Steel and Composite Structures, *Vol. 19, No. 4 (2015) 1011-1033*
DOI: http://dx.doi.org/10.12989/scs.2015.19.4.1011

Nonlinear flexural analysis of laminated composite flat panel under hygro-thermo-mechanical loading

Vishesh R. Kar [1a], Trupti R. Mahapatra [2b] and Subrata K. Panda [*1]

[1] *Department of Mechanical Engineering, National Institute of Technology, Rourkela, India*
[2] *School of Mechanical Engineering, KIIT University, Bhubaneswar, India*

(*Received November 13, 2014, Revised March 23, 2015, Accepted April 02, 2015*)

Abstract. In this article, large amplitude bending behaviour of laminated composite flat panel under combined effect of moisture, temperature and mechanical loading is investigated. The laminated composite panel model has been developed mathematically by introducing the geometrical nonlinearity in Green-Lagrange sense in the framework of higher-order shear deformation theory. The present study includes the degraded composite material properties at elevated temperature and moisture concentration. In order to achieve any general case, all the nonlinear higher order terms have been included in the present formulation and the material property variations are introduced through the micromechanical model. The nonlinear governing equation is obtained using the variational principle and discretised using finite element steps. The convergence behaviour of the present numerical model has been checked. The present proposed model has been validated by comparing the responses with those available published results. Some new numerical examples have been solved to show the effect of various parameters on the bending behaviour of laminated composite flat panel under hygro-thermo-mechanical loading.

Keywords: laminated composite flat panel; hygro-thermo-mechanical bending; HSDT; Green-Lagrange nonlinearity; Nonlinear FEM; flexural analysis

1. Introduction

In the weight sensitive industries, composite materials are much compatible due to their uniqueness in specific mechanical properties. However, the behaviour of laminated composites may alter adversely when exposed to thermal and moisture conditions. It is well known that, the deflection behaviour of composites has a great importance in the design and analysis of structural components. When these structures are exposed to mechanical loading under sever environmental conditions, their strength and stiffness changes due to change in thermal and mechanical properties. It is well known that for large deformation regime the basic geometry of panel is distorted and nonlinearity in geometry is induced. This in turn affects the structural behaviour of laminated structures.

In past, many researchers have already examined flexural behaviour of laminate structures

*Corresponding author, Assistant Professor, E-mail: call2subrat@gmail.com
[a] Ph.D. Student, E-mail: visheshkar@gmail.com
[b] Ph.D. Student, E-mail: trmahapatrafme@kiit.ac.in

http://www.techno-press.org/?journal=scs&subpage=8 ISSN: 1229-9367 (Print), 1598-6233 (Online)

either by developing new theories or by modifying existing theories based on analytical and/or numerical method. A selective review on nonlinear flexural behaviour of laminated structures is discussed briefly in the following lines to make the article self-standing. Sai Ram and Sinha (1991) studied bending characteristics of laminated composite plate under hygrothermal loading using finite element method (FEM). Liu and Huang (1996) investigated nonlinear free vibration behaviour of laminated composite plates under dissimilar temperature load using von Karman type nonlinear kinematics in the framework of the first order shear deformation theory (FSDT). Upadhyay and Lyons (2000) examined hygrothermal effect on the nonlinear bending behaviour of fiber reinforced polymer matrix composites based on von Karman plate theory. A quadratic isoparametric FE formulation based on the FSDT are presented by Parhi *et al*. (2001) to show free vibration and transient response behaviour of multiple delaminated plates and shells under hygrothermal loading. Patel *et al*. (2002) analysed static and dynamic characteristics of thick laminated composite plates using modified higher order shear deformation theory (HSDT) by hygrothermal dependent material properties of composites. Huang *et al*. (2004) studied nonlinear vibration and dynamic behaviour of shear deformable laminated plate under hygrothermal loading in the framework of the HSDT by taking von Karman type nonlinear kinematics. Shen *et al*. (2004) presented analytical solutions of dynamic behaviour of laminated plates resting on a two-parameter (Pasternak-type) elastic foundation under hygrothermal environment. They have developed a micromechanical model by taking the effect of volume fractions of individual constituents in the framework of HSDT mid-plane kinematics. Naidu and Sinha (2005) investigated large deflection bending behaviour of composite cylindrical shell panels subjected to hygrothermal environments by using FSDT and Green-Lagrange nonlinear kinematics. Zhang and Kim (2006) developed two displacement-based 4-noded quadrilateral elements (20 and 24 degrees of freedom) to analyse linear and geometrical nonlinear behaviour of thin to moderately thick laminated composite plates based on the FSDT and von Karman geometric nonlinearity. Zhang and Yang (2006) developed 4-noded flat quadrilateral element having 24 degrees of freedom to analyse linear and nonlinear bending behaviour of laminated composite plates. They have developed the model based on the FSDT mid-plane kinematics using Timoshenko's beam functions. Kundu *et al*. (2007) analysed numerically geometrical nonlinear bending behaviour of laminated composite shells in hygrothermal environment using finite element method (FEM) based on the FSDT mid-plane kinematics. Lo *et al*. (2010) developed a global–local HSDT to study the response of laminated plates exposed to hygrothermal environment. Hari Kishore *et al*. (2011) presented nonlinear static behaviour of the composite plates embedded with magnetostrictive materials based on the third order shear deformation theory by taking the geometric nonlinearity in von Karman sense. Baltacioglu *et al*. (2011) investigated nonlinear static deflections of rectangular laminated thick plates resting on elastic foundation using the discrete singular convolution method in the framework of the FSDT and von Karman large deformation equations. Zenkour (2012) examined hygrothermal bending of thick multilayered composite plates using a sinusoidal theory. Recently, Zenkour *et al*. (2014) have extended their work to study the static response of laminated plates resting on elastic foundations by assuming sinusoidal distribution of temperature and moisture. Sharma *et al*. (2013) presented the analytical solutions for flexural response of doubly curved laminated composite shells based on the FSDT mid plane kinematics. A micromechanical model to study the nonlinear vibration behaviour of laminated composite plates resting on elastic foundations has been presented by Kumar and Patil (2013). The formulation is based on the HSDT and von Karman nonlinear kinematics. Szekrenyes (2014) presented an analytical model for de-laminated orthotropic plates based on the Reddy's third-order shear deformable theory.

In addition to the above, we note that a very few work have been reported by Shen (2002), Upadhyay *et al*. (2010) and Lal *et al*. (2011) on hygro-thermo-mechanical static analysis of laminated composite plate based on micro-mechanical approach to show the effect of volume fraction of each constituents. However, in all the cases the mathematical model have been developed using HSDT mid-plane kinematics in conjunction with von Karman nonlinearity. To the best of the authors' knowledge, no work has been reported in literature on nonlinear bending analysis of laminated composite structure under hygro-thermo-mechanical loading by taking HSDT mid-plane kinematics with Green-Lagrange geometric nonlinearity. In this present work, the authors' attempt to develop a general nonlinear FEM model for laminated composite plate under combined hygro-thermo-mechanical loading based on micro-mechanics model. Here, the composite material properties are considered to be dependent on temperature and moisture. In addition to this, all the nonlinear higher order terms have been incorporated in the mathematical model to capture the original flexure of the structure. The nonlinear system governing equations are obtained using variational approach and discretised using suitable FEM. A direct iterative method is employed to solve the system equation to obtain the bending responses. The efficacy and accuracy of the model has been evaluated by comparing the responses with available published results. The effects of various parameters on hygro-thermo-mechanical bending response of laminated composite flat panel are examined and discussed in details.

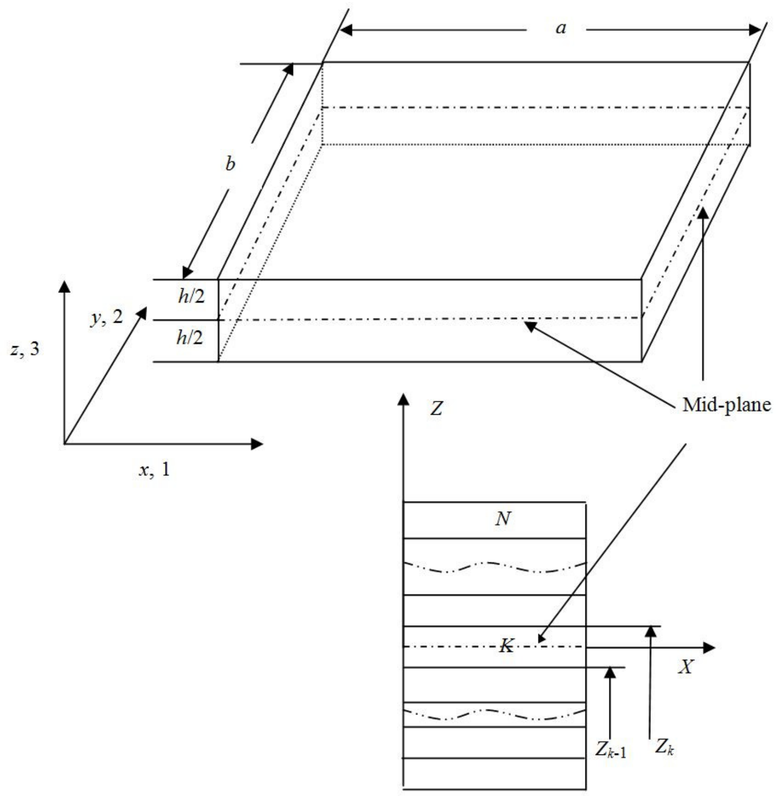


Fig. 1 Geometry and stacking sequence of laminated plate

2. Theoretical development and finite element formulation

A typical flat panel geometry has been considered for the present investigation as shown in Fig. 1. The laminated plate is consists of N number of equally thick orthotropic layers of length a, width b and thickness h. The proposed mathematical model has been developed based on the HSDT displacement field (Reddy 2004) and any arbitrary point on the laminated flat panel with respect to the mid-plane along x, y and z directions is given by

$$\bar{u} = u + \phi_1 z + \psi_1 z^2 + \theta_1 z^3, \quad v = v + \phi_2 z + \psi_2 z^2 + \theta_2 z^3, \quad w = w \tag{1}$$

where, $(\bar{u}, \bar{v}, \bar{w})$ indicate the displacements at any arbitrary point on the plate along the (x, y, z) respectively. Similarly, (u, v, w) represent the corresponding displacements of the points on the mid-plane, ϕ_1 and ϕ_2 are the rotations of normal to the mid-plane relating to y-axis and x-axis, respectively. This displacement field represents the transverse shear strains as quadratic function of thickness coordinate at any point within the shell and also account for the parabolic distribution of shear stress across the thickness represented by ψ_1, ψ_2, θ_1 and θ_2, which are the higher order terms of Taylor series expansion defined at the mid-plane.

The following Green-Lagrange type nonlinear strain–displacement relations have been used to express the deformation behaviour (Panda and Mahapatra 2014).

$$\left.\begin{aligned}
\varepsilon_{xx} &= \bar{u},_x + \frac{1}{2}\left\{(\bar{u},_x)^2 + (\bar{v},_x)^2 + (\bar{w},_x)^2\right\} \\
\varepsilon_{yy} &= \bar{v},_y + \frac{1}{2}\left\{(\bar{u},_y)^2 + (\bar{v},_y)^2 + (\bar{w},_y)^2\right\} \\
\gamma_{xy} &= (\bar{u},_y + \bar{v},_x) + \left\{\bar{u},_x \bar{u},_y + \bar{v},_x \bar{v},_y + \bar{w},_x \bar{w},_y\right\} \\
\gamma_{xz} &= (\bar{u},_z + \bar{w},_x) + \left\{\bar{u},_x \bar{u},_z + \bar{v},_x \bar{v},_z + \bar{w},_x \bar{w},_z\right\} \\
\gamma_{yz} &= (\bar{v},_z + \bar{w},_y) + \left\{\bar{u},_y \bar{u},_z + \bar{v},_y \bar{v},_z + \bar{w},_y \bar{w},_z\right\}
\end{aligned}\right\} \tag{2}$$

Now, substituting Eq. (1) in Eq. (2) the strain displacement relation of the laminated flat panel is expressed as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix}
\left[\varepsilon_1^0 + \frac{1}{2}\varepsilon_1^4 + zk_1^1 + \frac{1}{2}zk_1^5 + z^2k_1^2 + \frac{1}{2}zk_1^6 + z^3k_1^3 + \frac{1}{2}z^3k_1^7 + \frac{1}{2}z^4k_1^8 + \frac{1}{2}z^5k_1^9 + \frac{1}{2}z^6k_1^{10}\right] \\
\left[\varepsilon_2^0 + \frac{1}{2}\varepsilon_2^4 + zk_2^1 + \frac{1}{2}zk_2^5 + z^2k_2^2 + \frac{1}{2}z^2k_2^6 + z^3k_2^3 + \frac{1}{2}z^3k_2^7 + \frac{1}{2}z^4k_2^8 + \frac{1}{2}z^5k_2^9 + \frac{1}{2}z^6k_2^{10}\right] \\
\left[\varepsilon_6^0 + \varepsilon_6^4 + zk_6^1 + zk_6^5 + z^2k_6^2 + z^2k_6^6 + z^3k_6^3 + z^3k_6^7 + z^4k_6^8 + z^5k_6^9 + z^6k_6^{10}\right] \\
\left[\varepsilon_5^0 + \varepsilon_5^4 + zk_5^1 + zk_5^5 + z^2k_5^2 + z^2k_5^6 + z^3k_5^3 + z^3k_5^7 + z^4k_5^8 + z^5k_5^9\right] \\
\left[\varepsilon_4^0 + \varepsilon_4^4 + zk_4^1 + zk_4^5 + z^2k_4^2 + z^2k_4^6 + z^3k_4^3 + z^3k_4^7 + z^4k_4^8 + z^5k_4^9\right]
\end{Bmatrix} \tag{3}$$

Now, Eq. (3) can be rewritten as

$$\{\varepsilon\} = [H]_L \{\bar{\varepsilon}_L\} + \frac{1}{2}[H]_{NL} \{\bar{\varepsilon}_{NL}\} \tag{4}$$

where, $[H]_L$ and $[H]_{NL}$ are the functions of thickness coordinate and represents linear and the nonlinear thickness coordinate matrices. $\{\bar{\varepsilon}_L\}$ and $\{\bar{\varepsilon}_{NL}\}$ are the functions of x and y and represent mid-plane linear and nonlinear strains, respectively. The superscripts 0-3 and 4-10 accounts for the extension, bending, curvature and higher order strain terms in linear and nonlinear strain vectors, respectively. The detail terms of $\{\bar{\varepsilon}_L\}$ and $\{\bar{\varepsilon}_{NL}\}$, $[H]_L$ and $[H]_{NL}$ can be seen in Panda and Mahapatra (2014).

In the present analysis, it is assumed that the deformation occurring due to unlike temperature and moisture change are not coupled. Hence, the constitutive matrix equation of generalized stress tensor for any general k^{th} orthotropic composite lamina with any fibre orientation angle θ is given by

$$\{\sigma_{ij}\}^k = [\bar{Q}_{ij}]^k \{\varepsilon_{ij} - \alpha_{ij}\Delta T - \beta_{ij}\Delta C\}^k \tag{5}$$

where, $\{\sigma_{ij}\}^k = \{\sigma_1\ \sigma_2\ \sigma_6\ \sigma_5\ \sigma_4\}^T$ and $\{\varepsilon_{ij}\}^k = \{\varepsilon_1\ \varepsilon_2\ \varepsilon_6\ \varepsilon_5\ \varepsilon_4\}^T$ are the stress and strain vectors respectively for the k^{th} layer. $[\bar{Q}_{ij}]^k$ is the transferred reduced stiffness matrix for the k^{th} layer, $\{\alpha_{ij}\}^k = \{\alpha_1\ \alpha_2\ 2\alpha_{12}\}^T$ is the thermal expansion/contraction coefficient vector and $\{\beta_{ij}\}^k = \{\beta_1\ \beta_2\ 2\beta_{12}\}^T$ is the moisture expansion/contraction coefficient vector. Here, $\Delta T = T - T_0$ is the temperature difference, where T is applied and T_0 is reference temperatures, respectively. Similarly, $\Delta C = C - C_0$ is the moisture difference between applied (C) and reference (C_0) values of weight percentage of moisture.

Now, Eq. (5) can be expanded as

$$\{\sigma_{ij}\}^k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{55} & \bar{Q}_{54} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{44} \end{bmatrix}^k \left[\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_5 \\ \varepsilon_4 \end{Bmatrix}_l^k - \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ 2\alpha_{12} \\ 0 \\ 0 \end{Bmatrix}^k \Delta T - \begin{Bmatrix} \beta_1 \\ \beta_2 \\ 2\beta_{12} \\ 0 \\ 0 \end{Bmatrix}^k \Delta C + \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_5 \\ \varepsilon_4 \end{Bmatrix}_{nl}^k \right] \tag{6}$$

In the present formulation, a micromechanical material model is employed to evaluate the mechanical and thermal properties of composite laminate. Since, the properties of the polymer based matrix material are predominantly affected by hygrothermal conditions the degradation of the composite material properties are estimated by degrading the matrix properties only. This is achieved through the "matrix mechanical property retention ratio" expressed as (Chamis and Sinclair 1982)

$$F_m = \left[\frac{T_{gw} - T}{T_{g0} - T_0} \right]^{\frac{1}{2}} \tag{7}$$

where, $T = T_0 + \Delta T$, T_{gw} and T_{g0} are the glass transition temperature for hygrothermal and reference dry conditions respectively. The glass transition temperature under hygrothermal condition can be obtained as (Chamis 1987)

$$T_{gw} = (0.005C^2 - 0.10C + 1.0)T_{g0} \tag{8}$$

Now, the hygro-thermo elastic constants are evaluated using the steps as in Upadhyay *et al*. (2010).

$$\left.\begin{aligned}
E_{11} &= E_{f1}V_f + F_m E_m V_m \\
E_{22} &= \left(1.0 - \sqrt{V_f}\right)F_m E_m + \frac{F_m E_m \sqrt{V_f}}{1.0 - \sqrt{V_f}\left(1.0 - \dfrac{F_m E_m}{E_{f2}}\right)} \\
G_{12} &= \left(1.0 - \sqrt{V_f}\right)F_m G_m + \frac{F_m G_m \sqrt{V_f}}{1.0 - \sqrt{V_f}\left(1.0 - \dfrac{F_m G_m}{G_{f12}}\right)} \\
\upsilon_{12} &= \upsilon_{f12}V_f + \upsilon_m V_m
\end{aligned}\right\} \tag{9}$$

where, V is the volume fraction and the subscripts "*f*" and "*m*" are used for fiber and matrix materials, respectively.

The modified coefficients of thermal and moisture expansion/contraction are also obtained by neglecting the moisture effect on fiber ($\beta_f = 0$). For the same, the matrix hygrothermal property retention ratio is approximated as

$$F_h = \frac{1}{F_m} \tag{10}$$

The longitudinal and transverse coefficients of thermal and moisture expansions are conceded as Upadhyay *et al*. (2010)

$$\left.\begin{aligned}
\alpha_{11} &= \frac{E_{f1}V_f\alpha_{f1} + F_m E_m V_m F_h \alpha_m}{E_{f1}V_f + F_m E_m V_m} \\
\alpha_{22} &= \alpha_{f2}V_f + V_m F_h \alpha_m + \frac{V_f V_m(\upsilon_{f12}F_m E_m - \upsilon_m E_{f1})}{E_{f1}V_f + F_m E_m V_m}(\alpha_{f1} - F_h\alpha_m)
\end{aligned}\right\} \tag{11}$$

$$\left.\begin{aligned}
\beta_{11} &= \frac{E_{f1}V_f\beta_{f1} + F_m E_m V_m F_h \beta_m}{E_{f1}V_f + F_m E_m V_m} \\
\beta_{22} &= \frac{V_m F_h \beta_m\left[\left(1+\upsilon_m\right)\left(E_{f1}V_f + F_m E_m V_m\right) - \left(\upsilon_{f12}V_f + \upsilon_m V_m\right)E_m F_m\right]}{E_{f1}V_f + F_m E_m V_m}
\end{aligned}\right\} \tag{12}$$

Now, Eqs. (7)-(12), are used to evaluate the coefficients of transverse reduced stiffness matrix, thermal and hygroscopic expansion coefficients in Eq. (6).

Here, a nine noded isoparametric quadrilateral Lagrangian element with nine degrees of freedom per node has been taken to discretize the present laminate model. The field displacement vector corresponding to any nodal point of the element can be expressed as

$$\left\{\delta^*\right\}=\begin{bmatrix}u & v & w & \phi_1 & \phi_2 & \psi_1 & \psi_2 & \theta_1 & \theta_2\end{bmatrix}^{\mathrm{T}}=\sum_{i=1}^{9}[N_i]\{\delta_i\} \tag{13}$$

where, $[N_i]$ and $\{\delta_i\}$ are the interpolation function and displacement vector for the ith node, respectively and the details of interpolation functions can be seen in Cook *et al*. (2009).

Now, the mid-plane strain vector can be written as

$$\{\varepsilon\}_i=[B_i]\{\delta_i\} \tag{14}$$

where, $[B_i]$ is the strain displacement relation matrix.

The strain energy of the panel can be expressed as

$$U=\frac{1}{2}\int_v\{\varepsilon\}_i^{\mathrm{T}}\{\sigma_i\}\,dV \tag{15}$$

Using the expression of strain vectors and resultant stress from Eq. (4) and Eq. (5) and putting into Eq. (15) the strain energy can be expressed as

$$\begin{aligned}U&=\frac{1}{2}\int_V\{\varepsilon\}_i^{\mathrm{T}}\left[\bar{Q}\right]\{\varepsilon\}_i dV=\frac{1}{2}\iiint\{\varepsilon_{\mathrm{L}}+\varepsilon_{\mathrm{NL}}\}^T\left[\bar{Q}\right]\{\varepsilon_{\mathrm{L}}+\varepsilon_{\mathrm{NL}}\}\,dx\,dy\,dz\\&=\frac{1}{2}\int_A\left(\{\varepsilon_{\mathrm{L}}\}_i^{\mathrm{T}}[\mathrm{D}_1]\{\varepsilon_{\mathrm{L}}\}_i+\frac{1}{2}\{\varepsilon_{\mathrm{L}}\}_i^{\mathrm{T}}[\mathrm{D}_2]\{\varepsilon_{\mathrm{NL}}\}_i+\frac{1}{2}\{\varepsilon_{\mathrm{NL}}\}_i^{\mathrm{T}}[\mathrm{D}_3]\{\varepsilon_{\mathrm{L}}\}_i+\frac{1}{4}\{\varepsilon_{\mathrm{NL}}\}_i^{\mathrm{T}}[\mathrm{D}_4]\{\varepsilon_{\mathrm{NL}}\}_i\right)dA\end{aligned} \tag{16}$$

where, $[D_1]=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[H]_L^T[\bar{Q}][H]_L\,dz$, $[D_2]=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[H]_L^T[\bar{Q}][H]_{NL}\,dz$, $[D_3]=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[H]_{NL}^T[\bar{Q}][H]_L\,dz$ and $[D_4]=\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}[H]_{NL}^T[\bar{Q}][H]_{NL}\,dz$.

Substituting Eq. (4) into Eq. (16) the expression for strain energy becomes

$$U=\frac{1}{2}\int_A\begin{pmatrix}\left\{\delta^*\right\}^T[\mathrm{B}_L]_i^T[\mathrm{D}_1][\mathrm{B}_L]_i\left\{\delta^*\right\}\\+\frac{1}{2}\left\{\delta^*\right\}^T[\mathrm{B}_L]_i^T[\mathrm{D}_2][\mathrm{A}]_i[\mathrm{G}]_i\left\{\delta^*\right\}\\+\frac{1}{2}\left\{\delta^*\right\}^T[\mathrm{G}]_i^T[\mathrm{A}]_i^T[\mathrm{D}_3][\mathrm{B}_L]_i\left\{\delta^*\right\}\\+\frac{1}{4}\left\{\delta^*\right\}^T[\mathrm{G}]_i^T[\mathrm{A}]_i^T[\mathrm{D}_4][\mathrm{A}]_i[\mathrm{G}]_i\left\{\delta^*\right\}\end{pmatrix}dA \tag{17}$$

where, $\{\varepsilon_L\}_i=[B_L]_i\{\delta^*\}$ and $\{\varepsilon_{NL}\}_i=\frac{1}{2}[B_{NL}(\delta)]_i\{\delta^*\}=\frac{1}{2}[A(\delta)]_i[G]_i\{\delta^*\}$.

$[B_L]$ is the product form of the differential operator and nodal interpolation function in the linear strain terms. $[A]$ is function of the displacements and $[G]$ is the product form of differential operator and shape function in the nonlinear strain terms. The expressions of $[A]$ and $[G]$ arising due to the Green-Lagrange nonlinearity in the nonlinear stiffness matrices are given in the appendix.

The work done due to the external applied distributed transverse static load "*q*" can be expressed as

$$W = \int_A \{\delta\}^T q dA \tag{18}$$

where, the intensity of transverse static load is expressed in terms of the applied uniform lateral pressure as

$$q = \frac{Qh^4 E_{22}}{a^4} \tag{19}$$

3. System governing equation and solution approach

The governing equation of nonlinear bending for laminated composite flat panel is obtained by minimizing the total energy expression. This result in

$$\delta \Pi = 0 \tag{20}$$

where, $\Pi = (U - W)$

Using, Eqs. (17) and (18) in Eq. (20) and applying finite element approximation, the system governing expression can be obtained as

$$[K]\{\delta\} = \{q\} \quad \text{or} \quad [K_L + K_{NL}]\{\delta\} = \{q\} \tag{21}$$

where, $[K_L]$ and $[K_{NL}]$ are the global linear and nonlinear stiffness matrices. The nonlinear stiffness matrix depends on the displacement vector linearly and quadratically, respectively.

Now, Eq. (21) is solved using a direct iterative method and solutions scheme are depicted in Fig. 2.

4. Results and discussion

A nonlinear finite element computer code have been developed in MATLAB 7.10.0 and based on the present formulation in order to obtain the hygro-thermo-mechanical linear/nonlinear bending responses of laminated composite flat panel. As a first step, the validation behaviour of the present model has been established. Subsequently, different parametric studies are also been carried out and their significance have been discussed in details. For computation, the material properties of the composite material are considered to be dependent on hygrothermal conditions. Their values corresponding to reference temperature 21°C and moisture concentration 0% are given below as in Upadhyay *et al*. (2010) and remain unchanged for each case if not specified otherwise.

E_{f1} = 220 GPa, E_{f2} = 13.79 GPa, E_m = 3.45 GPa, G_{f2} = 8.97 GPa, υ_{f12} = 0.2, υ_m = 0.35,
$\alpha_{f1} = -0.99 \times 10^{-6}$/°C, $\alpha_{f2} = 10.08 \times 10^{-6}$/°C, $\alpha_m = 72 \times 10^{-6}$/°C, β_m = 0.33, T_{g0} = 216°C

The boundary conditions used for analysis are given below:

(a) All edges simply support (SSSS):

$v = w = \Phi_2 = \Psi_2 = \theta_2 = 0$ at $x = 0, a$ and $u = w = \Phi_1 = \Psi_1 = \theta_1 = 0$ at $y = 0, b$.

(b) All edges clamped (CCCC):
$u = v = w = \Phi_1 = \Phi_2 = \Psi_1 = \Psi_2 = \theta_1 = \theta_2 = 0$ for both $x = 0, a$ and $y = 0, b$.

(c) All edges hinged (HHHH):
$u = v = w = \Phi_2 = \Psi_2 = \theta_2 = 0$ at $x = 0, a$ and $u = v = w = \Phi_1 = \Psi_1 = \theta_1 = 0$ at $y = 0, b$.

Unless defined otherwise, the transversely applied load parameter (Q = 100, 200, 300, 400 and 500) and the linear/nonlinear transverse central deflections are non-dimensionalized using the relations $Q = (q / E_2)*(a/h)^4$ and $w_{central} = \frac{w_{\max}}{h}$, respectively.

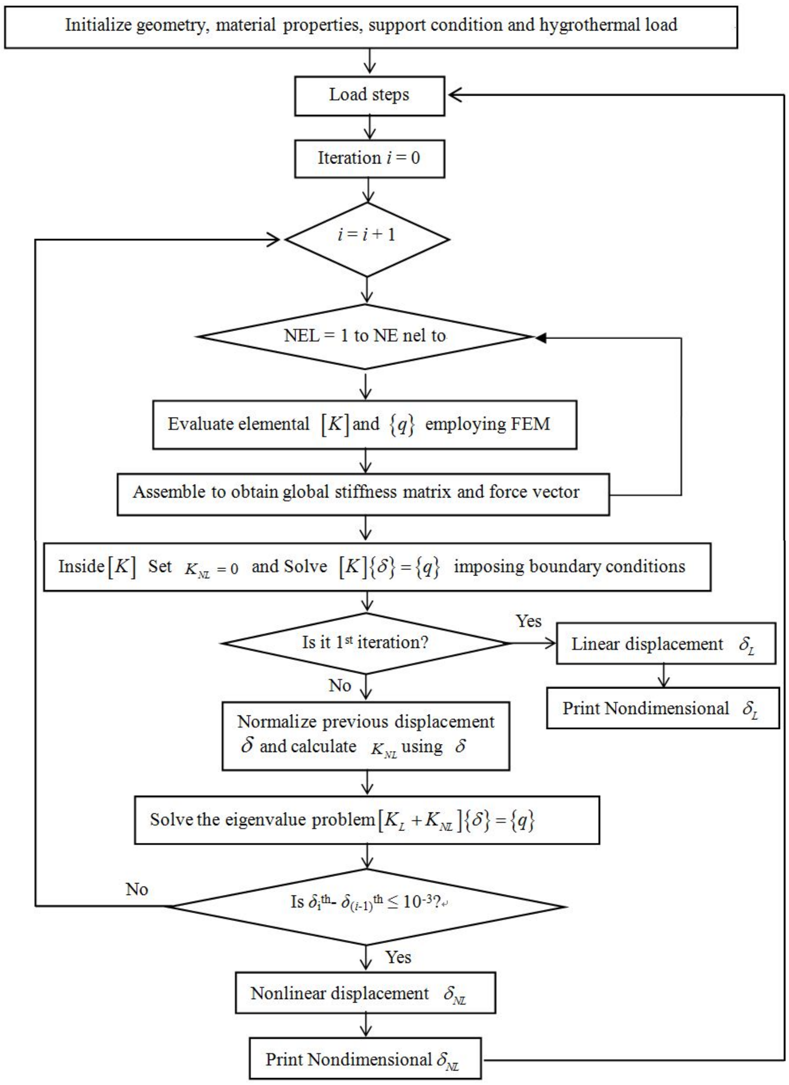


Fig. 2 Solution steps for system governing equation

4.1 Convergence study

In this section, the convergence behaviour of the present model is performed by analysing the bending responses of simply supported, symmetric cross-ply ($[0/90]_S$) and anti-symmetric angle-ply ($[\pm 45]_2$) laminated square plates (a/h = 10, V_f = 0.6) under unlike environmental conditions (ΔT = 0°C, ΔC = 0% and ΔT = 300°C, ΔC = 3%). The geometrical and material properties of the laminates have been taken same as Shen (2002). The present responses are plotted with various mesh refinements as shown in Fig. 3. The corresponding nonlinear bending responses of Shen (2002) are also mentioned in the figure for comparison purpose. It is clearly understood that, good convergence rate is achieved for both the linear and nonlinear central deflections obtained using present model (the HSDT and Green-Lagrange nonlinearity) under different hygrothermal environments. It is also noted that, the present nonlinear response values

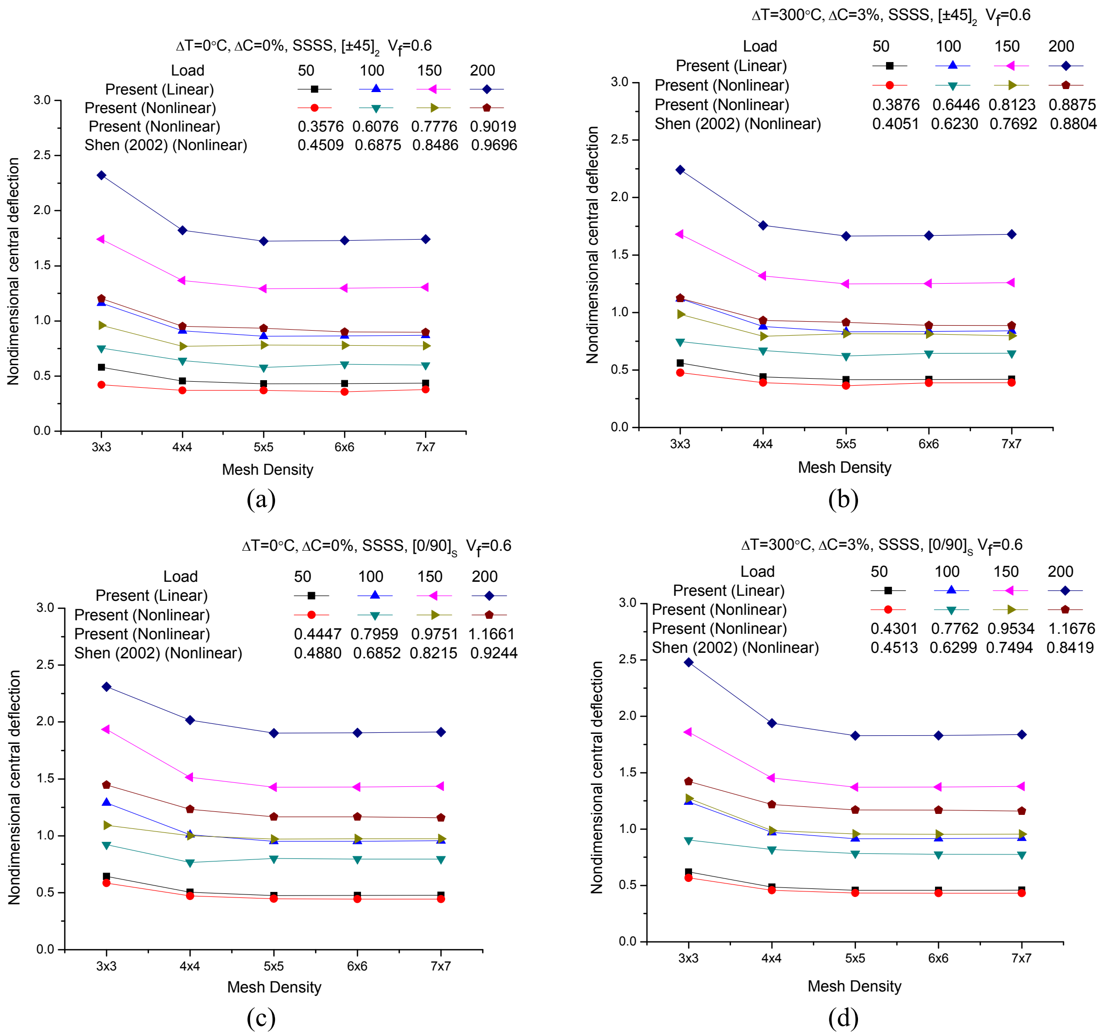


Fig. 3 Convergence behaviour of laminated composite flat panel

are in good agreement with Shen (2002) corresponding to (6×6) mesh size. Based on the convergence study, a (6×6) mesh has been used throughout the present analysis.

4.2 Comparison study

In order to extend the validity and the accuracy of the present formulation, the results obtained using the present model corresponding to different environmental conditions, are checked with Chebyshev series based analytical solutions of Upadhyay *et al*. (2010) and perturbation based analytical solution of Shen (2002). The results for simply supported anti-symmetric angle-ply ($[45/\text{-}45]_2$) laminated flat panel ($a/h=10$, $a/b=1$, $V_f=0.6$) under various hygro-thermo-mechanical loading are plotted in Fig. 4. It is evident that the present FEM results are in good agreement with the analytical results of the references. The nominal difference in results is due to the fact that the

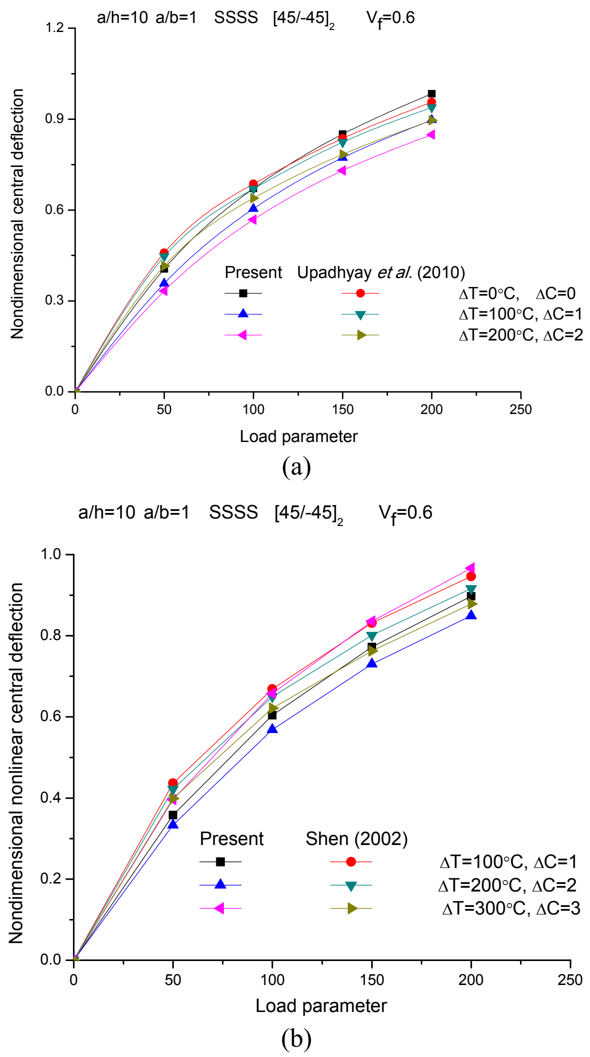
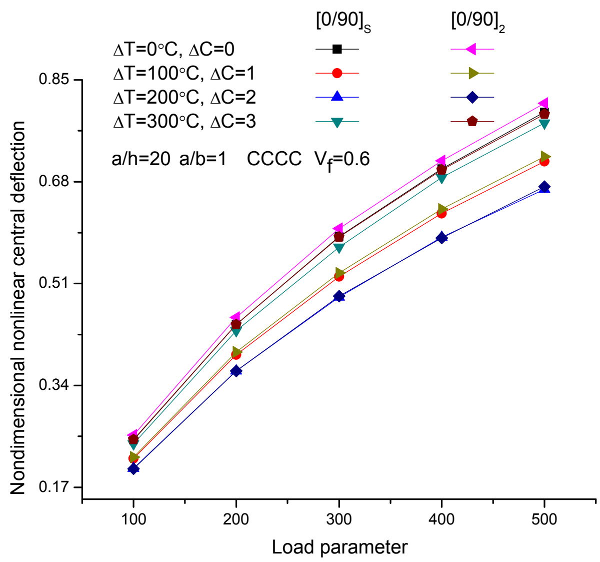
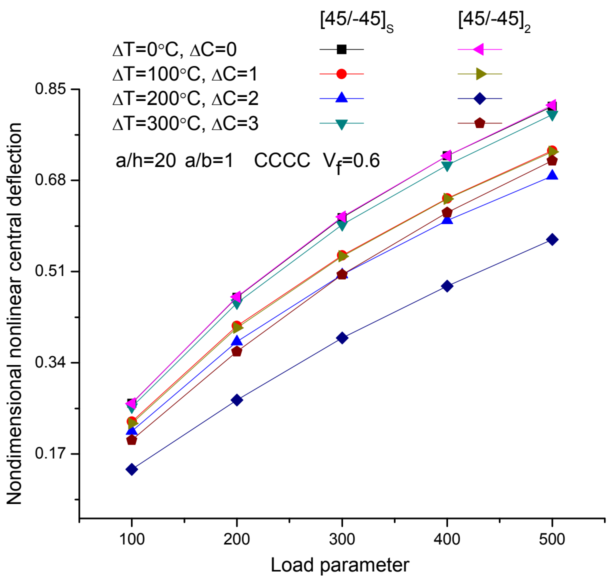


Fig. 4 Comparison study of nonlinear bending of laminated composite flat panel

present mathematical model is developed based on the HSDT mid-plane kinematics and Green-Lagrange nonlinearity including all the nonlinear higher order terms in the formulation, whereas the stated references used HSDT mid-plane kinematics and von Karman nonlinearity. However, it is worthy to mention that the present nonlinear responses are showing closer conformation for higher mechanical and hygrothermal load values in comparison to lower load cases. This demonstrates the significance and necessity of the present mathematical model.



(a)

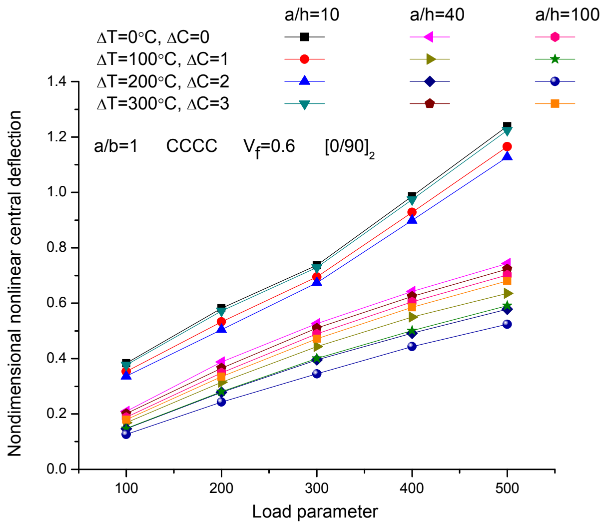


(b)

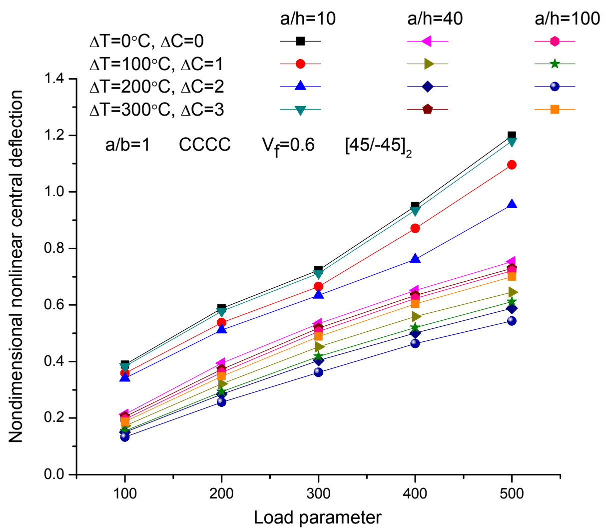
Fig. 5 Effect of lamination scheme and hygrothermal conditions on the nonlinear centre deflection of laminated composite flat panel

4.3 Additional examples

In order to evaluate the effect of hygrothermal conditions, geometrical and material parameters and boundary conditions on the nonlinear hygro-thermo-mechanical bending responses of laminated composite flat panels, some new examples are solved. Each illustration is worked out for anti-symmetric cross-ply ($[0/90]_2$) and anti-symmetric angle-ply $[\pm 45]_2$ laminated plates having volume fraction $V_f = 0.6$. In addition to that four sets of unlike environmental conditions ($\Delta T = 0$°C and $\Delta C = 0$%, $\Delta T = 100$°C and $\Delta C = 1$%, $\Delta T = 200$°C and $\Delta C = 2$%, $\Delta T = 300$°C and $\Delta C = 3$%) are considered in each case. Results are plotted and discussed in details.

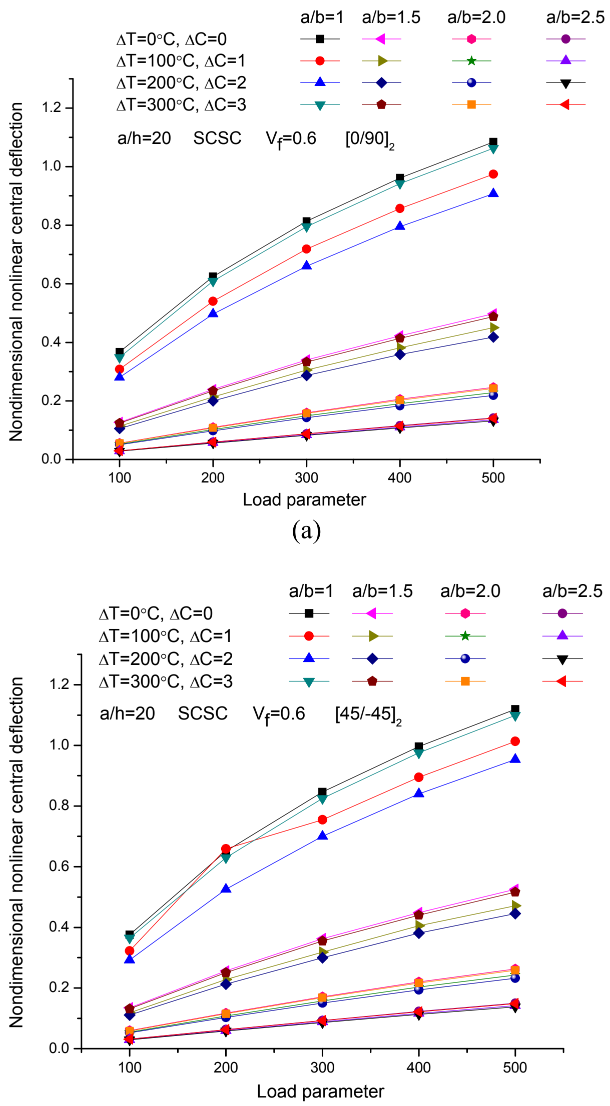


(a)



(b)

Fig. 6 Effect of thickness ratio (a/h) on nonlinear central deflection of laminated composite flat panel



(a)

(b)

Fig. 7 Effect of aspect ratio (a/b) on nonlinear central deflection of laminated composite flat panel

4.3.1 Effect of lamination scheme

It is true that the lamination schemes are one of the key factors in strength design of fiber reinforced composite structure. Hence, in this example the effect of lamination scheme under elevated hygrothermal environment is investigated and presented in Fig. 5. The figure shows the nonlinear central deflection of fully clamped square laminated composite plate (a/h = 20) for four different lamination schemes (symmetric/anti-symmetric cross-ply/angle-ply) under hygrothermal load. It is observed from the results that the symmetric laminations are less affected under hygrothermal loads in comparison to anti-symmetric cases. It is also noted that the deflections are lower for symmetric cross-ply cases than the anti-symmetric cross-ply plate whereas a reverse trend is observed for angle-ply laminations under higher hygrothermal loads. Nonlinear central deflections are decreasing with increasing in hygrothermal load irrespective of lamination schemes. However, sudden increase in deflection values are noticed beyond the glass transition temperature i.e., $\Delta T = 300°C$ and $\Delta C = 3\%$.

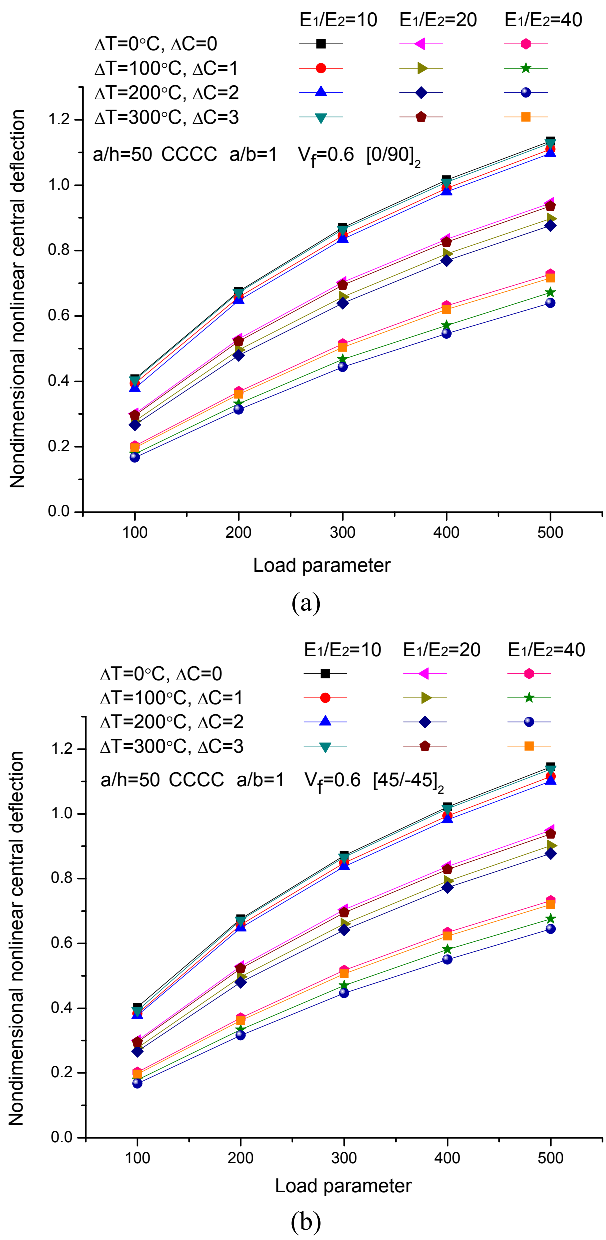
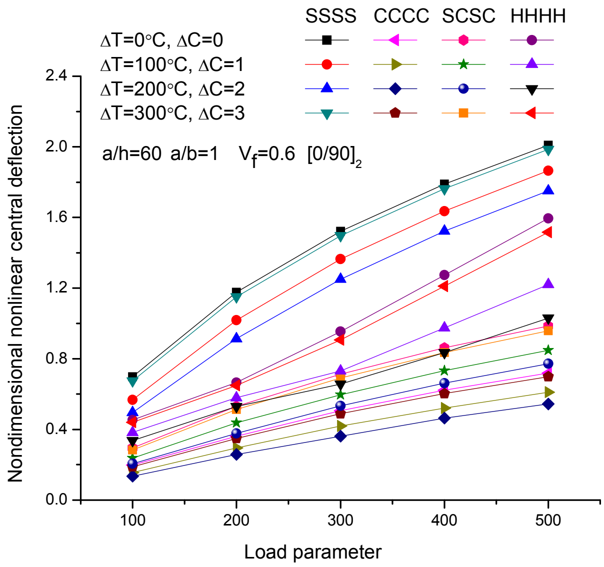
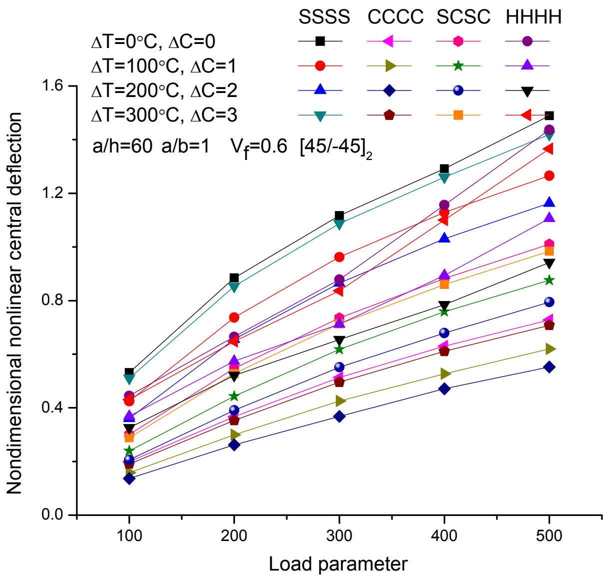


Fig. 8 Effect of modular ratio (E_1/E_2) on nonlinear deflection of laminated composite flat panel



(a)



(b)

Fig. 9 Effect of support conditions on nonlinear deflection of laminated composite flat panel

4.3.2 Effect of thickness ratio

The stiffness of laminated plate depends greatly on the thickness ratio, which in turn affect the bending responses. Fig. 6 presents the effect of thickness ratio (a/h = 10, 40 and 100) on nondimensional central deflection for clamped square anti-symmetric cross-ply and angle ply laminated composite flat panel subjected to hygro-thermo-mechanical loading. It is observed that the nonlinear central deflections decrease with increase in thickness ratio. However, it is noted that the thick shells are showing hardening type of behaviour at higher hygrothermal loads, whereas the thin shells are showing softening type of behaviour for every hygrothermal load considered in this analysis.

4.3.3 Effect of aspect ratio

The variation of non-dimensional central deflection of anti-symmetric cross-ply/angle-ply

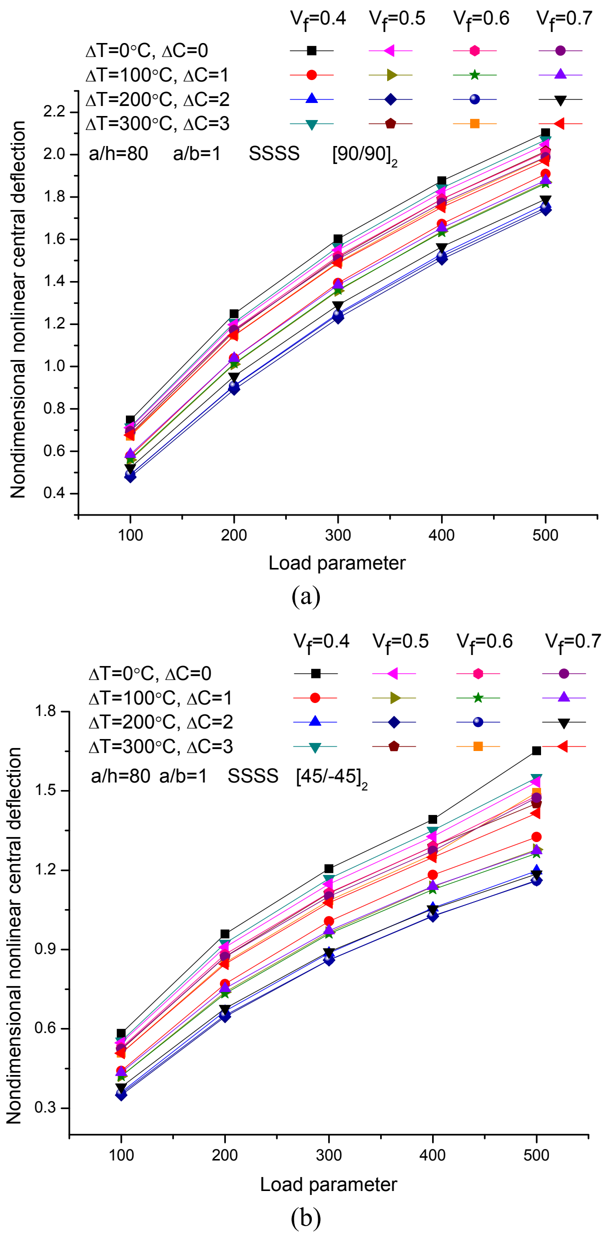


Fig. 10 Effect of volume fraction on the nonlinear centre deflection of laminated composite flat panel

laminated composite flat panel (a/h = 20) for four different aspect ratios (a/b = 1, 1.5, 2 and 2.5) under hygro-thermo-mechanical loading is presented in Fig. 7. It is observed that, the nondimensional nonlinear central deflections decrease with increase in aspect ratios. This is because of the fact that as the aspect ratio increases the load per unit area decreases; hence the responses are within the expected line. However, higher values of deflections are conceded at ΔT = 300°C, ΔC = 3% for each aspect ratios. This is due to the fact the stiffness of the structure abruptly reduced due to moisture absorption at higher temperature and the flexural strength of the laminated structure decreases, subsequently.

4.3.4 Effect of modular ratio

The flexural behaviour of laminated structures is predominantly affected due to its orthotropicity. Fig. 8 shows the effect of modular ratio on the nonlinear static behaviour of laminated composite flat panel under hygro-thermo-mechanical loading. For this investigation, anti-symmetric cross-ply/angle-ply clamped laminated square (a/b = 1) plate (a/h = 50) is considered for three different modular ratios (E_1/E_2 = 10, 20 and 40). It is observed from the figure that the panel is showing softening type of behaviour with increase in modular ratio. However, the deflections of anti-symmetric cross-ply laminations are lower than the anti-symmetric angle-ply laminated plates.

4.3.5 Effect of support condition

Fig. 9 presents the nondimensional central deflection of anti-symmetric cross-ply/angle-ply laminated square (a/b = 1) plates (a/h = 60) for four different support conditions (SSSS, CCCC, SCSC and HHHH) under hygro-thermo-mechanical loading. It is observed that the non-dimensional central deflections are lowest for clamped and highest for simply-supported case. This is due to the fact that as the number of constraints decreases, the stiffness of the structure decreases and the deflection value increases monotonically. It is interesting to note that, both types of laminated plates are showing softening type of behaviour under all support conditions except for the hinged case.

4.3.6 Effect of fiber volume fraction

The effect of fiber volume fraction on the nonlinear bending behaviour of simply supported thin (a/h = 80) anti-symmetric cross-ply/angle-ply laminated square (a/b = 1) plate is examined for four sets of hygrothermal loading and presented in Fig. 10. The results show that with increase in fiber volume fraction the nonlinear transverse central deflection parameter decreases. This is expected as the stiffness of the flat panel increase with increase in fiber volume fraction.

5. Conclusions

The nonlinear bending behaviour of laminated composite flat panels under hygro-thermo-mechanical loading have been analysed by taking the geometrical nonlinearity in Green-Lagrange sense in the framework of the HSDT mid plane kinematics. The formulation is unique in the sense that the effective material properties of the composite lamina are computed through a micromechanical model. In addition to that all the nonlinear higher order terms are considered in the present nonlinear model to capture realistic response. The system governing equations are obtained using variational principle and discretised using suitable FEM. The convergence and validation of the present model has been established. Finally the efficacy and applicability of the developed nonlinear model has been checked by illustrating few parametric studies. Based on this the following useful conclusions are drawn.

- Hygrothermal dependent thermal and mechanical properties greatly affect the flexural behaviour of laminated flat panel and in small strain and large deformation problems. Green-Lagrange type of nonlinearity is more practical in comparison to von Karman type analysis.
- The nonlinear bending response of laminated composite flat panel is influenced considerably with increase in hygrothermal load. The effect is predominant at higher temperature, particularly when the applied temperature is more than the glass transition temperature.
- The nonlinear transverse central deflection of laminated composite flat panel under hygro-thermo-mechanical loading decrease with increase in thickness ratio, aspect ratio, modular ratio and fiber volume fraction. Hence, suitable assortment of these parameters is to be done for optimal design of the laminated structures.
- It is interesting to note that as the number of restraints increases, the nonlinear central deflection decreases under hygro-thermo-mechanical loading.
- Nonlinearity due to moisture absorption at high temperature is severe for anti-symmetric angle-ply in comparison to symmetric laminations.

References

Baltacioglu, A.K., Civalek, O., Akgoz, B. and Demir, F. (2011), "Large deflection analysis of laminated composite plates resting on nonlinear elastic foundations by the method of discrete singular convolution", *Int. J. Press. Vessels Piping*", **88**(8-9), 290-300.

Chamis, C.C. (1987), "*Simplified Composite Micromechanics Equations for Mechanical, Thermal and Moisture Related Properties, Engineers Guide to Composite Materials*, ASM International; Materials Park, OH, USA.

Chamis, C.C. and Sinclair, J.H. (1982), "Durability/life of fibre composites in hygro-thermo-mechanical environments", *Proceedings of the Composite Materials: Testing and Design (6th Conference)*, ASTM, Phoenix, Arizona, May, Volume 787, pp. 498-512.

Cook, R.D., Malkus, D.S., Plesha, M.E. and Witt, R.J. (2009), *Concepts and Applications of Finite Element Analysis*, (4th Edition), John Wiley & Sons Pvt. Ltd., Singapore.

Hari Kishore, M.D.V., Singh, B.N. and Pandit, M.K. (2011), "Nonlinear static analysis of smart laminated composite plate", *Aerosp. Sci. Technol.*, **15**(3), 224-235.

Huang, X.L., Shen, H.S. and Zheng, J.J. (2004), "Nonlinear vibration and dynamic response of shear deformable laminated plates in hygrothermal environments", *Compos. Sci. Technol.*, **64**(10-11), 1419-1435.

Kumar, R. and Patil, H.S. (2013), "Hygrothermally induced nonlinear free vibration response of nonlinear elastically supported laminated composite plates with random system properties", *Frontiers Aerosp. Eng.*, **2**(2), 143-156.

Kundu, C.K., Maiti, D.K and Sinha, P.K. (2007), "Nonlinear finite element analysis of laminated composite doubly curved shells in hygrothermal environment", *J. Reinf. Plast. Comp.*, **26**(14), 1461.

Lal, A. Singh, B.N. and Anand, S. (2011), "Nonlinear bending response of laminated composite spherical shell panel with system randomness subjected to hygro-thermo-mechanical loading", *Int. J. Mech. Sci.*, **53**(10), 855-866.

Liu, C.F. and Huang, C.H. (1996), "Free vibration of composite laminated plates subjected to temperature

changes", *Comput. Struct.*, **60**(1), 95-101.
Lo, S.H., Zhen, W., Cheung, Y.K. and Wanji, C. (2010), "Hygrothermal effects on multilayered composite plates using a refined higher order theory", *Compos. Struct.*, **92**(3), 633-646.
Naidu, N.V.S. and Sinha, P.K. (2005), "Nonlinear finite element analysis of laminated composite shells in hygrothermal environments", *Compos. Struct.*, **69**(4), 387-395.
Panda, S.K. and Mahapatra, T.R. (2014), "Nonlinear finite element analysis of laminated composite spherical shell vibration under uniform thermal loading", *Meccanica*, **49**(1), 191-213.
Parhi, P.K., Bhattacharyya, S.K. and Sinha, P.K. (2001), "Hygrothermal effects on the dynamic behavior of multiple delaminated composite plates and shells", *J. Sound Vib.*, **248**(2), 195-214.
Patel, B.P., Ganapathi, M. and Makhecha, D.P. (2002), "Hygrothermal effects on the structural behaviour of thick composite laminates using higher-order theory", *Compos. Struct.*, **56**(1), 25-34.
Qatu, M.S. (2004), *Vibration of Laminated Shells and Plates*, Academic Press, Oxford, UK**.**
Reddy, J.N. (2004), *Mechanics of Laminated Composite: Plates and Shells-Theory and Analysis*, (2nd Edition), CRC press, Boca Raton, FL, USA.
Sai Ram, K.S. and Sinha, P.K. (1991), "Hygrothermal effects on the bending characteristics of laminated composite plates", *Comput. Struct.*, **40**(4), 1009-1015.
Sharma, A., Upadhyay, A.K. and Shukla, K.K. (2013), "Flexural response of doubly curved laminated composite shells", *Sci. China Phys. Mech. Astron.*, **56**(4), 812-817.
Shen, H.S. (2002), "Hygrothermal effects on the nonlinear bending of shear deformable laminated plates", *J. Eng. Mech.*, **128**(4), 493-496.
Shen, H.S, Zheng, J.J. and Huang, X.L. (2004), "The effects of hygrothermal conditions on the dynamic response of shear deformable laminated plates resting on elastic foundations", *J. Reinf. Plast. Comp.*, **23**(10), 1095.
Sundaramoorthy, R., David, W. and Murray, M. (1973), "Incremental finite element matrices", *J. Struct. Div.*, **99**(12), 2423-2438.
Szekrenyes, A. (2014), "Bending solution of third-order orthotropic Reddy plates with asymmetric interfacial crack", *Int. J. Solid. Struct.*, **51**(14), 2598-2619.
Upadhyay, P.C. and Lyons, J.S. (2000), "Effect of hygrothermal environment on the bending of PMC laminates under large deflection", *J. Reinf. Plast. Comp.*, **19**(6), 465.
Upadhyay, A.K., Pandey, R. and Shukla, K.K. (2010), "Nonlinear flexural response of laminated composite plates under hygro-thermo-mechanical loading", *Commun. Nonlinear Sci. Numer. Simul.*, **15**(9), 2634-2650.
Zenkour, A.M. (2012), "Hygrothermal effects on the bending of angle-ply composite plates using a sinusoidal theory", *Compos. Struct.*, **94**(12), 3685-3696
Zenkour, A.M., Alam, M.N.M. and Radwan, A.F. (2014), "Effects of hygrothermal conditions on cross-ply laminated plates resting on elastic foundations", *Arch. Civil Mech. Eng.*, **14**(1), 144-159.
Zhang, Y.X. and Kim, K.S. (2006), "Geometrically nonlinear analysis of laminated composite plates by two new displacement-based quadrilateral plate elements", *Compos. Struct.*, **72**(3), 301-310.
Zhang, Y.X. and Yang, C.H. (2006), "A family of simple and robust finite elements for linear and geometrically nonlinear analysis of laminated composite plates", *Compos. Struct.*, **75**(1-4), 545-552.

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Appendix

Individual terms of the matrix [A] as appeared in the Eq. (17)

$[\mathrm{A}]_{1\_1}=u_{,x}$, $[\mathrm{A}]_{1\_3}=v_{,x}$, $[\mathrm{A}]_{1\_5}=w_{,x}$, $[\mathrm{A}]_{2\_2}=u_{,y}$, $[\mathrm{A}]_{2\_4}=v_{,y}$, $[\mathrm{A}]_{2\_6}=w_{,y}$

$[\mathrm{A}]_{3\_1}=u_{,y}$, $[\mathrm{A}]_{3\_2}=u_{,x}$, $[\mathrm{A}]_{3\_3}=v_{,y}$, $[\mathrm{A}]_{3\_4}=v_{,x}$, $[\mathrm{A}]_{3\_5}=w_{,y}$, $[\mathrm{A}]_{3\_6}=w_{,x}$

$[\mathrm{A}]_{4\_1}=\phi_1$, $[\mathrm{A}]_{4\_3}=\phi_2$, $[\mathrm{A}]_{4\_22}=u_{,x}$, $[\mathrm{A}]_{4\_23}=v_{,x}$, $[\mathrm{A}]_{5\_2}=\phi_1$, $[\mathrm{A}]_{5\_4}=\phi_2$,

$[\mathrm{A}]_{5\_22}=u_{,y}$, $[\mathrm{A}]_{5\_23}=v_{,y}$, $[\mathrm{A}]_{6\_1}=\phi_{1,x}$, $[\mathrm{A}]_{6\_3}=\phi_{2,x}$, $[\mathrm{A}]_{6\_7}=u_{,x}$, $[\mathrm{A}]_{6\_9}=v_{,x}$, $[\mathrm{A}]_{7\_2}=\phi_{1,y}$

$[\mathrm{A}]_{7\_4}=\phi_{2,y}$, $[\mathrm{A}]_{7\_8}=u_{,y}$, $[\mathrm{A}]_{7\_10}=v_{,y}$, $[\mathrm{A}]_{8\_1}=\phi_{1,y}$, $[\mathrm{A}]_{8\_2}=\phi_{1,x}$, $[\mathrm{A}]_{8\_3}=\phi_{2,y}$, $[\mathrm{A}]_{8\_4}=\phi_{2,x}$,

$[\mathrm{A}]_{8\_7}=u_{,y}$, $[\mathrm{A}]_{8\_8}=u_{,x}$, $[\mathrm{A}]_{8\_9}=v_{,y}$, $[\mathrm{A}]_{8\_10}=v_{,x}$, $[\mathrm{A}]_{9\_1}=2\psi_1$, $[\mathrm{A}]_{9\_3}=2\psi_2$,

$[\mathrm{A}]_{9\_7}=\phi_1$, $[\mathrm{A}]_{9\_9}=\phi_2$, $[\mathrm{A}]_{9\_22}=\phi_{1,x}$, $[\mathrm{A}]_{9\_23}=\phi_{2,x}$, $[\mathrm{A}]_{9\_24}=2u_{,x}$, $[\mathrm{A}]_{9\_25}=2v_{,x}$

$[\mathrm{A}]_{10\_2}=2\psi_1$, $[\mathrm{A}]_{10\_4}=2\psi_2$, $[\mathrm{A}]_{10\_8}=\phi_1$, $[\mathrm{A}]_{10\_10}=\phi_2$, $[\mathrm{A}]_{10\_22}=\phi_{1,y}$,

$[\mathrm{A}]_{10\_24}=2u_{,y}$, $[\mathrm{A}]_{10\_25}=2v_{,y}$, $[\mathrm{A}]_{11\_1}=\psi_{1,x}$, $[\mathrm{A}]_{11\_3}=\psi_{2,x}$, $[\mathrm{A}]_{11\_7}=\phi_{1,x}$, $[\mathrm{A}]_{11\_9}=\phi_{2,x}$

$[\mathrm{A}]_{11\_11}=u_{,x}$, $[\mathrm{A}]_{11\_13}=v_{,x}$, $[\mathrm{A}]_{12\_2}=\psi_{1,y}$, $[\mathrm{A}]_{12\_4}=\psi_{2,y}$, $[\mathrm{A}]_{12\_8}=\phi_{1,y}$, $[\mathrm{A}]_{12\_10}=\phi_{2,y}$

$[\mathrm{A}]_{12\_12}=u_{,y}$, $[\mathrm{A}]_{12\_14}=v_{,y}$, $[\mathrm{A}]_{13\_1}=\psi_{1,y}$, $[\mathrm{A}]_{13\_2}=\psi_{1,x}$, $[\mathrm{A}]_{13\_3}=\psi_{2,y}$, $[\mathrm{A}]_{13\_4}=\psi_{2,x}$

$[\mathrm{A}]_{13\_7}=\phi_{1,y}$, $[\mathrm{A}]_{13\_8}=\phi_{1,x}$, $[\mathrm{A}]_{13\_9}=\phi_{2,y}$, $[\mathrm{A}]_{13\_10}=\phi_{2,x}$, $[\mathrm{A}]_{13\_11}=u_{,y}$, $[\mathrm{A}]_{13\_12}=u_{,x}$

$[\mathrm{A}]_{13\_13}=v_{,y}$, $[\mathrm{A}]_{13\_14}=v_{,x}$, $[\mathrm{A}]_{14\_1}=3\theta_1$, $[\mathrm{A}]_{14\_3}=3\theta_2$, $[\mathrm{A}]_{14\_7}=2\psi_1$, $[\mathrm{A}]_{14\_9}=2\psi_2$

$[\mathrm{A}]_{14\_11}=\phi_1$, $[\mathrm{A}]_{14\_13}=\phi_2$, $[\mathrm{A}]_{14\_22}=\psi_{1,x}$, $[\mathrm{A}]_{14\_23}=\psi_{2,x}$, $[\mathrm{A}]_{14\_24}=2\phi_{1,x}$, $[\mathrm{A}]_{14\_25}=2\phi_{2,x}$

$[\mathrm{A}]_{14\_26}=3u_{,x}$, $[\mathrm{A}]_{14\_27}=3v_{,x}$, $[\mathrm{A}]_{15\_2}=3\theta_1$, $[\mathrm{A}]_{15\_4}=3\theta_2$, $[\mathrm{A}]_{15\_8}=2\psi_1$, $[\mathrm{A}]_{15\_10}=2\psi_2$,

$[\mathrm{A}]_{15\_12}=\phi_1$, $[\mathrm{A}]_{15\_14}=\phi_2$, $[\mathrm{A}]_{15\_22}=\psi_{1,y}$, $[\mathrm{A}]_{15\_23}=\psi_{2,y}$, $[\mathrm{A}]_{15\_24}=2\phi_{1,y}$, $[\mathrm{A}]_{15\_25}=2\phi_{2,y}$

$[\mathrm{A}]_{15\_26}=3u_{,y}$, $[\mathrm{A}]_{15\_27}=3v_{,y}$, $[\mathrm{A}]_{16\_1}=\theta_{1,x}$, $[\mathrm{A}]_{16\_3}=\theta_{2,x}$, $[\mathrm{A}]_{16\_7}=\psi_{1,x}$, $[\mathrm{A}]_{16\_9}=\psi_{2,x}$

$[\mathrm{A}]_{16\_11}=\phi_{1,x}$, $[\mathrm{A}]_{16\_13}=\phi_{2,x}$, $[\mathrm{A}]_{16\_15}=u_{,x}$, $[\mathrm{A}]_{16\_17}=v_{,x}$, $[\mathrm{A}]_{17\_2}=\theta_{1,y}$, $[\mathrm{A}]_{17\_4}=\theta_{2,y}$,

$[\mathrm{A}]_{17\_8}=\psi_{1,y}$, $[\mathrm{A}]_{17\_10}=\psi_{2,y}$, $[\mathrm{A}]_{17\_12}=\phi_{1,y}$, $[\mathrm{A}]_{17\_14}=\phi_{2,y}$, $[\mathrm{A}]_{17\_16}=u_{,y}$, $[\mathrm{A}]_{17\_18}=v_{,y}$,

$[\mathrm{A}]_{18\_1}=\theta_{1,y}$, $[\mathrm{A}]_{18\_2}=\theta_{1,x}$, $[\mathrm{A}]_{18\_3}=\theta_{2,y}$, $[\mathrm{A}]_{18\_4}=\theta_{2,x}$, $[\mathrm{A}]_{18\_7}=\psi_{1,y}$, $[\mathrm{A}]_{18\_8}=\psi_{1,x}$,

$[\mathrm{A}]_{18\_9}=\psi_{2,y}$, $[\mathrm{A}]_{18\_10}=\psi_{2,x}$, $[\mathrm{A}]_{18\_11}=\phi_{1,y}$, $[\mathrm{A}]_{18\_12}=\phi_{1,x}$, $[\mathrm{A}]_{18\_13}=\phi_{2,y}$, $[\mathrm{A}]_{18\_14}=\phi_{2,x}$

$[\mathrm{A}]_{18\_15}=u_{,y}$, $[\mathrm{A}]_{18\_16}=u_{,x}$, $[\mathrm{A}]_{18\_17}=v_{,y}$, $[\mathrm{A}]_{18\_18}=v_{,x}$, $[\mathrm{A}]_{19\_7}=3\theta_1$, $[\mathrm{A}]_{19\_9}=3\theta_2$

$[\mathrm{A}]_{19\_11}=2\psi_1$, $[\mathrm{A}]_{19\_13}=2\psi_2$, $[\mathrm{A}]_{19\_15}=\phi_1$, $[\mathrm{A}]_{19\_17}=\phi_2$, $[\mathrm{A}]_{19\_22}=\theta_{1,x}$, $[\mathrm{A}]_{19\_23}=\theta_{2,x}$,
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$[\mathrm{A}]_{24\_17}=2\psi_2$, $[\mathrm{A}]_{24\_24}=2\theta_{1,x}$, $[\mathrm{A}]_{24\_25}=2\theta_{2,x}$, $[\mathrm{A}]_{24\_26}=3\psi_{1,x}$, $[\mathrm{A}]_{24\_27}=3\psi_{2,x}$,
$[\mathrm{A}]_{25\_12}=3\theta_1$, $[\mathrm{A}]_{25\_14}=3\theta_2$, $[\mathrm{A}]_{25\_16}=2\psi_1$, $[\mathrm{A}]_{25\_18}=2\psi_2$, $[\mathrm{A}]_{25\_24}=2\theta_{1,y}$,
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$[\mathrm{A}]_{28\_11}=\theta_{1,y}$, $[\mathrm{A}]_{28\_12}=\theta_{1,x}$, $[\mathrm{A}]_{28\_13}=\theta_{2,y}$, $[\mathrm{A}]_{28\_14}=\theta_{2,x}$, $[\mathrm{A}]_{28\_15}=\psi_{1,y}$,
$[\mathrm{A}]_{28\_16}=\psi_{1,x}$, $[\mathrm{A}]_{28\_17}=\psi_{2,y}$, $[\mathrm{A}]_{28\_18}=\psi_{2,x}$, $[\mathrm{A}]_{29\_15}=3\theta_1$, $[\mathrm{A}]_{29\_17}=3\theta_2$,
$[\mathrm{A}]_{29\_26}=3\theta_{1,x}$, $[\mathrm{A}]_{29\_27}=3\theta_{2,x}$, $[\mathrm{A}]_{30\_16}=3\theta_1$, $[\mathrm{A}]_{30\_18}=3\theta_2$, $[\mathrm{A}]_{30\_26}=3\theta_{1,y}$,
$[\mathrm{A}]_{30\_27}=3\theta_{2,y}$, $[\mathrm{A}]_{31\_15}=\theta_{1,x}$, $[\mathrm{A}]_{31\_17}=\theta_{2,x}$, $[\mathrm{A}]_{32\_16}=\theta_{1,y}$, $[\mathrm{A}]_{32\_18}=\theta_{2,y}$,
$[\mathrm{A}]_{33\_15}=\theta_{1,y}$, $[\mathrm{A}]_{33\_16}=\theta_{1,x}$, $[\mathrm{A}]_{33\_17}=\theta_{2,y}$, $[\mathrm{A}]_{33\_18}=\theta_{2,x}$.

Individual terms of the [G] matrix

$[\mathrm{G}]_{1\_1}=\frac{\partial}{\partial x}$, $[\mathrm{G}]_{2\_1}=\frac{\partial}{\partial y}$, $[\mathrm{G}]_{3\_2}=\frac{\partial}{\partial x}$, $[\mathrm{G}]_{4\_2}=\frac{\partial}{\partial y}$, $[\mathrm{G}]_{5\_3}=\frac{\partial}{\partial x}$, $[\mathrm{G}]_{6\_3}=\frac{\partial}{\partial y}$, $[\mathrm{G}]_{7\_4}=\frac{\partial}{\partial x}$,
$[\mathrm{G}]_{8\_4}=\frac{\partial}{\partial y}$, $[\mathrm{G}]_{9\_5}=\frac{\partial}{\partial x}$, $[\mathrm{G}]_{10\_5}=\frac{\partial}{\partial y}$, $[\mathrm{G}]_{11\_6}=\frac{\partial}{\partial x}$, $[\mathrm{G}]_{12\_6}=\frac{\partial}{\partial y}$, $[\mathrm{G}]_{13\_7}=\frac{\partial}{\partial x}$

$[\mathrm{G}]_{14\_7}=\dfrac{\partial}{\partial y}$, $[\mathrm{G}]_{15\_8}=\dfrac{\partial}{\partial x}$, $[\mathrm{G}]_{16\_8}=\dfrac{\partial}{\partial y}$, $[\mathrm{G}]_{18\_9}=\dfrac{\partial}{\partial y}$, $[\mathrm{G}]_{19\_1}=1$, $[\mathrm{G}]_{20\_2}=1$,

$[\mathrm{G}]_{21\_3}=1$, $[\mathrm{G}]_{22\_4}=1$, $[\mathrm{G}]_{23\_5}=1$, $[\mathrm{G}]_{24\_6}=1$, $[\mathrm{G}]_{25\_7}=1$, $[\mathrm{G}]_{26\_8}=1$, $[\mathrm{G}]_{27\_9}=1$.