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A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation

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Abstract. The objective of this work is to present a zeroth-order shear deformation theory for free vibration analysis of functionally graded (FG) nanoscale plates resting on elastic foundation. The model takes into consideration the influences of small scale and the parabolic variation of the transverse shear strains across the thickness of the nanoscale plate and thus, it avoids the employ use of shear correction factors. Also, in this present theory, the effect of transverse shear deformation is included in the axial displacements by using the shear forces instead of rotational displacements as in available high order plate theories. The material properties are supposed to be graded only in the thickness direction and the effective properties for the FG nanoscale plate are calculated by considering Mori–Tanaka homogenization scheme. The equations of motion are obtained using the nonlocal differential constitutive expressions of Eringen in conjunction with the zeroth-order shear deformation theory via Hamilton's principle. Numerical results for vibration of FG nanoscale plates resting on elastic foundations are presented and compared with the existing solutions. The influences of small scale, shear deformation, gradient index, Winkler modulus parameter and Pasternak shear modulus parameter on the vibration responses of the FG nanoscale plates are investigated.

Keywords: nonlocal elasticity theory; nanoscale-plates; free vibration; plate theory; functionally graded materials

1. Introduction

The local structural theories (classical theories) are utilized by employing the constitutive suppositions that the stress at a point is related only on the strain at that point. Whereas the nonlocal (non-classical) continuum mechanics proposed by Eringen (1972, 1983) assume that the stress at a point depends on strains at all points in the continuum. In non-classical elasticity theory,

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forces between atoms and internal length scale are included in the expressions of constitutive equations (Reddy and Pang 2008, Lu *et al*. 2008, Heireche *et al*. 2008a, b, Benzair *et al*. 2008, Amara *et al*. 2010, Hashemi and Samaei 2011, Berrabah *et al*. 2013, Benguediab *et al*. 2014, Zidour *et al*. 2014, Samaei *et al*. 2015, Adda Bedia *et al*. 2015, Besseghier *et al*. 2015, Aissani *et al*. 2015).

In recent years, nanostructures, such as nanoplates and nanobeams, are being employed in the nano-electro-mechanical (NEMS) and microelectro-mechanical (MEMS) devices and are nowadays engineering structure. Thus, a lot of researches have been carried out for both experimental and theoretical studies. Katsnelson and Novoselov (2007) investigated the electronic characteristics of graphene sheets. Bunch *et al*. (2007) discussed some experimental results by utilizing electromechanical resonators manufactured from single- and multi-layered graphene sheets. Aghbabaei and Reddy (2009) used a third order shear deformation plate theory to investigate analytically the bending ad free vibration of a simply supported rectangular nanoplate. Pradhan and Phadikar (2009) employed the nonlocal classical plate theory to investigate the vibration of embedded multi-layered graphene sheets considering the small scale effects. Based on an efficient higher-order nonlocal beam theory, Pradhan (2009) used the higher order shear deformation theory (HSDT) in conjunction with the nonlocal differential constitutive relations of Eringen to study buckling response of isotropic nanoplates. Pradhan and Kumar (2010) discussed the small scale influence on the vibration behavior of orthotropic single-layered graphene sheets embedded in an elastic medium. Samaei *et al*. (2011) examined the stability response of a single-layered graphene sheets embedded in a Pasternak's elastic medium by employing a nonlocal Mindlin plate theory. Tounsi *et al*. (2013a) studied the thermal stability of nanoscale beams. Tounsi *et al*. (2013b) analyzed the nonlocal effects on thermal buckling properties of double-walled carbon nanotubes. Nami and Janghorban (2013) investigated the static behavior of rectangular nanoplates using nonlocal trigonometric shear deformation theory. By incorporating Eringen's nonlocal elasticity equations in two-variable plate theories, Sobhy (2014) studied the free vibration, mechanical buckling and thermal buckling responses of multi-layered graphene sheets.

Due to their new thermo-mechanical characteristics, the applications of functionally graded materials (FGMs) have been speared in various engineering applications (El Meiche *et al*. 2011, Bourada *et al*. 2012, Tounsi *et al*. 2013c, Bouderba *et al*. 2013, Yaghoobi and Torabi 2013, Ould Larbi *et al*. 2013, Chakraverty and Pradhan 2014, Liang *et al*. 2014, Zidi *et al*. 2014, Khalfi *et al*. 2014, Fekrar *et al*. 2014, Bousahla *et al*. 2014, Belabed *et al*. 2014, Ait Amar Meziane *et al*. 2014, Hebali *et al*. 2014, Hamidi *et al*. 2015, Ait Yahia *et al*. 2015, Ait Atmane *et al*. 2015, Ziane *et al*. 2015, Bennai *et al*. 2015, Bouchafa *et al*. 2015). Nowadays, functionally graded micro/nano structures become considerably useful in many modern engineering applications such as aircraft fuselages, microelectronic industry, building blocks for ultrasensitive and steam and gas turbine rotors. It seem that functionally graded nanoscale structures have many advantages over the isotropic nanoscale structures, such as smaller thermal stresses, stress concentrations, attenuation of stress waves, etc. Increasing of the material technology has conducted to use of FGMs in micro and nano-sized system and devices such as sensors, nanowires, atomic force microscopes, actuators, thin films to improve their performances (Fu *et al*. 2003, Lee *et al*. 2006, Lu *et al*. 2011, Lun *et al*. 2006, Moser and Gijs 2007, Rahaeifard *et al*. 2009, Stölken and Evans 1998, Witvrouw and Mehta 2005). Jung and Han (2013) developed a model for vibration behavior of sigmoid functionally graded material nanoplate using first-order shear deformation theory. Natarajan *et al*. (2012) studied the free flexural vibration behavior of FG nanoplates using the iso-geometric based finite element method. Hosseini-Hashemi *et al*. (2013) presented an exact analytical solution for free vibration of FG circular/annular Mindlin nanoplates using nonlocal elasticity. Recently, Larbi Chaht *et al*. (2015) investigated the bending and stability behavior of FG size-dependent nanobeams incorporating the thickness stretching effect. Belkorissat *et al*. (2015) studied the vibration properties of nanoplates using a new nonlocal hyperbolic refined plate model. Ansari *et al*. (2015) examined the vibration and buckling characteristics of FG nanoplates subjected to thermal loading based on surface elasticity theory. Zemri *et al*. (2015) presented a refined nonlocal shear deformation theory beam theory for mechanical response of FG nanoscale beam.

In this paper, the zeroth-order shear deformation theory (ZSDT) is extended for the first time for vibration analysis of FG nanoplates embedded in an elastic medium. This theory (ZSDT) is used by Ray (2003) for laminated composite plates and incorporates the transverse shear deformation effect through the employ of shear forces instead of rotational displacements as in existing shear deformation theories. The ZSDT utilizes the same five unknowns as in the FSDT, but respects the traction-free boundary conditions on the top and bottom surfaces of the plate without introducing of any shear correction factor. In addition, the small scale effect is taken into account by using the nonlocal constitutive relations of Eringen. Some numerical results are also computed to check the validity of the present theory.

2. Mathematical formulation

2.1 Functionally graded material

Consider FG nano-plates manufactured from a mixture of two material phases, for example, a metal and a ceramic as indicated in Fig. 1. According to Mori–Tanaka homogenization scheme, the effective Bulk Modulus (K) and the effective shear modulus (G) are given by (Belabed *et al*. 2014, Valizadeh *et al*. 2013, Cheng and Batra 2000, Qian *et al*. 2004)

$$\frac{K-K_m}{K_c-K_m}=\frac{V_c}{1+(1-V_c)\dfrac{3(K_c-K_m)}{3K_m+4G_m}} \tag{1a}$$

$$\frac{G-G_m}{G_c-G_m}=\frac{V_c}{1+(1-V_c)\dfrac{(G_c-G_m)}{G_m+f_1}} \tag{1b}$$

where

$$f_1=\frac{G_m(9K_m+8G_m)}{6(K_m+2G_m)} \tag{2}$$

where, V_i (i = c, m) is the volume fraction of the phase material. The subscripts c and m represent the ceramic and metal phases, respectively. The volume fractions of the ceramic and metal phases are related by $V_c + V_m = 1$, and V_c is written as

$$V_c(z)=\left(\frac{2z+h}{2h}\right)^n, \qquad n\geq 0 \tag{3}$$

with n in Eq. (3) is the volume fraction exponent. Fig. 2 plots the distribution of the volume fraction of the ceramic phase within the thickness direction z for the FG plate. The effective Young's modulus E and Poisson's ratio ν can be calculated from the following equations

$$E = \frac{9KG}{3K + G} \tag{4a}$$

$$\nu = \frac{3K - 2G}{2(3K + G)} \tag{4b}$$

The effective mass density ρ is computed from the rule of mixtures as (Benachour *et al*. 2011, Natarajan *et al*. 2011, Hebali *et al*. 2014)

$$\rho = \rho_c V_c + \rho_m V_m \tag{5}$$

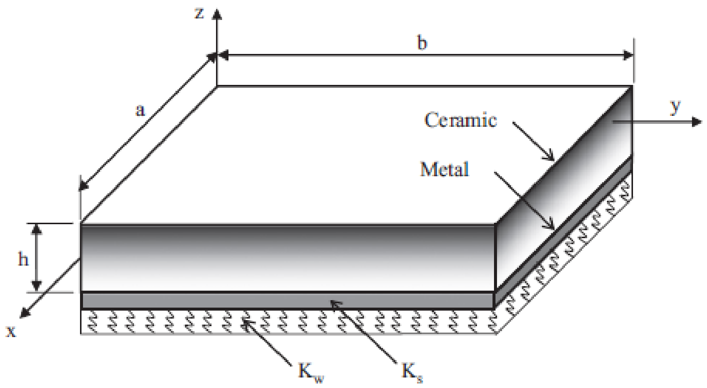

Fig. 1 Schematic representation of a rectangular FG plate resting on elastic foundation

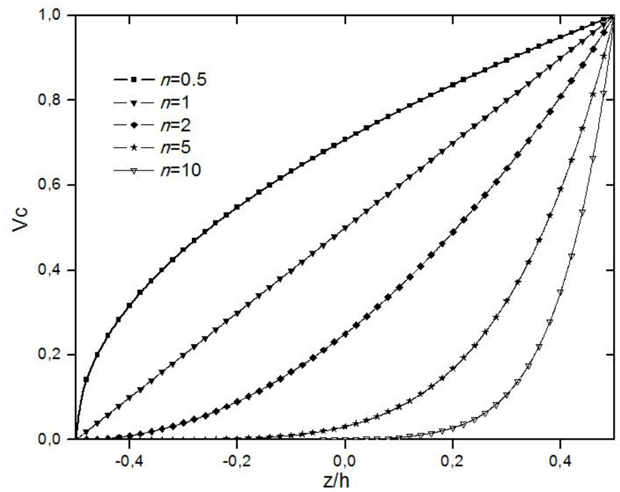

Fig. 2 Variation of ceramic phase through the thickness of the plate

2.2 Kinematics

The displacement field of the ZSDT is considered based on the supposition that the transverse shear stresses change according to a parabolic variation within the plate thickness and vanish on the plate surfaces, and hence, there is no require to utilize shear correction factor. Based on this supposition, the following displacement field can be determined (Ray 2003)

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_0}{\partial x} + \frac{1}{\lambda_x}\left[\frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_x(x,y,t) \tag{6a}$$

$$v(x,y,z,t) = v_0(x,y,t) - z\frac{\partial w_0}{\partial y} + \frac{1}{\lambda_x}\left[\frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_y(x,y,t) \tag{6b}$$

$$w(x,y,z,t) = w_0(x,y,t) \tag{6c}$$

where u_0 and v_0 represent the displacements along the x and y coordinate directions of a point on the mid-plane of the plate; w_0 is the transverse displacement; and h is the plate thickness. Q_x and Q_y are the transverse shear forces; and λ_x and λ_y are unknown constants obtained based on the definition of the transverse shear forces as

$$Q_i = \int_{-h/2}^{h/2} \tau_{iz}\, dz\,, \qquad (i = x, y) \tag{7}$$

The only nonzero strains related to the displacement field in Eqs. (6) are

$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix} = \begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} + z\begin{Bmatrix}k_x\\ k_y\\ k_{xy}\end{Bmatrix} + f(z)\begin{Bmatrix}\eta_x\\ \eta_y\\ \eta_{xy}\end{Bmatrix}, \qquad \begin{Bmatrix}\gamma_{yz}\\ \gamma_{xz}\end{Bmatrix} = g(z)\begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix}, \tag{8}$$

where

$$\begin{Bmatrix}\varepsilon_x^0\\ \varepsilon_y^0\\ \gamma_{xy}^0\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial u_0}{\partial x}\\ \dfrac{\partial v_0}{\partial x}\\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}k_x\\ k_y\\ k_{xy}\end{Bmatrix} = \begin{Bmatrix}-\dfrac{\partial^2 w_0}{\partial x^2}\\ -\dfrac{\partial^2 w_0}{\partial y^2}\\ -2\dfrac{\partial^2 w_0}{\partial x \partial y}\end{Bmatrix}, \quad \begin{Bmatrix}\eta_x\\ \eta_y\\ \eta_{xy}\end{Bmatrix} = \begin{Bmatrix}\dfrac{1}{\lambda_x}\dfrac{\partial Q_x}{\partial x}\\ \dfrac{1}{\lambda_y}\dfrac{\partial Q_y}{\partial y}\\ \dfrac{1}{\lambda_x}\dfrac{\partial Q_x}{\partial y} + \dfrac{1}{\lambda_y}\dfrac{\partial Q_y}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}\gamma_{yz}^0\\ \gamma_{xz}^0\end{Bmatrix} = \begin{Bmatrix}\dfrac{Q_x}{\lambda_x}\\ \dfrac{Q_y}{\lambda_y}\end{Bmatrix}, \tag{9a}$$

and

$$f(z) = \left[\frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right] \quad \text{and} \quad g(z) = \frac{df}{dz} = \frac{3}{2h}\left[1 - 4\left(\frac{z}{h}\right)^2\right] \tag{9b}$$

2.3 Equations of motion

In this section, equations of motion are determined by utilizing Hamilton's principle. In analytical form, this principle can be stated by (Reddy 2007, Draiche *et al.* 2014, Ait Amar Meziane *et al.* 2014, Nedri *et al.* 2014, Mahi *et al.* 2015, Bourada *et al.* 2015, Al-Basyouni *et al.* 2015)

$$0=\int_0^t (\delta U_p+\delta U_f-\delta K)\,dt \tag{10}$$

where δU_p and δU_f are the variations of strain energy of the plate and foundation, respectively; and δK is the variation of kinetic energy.

The variation of strain energy of the plate is expressed by

$$\begin{aligned}\delta U_p &=\int_V\left[\sigma_x\,\delta\varepsilon_x+\sigma_y\,\delta\varepsilon_y+\tau_{xy}\,\delta\gamma_{xy}+\tau_{yz}\,\delta\gamma_{yz}+\tau_{zx}\,\delta\gamma_{zx}\right]dAdz\\ &=\int_A\left[N_x\,\delta\varepsilon_x^0+N_y\,\delta\varepsilon_y^0+N_{xy}\,\delta\varepsilon_{xy}^0+M_x\,\delta k_x+M_y\,\delta k_y+M_{xy}\,\delta k_{xy}+P_x\,\delta\eta_x\right.\\ &\left.\quad+P_y\,\delta\eta_y+P_{xy}\,\delta\eta_{xy}+R_y\,\delta\gamma_{yz}^0+R_x\,\delta\gamma_{xz}^0\right]dA\end{aligned} \tag{11}$$

where the stress resultants N, M, P and R are expressed by

$$(N_i,M_i,P_i)=\int_{-h/2}^{h/2}(1,z,f)\sigma_i dz\,,\quad (i=x,y,xy)\qquad\text{and}\qquad (R_x,R_y)=\int_{-h/2}^{h/2}g(\tau_{xz},\tau_{yz})dz \tag{12}$$

The variation of strain energy of the elastic medium is computed by

$$\delta U_f=\int_A\left[K_w w\delta w+K_s\left(\frac{\partial w}{\partial x}\frac{\partial\delta w}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial\delta w}{\partial y}\right)\right]dxdy \tag{13}$$

where K_w and K_s are the transverse and shear stiffness coefficients of the elastic medium, respectively.

The variation of kinetic energy of the plate is expressed as

$$\begin{aligned}\delta K &=\int_{-h/2}^{h/2}\int_A\left[\dot u\delta\dot u+\dot v\delta\dot v+\dot w\delta\dot w\right]\rho(z)\,dA\,dz\\ &=\int_A\Big\{I_0\left[\dot u_0\delta\dot u_0+\dot v_0\delta\dot v_0+\dot w_0\delta\dot w_0\right]\\ &\quad-I_1\left(\dot u_0\frac{\partial\delta\dot w_0}{\partial x}+\frac{\partial\dot w_0}{\partial x}\delta\dot u_0+\dot v_0\frac{\partial\delta\dot w_0}{\partial y}+\frac{\partial\dot w_0}{\partial y}\delta\dot v_0\right)\\ &\quad+J_1\left(\dot u_0\frac{\delta\dot Q_x}{\lambda_x}+\frac{\dot Q_x}{\lambda_x}\delta\dot u_0+\dot v_0\frac{\delta\dot Q_y}{\lambda_y}+\frac{\dot Q_y}{\lambda_y}\delta\dot v_0\right)\end{aligned} \tag{14}$$

$$+I_2\left(\frac{\partial\dot{w}_0}{\partial x}\frac{\partial\delta\,\dot{w}_0}{\partial x}+\frac{\partial\dot{w}_0}{\partial y}\frac{\partial\delta\,\dot{w}_0}{\partial y}\right)+K_2\left(\frac{\dot{Q}_x\delta\dot{Q}_x}{\lambda_x^2}+\frac{\dot{Q}_y\delta\dot{Q}_y}{\lambda_y^2}\right)$$
$$\left.-J_2\left(\frac{\partial\dot{w}_0}{\partial x}\frac{\delta\dot{Q}_x}{\lambda_x}+\frac{\dot{Q}_x}{\lambda_x}\frac{\partial\delta\,\dot{w}_0}{\partial x}+\frac{\partial\dot{w}_0}{\partial y}\frac{\delta\dot{Q}_y}{\lambda_y}+\frac{\dot{Q}_y}{\lambda_y}\frac{\partial\delta\,\dot{w}_0}{\partial y}\right)\right\}dA \tag{14}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; and (I_0, I_1, J_1, I_2, J_2, K_2) are mass inertias defined as

$$\left(I_0,I_1,J_1,I_2,J_2,K_2\right)=\int_{-h/2}^{h/2}\left(1,z,f,z^2,z\,f,f^2\right)\rho(z)dz \tag{15}$$

Substituting the expressions for δU_p, δU_f and δK from Eqs. (11), (13) and (14) into Eq. (10) and integrating by parts, and collecting the coefficients of δu_0, δv_0, δw_0, δQ_x and δQ_y, the following equations of motion of the present theory are obtained

$$\begin{aligned}
&\delta\,u_0:\ \frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=I_0\ddot{u}_0-I_1\frac{\partial\ddot{w}_0}{\partial x}+J_1\frac{\ddot{Q}_x}{\lambda_x}\\
&\delta\,v_0:\ \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=I_0\ddot{v}_0-I_1\frac{\partial\ddot{w}_0}{\partial y}+J_1\frac{\ddot{Q}_y}{\lambda_y}\\
&\delta\,w_0:\ \frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}-K_w w_0+K_s\nabla^2 w_0\\
&\qquad =I_0\ddot{w}_0+I_1\left(\frac{\partial\ddot{u}_0}{\partial x}+\frac{\partial\ddot{v}_0}{\partial y}\right)-I_2\nabla^2\ddot{w}_0+J_2\left(\frac{\partial\ddot{Q}_x}{\lambda_x\partial x}+\frac{\partial\ddot{Q}_y}{\lambda_y\partial y}\right)\\
&\delta\,Q_x:\ \frac{\partial P_x}{\partial x}+\frac{\partial P_{xy}}{\partial y}-R_x=J_1\ddot{u}_0-J_2\frac{\partial\ddot{w}_0}{\partial x}+K_2\frac{\ddot{Q}_x}{\lambda_x}\\
&\delta\,Q_y:\ \frac{\partial P_{xy}}{\partial x}+\frac{\partial P_y}{\partial y}-R_y=J_1\ddot{v}_0-J_2\frac{\partial\ddot{w}_0}{\partial y}+K_2\frac{\ddot{Q}_y}{\lambda_y}
\end{aligned} \tag{16}$$

where $\nabla^2=(\partial^2/\partial x^2)+(\partial^2/\partial y^2)$ is the Laplacian operatorin2D Cartesian coordinate system.

2.4 Nonlocal theory and Constitutive relations

Contrary to the classical (local) theory, the non-classical (nonlocal) theory considers that the stress at a point is related not only to the strain at that point but also to strains at all other points of the body. Based on work presented by Eringen (1983), the nonlocal stress tensor σ at point x is given by

$$\sigma-\mu\nabla^2\sigma=\tau \tag{17}$$

where τ is local stress tensor at a point x expressed versus the strain by the Hooke's law; $\mu=(e_0 a)^2$

is the nonlocal parameter which includes the small scale effect, a is the internal characteristic length and e_0 is a constant appropriate to each material.

2.5 Stress resultants

For a functionally graded material in the two-dimensional case, the nonlocal constitutive relation in Eq. (17) takes the following forms

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} - \mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & C_{66} & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{18}$$

where (σ_x, σ_y, τ_{xy}, τ_{yz}, τ_{yx}) and (ε_x, ε_y, γ_{xy}, γ_{yz}, γ_{yx}) are the stress and strain components, respectively. Using the material properties defined in Eq. (4), stiffness coefficients, C_{ij}, can be expressed as

$$C_{11} = C_{22} = \frac{E(z)}{1-\nu(z)^2}, \tag{19a}$$

$$C_{12} = \frac{\nu\, E(z)}{1-\nu(z)^2}, \tag{19b}$$

$$C_{44} = C_{55} = C_{66} = \frac{E(z)}{2[1+\nu(z)]}, \tag{19c}$$

By substituting Eq. (8) into Eq. (18) and the subsequent results into Eq. (12), the stress resultants are determined as

$$\begin{Bmatrix} N \\ M \\ P \end{Bmatrix} - \mu\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\begin{Bmatrix} N \\ M \\ P \end{Bmatrix} = \begin{bmatrix} A & B & B^a \\ B & D & D^a \\ B^a & D^a & H^a \end{bmatrix}\begin{Bmatrix} \varepsilon \\ k \\ \eta \end{Bmatrix}, \qquad R - \mu\left(\frac{\partial^2 R}{\partial x^2}+\frac{\partial^2 R}{\partial y^2}\right) = A^a \gamma, \tag{20a}$$

where

$$N = \{N_x, N_y, N_{xy}\}^t, \qquad M^b = \{M_x^b, M_y^b, M_{xy}^b\}^t, \qquad M^s = \{M_x^s, M_y^s, M_{xy}^s\}^t, \tag{20b}$$

$$\varepsilon = \{\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0\}^t, \qquad k = \{k_x, k_y, k_{xy}\}^t, \qquad \eta = \{\eta_x, \eta_y, \eta_{xy}\}^t, \tag{20c}$$

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \qquad B = \begin{bmatrix} B_{11} & B_{12} & 0 \\ B_{12} & B_{22} & 0 \\ 0 & 0 & B_{66} \end{bmatrix}, \qquad D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \tag{20d}$$

$$B^a = \begin{bmatrix} B_{11}^a & B_{12}^a & 0 \\ B_{12}^a & B_{22}^a & 0 \\ 0 & 0 & B_{66}^a \end{bmatrix}, \quad D^a = \begin{bmatrix} D_{11}^a & D_{12}^a & 0 \\ D_{12}^a & D_{22}^a & 0 \\ 0 & 0 & D_{66}^a \end{bmatrix}, \quad H^a = \begin{bmatrix} H_{11}^a & H_{12}^a & 0 \\ H_{12}^a & H_{22}^a & 0 \\ 0 & 0 & H_{66}^a \end{bmatrix}, \tag{20e}$$

$$S = \left\{S_{xz}^s, S_{yz}^s\right\}^t, \quad \gamma = \left\{\gamma_{xz}, \gamma_{yz}\right\}^t, \quad A^a = \begin{bmatrix} A_{44}^a & 0 \\ 0 & A_{55}^a \end{bmatrix}, \tag{20f}$$

where A_{ij}, B_{ij}, D_{ij}, etc., are the plate stiffness, defined by

$$\begin{Bmatrix} A_{11} & B_{11} & D_{11} & B_{11}^a & D_{11}^a & H_{11}^a \\ A_{12} & B_{12} & D_{12} & B_{12}^a & D_{12}^a & H_{12}^a \\ A_{66} & B_{66} & D_{66} & B_{66}^a & D_{66}^a & H_{66}^a \end{Bmatrix} = \int_{-h/2}^{h/2} C_{11}\left(1, z, z^2, f(z), z\,f(z), f^2(z)\right) \begin{Bmatrix} 1 \\ \nu \\ \dfrac{1-\nu}{2} \end{Bmatrix} dz, \tag{21a}$$

$$\left(A_{22}, B_{22}, D_{22}, B_{22}^a, D_{22}^a, H_{22}^a\right) = \left(A_{11}, B_{11}, D_{11}, B_{11}^a, D_{11}^a, H_{11}^a\right), \tag{21b}$$

$$A_{44}^a = A_{55}^a = \int_{-h/2}^{h/2} C_{44}\left[g(z)\right]^2 dz, \tag{21c}$$

2.6 Equations of motion in terms of displacements

The nonlocal equations of motion of the present plate theory can be written in terms of displacements (u_0, v_0, w_0, Q_x, Q_y) by substituting stress resultants in Eq. (20) into Eq. (16) as

$$\begin{aligned} & A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{12}\frac{\partial^2 v_0}{\partial x \partial y} - B_{11}\frac{\partial^3 w_0}{\partial x^3} - B_{12}\frac{\partial^3 w_0}{\partial y^2 \partial x} + \frac{B_{11}^a}{\lambda_x}\frac{\partial^2 Q_x}{\partial x^2} + \frac{B_{12}^a}{\lambda_y}\frac{\partial^2 Q_y}{\partial x \partial y} + \\ & A_{66}\left(\frac{\partial^2 u_0}{\partial y^2} + \frac{\partial^2 v_0}{\partial x \partial y}\right) - 2B_{66}\frac{\partial^3 w_0}{\partial y^2 \partial x} + B_{66}^a\left(\frac{1}{\lambda_x}\frac{\partial^2 Q_x}{\partial y^2} + \frac{1}{\lambda_y}\frac{\partial^2 Q_x}{\partial x \partial y}\right) = \left(1 - \mu\nabla^2\right)\left(I_0 \ddot{u}_0 - I_1\frac{\partial \ddot{w}_0}{\partial x} + J_1\frac{\ddot{Q}_x}{\lambda_x}\right) \end{aligned} \tag{22a}$$

$$\begin{aligned} & A_{11}\frac{\partial^2 v_0}{\partial y^2} + A_{12}\frac{\partial^2 u_0}{\partial x \partial y} - B_{11}\frac{\partial^3 w_0}{\partial y^3} - B_{12}\frac{\partial^3 w_0}{\partial y \partial x^2} + \frac{B_{11}^a}{\lambda_x}\frac{\partial^2 Q_y}{\partial y^2} + \frac{B_{12}^a}{\lambda_x}\frac{\partial^2 Q_x}{\partial x \partial y} + \\ & A_{66}\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial x \partial y}\right) - 2B_{66}\frac{\partial^3 w_0}{\partial y \partial x^2} + B_{66}^a\left(\frac{1}{\lambda_y}\frac{\partial^2 Q_y}{\partial x^2} + \frac{1}{\lambda_x}\frac{\partial^2 Q_x}{\partial x \partial y}\right) = \left(1 - \mu\nabla^2\right)\left(I_0 \ddot{v}_0 - I_1\frac{\partial \ddot{w}_0}{\partial y} + J_1\frac{\ddot{Q}_y}{\lambda_y}\right) \end{aligned} \tag{22b}$$

$$\begin{aligned} & B_{11}\left(\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 v_0}{\partial y^3}\right) + \left(B_{12} + 2B_{66}\right)\left(\frac{\partial^3 u_0}{\partial y^2 \partial x} + \frac{\partial^3 v_0}{\partial x^2 \partial y}\right) - D_{11}\left(\frac{\partial^4 w_0}{\partial x^4} + \frac{\partial^4 w_0}{\partial y^4}\right) \\ & - 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 w_0}{\partial y^2 \partial x^2} + D_{11}^a\left(\frac{1}{\lambda_x}\frac{\partial^3 Q_x}{\partial x^3} + \frac{1}{\lambda_y}\frac{\partial^3 Q_y}{\partial y^3}\right) \end{aligned} \tag{22c}$$

$$
\begin{aligned}
&+D_{12}^{a}\left(\frac{1}{\lambda_x}\frac{\partial^3 Q_x}{\partial y^2 \partial x}+\frac{1}{\lambda_y}\frac{\partial^3 Q_y}{\partial x^2 \partial y}\right)+B_{66}^{a}\left(\frac{1}{\lambda_x}\frac{\partial^2 Q_x}{\partial y^2}+\frac{1}{\lambda_y}\frac{\partial^2 Q_y}{\partial x \partial y}\right)\\
&+2D_{66}^{a}\left(\frac{1}{\lambda_x}\frac{\partial^3 Q_x}{\partial y^2 \partial x}+\frac{1}{\lambda_y}\frac{\partial^2 Q_y}{\partial x \partial y}\right)+\left(1-\mu\nabla^2\right)\left(-K_w w_0+K_s\nabla^2 w_0\right)\\
&=\left(1-\mu\nabla^2\right)\left(I_0\ddot{w}_0+I_1\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\partial \ddot{v}_0}{\partial y}\right)-I_2\nabla^2\ddot{w}_0+J_2\left(\frac{1}{\lambda_x}\frac{\partial \ddot{Q}_x}{\partial x}+\frac{1}{\lambda_y}\frac{\partial \ddot{Q}_y}{\partial y}\right)\right)
\end{aligned}
\tag{22c}
$$

$$
\begin{aligned}
&B_{11}^{a}\frac{\partial^2 u_0}{\partial x^2}+B_{12}^{a}\frac{\partial^2 v_0}{\partial x \partial y}-D_{11}^{a}\frac{\partial^3 w_0}{\partial x^3}-D_{12}^{a}\frac{\partial^3 w_0}{\partial y^2 \partial x}+\frac{H_{11}^{a}}{\lambda_x}\frac{\partial^2 Q_x}{\partial x^2}+\frac{H_{12}^{a}}{\lambda_y}\frac{\partial^2 Q_y}{\partial x \partial y}+\\
&B_{66}^{a}\left(\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial x \partial y}\right)-2D_{66}^{a}\frac{\partial^3 w_0}{\partial y^2 \partial x}+H_{66}^{a}\left(\frac{1}{\lambda_x}\frac{\partial^2 Q_x}{\partial y^2}+\frac{1}{\lambda_y}\frac{\partial^2 Q_y}{\partial x \partial y}\right)-A_{44}^{a}\frac{Q_x}{\lambda_x}\\
&=\left(1-\mu\nabla^2\right)\left(J_1\ddot{u}_0-J_2\frac{\partial \ddot{w}_0}{\partial x}+K_2\frac{\ddot{Q}_x}{\lambda_x}\right)
\end{aligned}
\tag{22d}
$$

$$
\begin{aligned}
&B_{11}^{a}\frac{\partial^2 v_0}{\partial y^2}+B_{12}^{a}\frac{\partial^2 u_0}{\partial x \partial y}-D_{11}^{a}\frac{\partial^3 w_0}{\partial y^3}-D_{12}^{a}\frac{\partial^3 w_0}{\partial x^2 \partial y}+\frac{H_{11}^{a}}{\lambda_y}\frac{\partial^2 Q_y}{\partial y^2}+\frac{H_{12}^{a}}{\lambda_x}\frac{\partial^2 Q_x}{\partial x \partial y}+\\
&B_{66}^{a}\left(\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial x \partial y}\right)-2D_{66}^{a}\frac{\partial^3 w_0}{\partial x^2 \partial y}+H_{66}^{a}\left(\frac{1}{\lambda_x}\frac{\partial^2 Q_x}{\partial x \partial y}+\frac{1}{\lambda_y}\frac{\partial^2 Q_y}{\partial x^2}\right)-A_{55}^{a}\frac{Q_y}{\lambda_y}\\
&=\left(1-\mu\nabla^2\right)\left(J_1\ddot{v}_0-J_2\frac{\partial \ddot{w}_0}{\partial y}+K_2\frac{\ddot{Q}_y}{\lambda_y}\right)
\end{aligned}
\tag{22e}
$$

3. Closed-form solution for simply supported FG nanoplates

A simply supported rectangular nanoplate with length a and width b is considered here. Based on Navier method, the following expansions of generalized displacements are chosen to automatically satisfy the simply supported boundary conditions

$$
\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \\ Q_x \\ Q_y \end{Bmatrix}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\begin{Bmatrix} U_{mn}e^{i\omega t}\cos(\alpha\, x)\sin(\beta\, y) \\ V_{mn}e^{i\omega t}\sin(\alpha\, x)\cos(\beta\, y) \\ W_{mn}e^{i\omega t}\sin(\alpha\, x)\sin(\beta\, y) \\ X_{mn}e^{i\omega t}\cos(\alpha\, x)\sin(\beta\, y) \\ Y_{mn}e^{i\omega t}\sin(\alpha\, x)\cos(\beta\, y) \end{Bmatrix}
\tag{23}
$$

where U_{mn}, V_{mn}, W_{mn}, X_{mn} and Y_{mn} are arbitrary coefficients to be determined, ω is the eigenfrequency associated with (m, n)th eigenmode, and $\alpha = m\pi / a$ and $\beta = n\pi / b$.

Substituting Eqs. (23) into Eq. (22), the analytical solutions can be determined from

$$\left(\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} \end{bmatrix} - \lambda\omega^2 \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} & 0 \\ 0 & m_{22} & m_{23} & 0 & m_{25} \\ m_{13} & m_{23} & m_{33} & m_{34} & m_{35} \\ m_{14} & 0 & m_{43} & m_{44} & 0 \\ 0 & m_{52} & m_{53} & 0 & m_{55} \end{bmatrix}\right) \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{24}$$

where

$$S_{11} = A_{11}\alpha^2 + A_{66}\beta^2, \quad S_{12} = \alpha\beta(A_{12} + A_{66}), \quad S_{13} = -\alpha\left(B_{11}\alpha^2 + B_{12}\beta^2 + 2B_{66}\beta^2\right)$$

$$S_{14} = \frac{B^a{}_{11}\alpha^2 + B^a{}_{66}\beta^2}{\lambda_x}, \quad S_{15} = \frac{\alpha\beta\left(B^a{}_{12} + B^a{}_{66}\right)}{\lambda_y}, \quad S_{22} = A_{66}\alpha^2 + A_{11}\beta^2$$

$$S_{23} = -\beta\left(B_{11}\beta^2 + B_{12}\alpha^2 + 2B_{66}\alpha^2\right), \quad S_{24} = \frac{\alpha\beta\left(B^a{}_{12} + B^a{}_{66}\right)}{\lambda_x}, \quad S_{25} = \frac{B^a{}_{66}\alpha^2 + B^a{}_{11}\beta^2}{\lambda_y}$$

$$S_{33} = D_{11}\left(\alpha^4 + \beta^4\right) + 2D_{12}\alpha^2\beta^2 + 4D_{66}\alpha^2\beta^2 + \lambda\left(K_w + K_s\left(\alpha^2 + \beta^2\right)\right),$$

$$S_{34} = \frac{\alpha\left(-2D^a{}_{66}\beta^2 - D^a{}_{11}\alpha^2 - D^a{}_{12}\beta^2\right)}{\lambda_x}, \quad S_{35} = \frac{\beta\left(-D^a{}_{11}\beta^2 - D^a{}_{12}\alpha^2\right)}{\lambda_y}, \tag{25a}$$

$$S_{41} = B^a{}_{11}\alpha^2 + B^a{}_{66}\beta^2, \quad S_{42} = \alpha\beta\left(B^a{}_{12} + B^a{}_{66}\right), \quad S_{43} = \alpha\left(-2D^a{}_{66}\beta^2 - D^a{}_{11}\alpha^2 - D^a{}_{12}\beta^2\right),$$

$$S_{44} = \frac{H^a{}_{11}\alpha^2 + H^a{}_{66}\beta^2 + A^a{}_{44}}{\lambda_x}, \quad S_{45} = \frac{\alpha\beta\left(H^a{}_{12} + H^a{}_{66}\right)}{\lambda_y},$$

$$S_{51} = \alpha\beta\left(B^a{}_{12} + B^a{}_{66}\right), \quad S_{52} = B^a{}_{66}\alpha^2 + B^a{}_{11}\beta^2, \quad S_{53} = \beta\left(-D^a{}_{11}\beta^2 - D^a{}_{12}\alpha^2\right),$$

$$S_{54} = \frac{\alpha\beta\left(H^a{}_{12} + H^a{}_{66}\right)}{\lambda_x}, \quad S_{55} = \frac{H^a{}_{66}\alpha^2 + H^a{}_{11}\beta^2 + A^a{}_{55}}{\lambda_y}$$

and

$$\begin{aligned} & m_{11} = m_{22} = I_0, \quad m_{13} = -\alpha I_1, \quad m_{14} = \frac{J_1}{\lambda_x}, \\ & m_{23} = -\beta I_1, \quad m_{25} = \frac{J_1}{\lambda_y}, \quad m_{33} = I_0 + I_2(\alpha^2 + \beta^2) \\ & m_{34} = -\alpha\frac{J_2}{\lambda_x}, \quad m_{35} = -\beta\frac{J_2}{\lambda_y}, \quad m_{41} = J_1, \quad m_{43} = -\alpha J_2, \quad m_{44} = \frac{K_2}{\lambda_x}, \\ & m_{52} = J_1, \quad m_{53} = -\beta J_2, \quad m_{55} = \frac{K_2}{\lambda_y}, \quad \lambda = 1 + \mu\left(\alpha^2 + \beta^2\right) \end{aligned} \tag{25b}$$

The frequency ratio serves as an index to evaluate quantitatively the nonlocal parameter effect on vibration response of FG nano-plate. The frequency ratio is defined as

$$\text{Frequency ratio} = \frac{\omega_{NL}}{\omega_L} \tag{26}$$

where ω_{Nl} and ω_L are the frequencies computed using the nonlocal model and the local model, respectively.

4. Results and discussion

In this section, the size-dependent free vibration behavior of a simply supported FG nano-plate resting on elastic foundation is discussed. The free vibration analysis is carried out by supposing the top surface of the plate is ceramic rich (Si_3N_4) and the bottom surface is metal rich (SUS304). The mass density ρ and the Young's modulus E are: ρ_c = 2370 kg/m^3, E_c = 348.43e^9 N/m^2 for Si_3N_4 and ρ_m = 8166 kg/m^3, E_m = 201.04e^9 N/m^2 for SUS304. Poisson's ratio ν is considered to be constant and taken as 0.3 for the current study. For convenience, the following dimensionless quantities are employed in presenting the numerical results in graphical and tabular forms

$$\bar{\omega} = \omega h\sqrt{\frac{\rho_c}{G_c}}, \quad \hat{\omega} = \omega h\sqrt{\frac{\rho_m}{E_m}}, \quad k_w = \frac{K_s a^4}{D_m}, \quad k_s = \frac{K_s a^2}{D_m}, \quad D_m = \frac{E_m h^3}{12(1-\nu^2)} \tag{27}$$

As the first step, to confirm the accuracy of present model, plates without the presence of elastic foundations are considered and the computed results are compared with the results of Belkorissat *et al*. (2015) and Aghababaei and Reddy (2009) in Tables 1 and 2. In the first example, simply supported homogeneous nanoplates with different values of nonlocal parameter, the plate thickness and the plate aspect ratio are considered. The results tabulated in Table 1 are compared with those given by both Belkorissat *et al*. (2015) and Aghababaei and Reddy (2009). It can be seen that the present numerical results are in very good agreement with the results available in the literature. In the second example, FG nanoplates (n = 5) with different values of nonlocal parameter, the plate thickness and the plate aspect ratio are examined. The natural frequencies predicted via the present formulation, are compared with those of Belkorissat *et al*. (2015) in Table 2. Again, very good agreement is found between the results.

Table 1 Comparison of fundamental frequency $(\bar{\omega} = \omega h\sqrt{\rho / G})$ of nano-plate (a = 10, E = 30×10^6, ρ = 1, ν = 0.3)

a/b	a/h	μ	present	REF[(a)]	TSDT[(b)]	FSDT[(b)]	CPT[(b)]
1	10	0	0.0930	0.0930	0.0935	0.0930	0.0963
		1	0.0850	0.0850	0.0854	0.0850	0.0880
		2	0.0787	0.0787	0.0791	0.0788	0.0816
		3	0.0737	0.0737	0.0741	0.0737	0.0763
		4	0.0695	0.0695	0.0699	0.0696	0.0720
		5	0.0659	0.0659	0.0663	0.0660	0.0683

Table 1 Comparison of fundamental frequency $(\bar{\omega} = \omega h\sqrt{\rho / G})$ of nano-plate (a = 10, $E = 30\times10^6$, ρ = 1, ν = 0.3)

a/*b*	*a*/*h*	μ	present	REF(a)	TSDT(b)	FSDT(b)	CPT(b)
	20	0	0.0238	0.0238	0.0239	0.0239	0.0241
		1	0.0218	0.0218	0.0218	0.0218	0.0220
		2	0.0202	0.0202	0.0202	0.0202	0.0204
		3	0.0189	0.0189	0.0189	0.0189	0.0191
		4	0.0178	0.0178	0.0179	0.0178	0.0180
		5	0.0169	0.0169	0.0170	0.0169	0.0171
2	10	0	0.0588	0.0588	0.0591	0.0589	0.0602
		1	0.0555	0.0555	0.0557	0.0556	0.0568
		2	0.0527	0.0527	0.0529	0.0527	0.0539
		3	0.0503	0.0503	0.0505	0.0503	0.0514
		4	0.0481	0.0481	0.0483	0.0482	0.0493
		5	0.0463	0.0463	0.0464	0.0463	0.0473
	20	0	0.0149	0.0149	0.0150	0.0150	0.0150
		1	0.0141	0.0141	0.0141	0.0141	0.0142
		2	0.0134	0.0134	0.0134	0.0134	0.0135
		3	0.0127	0.0127	0.0128	0.0128	0.0129
		4	0.0122	0.0122	0.0123	0.0123	0.0123
		5	0.0117	0.0117	0.0118	0.0118	0.0118

(a) Belkorissat *et al*. (2015)
(b) Aghababaei and Reddy (2009)

Table 2 Comparison of natural frequency $(\bar{\omega})$ of FG nano-plate (a = 10, n = 5)

a/*b*	*a*/*h*	μ	Mode 1		Mode 2		Mode 3	
			REF(a)	Present	REF(a)	Present	REF(a)	Present
1	10	0	0.0432	0.0432	0.1029	0.1029	0.1915	0.1918
		1	0.0395	0.0395	0.0842	0.0842	0.1358	0.1360
		2	0.0366	0.0366	0.0730	0.0730	0.1110	0.1112
		4	0.0323	0.0323	0.0596	0.0596	0.0861	0.0862
	20	0	0.0111	0.0111	0.0274	0.0274	0.0536	0.0536
		1	0.0101	0.0101	0.0224	0.0224	0.0380	0.0380
		2	0.0094	0.0094	0.0194	0.0194	0.0310	0.0310
		4	0.0083	0.0083	0.0158	0.0158	0.0241	0.0241
2	10	0	0.1029	0.1029	0.1574	0.1576	0.2397	0.2402
		1	0.0842	0.0842	0.1177	0.1178	0.1587	0.1590
		2	0.0730	0.0730	0.0980	0.0981	0.1269	0.1272
		4	0.0596	0.0596	0.0772	0.0773	0.0968	0.0970

Table 2 Comparison of natural frequency $(\bar{\omega})$ of FG nano-plate (a = 10, n = 5)

a/b	a/h	μ	Mode 1		Mode 2		Mode 3	
			REF[(a)]	Present	REF[(a)]	Present	REF[(a)]	Present
	20	0	0.0274	0.0274	0.0432	0.0432	0.0688	0.0688
		1	0.0224	0.0224	0.0323	0.0323	0.0455	0.0455
		2	0.0194	0.0194	0.0269	0.0269	0.0364	0.0364
		4	0.0158	0.0158	0.0212	0.0212	0.0277	0.0277

(a) Belkorissat *et al*. (2015)

The non-dimensionalized fundamental frequency of square FG nanoplates are given in Table 3 for different values of nonlocal parameter, the plate thickness, the material distribution parameter (n) and foundation parameters (k_w, k_s). From the results illustrated in Table 3, it can be concluded that the non-dimensionalized fundamental frequency increases when foundation parameters (k_w, k_s) increase. Compared to the Winkler parameter k_w, the Pasternak foundation parameter k_s has dominant impact on increasing the non-dimensionalized frequency. It is also observed that with the presence of elastic foundations, the plate becomes stiffer, while, the nonlocal parameter makes the plate softer. In addition, it can be seen that the increase of the material distribution parameter (n) leads to a reduction of frequency. This is due to the fact that the material distribution parameter yields a decrease in the stiffness of the FG nano-plate.

Table 3 Dimensionless frequency $(\hat{\omega})$ of FG square nano-plate

k_w	k_s	a/h	μ	Material distribution parameter (n)						
				0	0.5	1	2	3	4	5
0	0	10	0	0.1409	0.0902	0.0793	0.0717	0.0686	0.0667	0.0655
			1	0.1288	0.0825	0.0725	0.0655	0.0626	0.0610	0.0599
			2	0.1193	0.0764	0.0672	0.0607	0.0580	0.0565	0.0555
			3	0.1117	0.0715	0.0629	0.0568	0.0543	0.0529	0.0519
			4	0.1053	0.0674	0.0593	0.0536	0.0512	0.0499	0.0490
		20	0	0.0361	0.0231	0.0203	0.0184	0.0176	0.0171	0.0168
			1	0.0330	0.0211	0.0186	0.0168	0.0161	0.0156	0.0153
			2	0.0306	0.0196	0.0172	0.0156	0.0149	0.0145	0.0142
			3	0.0286	0.0183	0.0161	0.0146	0.0139	0.0136	0.0133
			4	0.0270	0.0173	0.0152	0.0137	0.0131	0.0128	0.0125
0	20	10	0	0.1793	0.1198	0.1070	0.0980	0.0943	0.0922	0.0908
			1	0.1699	0.1141	0.1020	0.0936	0.0901	0.0881	0.0868
			2	0.1628	0.1098	0.0983	0.0903	0.0869	0.0851	0.0839
			3	0.1573	0.1064	0.0954	0.0877	0.0845	0.0827	0.0815
			4	0.1529	0.1037	0.0931	0.0856	0.0826	0.0808	0.0797

Table 3 Dimensionless frequency $(\hat{\omega})$ of FG square nano-plate

k_w	k_s	a/h	μ	Material distribution parameter (n)						
				0	0.5	1	2	3	4	5
		20	0	0.0456	0.0304	0.0272	0.0249	0.0239	0.0234	0.0231
			1	0.0432	0.0290	0.0259	0.0237	0.0228	0.0223	0.0220
			2	0.0413	0.0278	0.0249	0.0229	0.0220	0.0216	0.0212
			3	0.0399	0.0270	0.0242	0.0222	0.0214	0.0209	0.0206
			4	0.0388	0.0263	0.0236	0.0217	0.0209	0.0204	0.0202
100	0	10	0	0.1516	0.0986	0.0872	0.0792	0.0759	0.0740	0.0728
			1	0.1403	0.0915	0.0810	0.0736	0.0706	0.0689	0.0677
			2	0.1317	0.0861	0.0763	0.0694	0.0666	0.0649	0.0639
			3	0.1248	0.0818	0.0725	0.0660	0.0633	0.0618	0.0608
			4	0.1192	0.0782	0.0694	0.0633	0.0607	0.0593	0.0583
		20	0	0.0387	0.0252	0.0223	0.0202	0.0194	0.0189	0.0186
			1	0.0358	0.0234	0.0207	0.0188	0.0180	0.0176	0.0173
			2	0.0336	0.0220	0.0194	0.0177	0.0170	0.0166	0.0163
			3	0.0319	0.0208	0.0185	0.0168	0.0161	0.0158	0.0155
			4	0.0304	0.0199	0.0177	0.0161	0.0155	0.0151	0.0148
100	20	10	0	0.1877	0.1262	0.1130	0.1036	0.0998	0.0976	0.0962
			1	0.1788	0.1208	0.1083	0.0994	0.0958	0.0938	0.0924
			2	0.1721	0.1167	0.1048	0.0963	0.0928	0.0909	0.0896
			3	0.1669	0.1135	0.1021	0.0939	0.0906	0.0887	0.0875
			4	0.1627	0.1110	0.0999	0.0920	0.0887	0.0869	0.0858
		20	0	0.0477	0.0320	0.0287	0.0263	0.0253	0.0247	0.0244
			1	0.0454	0.0306	0.0274	0.0252	0.0243	0.0237	0.0234
			2	0.0436	0.0296	0.0265	0.0244	0.0235	0.0230	0.0227
			3	0.0423	0.0287	0.0258	0.0238	0.0229	0.0224	0.0221
			4	0.0412	0.0281	0.0253	0.0233	0.0224	0.0220	0.0217

Table 4 present the non-dimensionalized fundamental frequency for a square FG plate, and for different values of the plate thickness, the plate aspect ratio, the nonlocal parameter and the material distribution parameter based on the present theory. The foundation parameters (k_w, k_s) are taken to be 100. It can be seen that increasing the aspect ratio (a/b) will increase the fundamental frequencies, while, increasing the material distribution parameter will cause the fundamental frequency to decrease. These results are independent of the values of nonlocal parameter. Again, one can easily find from Tables 1 to 4 that the nonlocal parameter plays an important role in studying the vibration response of FG nano-plates and its effects can't be ignored.

To examine the influences of the elastic foundation parameters on the vibration behavior of FG nano-plates, variations of frequency ratio with both Winkler modulus and the shear modulus parameters are plotted. The elastic foundation is modeled as both (i) Winkler-type foundation; and

Table 4 Dimensionless frequency $(\breve{\omega} = \omega(a^2/h)\sqrt{\rho_m/E_m})$ of FG square nano-plate ($k_w = k_s = 100$, $a = 10$)

a/b	a/h	μ	Material distribution parameter (n)						
			0	0.5	1	2	3	4	5
0.5	10	0	22.2810	15.6101	14.1631	13.1234	12.6974	12.4624	12.3122
		1	22.0839	15.4947	14.0647	13.0366	12.6154	12.3832	12.2350
		2	21.9245	15.4014	13.9853	12.9665	12.5492	12.3193	12.1726
		3	21.7929	15.3245	13.9198	12.9088	12.4947	12.2666	12.1212
		4	21.6824	15.2599	13.8648	12.8603	12.4489	12.2225	12.0782
	20	0	22.4066	15.6998	14.2421	13.1933	12.7633	12.5259	12.3741
		1	22.2040	15.5812	14.1411	13.1041	12.6788	12.4443	12.2945
		2	22.0401	15.4855	14.0595	13.0321	12.6107	12.3785	12.2303
		3	21.9049	15.4065	13.9923	12.9727	12.5546	12.3243	12.1775
		4	21.7913	15.3403	13.9359	12.9230	12.5076	12.2788	12.1331
1	10	0	29.0449	20.1923	18.2786	16.9095	16.3490	16.0386	15.8392
		1	28.4751	19.8562	17.9918	16.6563	16.1096	15.8074	15.6136
		2	28.0595	19.6116	17.7831	16.4722	15.9357	15.6394	15.4497
		3	27.7428	19.4255	17.6245	16.3323	15.8036	15.5118	15.3253
		4	27.4934	19.2791	17.4997	16.2224	15.6997	15.4116	15.2276
	20	0	29.3436	20.3983	18.4597	17.0706	16.5011	16.1855	15.9825
		1	28.7497	20.0483	18.1608	16.8063	16.2509	15.9436	15.7465
		2	28.3163	19.7936	17.9435	16.6141	16.0691	15.7680	15.5751
		3	27.9859	19.5999	17.7782	16.4682	15.9311	15.6345	15.4449
		4	27.7257	19.4475	17.6484	16.3535	15.8226	15.5298	15.3427
2	10	0	51.6791	35.2203	31.6837	29.1759	28.1501	27.5777	27.2060
		1	47.9241	32.9674	29.7530	27.4691	26.5361	26.0177	25.6828
		2	45.9176	31.7707	28.7297	26.5665	25.6837	25.1945	24.8796
		3	44.6653	31.0259	28.0933	26.0059	25.1549	24.6843	24.3821
		4	43.8082	30.5161	27.6578	25.6228	24.7938	24.3362	24.0428
	20	0	53.3636	36.3229	32.6507	30.0438	28.9753	28.3768	27.9868
		1	49.2611	33.8619	30.5376	28.1685	27.1977	26.6567	26.3063
		2	47.0613	32.5530	29.4166	27.1757	26.2575	25.7475	25.4186
		3	45.6857	31.7387	28.7205	26.5599	25.6747	25.1842	24.8687
		4	44.7430	31.1828	28.2458	26.1404	25.2779	24.8006	24.4944

(ii) Pasternak-type foundation. The Winkler type and Pasternak foundations are described by foundation stiffness, k_w and k_s, respectively.

Fig. 3 shows the effect of the nonlocal parameter on the vibration response of FG nano-plates supported by elastic medium modeled as Winkler-type foundation. From the figure it is seen that there is significant effect of the nonlocal parameter on the vibration behavior of FG nano-plates

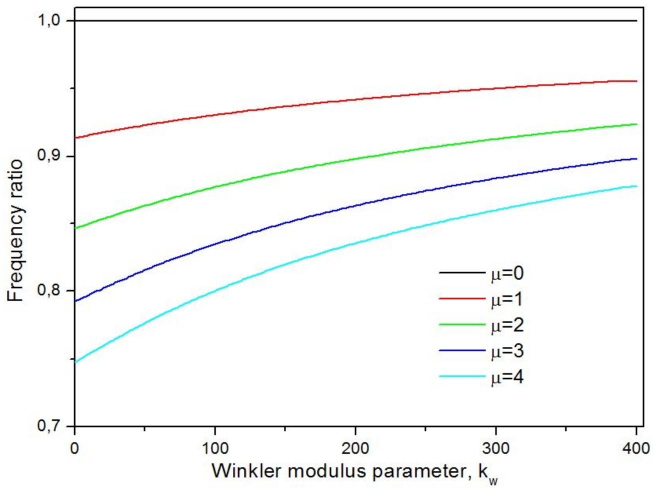

Fig. 3 Effect of Winkler modulus parameter on the frequency ratio of FG square nano-plate for various nonlocal parameters (k_s = 0, a/h = 10, n = 5)

supported by elastic foundation. The fundamental frequency ratios including nonlocal model are always smaller than the local model (μ = 0). This implies that the use of the local zeroth-order shear deformation theory for FG nano-plates investigation would lead to an over-prediction of the frequency. Further, with increase in nonlocal parameter (μ) values, the frequencies predicted by non-classical become smaller compared to classical model. Furthermore, it is observed that the increase of the Winkler modulus parameter leads to an increase in the frequency ratio. This increasing trend is related to the stiffness of the elastic foundation. With higher values of Winkler modulus the rate of increase of frequency ratio diminishes. This implies that nonlocal effect in vibration behavior of FG nano-plates looses its importance as the Winkler modulus values increase. Thus, although the nonlocal effect makes the nano-plates softer, the external elastic foundation "grips" the nano-plates and forces it to be stiffer. Hence, it can be concluded that the nonlocal effect becomes more significant in the case of plates without elastic foundation.

Fig. 4 presents the effect of the nonlocal parameter on the vibration behavior of FG nano-plate resting on elastic medium modeled as Pasternak-type foundation. The Winkler modulus parameter is supposed as k_w = 100. The evolution of frequency ratio for first mode with shear modulus parameter is plotted in Fig. 4. The frequency ratio increases with increasing the shear modulus parameter. However, the frequency ratios including the nonlocal model are always smaller than the local model. With higher nonlocal parameter (μ) values the frequencies becomes comparatively less. Contrary to the variation of frequency ratio with Winkler parameter, which is nonlinear, the variation of frequency ratio with Pasternak shear modulus parameter is linear in nature.

The variation of the nonlocal frequency versus the Winkler modulus parameter is presented in Fig. 5 for various aspect ratios a/h. It is observed that as the nonlocal frequency increases linearly with the increase of the Winkler modulus parameter and this for all the considered aspect ratios a/h. Moreover, it is seen that the change in nonlocal frequency of nano-plate is significantly influenced by the side-to-thickness ratio a/h. For a thin plate (a/h = 100) the effect of nonlocal scale parameter on frequency is less compared to thick plate (a/h = 10). Hence side-to-thickness ratio of nano-plate plays an important role in predicting true vibration behavior of nanoscale plates resting

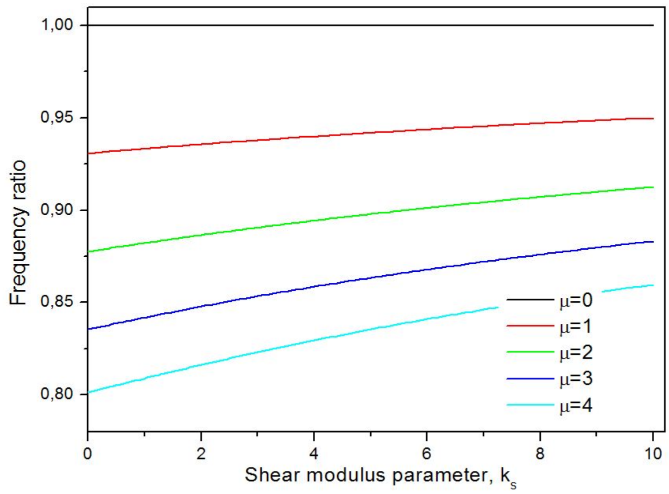

Fig. 4 Effect of Pasternak shear modulus parameter on the frequency ratio of FG square nano-plate for various nonlocal parameters (k_s = 100, a/h = 10, n = 5)

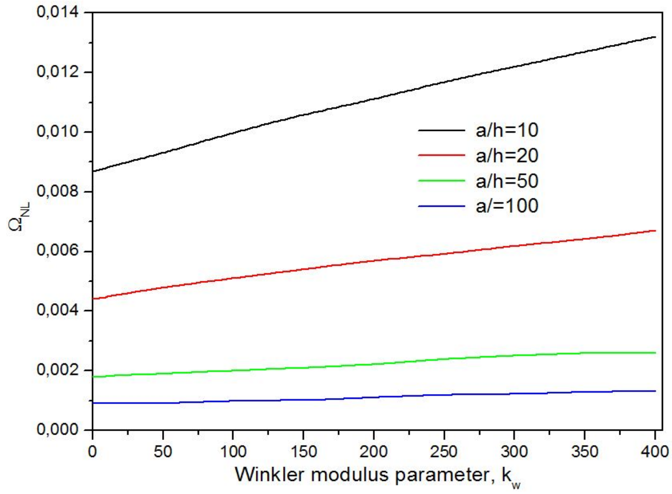

Fig. 5 Effect of Pasternak shear modulus parameter on the nonlocal frequency of FG square nano-plate for various aspect (k_s = 0, μ = 2, n = 5)

on elastic foundation.

Fig. 6 reveals the variation of nonlocal frequency versus shear modulus parameter for various side-to-thickness ratios (a/h).The inclusion of the Pasternak foundation produces results higher than those with the introduction of Winkler foundation. The nonlocal frequency increases with increasing the shear modulus parameter. The change is found to be nonlinear in nature. However, it is seen demonstrated, that change in nonlocal frequency is more influenced by low side-to-thickness ratios values (a/h = 10) as demonstrated in the figure.

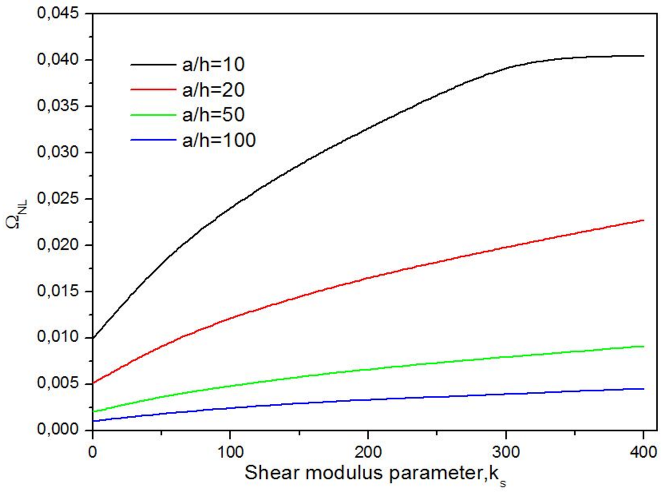

Fig. 6 Effect of Winkler modulus parameter on the nonlocal frequency of FG square nano-plate for various aspect (k_w = 100, μ = 2, n = 5)

5. Conclusions

In this work, vibration analysis of FG nanoscale plates resting on elastic foundation based on nonlocal zeroth-order shear deformation theory is presented. The present model considers the transverse shear deformation effect via the use of shear forces instead of rotational displacements as in existing shear deformation theories and it is capable of including the nonlocal scale parameter via the nonlocal Eringen's elasticity model. Accuracy of the results is investigated by utilizing available date in the literature. It is concluded that various factors such as nonlocal scale parameter, the volume fraction exponent, Winkler modulus parameter, Pasternak shear modulus parameter, and side-to-thickness ratios play considerable roles in dynamic response of FG nanoscale plates. The formulation lends itself particularly well to use novel structural element formulations (Phan-Dao *et al*. 2013, Zhuang *et al*. 2013, Thai *et al*. 2012) which will be considered in the near future.

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