,t}\,m^2\,\pi^2\,d^5}{a^2}\right.\\
&-\frac{242}{1215}\,\frac{Q_{11\,b}\,m^2\,\pi^2\,d^5}{a^2}+\frac{2186}{15309}\,\frac{Q_{11\,t}\,c1\;m^2\,\pi^2\,d^7}{a^2}\\
&\left.-\frac{2}{1215}\,\frac{Q_{11\,m}\,m^2\,\pi^2\,d^5}{a^2}+\frac{2}{15309}\,\frac{Q_{11\,m}\,c1\;m^2\,\pi^2\,d^7}{a^2}\right)
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
&-\frac{26}{81}\,\frac{Q_{66\,t}\,n^2\,\pi^2\,d^3}{b^2}+\frac{242}{1215}\,\frac{Q_{11\,t}\,c1\,m^2\,\pi^2\,d^5}{a^2}\\
&+\frac{26}{27}\,Q_{55\,t}\,c1\,d^3-\frac{2}{3}\,Q_{55\,t}\,d-\frac{26}{81}\,\frac{Q_{11\,t}\,m^2\,\pi^2\,d^3}{a^2}+c2\left(\vphantom{\frac{1}{1}}\right.\\
&-\frac{242}{405}\,Q_{55\,b}\,c1\,d^5+\frac{26}{81}\,Q_{55\,b}\,d^3-\frac{2}{405}\,Q_{55\,m}\,c1\,d^5\\
&\left.+\frac{2}{81}\,Q_{55\,m}\,d^3-\frac{242}{405}\,Q_{55\,t}\,c1\,d^5+\frac{26}{81}\,Q_{55\,t}\,d^3\right)
\end{aligned}
\tag{A19}
$$

$$
\begin{aligned}
K_{45}=&-\frac{26}{81}\,\frac{Q_{66\,b}\,m\,\pi^2\,n\,d^3}{a\,b}-\frac{2}{81}\,\frac{Q_{12\,m}\,m\,\pi^2\,n\,d^3}{a\,b}\\
&-\frac{26}{81}\,\frac{Q_{66\,t}\,m\,\pi^2\,n\,d^3}{a\,b}+\frac{242}{1215}\,\frac{Q_{66\,t}\,c1\,m\,\pi^2\,n\,d^5}{a\,b}\\
&-c2\left(\frac{2186}{15309}\,\frac{Q_{66\,b}\,c1\,m\,\pi^2\,n\,d^7}{a\,b}-\frac{242}{1215}\,\frac{Q_{66\,t}\,m\,\pi^2\,n\,d^5}{a\,b}\right.\\
&-\frac{242}{1215}\,\frac{Q_{66\,b}\,m\,\pi^2\,n\,d^5}{a\,b}+\frac{2186}{15309}\,\frac{Q_{66\,t}\,c1\,m\,\pi^2\,n\,d^7}{a\,b}\\
&\left.-\frac{2}{1215}\,\frac{Q_{66\,m}\,m\,\pi^2\,n\,d^5}{a\,b}+\frac{2}{15309}\,\frac{Q_{66\,m}\,c1\,m\,\pi^2\,n\,d^7}{a\,b}\right)\\
&+\frac{242}{1215}\,\frac{Q_{66\,b}\,c1\,m\,\pi^2\,n\,d^5}{a\,b}+\frac{242}{1215}\,\frac{Q_{12\,b}\,c1\,m\,\pi^2\,n\,d^5}{a\,b}\\
&-\frac{26}{81}\,\frac{Q_{12\,b}\,m\,\pi^2\,n\,d^3}{a\,b}+\frac{2}{1215}\,\frac{Q_{66\,m}\,c1\,m\,\pi^2\,n\,d^5}{a\,b}\\
&+\frac{2}{1215}\,\frac{Q_{12\,m}\,c1\,m\,\pi^2\,n\,d^5}{a\,b}\\
&-c1\left(\frac{2186}{15309}\,\frac{Q_{12\,b}\,c1\,m\,\pi^2\,n\,d^7}{a\,b}-\frac{242}{1215}\,\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^5}{a\,b}\right.\\
&-\frac{242}{1215}\,\frac{Q_{12\,b}\,m\,\pi^2\,n\,d^5}{a\,b}+\frac{2186}{15309}\,\frac{Q_{12\,t}\,c1\,m\,\pi^2\,n\,d^7}{a\,b}\\
&\left.-\frac{2}{1215}\,\frac{Q_{12\,m}\,m\,\pi^2\,n\,d^5}{a\,b}+\frac{2}{15309}\,\frac{Q_{12\,m}\,c1\,m\,\pi^2\,n\,d^7}{a\,b}\right)\\
&-\frac{2}{81}\,\frac{Q_{66\,m}\,m\,\pi^2\,n\,d^3}{a\,b}+\frac{242}{1215}\,\frac{Q_{12\,t}\,c1\,m\,\pi^2\,n\,d^5}{a\,b}\\
&-\frac{26}{81}\,\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^3}{a\,b}
\end{aligned}
\tag{A20}
$$

$$\begin{aligned}K_{51} = -c2 &\left(\frac{20}{81}\,\frac{Q_{12\,b}\, m\,\pi^2\, n\, d^4}{a\, b} - \frac{20}{81}\,\frac{Q_{12\,t}\, m\,\pi^2\, n\, d^4}{a\, b}\right)\\ &- \frac{4}{9}\,\frac{Q_{12\,t}\, m\,\pi^2\, n\, d^2}{a\, b} - \frac{4}{9}\,\frac{Q_{66\,t}\, m\,\pi^2\, n\, d^2}{a\, b}\\ &+ \frac{4}{9}\,\frac{Q_{12\,b}\, m\,\pi^2\, n\, d^2}{a\, b} + \frac{4}{9}\,\frac{Q_{66\,b}\, m\,\pi^2\, n\, d^2}{a\, b}\\ &- c1\left(\frac{20}{81}\,\frac{Q_{66\,b}\, m\,\pi^2\, n\, d^4}{a\, b} - \frac{20}{81}\,\frac{Q_{66\,t}\, m\,\pi^2\, n\, d^4}{a\, b}\right)\end{aligned} \tag{A21}$$

$$\begin{aligned}K_{52} = -c2 &\left(\frac{20}{81}\,\frac{Q_{22\,b}\, n^2\,\pi^2\, d^4}{b^2} - \frac{20}{81}\,\frac{Q_{22\,t}\, n^2\,\pi^2\, d^4}{b^2}\right)\\ &+ \frac{4}{9}\,\frac{Q_{66\,b}\, m^2\,\pi^2\, d^2}{a^2} - c1\left(\frac{20}{81}\,\frac{Q_{66\,b}\, m^2\,\pi^2\, d^4}{a^2}\right.\\ &\left.- \frac{20}{81}\,\frac{Q_{66\,t}\, m^2\,\pi^2\, d^4}{a^2}\right) + \frac{4}{9}\,\frac{Q_{22\,b}\, n^2\,\pi^2\, d^2}{b^2}\\ &- \frac{4}{9}\,\frac{Q_{22\,t}\, n^2\,\pi^2\, d^2}{b^2} - \frac{4}{9}\,\frac{Q_{66\,t}\, m^2\,\pi^2\, d^2}{a^2}\end{aligned} \tag{A22}$$

$$\begin{aligned}K_{53} = \frac{2}{1215} &\left(\frac{Q_{22\,m}\, c1\, n^3\,\pi^3}{b^3} + \frac{Q_{12\,m}\, c1\, m^2\,\pi^3\, n}{a^2\, b}\right) d^5 + c2\,\Bigg(\\ &- \frac{242}{405}\,\frac{Q_{44\,b}\, c1\, n\,\pi\, d^5}{b} + \frac{26}{81}\,\frac{Q_{44\,b}\, n\,\pi\, d^3}{b}\\ &- \frac{2}{405}\,\frac{Q_{44\,m}\, c1\, n\,\pi\, d^5}{b} + \frac{2}{81}\,\frac{Q_{44\,m}\, n\,\pi\, d^3}{b}\\ &- \frac{242}{405}\,\frac{Q_{44\,t}\, c1\, n\,\pi\, d^5}{b} + \frac{26}{81}\,\frac{Q_{44\,t}\, n\,\pi\, d^3}{b}\Bigg)\\ &+ \frac{242}{1215}\left(\frac{Q_{22\,t}\, c1\, n^3\,\pi^3}{b^3} + \frac{Q_{12\,t}\, c1\, m^2\,\pi^3\, n}{a^2\, b}\right) d^5\\ &+ \frac{26}{27}\,\frac{Q_{44\,b}\, c1\, n\,\pi\, d^3}{b} - c2\left(\frac{2186}{15309}\left(\frac{Q_{22\,b}\, c1\, n^3\,\pi^3}{b^3}\right.\right.\\ &\left.+ \frac{Q_{12\,b}\, c1\, m^2\,\pi^3\, n}{a^2\, b}\right) d^7 + \frac{2}{15309}\left(\frac{Q_{22\,m}\, c1\, n^3\,\pi^3}{b^3}\right.\\ &\left.+ \frac{Q_{12\,m}\, c1\, m^2\,\pi^3\, n}{a^2\, b}\right) d^7 + \frac{2186}{15309}\left(\frac{Q_{22\,t}\, c1\, n^3\,\pi^3}{b^3}\right.\end{aligned} \tag{A23}$$

$$
\begin{aligned}
&+ \frac{Q_{12\,t}\, c1\, m^2 \pi^3 n}{a^2 b} \Bigg) d^7 \Bigg) + \frac{242}{1215} \Bigg(\frac{Q_{22\,b}\, c1\, n^3 \pi^3}{b^3} \\
&+ \frac{Q_{12\,b}\, c1\, m^2 \pi^3 n}{a^2 b} \Bigg) d^5 + \frac{4}{1215} \frac{Q_{66\,m}\, c1\, m^2 \pi^3 n\, d^5}{a^2 b} \\
&- c1 \Bigg(\frac{4372}{15309} \frac{Q_{66\,b}\, c1\, m^2 \pi^3 n\, d^7}{a^2 b} \\
&+ \frac{4}{15309} \frac{Q_{66\,m}\, c1\, m^2 \pi^3 n\, d^7}{a^2 b} \\
&+ \frac{4372}{15309} \frac{Q_{66\,t}\, c1\, m^2 \pi^3 n\, d^7}{a^2 b} \Bigg) + \frac{484}{1215} \frac{Q_{66\,t}\, c1\, m^2 \pi^3 n\, d^5}{a^2 b} \\
&+ \frac{484}{1215} \frac{Q_{66\,b}\, c1\, m^2 \pi^3 n\, d^5}{a^2 b} - \frac{2}{3} \frac{Q_{44\,t}\, n \pi d}{b} \\
&- \frac{2}{3} \frac{Q_{44\,b}\, n \pi d}{b} + \frac{26}{27} \frac{Q_{44\,t}\, c1\, n \pi d^3}{b} - \frac{2}{3} \frac{Q_{44\,m}\, n \pi d}{b} \\
&+ \frac{2}{27} \frac{Q_{44\,m}\, c1\, n \pi d^3}{b}
\end{aligned}
\tag{A23}
$$

$$
\begin{aligned}
K_{54} = &-c1 \Bigg(\frac{2186}{15309} \frac{Q_{66\,b}\, c1\, m \pi^2 n\, d^7}{a b} - \frac{242}{1215} \frac{Q_{66\,t}\, m \pi^2 n\, d^5}{a b} \\
&- \frac{242}{1215} \frac{Q_{66\,b}\, m \pi^2 n\, d^5}{a b} + \frac{2186}{15309} \frac{Q_{66\,t}\, c1\, m \pi^2 n\, d^7}{a b} \\
&- \frac{2}{1215} \frac{Q_{66\,m}\, m \pi^2 n\, d^5}{a b} + \frac{2}{15309} \frac{Q_{66\,m}\, c1\, m \pi^2 n\, d^7}{a b} \Bigg) \\
&+ \frac{242}{1215} \frac{Q_{66\,b}\, c1\, m \pi^2 n\, d^5}{a b} + \frac{242}{1215} \frac{Q_{12\,t}\, c1\, m \pi^2 n\, d^5}{a b} \\
&+ \frac{242}{1215} \frac{Q_{12\,b}\, c1\, m \pi^2 n\, d^5}{a b} + \frac{2}{1215} \frac{Q_{12\,m}\, c1\, m \pi^2 n\, d^5}{a b} \\
&- \frac{26}{81} \frac{Q_{12\,t}\, m \pi^2 n\, d^3}{a b} - \frac{26}{81} \frac{Q_{12\,b}\, m \pi^2 n\, d^3}{a b} \\
&+ \frac{242}{1215} \frac{Q_{66\,t}\, c1\, m \pi^2 n\, d^5}{a b} - \frac{2}{81} \frac{Q_{12\,m}\, m \pi^2 n\, d^3}{a b} \\
&- \frac{2}{81} \frac{Q_{66\,m}\, m \pi^2 n\, d^3}{a b} - \frac{26}{81} \frac{Q_{66\,t}\, m \pi^2 n\, d^3}{a b} \\
&+ \frac{2}{1215} \frac{Q_{66\,m}\, c1\, m \pi^2 n\, d^5}{a b} - \frac{26}{81} \frac{Q_{66\,b}\, m \pi^2 n\, d^3}{a b}
\end{aligned}
\tag{A24}
$$

$$- c2\left(\frac{2186}{15309}\frac{Q_{12\,b}\,c1\;m\,\pi^2\,n\,d^7}{a\,b} - \frac{242}{1215}\frac{Q_{12\,t}\,m\,\pi^2\,n\,d^5}{a\,b} - \frac{242}{1215}\frac{Q_{12\,b}\,m\,\pi^2\,n\,d^5}{a\,b} + \frac{2186}{15309}\frac{Q_{12\,t}\,c1\;m\,\pi^2\,n\,d^7}{a\,b} - \frac{2}{1215}\frac{Q_{12\,m}\,m\,\pi^2\,n\,d^5}{a\,b} + \frac{2}{15309}\frac{Q_{12\,m}\,c1\;m\,\pi^2\,n\,d^7}{a\,b}\right) \tag{A24}$$

$$\begin{aligned}
K_{55} = {} & c2\left(-\frac{242}{405}Q_{44\,b}\,c1\;d^5 + \frac{26}{81}Q_{44\,b}\,d^3 - \frac{2}{405}Q_{44\,m}\,c1\;d^5 + \frac{2}{81}Q_{44\,m}\,d^3 - \frac{242}{405}Q_{44\,t}\,c1\;d^5 + \frac{26}{81}Q_{44\,t}\,d^3\right) \\
& - c1\left(\frac{2186}{15309}\frac{Q_{66\,b}\,c1\;m^2\,\pi^2\,d^7}{a^2} - \frac{242}{1215}\frac{Q_{66\,t}\,m^2\,\pi^2\,d^5}{a^2} - \frac{242}{1215}\frac{Q_{66\,b}\,m^2\,\pi^2\,d^5}{a^2} + \frac{2186}{15309}\frac{Q_{66\,t}\,c1\;m^2\,\pi^2\,d^7}{a^2} - \frac{2}{1215}\frac{Q_{66\,m}\,m^2\,\pi^2\,d^5}{a^2} + \frac{2}{15309}\frac{Q_{66\,m}\,c1\;m^2\,\pi^2\,d^7}{a^2}\right) \\
& + \frac{242}{1215}\frac{Q_{22\,b}\,c1\;n^2\,\pi^2\,d^5}{b^2} + \frac{2}{1215}\frac{Q_{22\,m}\,c1\;n^2\,\pi^2\,d^5}{b^2} \\
& - \frac{2}{3}Q_{44\,t}\,d + \frac{242}{1215}\frac{Q_{66\,b}\,c1\;m^2\,\pi^2\,d^5}{a^2} \\
& - \frac{26}{81}\frac{Q_{22\,b}\,n^2\,\pi^2\,d^3}{b^2} - \frac{2}{81}\frac{Q_{22\,m}\,n^2\,\pi^2\,d^3}{b^2} \\
& + \frac{2}{27}Q_{44\,m}\,c1\;d^3 + \frac{242}{1215}\frac{Q_{22\,t}\,c1\;n^2\,\pi^2\,d^5}{b^2} \\
& - \frac{26}{81}\frac{Q_{66\,t}\,m^2\,\pi^2\,d^3}{a^2} + \frac{26}{27}Q_{44\,t}\,c1\;d^3 - \frac{26}{81}\frac{Q_{22\,t}\,n^2\,\pi^2\,d^3}{b^2} \\
& - \frac{2}{3}Q_{44\,b}\,d + \frac{242}{1215}\frac{Q_{66\,t}\,c1\;m^2\,\pi^2\,d^5}{a^2} \\
& - \frac{2}{81}\frac{Q_{66\,m}\,m^2\,\pi^2\,d^3}{a^2} - \frac{2}{3}Q_{44\,m}\,d \\
& + \frac{2}{1215}\frac{Q_{66\,m}\,c1\;m^2\,\pi^2\,d^5}{a^2} - c2\left(\frac{2186}{15309}\frac{Q_{22\,b}\,c1\;n^2\,\pi^2\,d^7}{b^2}\right. \\
& \left. - \frac{242}{1215}\frac{Q_{22\,t}\,n^2\,\pi^2\,d^5}{b^2} - \frac{242}{1215}\frac{Q_{22\,b}\,n^2\,\pi^2\,d^5}{b^2}\right.
\end{aligned} \tag{A25}$$

$$
\begin{aligned}
&+ \frac{2186}{15309}\,\frac{Q_{22\,t}\,c1\,n^2\,\pi^2\,d^7}{b^2} - \frac{2}{1215}\,\frac{Q_{22\,m}\,n^2\,\pi^2\,d^5}{b^2} \\
&+ \frac{2}{15309}\,\frac{Q_{22\,m}\,c1\,n^2\,\pi^2\,d^7}{b^2}\Bigg) + \frac{26}{27}\,Q_{44\,b}\,c1\,d^3 \\
&- \frac{26}{81}\,\frac{Q_{66\,b}\,m^2\,\pi^2\,d^3}{a^2}
\end{aligned}
\tag{A25}
$$