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Buckling analysis of composite plates using differential quadrature method (DQM)

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Abstract. The differential quadrature method (DQM) is a numerical technique of rather recent origin, which by its continually increasing applications in different problems of engineering, is a competing alternative to the conventional numerical techniques for the solution of initial and boundary value problems. The work of this paper concerns the application of the DQM in the area of the buckling of multi layered orthotropic composite plates with various boundary conditions the buckling of multi layered composite plates with constant and variable thickness under axial compressive static loading is considered. The effects of fiber orientation and boundary conditions on static behavior of composite plates are presented .The comparison of results from the present method and those obtained from NISA II software shows the accuracy and reliability of this method.

Key words: buckling; composite; plate; differential; quadrature.

1. Introduction

The mechanical behavior of composite structures is of particular interest to engineers in modern technology. Among these structures, the rectangular plate is a common structural element. This is due to the introduction of high strength to weight ratio of fiber reinforced composite material which have led to greater flexibility in the design of advanced structures. One of the typical problem with composite plate specimens is presence of buckling. Designers and experimentalists are often interested in a quick and accurate estimate of the critical buckling load which normally requires a comprehensive development of a mathematical model.

The differential quadrature method (DQM) is a numerical technique of rather recent origin which by its continually increasing applications in different problems of engineering, is a competing alternative to the conventional numerical techniques for the solution of initial and boundary value problems. The work of this paper concerns the application of the DQM in the area of the buckling of multi layered

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orthotropic composite plates with different boundary conditions. This method was originally introduced by Bellman (1971 and 1972) as an accurate and fast computing numerical solution technique for nonlinear partial differential equations of initial value problems .The applications of the method to the problems of engineering and physical sciences became known with the work of Mingle (1973).

The application of the DQM in the area of structural mechanics concerning the static and dynamic behavior was developed by Bert (1988), Malik (1995), Bert (1996) and Moradi (1999). It was shown that the differential quadrature method offers, in terms of both the numerical accuracy and the computational efficiency, an alternative to the conventional numerical methods for the solution of boundary value problems.

The present work is undertaken with an objective of investigating further the ability of the differential quadrature method in the more complex problem of buckling of multi-layered fibrous composite plates with different boundary conditions. The buckling analysis of multi-layered composite plates with constant and variable thickness under axial compressive static loading is considered. The effects of fiber orientation and boundary conditions on static behavior of composite plates are presented. To show the accuracy and reliability of present method and its application to the composite plate with constant and variable thickness, the results are compared with the results from NISA II software which is based on finite element method.

2. The brief review of DQM

In the differential quadrature method the first order partial derivative of a function is defined as the calculus operator value of a function with respect to a coordinate direction at any discrete point as the weight linear sum of the values of the function at all the discrete points chosen in that direction.

In order to go in to the mathematical basis of the DQM consider a function $f(x_i, t)$. The first order partial derivative of such a function is defined as

$$f_x(x_i, t) = \sum_{i=1}^{N} C_{ij}^{(1)} f(x_j, t) \quad i = 1, 2, \ldots, N \tag{1}$$

where $C_{ij}^{(1)}$ are the respective weighting coefficients and N, is the number of grid points. In order to implement the DQM, one needs to know the weighting coefficients. This can be done by the functional approximations in the respective coordinate directions.The approximating functions are known as the test functions, and the primary requirement for the choice of the test functions is completeness in the same sense as one needs for the interpolation functions in finite element analysis. In order to have no restriction on the number of grid points used for the approximation and the weighting coefficients the test function is defined as (Loy 1997).

$$f_i(x) = \frac{M(x)}{(x - x_i)M^{(1)}(x_i)} \quad i = 1, 2, \ldots, N \tag{2}$$

Where $M(x) = \underset{j=1}{\overset{N}{\pi}}(x_i - x_j)$ and

$$M^{(1)}(x_i) = \underset{j=1, j \neq i}{\overset{N}{\pi}}(x_i - x_j) \tag{3}$$

By substitution of relation (2) in to (1) the weighting coefficients for the first-order derivative are given by

$$C_{ij}^{(1)} = \frac{M^{(1)}(x_i)}{(x_i - x_j)M^{(1)}(x_i)} \quad i \neq j \quad i, j = 1, 2, \dots N \tag{4}$$

$$C_{ij}^{(1)} = -\sum_{j=1, j \neq i}^{N} C_{ij}^{(1)} \quad i = 1, 2, \dots N \tag{5}$$

For the second and higher order derivatives, the weighting coefficients can be computed by using a recurrence relationship as follows.

$$C_{ij}^{(m)} = m\left(C_{ii}^{(m-1)}C_{ij} - \frac{C_{ij}^{(m-1)}}{x_i - x_j}\right) \quad \begin{array}{l} i \neq j,\ m = 2, 3, \dots N-1 \\ i, j = 1, 2, \dots N \end{array} \tag{6}$$

$$C_{ii}^{(m)} = -\sum_{j=1, j \neq i}^{N} C_{ij}^{(m)} \quad i = 1, 2, \dots N \tag{7}$$

And the grid points are chosen as Bert (1996).

$$x_i = \left(\frac{1 - \cos\left[\frac{i-1}{N-1}\right]\pi}{2}\right) \quad i = 1, 2, \dots N \tag{8}$$

And in the present analysis the number of grid points is equal to nine.

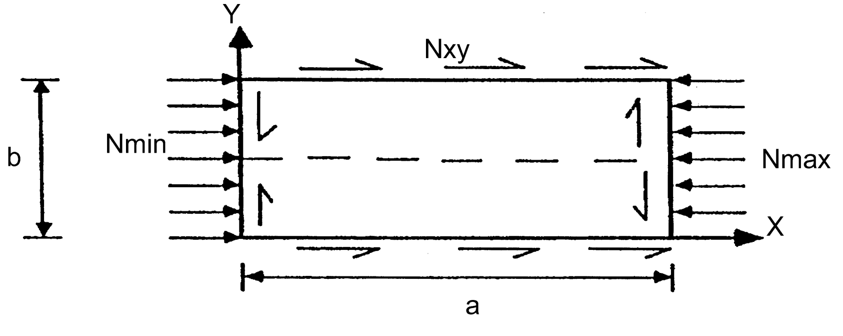


Fig. 1 Variable axial compressive force

3. Application of DQM to buckling problem of composite plates

To show the ability of DQ Method in solving the buckling problem of composite plate a rectangular plate as shown in (Fig. 1) is considered. First the thickness of plate is constant and the variable axial compressive force is applied according to the (Fig. 1). Second the constant axial compressive force is applied to a plate with variable thickness.

The governing differential equation for rectangular plate under variable axial compressive force is given as

$$D_{11}W_{,xxxx} + 2(D_{12} + 2D_{33})W_{,xxyy} + D_{22}W_{,yyyy} + 4D_{13}W_{,xxxy} + 4D_{23}\ W_{,xyyy} = N_x\ W_{,xx} + 2N_{xy}\ W_{,xy} \tag{9}$$

where W is the deflection of plate and D_{ij} are the stiffness coefficients as given in (Sharma 1999). The compressive force at a section X is defined as $N_x = 2N_{ave}(C_1X + C_2)$

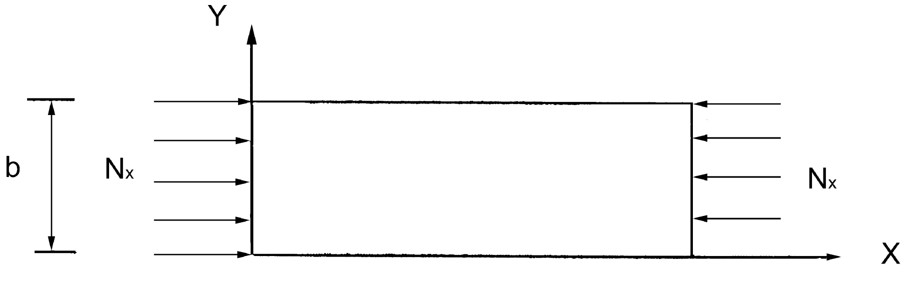
$$\text{Where} \qquad C_1 = \frac{1-r}{1+r} \quad C_2 = \frac{r}{1+r} \quad \text{and} \quad r = \frac{N_{\min}}{N_{\max}} \tag{10}$$

The shear stress at a section Y is defined as

$$N_{xy} = -\frac{N_{ave}}{\beta}C_1(2y-1) \quad \text{where} \quad \beta = \frac{a}{b} \tag{11}$$

By applying the differential quadrature method the Eq. (9) can be written as

$$D_{11}\sum_{k=1}^{N} H_{ik}W_{kj} + 2(D_{12} + 2D_{33})\beta^2 \sum_{m=1}^{N} B_{jm} \sum_{k=1}^{N} B_{ik}W_{km} + D_{22}\beta^4 \sum_{k=1}^{N} H_{jk}W_{jk} + 4D_{13}\beta \sum_{k=1}^{N} C_{ik}W_{Rm}$$
$$+ 4D_{23}\beta^3 \sum_{m=1}^{N} C_{jm} \sum_{k=1}^{N} A_{ik}W_{km} = 2N_{ave}a^2(C_1X_i + C_2)\,.$$



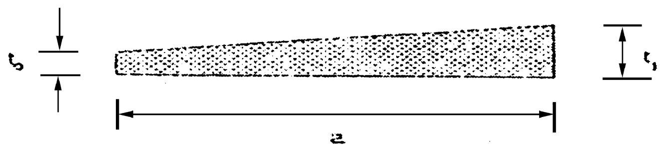


Fig. 2 Variable thickness with constant axial compressive force

$$\sum_{k=1}^{N} B_{ik}W_{kj} - 2N_{ave}a^2C_1(2y_j - 1)\sum_{m=1}^{N} A_{jm}\sum_{k=1}^{N} A_{ik}W_{km} \tag{12}$$

A rectangular plate with thickness variation, which is under constant axial compressive force, is shown in (Fig. 2). In this case the shear stress N_{xy} is assumed to be zero. The governing differential equation for an isotropic rectangular plate is given as

$$D\left(\frac{\delta^4 W}{\delta x^4} + 2\frac{\delta^4 W}{\delta x^2 \delta y^2} + \frac{\delta^4 W}{\delta y^2}\right) + 2\frac{\delta D}{\delta x}\left(\frac{\delta^3 W}{\delta x^3} + \frac{\delta^3 W}{\delta x \delta y^2}\right) + \frac{\delta^2 D}{\delta x^2}\left(\frac{\delta^2 W}{\delta x^2} + \frac{\delta^2 W}{\delta y^2}\right) = N_x\frac{\delta^2 W}{\delta X^2} \tag{13}$$

Since the thickness of an isotropic rectangular plate is varying linearly from t_0 to t_1 according to Fig. 2 the function representing this variation at each section X can be defined as, $t(x) = t_m = [1 + C\,(X–0.5)]$

Where $$t_m = \frac{(t_1 + t_0)}{2} \text{ and } C = \frac{2\left(\frac{t_1}{t_0} - 1\right)}{\frac{t_1}{t_0} + 1} \tag{14}$$

The bending rigidity D of the plate at each section X can also be defined as, $D(x) = D_m\,F^3(x)$

Where $D_m = \dfrac{Et_m^3}{12(1 - v^2)}$, $F(x) = 1 + C(X - 0.5)$

By substituting the bending rigidity in Eq. (13) and applying the Differential Quadrature Method one can have the following equation:

$$F^3(x)\left[\sum_{k=1}^{N} H_{ik}W_{kj} + 2\beta^2\sum_{m=1}^{N} B_{jm}\sum_{k=1}^{N} B_{ik}W_{km} + \beta^4\sum_{k=1}^{N} H_{jk}W_{ik}\right]$$

$$+6CF^2(x)\left[\sum_{k=1}^{N} C_{ik}W_{kj} + \beta^2\sum_{m=1}^{N} B_{jm}\sum_{k=1}^{N} A_{ik}W_{km}\right]$$

$$+6C^2F(x)\left[\sum_{k=1}^{N} B_{ik}W_{kj} + \upsilon\beta^2\sum B_{jk}W_{ik}\right] = K\left[\sum_{k=1}^{N} B_{ik}W_{kj}\right] \qquad i, j = 3, \ldots N - 2 \tag{15}$$

where $K = \dfrac{N_{ave}a^2}{D_m}$

4. Boundary conditions

For a rectangular plate six boundary conditions given as cccc, sscs, ccss, cccs and ssss are considered, where c and s are designated for clamped and simply supported respectively. The geometrical and natural boundary conditions along the edges of rectangular plate for clamped-clamped and simply supported are given as,

$$\begin{cases} W = 0 \text{ at } x = 0, \quad x = a \quad \text{for}(ssss) \\ M_x = D_{11}W_{,xx} + D_{12}W_{yy} + 2D_{13}W_{,xy} = 0 \end{cases} \tag{16}$$

$$\begin{cases} W = 0 \text{ at } y = 0, \quad Y = b \\ M_y = D_{11}W_{,xx} \;\; D_{22}W_{,yy} + 2D_{23}W_{,xy} = 0 \end{cases} \tag{17}$$

$$\begin{cases} W = 0,\; W_x = 0 \text{ at } x = 0, x = a \quad \text{for}(cccc) \\ W = 0,\; W_y = 0 \text{ at } y = 0, y = b \quad \text{for}(cccc) \end{cases} \tag{18}$$

By applying the Differential Quadrature Method the boundary conditions can be written as,

$W_{1j} = W_{Nj} = W_{i1} = W_{iN} = 0$ $\qquad i,j = 1,\ldots N$ for (ssss)

$$\begin{cases} X = 0 \\ X = 1 \end{cases} \rightarrow M_x = D_{11}\sum_{k=1}^{N} B_{ik}W_{kj} + D_{12}\beta^2\sum_{k=1}^{N} B_{jk}W_{ik}$$

$$+ 2D_{13}\beta \sum_{m=1}^{N} A_{jm}\sum_{k=1}^{N} A_{ik}W_{km} = 0 \qquad \begin{matrix} j = 2, \ldots N-1 \\ j = 3, \ldots N-1 \end{matrix} \tag{19}$$

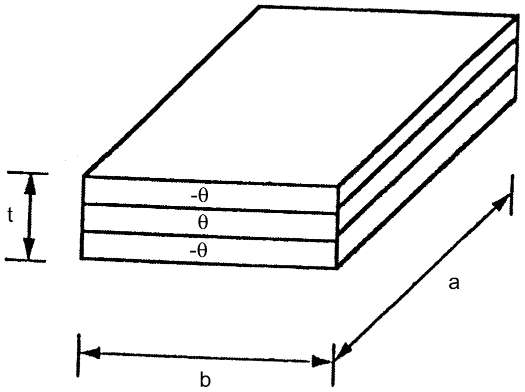


Fig. 3 Arrangement of fiber orientetion

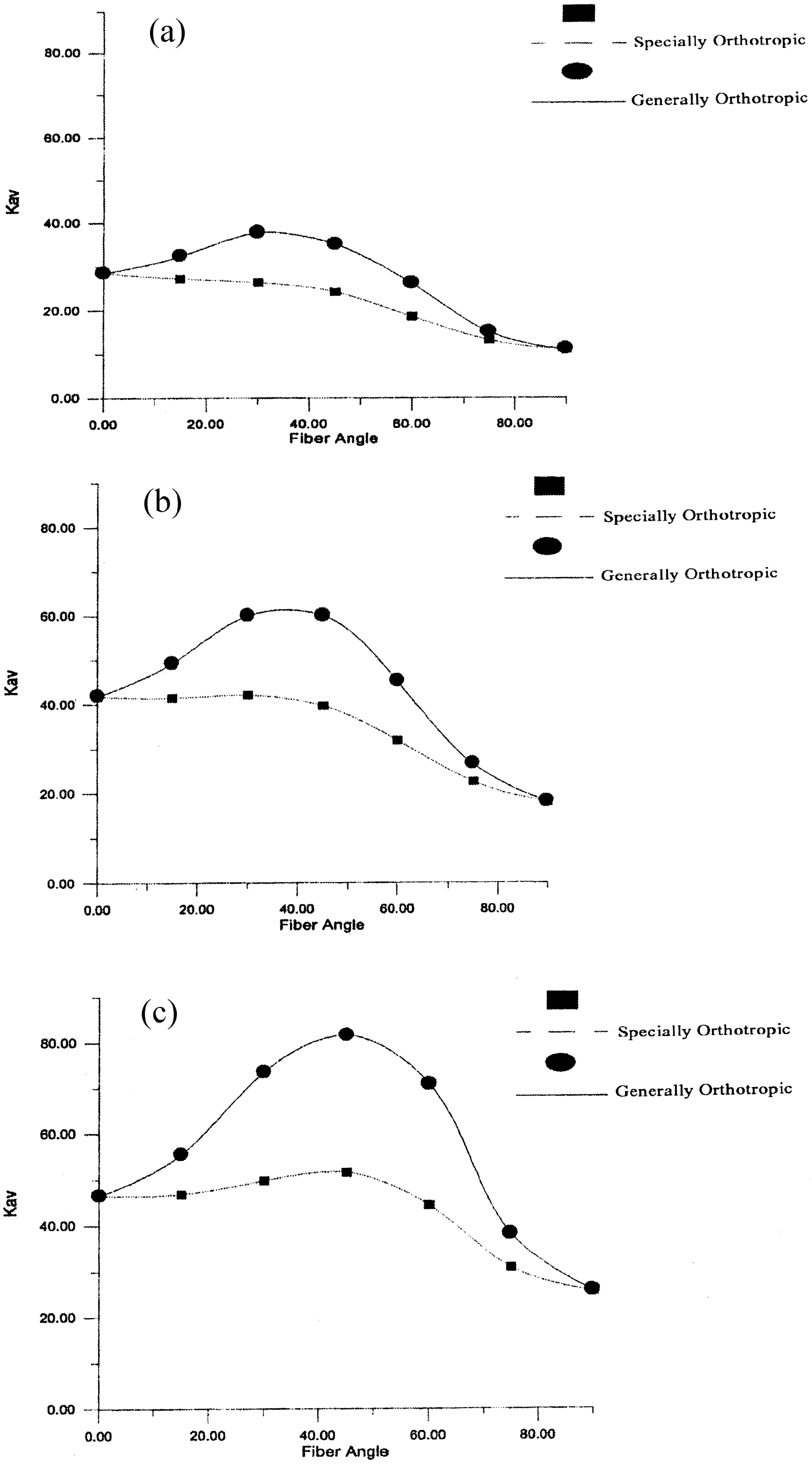


Fig. 4 Buckling of plate with ssss B.C.S: (a) r = -0.5 (b) r = 0 (c) r = 1

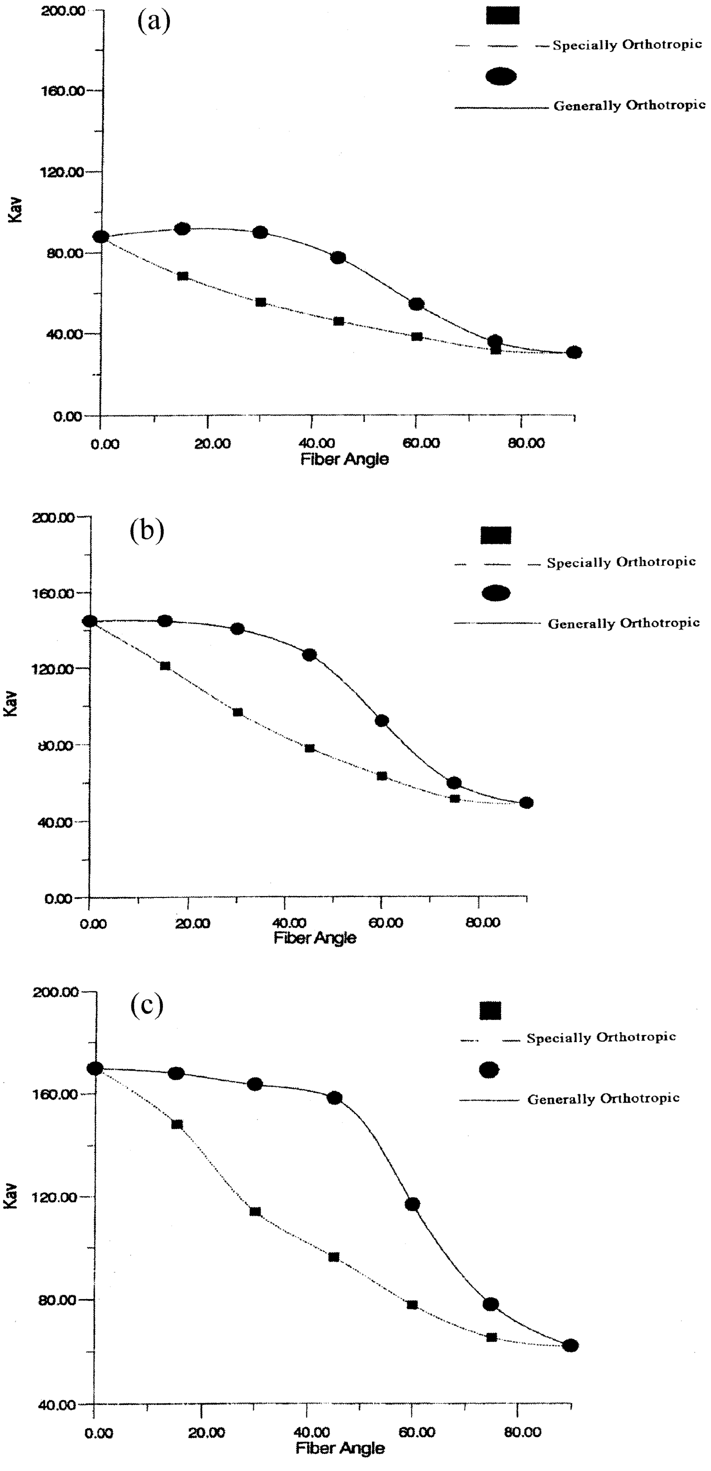


Fig. 5 Buckling of plates with cccc B.C.S: (a) r = -0.5 (b) r = 0 (c) r = 1

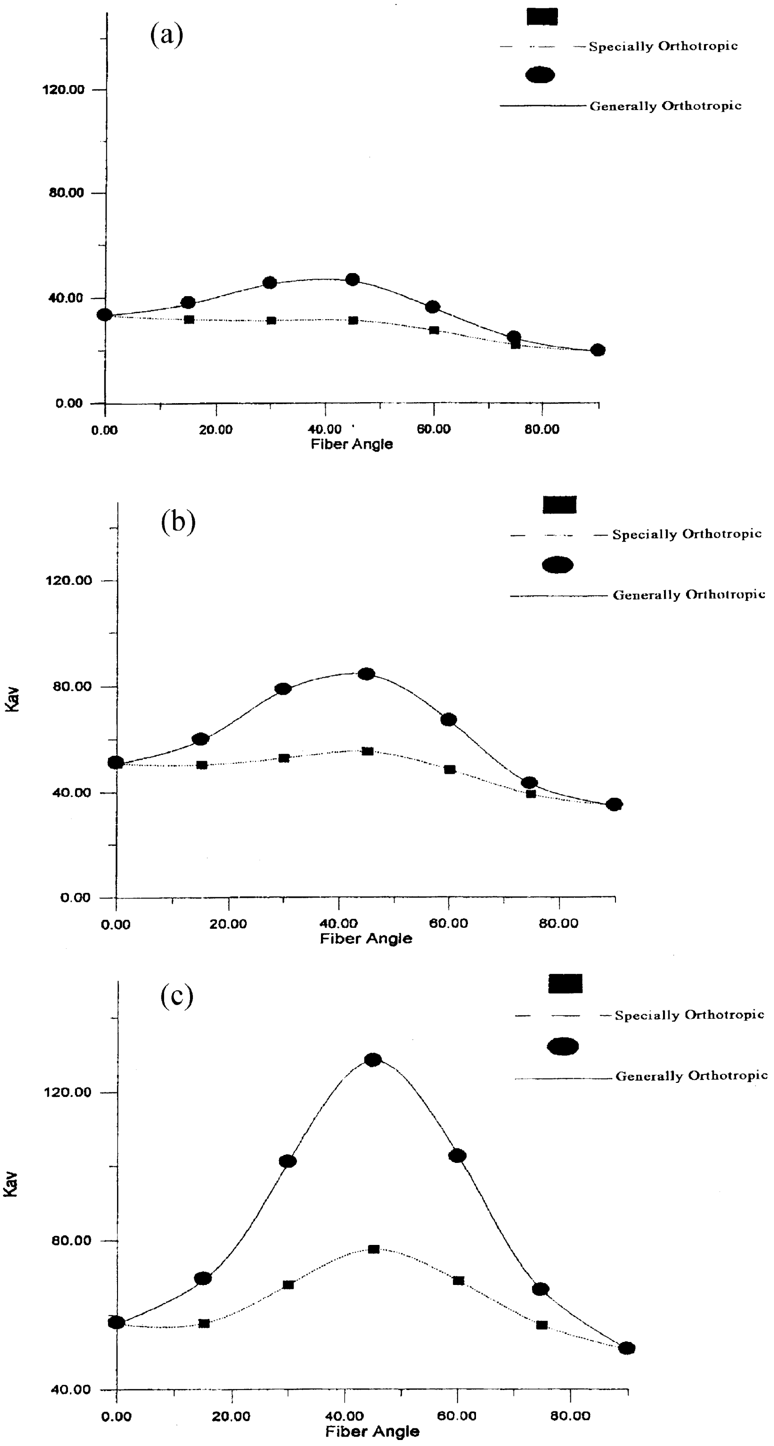


Fig. 6 Buckling of plates with sscc B.C.S: (a) r = -0.5 (b) r = 0 (c) r = 1

$$\begin{cases} Y = 0 \\ Y = 1 \end{cases} \rightarrow M_y = D_{12}\sum_{k=1}^{N} B_{ik}W_{kj} + D_{22}\beta^2\sum_{k=1}^{N} B_{jk}W_{ik}$$

$$+ 2D_{23}\beta\sum_{m=1}^{N} A_{jm}\sum_{k=1}^{N} A_{ik}W_{km} = 0 \quad \begin{matrix} j = 3, \ldots N-2 \\ j = 3, \ldots N-2 \end{matrix} \tag{20}$$

$W_{1j} = W_{Nj} = W_{i1} = W_{iN} = 0$ $\quad i, j = 1,$ $\quad$ for (cccc)

$$\begin{cases} X = 0 \\ X = 1 \end{cases} \rightarrow W_{,x} = \sum_{k=1}^{N} A_{ik}W_{kj} = 0 \quad \begin{matrix} i = 2, \ldots N-1 \\ i = 2, \ldots N-1 \end{matrix}$$

$$\begin{cases} Y = 0 \\ Y = 1 \end{cases} \rightarrow W_{,y} = \sum_{k=1}^{N} A_{jk}W_{ik} = 0 \quad \begin{matrix} i = 3, \ldots N-2 \\ i = 3, \ldots N-2 \end{matrix} \tag{21}$$

5. Results and discussion

To show the ability of differential quadrature method in handling the buckling analysis of multi-layered composite plates results are presented in the form of figures and tables. A Graphite Epoxy rectangular plate with fiber orientation as shown in (Fig. 3) under variable axial compressive force is considered. In (Fig. 4) buckling of a three layered rectangular plate with simply-supported boundary conditions corresponding to different values of axial compressive forcing ratios given as $r = -0.5$, 0 and is considered. The effect of fiber orientation on buckling parameter $K_{ave} = \frac{N_{ave}a^2}{\sqrt{D_{11}D_{22}}}$ for the cases of specially orthotropic plate where $D_{13} = D_{23}$ and generally orthotropic plate where $D_{13} \neq 0$, $D_{23} \neq 0$ is quit significant

It is observed that generally orthotropic plates made of Graphite Epoxy with material properties given in (S-Abrate 1999) are more prompt to buckling than specially orthotropic plates. It is observed that for simply-supported boundary conditions maximum buckling loads for $r = 0.5$, 0.0 and 1 corressponds to fiber angles $\theta = 30^o$, 43^o and 47^o respectively. One can also conclude that for $r = 1$ the differences between buckling parameter corresponding to different fiber angles for specially orthotropic and generally orthotropic is more significant than $r = -0.5$ and $r = 0$.

Fig. 5 represents the buckling load corresponding to fibrous composite plate made of Graphite Epoxy with clamped-clamped boundary conditions. In here also the axial compressive forcing ratios are given as $r = -0.5$, 0 and 1. It is clearly observed that for this boundary conditions the increament in the values of fiber angle causes decreament of the buckling load. This phenomenon is more significant for $r = 1$. For both cases of specially and generally orthotropic clamped plate, the optimum fiber orientation for tolerating the maximum buckling load is nearly zero angle. It is also observed that the buckling loads for

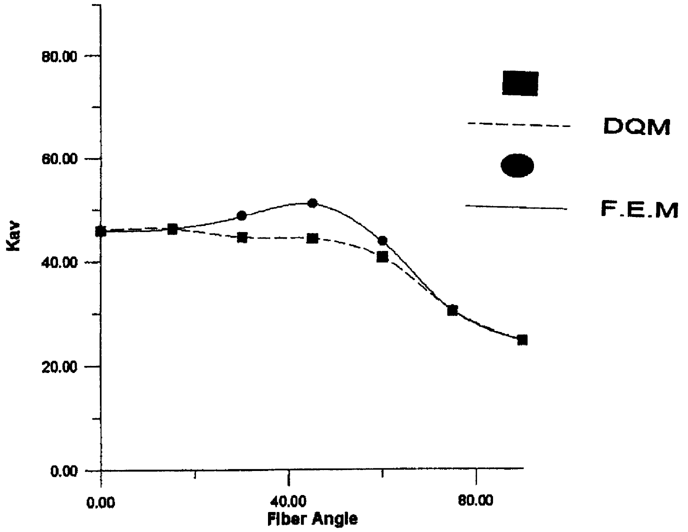


Fig. 7 Comparision of DQMwith FEM ssss B.C.S: (general orthotropic) $r = 1$

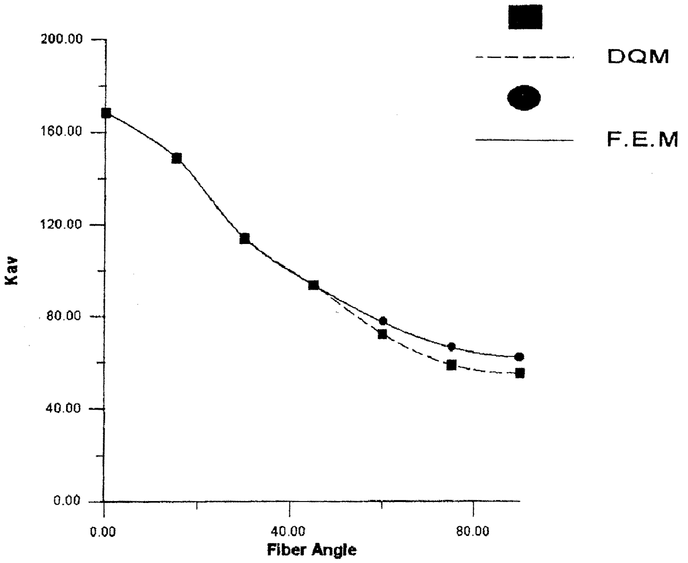


Fig. 8 Comparision of DQM with FEM cccc B.C.S: (general orthotropic) $r = 1$

specially and generally orthotropic plates corresponding to three values of $r = -0.5$, 0 and 1 diverges for the values of $0 < \theta < 45^o$ and converges for the values of the values of of $\theta > 45^o$.

The variations of buckling load with respect to different fiber angles are presented in (Fig. 6) for three values of $r = -0.5$, 0 and 1 for a Graphite Epoxy plate with (sscc) boundary conditions. In here it is found that buckling load corresponding to the generally orthotropic plate is effected more significantly by the variation of fiber orientation than specially orthotropic plate. This behavior which happens for the values of is more significant for $r = 1$.

To show the numerical accuracy as well as computational efficiency over the finite element method comparison of results between present method and those obtained from NISA II software are made in (Fig. 7), (Fig. 8) and (Fig. 9) corresponding to (ssss), (cccc) and (sscc) boundary conditions respectively for $r = 1$. The results from present analysis are also compared with those from (Wittrick 1962) for an isotropic plate with thickness variation which is based on exact analysis. These results are presented in Tables 1 and 2 which corresponds to (ssss) and (ccss) boundary conditions. It is observed that results from differential quadrature method used in the present analysis for the number of grid points 9 agree very well with those obtained from exact analysis. This shows the reliability of the differential quadrature method.

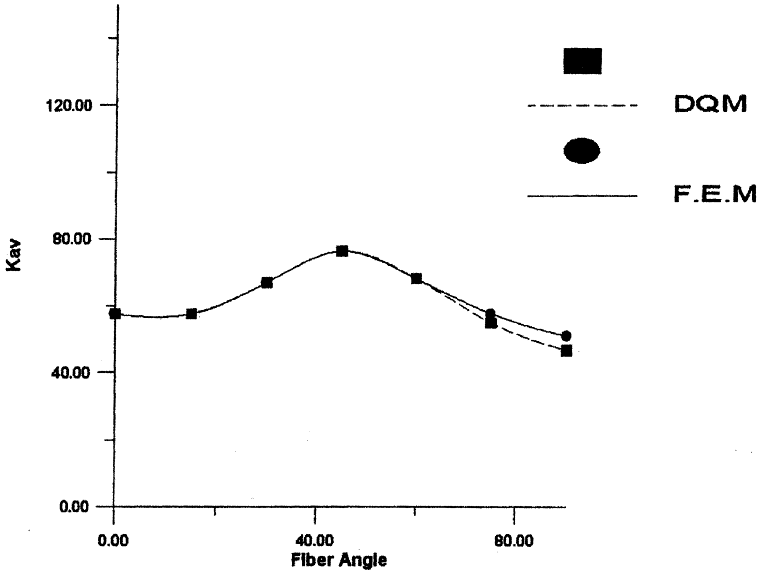


Fig. 9 Comparision of DQM with FEM sscc B.C.s (general orthotropic case) $r = 1$

Table 1 Bukling parameter kave km for quadretic plates with variable thikness, SSSS, B.C.s

No.	t_1/t_0	Wittrick (1962)	DQM [N = 9]
1	1.125	3.966	3.94
2	1.25	3.882	3.85
3	1.5	3.038	3.60
4	1.75	3.364	3.31
5	2	3.1	3.01

Table 2 Bukling parameter kave km for quadretic plates with variable thikness, CCSS, B.C.s

No.	t_1/t_0	Wittrick (1962)	DQM [N = 9]
1	1.125	6.678	6.619
2	1.25	6.51	6.23
3	1.5	6.036	6.150
4	1.75	5.466	5.51
5	2	4.912	5.11

6. Conclusions

The static behavior of fibrous composite plates with constant and variable thickness under axial compressive force is investigated in here. It is believed that the present work, which is the very important on the application of the differential quadrature method to the analysis of fibrous composite plates. As demonstarted by the applications, the differential quadrature method is a practical technique for the static analysis of composite plates which can effectively be used for dynamic analysis of composite plates and shell type structures. This solution is in close agreement with those based on finite element method and exact analysis. Due to the compact procedure of differential quadrature method, which for a given grid, the weighting coefficients need only be calculated once, boundary value problem in structural engineering can efficiently be solved.

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