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Vibration analysis of a pre-stressed laminated composite curved beam

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Abstract. In this study, natural frequency analysis of a large deflected cantilever laminated composite beam fixed at both ends, which forms the case of a pre-stressed curved beam, is investigated. The laminated beam is considered to have symmetric and asymmetric lay-ups and the effective flexural modulus of the beam is used in the analysis. In order to obtain the pre-stressed composite curved beam case, an external vertical concentrated load is applied at the free end of a cantilever laminated composite beam and then the loading point of the deflected beam is fixed. The non-linear deflection curve of the flexible beam undergoing large deflection is obtained by the Reversion Method. The curved laminated composite beam is modeled by using the Finite Element Method with a straight-beam element approach. The effects of orientation angle and vertical load on the natural frequency parameter for the first four modes are examined and the results obtained are given in graphics. It has been found that the effect of the load parameter, which forms the curved laminated beam, on the natural frequency parameter, almost disappears after a certain value of the load parameter. This certain value differs for each laminated curved beam and each vibration mode.

Keywords: large deflection; laminated curved beam; vibration; non-linear deflection; finite element method

1. Introduction

Laminated composite beams and plates have been widely used in naval, aircraft, lightweight structures, aerospace exploration, solar sail etc., where high strength and high stiffness to weight ratios are desired. In some instances, composite beams can be assembled under pre-stressed condition depending on the initial displacements without exceeding their elastic limits. For example, aircraft, civil and submarine structures are often designed to work under postbuckling conditions, because of weight considerations. Moreover, the beam can be used to model components of plated structures. Large deflection problem of composite beams evinces itself in the aforementioned areas and similar examples. The large deflection in these areas is defined as a non-linear elastic problem.

Although not in a large number, a number of studies are reported in literature dealing with the large deflection analysis of isotropic and laminated composite beams.

The large deflection of a cantilever beam loading consisting of a vertical load at the free end

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has been investigated with elliptic integral solution by Bisshopp and Drucker (1945). Wang (1968, 1969) has proposed an analysis for the non-linear bending of cantilever and simply supported beams of constant cross-section carrying uniformly concentrated and distributed loads. Holden (1972) has presented a numerical solution to the three problems of uniform beams of finite deflections using the Euler-Bernoulli theory. A matrix displacement approach has been developed for the analysis of beams and frames with large displacement by Yang (1973). Schmidt (1978) has developed a finite element formulation for determining the finite deflection of thin bars and applied it to the problem of a cantilever beam loaded by a point force at the free end. Rao and Rao (1986) have studied the large deflection of a cantilever beam subjected to a concentrated tip load which rotates in relation to the tip rotation of the beam. Ang *et al*. (1993) have developed a new and simple numerical method for obtaining the beam deflection curve of very flexible beams undergoing large deflections and reviewed two general methods called the elliptic integrals and reversion. Belendez *et al*. (2002, 2003) have experimentally and numerically analyzed the classical problem of the deflection of a cantilever beam, in case of both large and small deflections under the action of an external vertical concentrated load at the free end. Large deflection of cantilever beams made of Ludwick type material subjected to combined loading, consisting of a uniformly distributed load and one vertical load at the free end has been investigated by Lee (2002). Addessi *et al*. (2005) have investigated the natural frequencies and mode shapes of planar shear undeformable beams around their curved pre-stressed post-buckling configurations using the finite element and semi-analytical methods. Pulngern *et al*. (2005) have studied large amplitude vibrations of horizontal variable arc- length beams, considering the effect of large initial static sag deflections due to self-weight. In their paper, analytical and experimental studies have been conducted. Holland *et al*. (2008) have described the behavior of a slender, tapered, cantilever beam loaded through a cable attached to its free end and have computed large static deflections together with natural frequencies and mode shapes for small-amplitude vibrations around equilibrium. In their paper, the shooting method with an experimental study was used. A new integral approach to solve the large deflection of cantilever beam has been proposed by Chen (2010). The proposed method (2010) uses the integral of the bending moment and thus, can be applied to arbitrary loads and variable beam properties such as the cross sectional area or elasticity of the material. Nallathambi *et al*. (2010) have described a method to analyze the large deflections of curved prismatic cantilever beams with uniform curvature subjected to a follower load at the tip and fourth order Runge–Kutta method along with one parameter reverse shooting method was applied to the numerical solution to the problem. Ozturk (2011) has studied the in-plane free vibration analysis of a fixed– fixed pre-stressed curved beam obtained from a large deflected cantilever beam using the reversion and the finite element methods. Bayat *et al*. (2013) have used Hamiltonian Approach to analysis the nonlinear free vibration of simply-supported and for the clamped-clamped Euler-Bernoulli beams fixed at one end subjected to the axial loads.

Vibration analysis of laminated composite beams has been subject of intense research from past to present. A large number of studies related to vibration problems have been published. Since it is impossible to mention all the publications, a few exemplary studies can be listed as follows: Murty and Shimpi (1974), have derived the governing equations in the form of simultaneous ordinary differential equations for natural vibration analysis of isotropic laminated beams. Rikards *et al*. (1993), have investigated damped vibration of laminated composites by using the finite element analysis. Khdeir and Reddy (1994) have developed analytical solutions of refined beam theories for the free vibration behavior of cross-ply rectangular beams with arbitrary boundary conditions. Rao *et al*. (2001) have developed an analytical method for evaluating the natural frequencies of composite and sandwich beams using higher-order mixed theory. A higher order shear deformation beam theory has been developed by Hadji *et al*. (2014) for static and free vibration analysis of functionally graded beams and different higher order shear deformation theories and classical beam theories have been use.d in their analyses.

As seen from the aforementioned references, existing studies on the large deflection of a beam problem are related to isotropic beams. Because of the importance of composite beams, although not too many, the large deflection behavior of composite beams has been the subject of many investigations. Chen and Sun (1985) have investigated the dynamic large deflection response of composite laminates subjected to impact loading using the finite element method. Effects of large deflection on the static and dynamic behaviors of unsymmetric cross-ply laminates in cylindrical bending have been investigated by Sun and Chin (1988) using von Karman large deflection theory. Bauchau and Hong (1988) have presented the nonlinear analysis of naturally curved and twisted beams undergoing arbitrarily large deflections and rotations using the finite element method. Stemple and Lee (1989) have developed a finite element formulation to take into account the warping effect of composite beams undergoing large deflection using the finite element method. Kant and Kommineni (1994) have studied the large amplitude free vibration analysis of cross-ply composite and sandwich laminates with a refined theory and C^0 finite elements. Jeon *et al*. (1995) have investigated the static and dynamic behavior of composite box beams using a large deflection beam theory. The nonlinear finite element equations of motion were obtained from Hamilton's principle and solved iteratively by the Newton-Raphson technique. The effects of hygrothermal conditions on the large deflection behavior of fiber-reinforced polymer matrix composite laminates have been studied by Upadhyay and Lyons (2000). Zhang *et al*. (2003) have developed a B-spline finite strip model to simulate large deflection and failure behaviour of laminated composite plates subjected to transverse loading. Agarwal *et al*. (2006) have studied the large deformation effects on static and dynamic responses in isotropic, composite and functionally graded material beams using the existing statically exact beam finite element based on the first order shear deformation theory. Kien (2013) has investigated the large displacement response of tapered cantilever beams made of axially functionally graded material using the finite element method.

As can be seen from existing literature, many methods and techniques have been used for the large deflection isotropic and laminated composite beams analysis as presented in Table 1. Furthermore, a large number of studies related to vibration analysis of laminated composite curved beams have been published. For a detailed literature survey, the readers can refer to the survey papers (Tseng *et al*. 2000) and (Hajianmaleki and Qatu 2013).

This study presents the in-plane free vibration analysis of a fixed-fixed pre-stressed laminated composite curved beam obtained from a large deflected cantilever laminated composite beam. The laminated beam is considered to have symmetric and asymmetric lay-ups and the effective flexural modulus of the beam is used in the analysis. In order to obtain a pre-stressed laminated composite curved beam, an external vertical concentrated load at the free end of the cantilever laminated composite beam is applied, then the loading point of the deflected beam is fixed. For each laminated beam having a different orientation angle, the maximum load is taken as approximately 75% of the maximum stress of the composite materials according to Tsai-Hill Failure Criterion (Jones 1999), since pre-stressed composite laminated curved beams will be loaded by additional forces depending on the operating conditions. A review of literature so far shows that there have not been any published papers on this study. There is one similar study (Ozturk 2011) belonging to the author in literature and this study (Ozturk 2011) is concerned with the isotropic pre-stressed curved beam. The non-linear deflection curve of the flexible beam undergoing large deflection is

Table 1 Historical representation of references for methods used in the large deflection isotropic and laminated composite beams analysis

Elliptic integrals method: Bisshopp and Drucker 1945 Rao and Rao 1986 Ang *et al*. 1993 Belendez *et al*. 2002	**Exact solution**: Wang 1969	**Newton-Raphson method:** Wang 1969 Bauchau and Hong 1988 Jeon *et al*. 1995
Shooting method: Holden 1972 Nallathambi *et al*. 2010	**Runge-Kutta method:** Holden 1972 Lee 2002 Nallathambi *et al*. 2010	**Matrix displacement approach:** Yang 1973
Finite element method: Schmidt 1978 Chen and Sun 1985 Bauchau and Hong 1988 Stemple and Lee 1989 Kant and Kommineni 1994 Jeon *et al*. 1995 Addessi *et al*. 2005 Pulngern *et al*. 2005 Agarwal *et al*. 2006 Kien 2013	**Von Karman large deflection theory:** Sun and Chin 1988 Kant and Kommineni1994 Upadhyay and Lyons 2000	**Reversion method:** Ang *et al*. 1993 Ozturk 2011
Experimental: Belendez *et al*. 2002, 2003 Pulngern *et al*. 2005	**Finite strip method:** Zhang 2003	**ANSYS and MSC/NASTRAN programs:** Belendez *et al*. 2003 Ozturk 2011
Semi-analytical method: Addessi *et al*. 2005	**Integral approach:** Chen 2010	**Hamiltonian Approach:** Bayat *et al*. 2013

obtained by the reversion method. The curved laminated composite beam is modeled by using the Finite Element Method with a straight- beam element approach. The effects of orientation angle and vertical load on the natural frequency parameter for the first four modes are examined and the results obtained are given in graphics.

2. Theoretical analysis

An external vertical concentrated load is applied to the cantilever laminated composite beam at its free end as seen in Fig. 1(a), in order to obtain a fixed-fixed pre-stressed curved laminated composite beam from a cantilever beam, Fig. 1(b). The effective flexural modulus of the laminated beam is used in the analysis and the equation of the elastic curve, $z(x)$, is obtained from large deflection analysis. Moreover, it is assumed that the deflection due to the beam's weight is zero.

2.1 The effective flexural moduli of the laminated composite beam

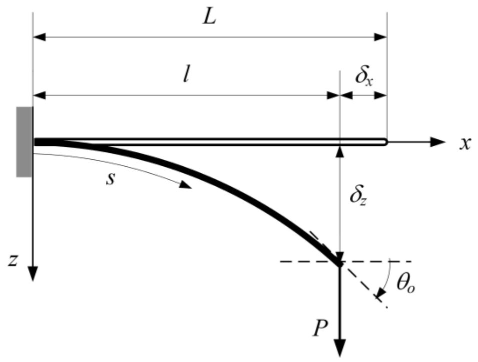
The beam is considered to have the bending of symmetrically laminated beams according to the classical laminated theory (CLT). For symmetric laminates, the equations for bending deflection are uncoupled from those of the longitudinal displacements. The classical laminated theory constitutive equation for symmetric laminates, in the absence of in-plane forces, is given by (Reddy 1997)

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix} \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \tag{1}$$

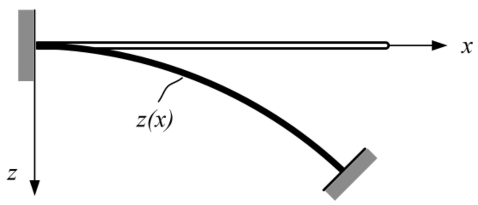
for symmetric laminates where M_x, M_y are the bending moments per unit length and M_{xy} is the twisting moment per unit length, κ_x, κ_y, κ_{xy} are the midplane curvatures. As shown in Fig. 2, the laminates have the same thickness and x axis is the axis of symmetry. **D** is the bending stiffness matrix and its elements D_{ij} are given by

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} \overline{Q}_{ij}^{k} (z_{k+1}^3 - z_k^3) \qquad i, j = 1, 2, 6 \tag{2}$$

where N is the number of laminates, z is the coordinate variable of the k^{th} laminate in the cross section, $\overline{Q}_{ij}^{k}$ are the stiffnesses, which are given by equation (A1) in the Appendix. It should be noted that, in case of antisymmetric lay-ups, $D_{16} = D_{26} = 0$. (Ozturk and Sabuncu 2005, Karaagac *et al*. 2013)



(a)



(b)

Fig. 1 (a) Large deflection of a laminated cantilever beam; (b) A pre-stressed laminated curved beam obtained from large deflection of a cantilever with fixed-fixed end conditions

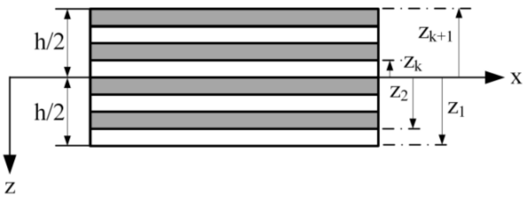


Fig. 2 Layer numbering used for a typical laminated beam

By taking the inverse of Eq. (1)

$$\begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{21}^* & D_{22}^* & D_{26}^* \\ D_{61}^* & D_{62}^* & D_{66}^* \end{bmatrix} \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{3}$$

is obtained, where $D_{ij}^* = D_{ij}^{-1}$

Eq. (3) is used to define the effective flexural moduli in terms of the bending compliance matrix as follows: First applying $M_x \neq 0$, $M_y = 0$, $M_{xy} = 0$ and then substituting in Eq. (3) gives

$$\kappa_x = D_{11}^* M_x \tag{4}$$

by using Eq. (4) along with known moment $M(x)$ expressions

$$M_y(x) = h M_x \tag{5}$$

$$M_y(x) = E_x I_y \kappa_x \tag{6}$$

and taking $I_y = hb^3 / 12$ for rectangular cross section, the effective flexural longitudinal modulus is

$$E_{ef} = \frac{12}{b^3 D_{11}^*} \tag{7}$$

2.2 Large deflection analysis

The curvature of the deflection curve of a beam under loading at any point depends only on the magnitude of the bending moment at that point under the assumption that the material of the beam remains linearly elastic. This relationship is (Ang *et al*. 1993, Silva 2006).

$$\frac{1}{\rho} = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{M(s)}{EI} \tag{8}$$

where $1/\rho$ is the curvature of the beam. Angle θ may be related to Cartesian coordinates of the deflection curve, yielding

$$\theta = \arctan\frac{\mathrm{d}z}{\mathrm{d}x} \Rightarrow \frac{1}{\rho} = \frac{\mathrm{d}\theta}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\arctan\frac{\mathrm{d}z}{\mathrm{d}x}\right)\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\dfrac{\mathrm{d}^2 z}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]^{3/2}} \tag{9}$$

The relation between the bending moment M and the cartesian coordinates (Fig. 1) of the deflection curve is defined by the differential equation for the linear elastic plane bending (Silva 2006)

$$\frac{\dfrac{\mathrm{d}^2 z}{\mathrm{d}x^2}}{\left[1+\left(\dfrac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]^{3/2}} = \frac{M(x)}{EI} \tag{10}$$

Eq. (10) is used in the large deflection analysis and is quite complex. As mentioned above, there are several methods given in literature for the solution of this equation; exact solution (Wang 1969), elliptic integrals (Bisshopp and Drucker 1945, Rao and Rao 1986, Ang *et al*. 1993, Belendez *et al*. 2002), the Newton-Raphson method (Wang 1969, Bauchau and Hong 1988, Jeon *et al*. 1995), the reversion method (Ang *et al*. 1993, Ozturk 2011), the Runge-Kutta method (Holden 1972, Lee 2002, Nallathambi *et al*. 2010) and the integral approach (Chen 2010) etc. In Ref. (Ang *et al*. 1993), the large elastic curve function, $z(x)$, has been obtained from the solution of Eq. (10) by using the reversion method and it is given as

$$\begin{aligned} z(x) = {} & \frac{P}{2EI}\left(-\frac{x^3}{3}+lx^2\right)+\frac{1}{2}\left(\frac{P}{2EI}\right)^3\left(-\frac{x^7}{7}+lx^6-\frac{12}{5}l^2x^5+2l^3x^4\right)+ \\ & \frac{3}{8}\left(\frac{P}{2EI}\right)^5\left(-\frac{x^{11}}{11}+lx^{10}-\frac{40}{9}l^2x^9+10l^3x^8-\frac{80}{7}l^4x^7+\frac{16}{3}l^5x^6\right)+\dots \end{aligned} \tag{11}$$

On the other hand, the projected beam length l is still an unknown. For obtaining the curved length, the projected length l can be calculated from the knowledge of the beam length L using the following formula (Ang *et al*. 1993)

$$L = \int_0^l \sqrt{1+\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{12}$$

Where the term $\mathrm{d}z/\mathrm{d}x$ in Eq. (12) is derived from Eq. (11). It is noted that Eq. (12) has only one unknown l and can be numerically solved by using an iterative approach starting with an initial estimate of l close to L. After substituting l in Eq. (11), a function representing the elastic curve can be obtained (Ozturk 2011).

2.3 The finite element model

A finite element model is developed to represent the laminated composite pre-stressed curved

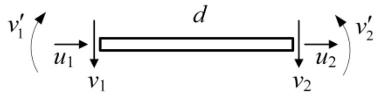


Fig. 3 Finite element model

beam element with straight-beam elements approach (Thomas and Wilson 1973, Yang *et al*. 1991, Ozturk 2011). The use of straight beam element approach is convenient since the radius of the curvature of the pre-stressed curved beam is large. The local coordinates of the straight beam element are transformed to global coordinates by using its angle of rotation obtained from the elastic curve function equation.

As shown in Fig. 3, an elemental finite element has six (u_1, v_1, v'_1, u_2, v_2, v'_2) degrees of freedom. They are called bending displacements (v_1, v_2), axial displacements (u_1, u_2) and slopes (v'_1, v'_2), where prime ($'$) denotes differentiation with respect to the axial coordinate x. The displacement $v(\bar{x})$ due to bending is approximated by cubic polynomials, while the axial displacement $u(\bar{x})$ can be expressed by linear functions of the beam segment and can be written as (Ozturk 2011)

$$v(\bar{x}) = a_1 + a_2\bar{x} + a_3\bar{x}^2 + a_4\bar{x}^3\,, \qquad u(\bar{x}) = b_1 + b_2\bar{x} \tag{13}$$

The generalized displacement vector with respect to local reference coordinates can be expressed as

$$\bar{\mathbf{q}}_{\mathbf{e}} = \begin{bmatrix} u_1 & v_1 & v'_1 & u_2 & v_2 & v'_2 \end{bmatrix} \tag{14}$$

As seen in Fig. 4, the relation between local and global reference coordinates can be written as

$$\bar{\mathbf{q}}_{\mathbf{e}} = \mathbf{T}\mathbf{q}_{\mathbf{e}} \tag{15}$$

where T is the transformation matrix, which is given by Eq. (A2) in the Appendix.

Energy equations should be expressed for the Euler beam element with an elemental length d. The elastic potential energy U_e is given as

$$U_e = \frac{1}{2}\int_0^d EI\left(\frac{\mathrm{d}^2 v}{\mathrm{d}\bar{x}^2}\right)^2 \mathrm{d}\bar{x} + \frac{1}{2}\int_0^d EA\left(\frac{\mathrm{d}u}{\mathrm{d}\bar{x}}\right)^2 \mathrm{d}\bar{x} \tag{16}$$

Eq. (16) can be written in matrix form

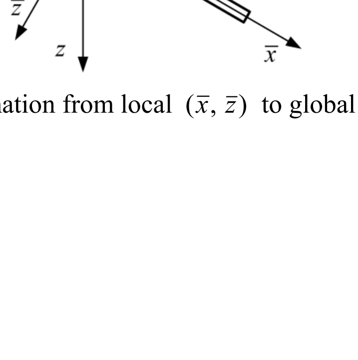


Fig. 4 Transformation from local $(\bar{x}, \bar{z})$ to global (x, z) coordinates

$$U_e = \frac{1}{2}\bar{\mathbf{q}}_{\mathbf{e}}\bar{\mathbf{k}}_{\mathbf{e}}\bar{\mathbf{q}}_{\mathbf{e}} \tag{17}$$

The kinetic energy of the beam element is

$$T_e = \frac{1}{2}\int_0^d \rho A\left(\dot{u}^2 + \dot{v}^2\right)\mathrm{d}\bar{x} \tag{18}$$

Eq. (18) can be written in matrix form as

$$T_e = \frac{1}{2}\bar{\mathbf{q}}_{\mathbf{e}}\bar{\mathbf{m}}_{\mathbf{e}}\bar{\mathbf{q}}_{\mathbf{e}} \tag{19}$$

The laminated composite curved beam, which is obtained from the large deflection theory has initial stress because of the vertical load. The initial stress for the finite element model is called the initial stress matrix (Yang 1973, Ozturk 2011), and obtained from Eq. (20) as

$$V_e = \frac{1}{2}S\int_0^d \left(\frac{\mathrm{d}v}{\mathrm{d}\bar{x}}\right)^2 \mathrm{d}\bar{x} \tag{20}$$

where S is an initial axial force acting in the local $\bar{x}$ direction. As seen in Fig. 5, the axial force S for each initial stress matrix is obtained as

$$S = F_x \cos\theta + F_z \sin\theta \tag{21}$$

where F_x and F_z are the nodal forces. These forces can be obtained from statics so that the vertical load P, at $x = L$ is equal to F_z. (Yang 1973, Ozturk 2011)

If Eq. (20) is written in matrix form

$$V_e = \frac{1}{2}\bar{\mathbf{q}}_{\mathbf{e}}\bar{\mathbf{s}}_{\mathbf{e}}\bar{\mathbf{q}}_{\mathbf{e}} \tag{22}$$

In this way, the elastic stiffness matrix $\bar{\mathbf{k}}_e$, mass matrix $\bar{\mathbf{m}}_e$ and initial stress matrix $\bar{\mathbf{s}}_e$, are obtained for a finite element. If these matrices are transformed in terms of reference coordinates,

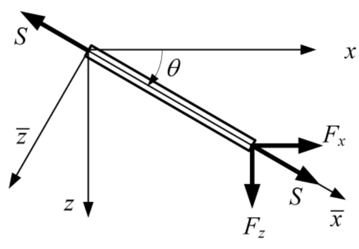


Fig. 5 Axial force in a beam element

they are obtained as

$$\mathbf{k}_e = \mathbf{T}^T \overline{\mathbf{k}}_e \mathbf{T}, \qquad \mathbf{m}_e = \mathbf{T}^T \overline{\mathbf{m}}_e \mathbf{T}, \quad \text{and} \quad \mathbf{s}_e = \mathbf{T}^T \overline{\mathbf{s}}_e \mathbf{T} \tag{23}$$

The dynamic response of a beam for a conservative system can be formulated by means of Lagrange's equation of motion in the matrix form as

$$\mathbf{M}_e \ddot{\mathbf{q}}_e + \left(\mathbf{K}_e + \mathbf{S}_e\right)\mathbf{q}_e = 0 \tag{24}$$

3. Results and discussion

In this study, in-plane free vibration of a fixed-fixed pre-stressed laminated composite curved beam, which is obtained from a large deflected cantilever laminated composite beam is investigated. The effective flexural modulus, Eq. (7) of the laminated beam is used in the analysis such as $E = E_{ef}$. The angle-ply laminated beams have six different configurations, which for simplicity, are denoted by C1, C2, C3, C4, C5 and C6. The explanation of these laminated beams is as follows

C1 = [0 0 0 0], C2 = [90 90 90 90], C3 = [0 90 90 0],
C4 = [90 0 0 90], C5 = [0 45 -45 0], C6 = [0 60 -60 0]

where C3 and C4 are called cross-ply laminates, C5 and C6 are called angle-ply laminates. The composite material properties and the beam geometry are given in Table 2.

In order to derive the elastic curve function equation z(x) obtained from the reversion method which has only one unknown denoted as l, Eq. (11) and Eq. (12) can be solved numerically together using the computer code developed by MATLAB software. An iterative approach starting with an initial estimate of l close to L is used for the solution and the error in the iterative approach

Table 2 Material properties and geometry of the beams

The properties of the glass-fiber epoxy unidirectional ply at 60% fiber volume fraction (Gay *et al*. 2003)		
Property	Symbol	Quantity
Longitudinal elastic modulus (GPa)	E_1	45
Transverse elastic modulus (GPa)	E_2	12
Longitudinal tensile fracture (MPa)	X_t	1250
Transverse tensile fracture (MPa)	Y_t	35
In plane shear strength (MPa)	S_i	63
Poisson's ratio	ν_c	0.3
Mass density (kg/m^3)	ρ_c	2080
Geometric properties of the beam		
Cross-section	h	2 mm
	b	30 mm
Beam length	750 mm	

Table 3 Convergence analysis of the straight beam element used in analysis of the composite curved beam for C1 and $\beta = 58$

Natural frequency parameters	Number of elements					
	4	8	12	16	20	30
λ_1	17.362	17.303	17.269	17.247	17.231	17.209
λ_2	35.787	35.544	35.563	35.582	35.594	35.609
λ_3	71.938	60.520	60.305	60.282	60.281	60.243
λ_4	115.11	91.184	90.428	90.321	90.309	90.291
λ_5	190.13	129.66	127.63	127.20	127.08	127.02

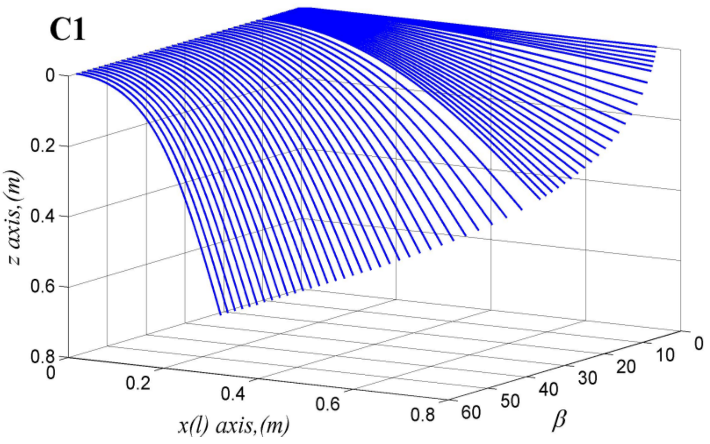
is taken as 10^{-7}. As a result of this process, a function representing the elastic curve can be obtained by substituting l into Eq. (11). This numerical process is repeated for all pre-stressed laminated composite curved beams formed by applying different loads.

A numerical comparison including the natural frequency analysis to verify the reliability and validity of the present model, which used the effective flexural modulus and the straight beam element approaches, are performed in references (Ozturk and Sabuncu 2005, Ozturk 2011). Therefore, these comparisons are not shown in this study again. In addition, Table 3 shows the convergence of the straight beam element used in analysis of the composite curved beam for C1. As seen from the Table, using number of 12 elements gives good results and satisfactory convergence. In this study, the curved beam is discretized with 30 finite elements.

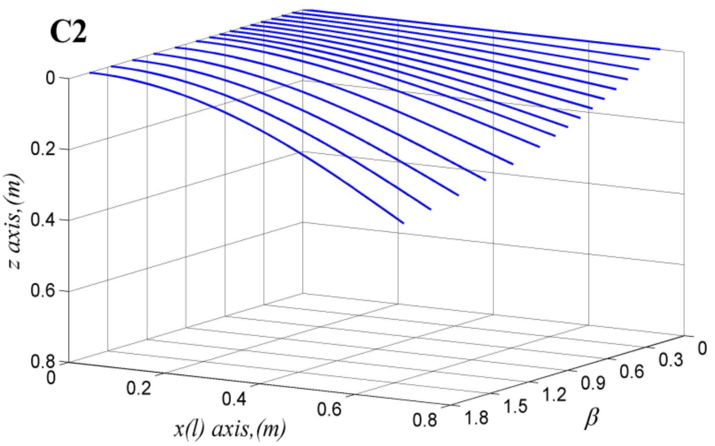
The effects of vertical non-dimensional load parameter on the natural frequency parameter λ given in Eq. (25) and mode shapes for the first four modes are investigated. The maximum load is taken as approximately 75% of the maximum stress of the composite materials according to Tsai-Hill Failure Criterion (Jones 1999) for each laminated beam having different orientation angle, because pre-stressed curved beams will be loaded by additional forces depending on the operating conditions. Therefore, the non-dimensional load parameter β given in Eq. (25) is considered in the range of different values for each laminated beam as follows: 0-58 (C1), 0-1.64 (C2), 0-3.28 (C3), 0-1.64 (C4), 0-5.73 (C5) and 0-4.16 (C6).

$$\lambda=\omega\sqrt{\frac{\rho AL^4}{E_2 I}} \quad \text{and} \quad \beta=\frac{PL^2}{E_2 I} \tag{25}$$

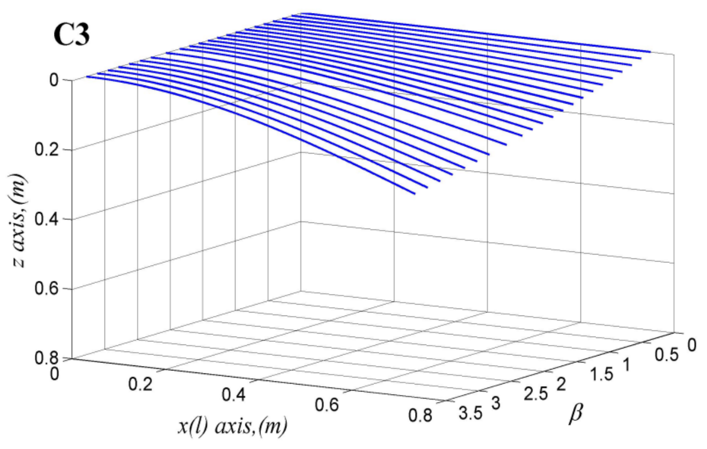
Figs. 6 and 7 show the deformations of cantilever laminated composite beams (C1, C2, C3, C4, C5 and C6) subjected to a vertical end load P varying according to the non-dimensional load parameter β. In accordance with the obtained numerical results, the vertical deflection at the beam end, δ_y increases linearly until about the load parameter value of 4, 0.8, 1.5, 0.8, 1 and 2 for C1, C2, C3, C4, C5 and C6, respectively. It increases nonlinearly after these values. On the other hand, while the horizontal deflection at the beam end, δ_x increases nonlinearly until about the load parameter value of 37, 0.8, 2.2, 0.8, 2.5 and 2.5 for C1, C2, C3, C4, C5 and C6, respectively, δ_x increases linearly beyond these load parameter values. As seen in Fig. 6(a), the deformation of C1 is greater than the other laminated composite beams, since C1 has a higher maximum stress capacity according to Tsai-Hill Failure Criterion (Jones 1999).



(a)



(b)



(c)

Fig. 6 Deformations of cantilever laminated beams (C1, C2 and C3) subjected to a vertical end load

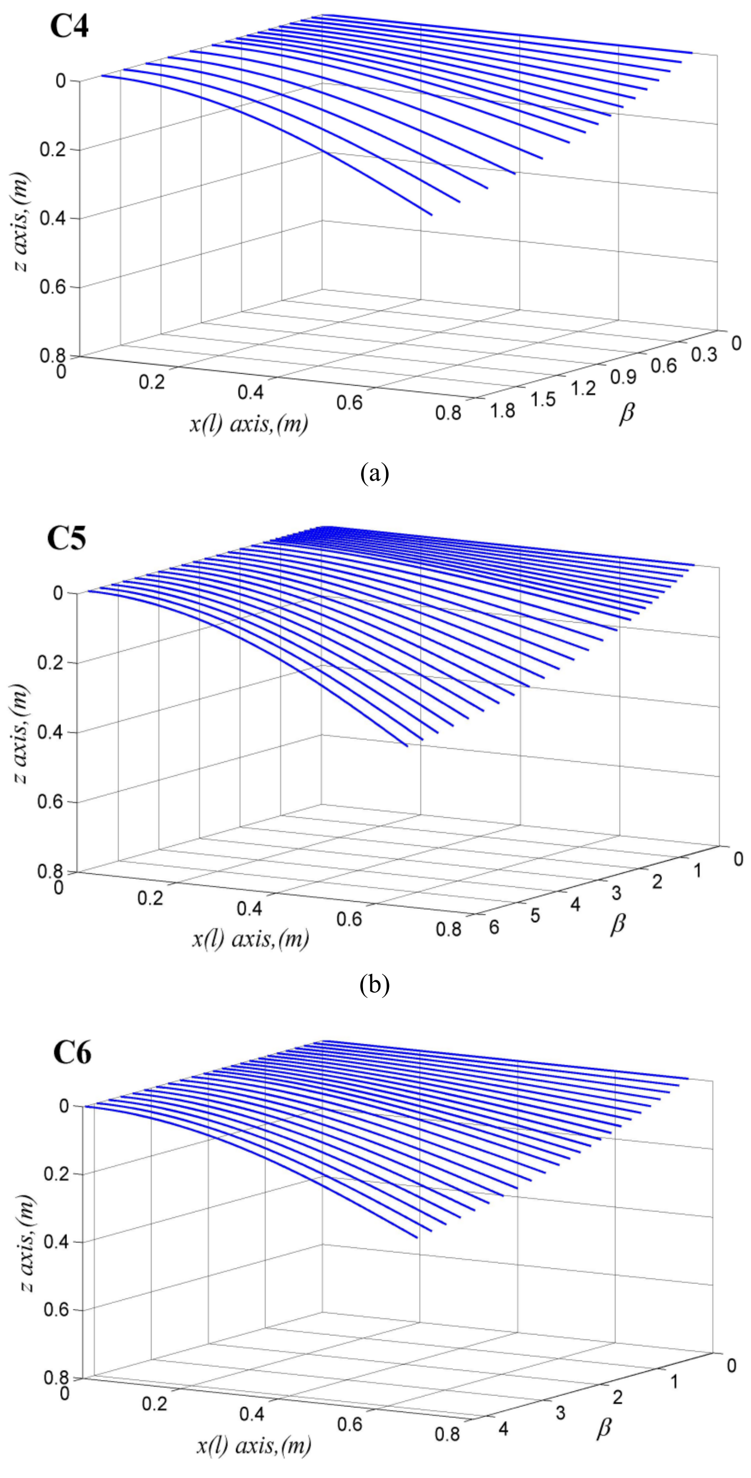


Fig. 7 Deformations of cantilever laminated beams (C4, C5 and C6) subjected to a vertical end load

After the cantilever laminated beam is deflected by vertical loading using large deflection theory, the large deflected cantilever laminated beam is fixed at the loading point (at the free end) as seen in Fig. 1(b). Thus, the pre-stressed laminated curved beam is obtained. Figs. 8-13 present the effect of non-dimensional load parameter on the first five natural frequency parameters of the fixed-fixed pre-stressed laminated curved beams. In addition, increases in the natural frequency parameters of the laminated curved beams with respect to the non-dimensional load parameter of 0 and β_{max} are given Table 4 which is obtained from Figs. 8-13. From these figures and Table 4, it can be said that the first five natural frequency parameters of laminated curved beams having six different orientation angles increase in the order as C1, C3, C5, C6, C4 and C2. If the non-dimensional load parameter, which is applied to the cantilever laminated beam to obtain the laminated curved beam, increases, the first natural frequency parameter λ_1 also increases until

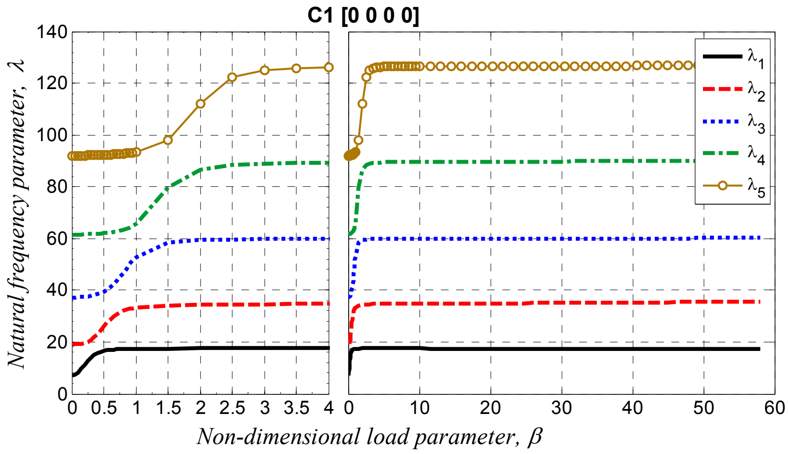


Fig. 8 Effect of load parameter on the first five natural frequency parameters of a fixed-fixed pre-stressed laminated curved beam, C1

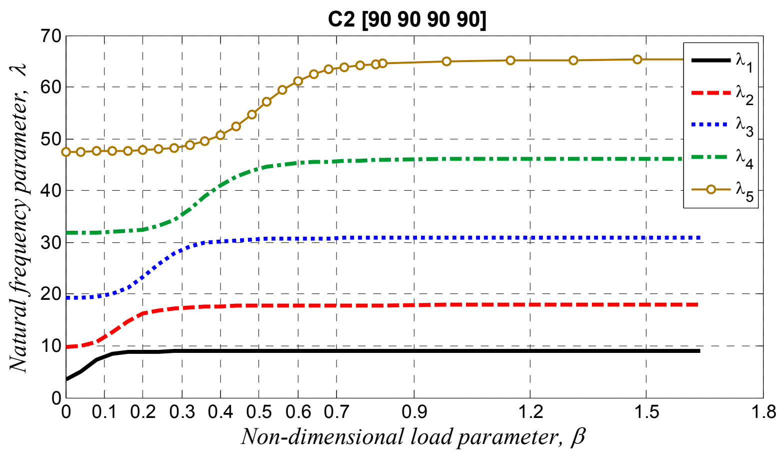


Fig. 9 Effect of load parameter on the first five natural frequency parameters of a fixed-fixed pre-stressed laminated curved beam, C2

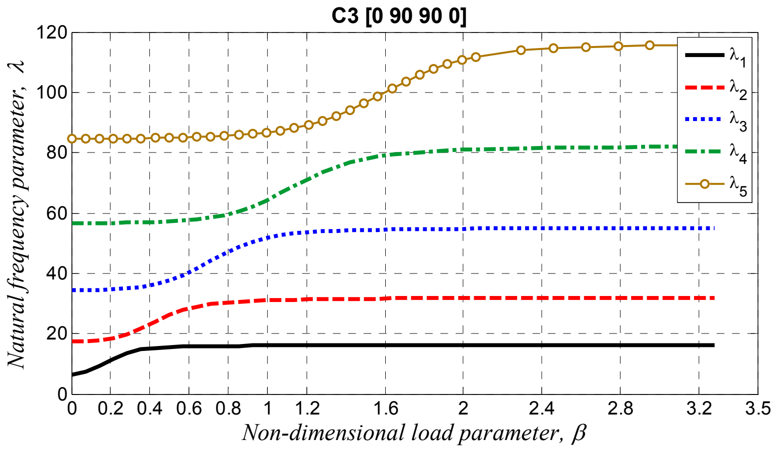


Fig. 10 Effect of load parameter on the first five natural frequency parameters of a fixed-fixed pre-stressed laminated curved beam, C3

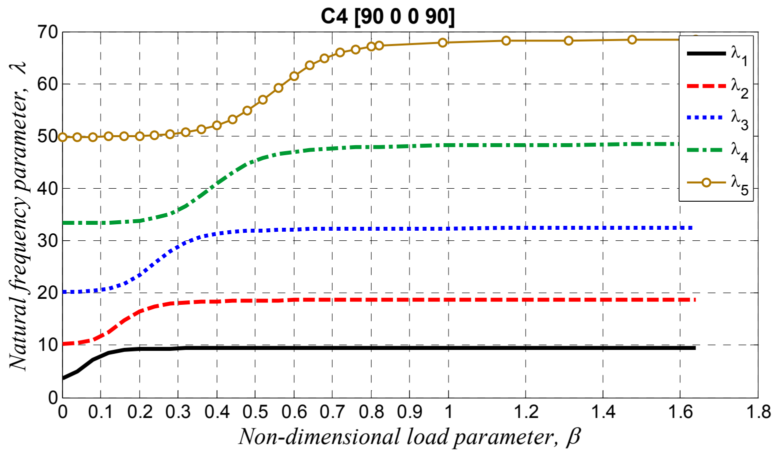


Fig. 11 Effect of load parameter on the first five natural frequency parameters of a fixed-fixed pre-stressed laminated curved beam, C4

about 0.5, 0.15, 0.4, 0.15, 0.4 and 0.4 for C1, C2, C3, C4, C5 and C6, respectively. Beyond these values, although the initial stress raises the stiffness of the laminated curved beams depending on the load parameter increment, λ_1 has a very small increment for all the laminated composite curved beams. As seen in Fig. 8, the second natural frequency parameter λ_2 of C1 increases slowly, rapidly and very slowly for 0-0.2, 0.2-0.9 and 0.9-58 values of the non-dimensional load parameters, respectively. For the other curved beams, this situation is seen about C2 (0-0.04, 0.04-0.24 and 0.24-1.64), C3 (0-0.2, 0.2-0.7 and 0.7-3.28), C4 (0-0.04, 0.04-0.24 and 0.24-1.64), C5 (0-0.2, 0.2-0.7 and 0.7-5.73) and C6 (0-0.2, 0.2-0.8 and 0.8-4.16) values as seen in Figs. 8-13. The variation of the other natural frequencies λ_3, λ_4 and λ_5 also show the similarity with the variation of the second natural frequency parameter for different values. It can be noticed from Figs. 8-13 that there is a relationship, which is valid for all laminated curved beams, between the

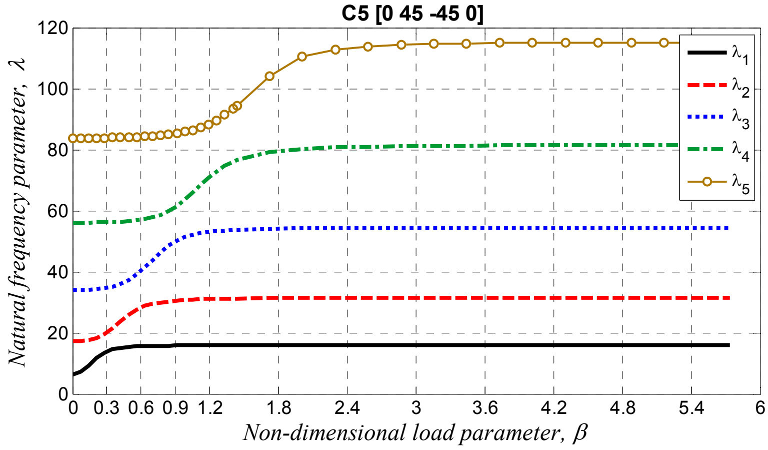


Fig. 12 Effect of load parameter on the first five natural frequency parameters of a fixed-fixed pre-stressed laminated curved beam, C5

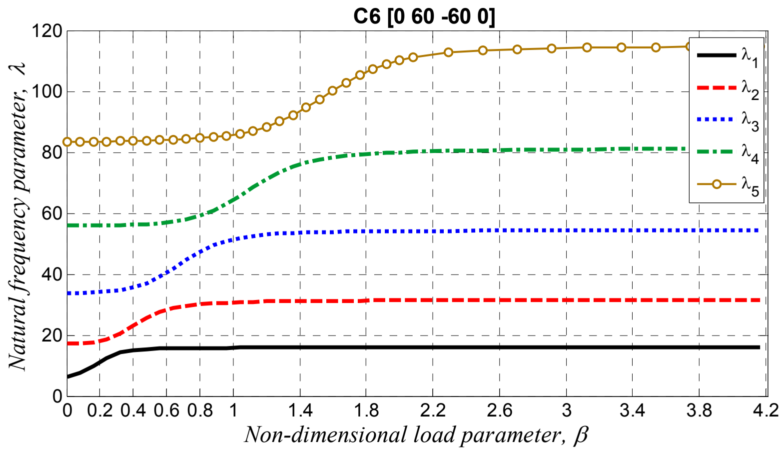


Fig. 13 Effect of load parameter on the first five natural frequency parameters of a fixed-fixed pre-stressed laminated curved beam, C6

variations of the first five natural frequency parameters. This relationship is that when the increase rate of the first natural parameter, λ_1 changes from rapid to very slow, the increase rate of the second natural parameter, λ_2 changes from slow to rapid. This phenomenon also occurs between the other frequency parameters such as λ_2-λ_3, λ_3-λ_4 and λ_4-λ_5.

It is observed from Table 4 that the percentage increments in the first five natural frequency parameters of all laminated curved beams between 0 and β_{max} values are almost close to each other except for that of λ_1 and λ_2 of C1. In addition, while the maximum percentage increment in the natural frequency parameter occurs in λ_1, the minimum percentage increment in the natural frequency parameter occurs in λ_5.

Figs. 14-20 show the first four modes of fixed-fixed pre-stressed laminated curved beam obtained from large deflected cantilever laminated beam with respect to load parameters for C1,

Table 4 Increases in the natural frequency parameters of the fixed-fixed pre-stressed laminated curved beams with respect to the non-dimensional load p arameter of 0 and β_{max}

	C1 [0 0 0 0]			**C2 [90 90 90 90]**			**C3 [0 90 90 0]**		
Natural frequency parameters	Load parameter 0	Load parameter β_{max} = 58	Increment (%)	Load parameter 0	Load parameter β_{max} = 1.64	Increment (%)	Load parameter 0	Load parameter β_{max} = 3.28	Increment (%)
λ_1	6.90	17.21	149.42	3.56	9.09	155.32	6.34	16.17	155.09
λ_2	19.00	35.61	87.42	9.82	17.91	82.46	17.47	31.85	82.31
λ_3	37.28	60.29	61.72	19.24	30.92	60.71	34.26	55.00	60.54
λ_4	61.61	90.33	46.62	31.81	46.19	45.21	56.63	82.03	44.85
λ_5	92.00	126.90	37.93	47.52	65.31	37.44	84.60	115.80	36.88
	C4 [90 0 0 90]			**C5 [0 45 -45 0]**			**C6 [0 60 -60 0]**		
Natural frequency parameters	Load parameter 0	Load parameter β_{max} = 1.64	Increment (%)	Load parameter 0	Load parameter β_{max} = 5.73	Increment (%)	Load parameter 0	Load parameter β_{max} = 4.16	Increment (%)
λ_1	3.73	9.53	155.35	6.28	16.02	155.30	6.26	15.97	155.32
λ_2	10.28	18.76	82.49	17.30	31.58	82.54	17.24	31.45	82.42
λ_3	20.16	32.40	60.71	33.91	54.49	60.69	33.80	54.31	60.68
λ_4	33.33	48.40	45.21	56.06	81.45	45.29	55.88	81.11	45.15
λ_5	49.79	68.40	37.38	83.74	115.10	37.45	83.48	114.60	37.28

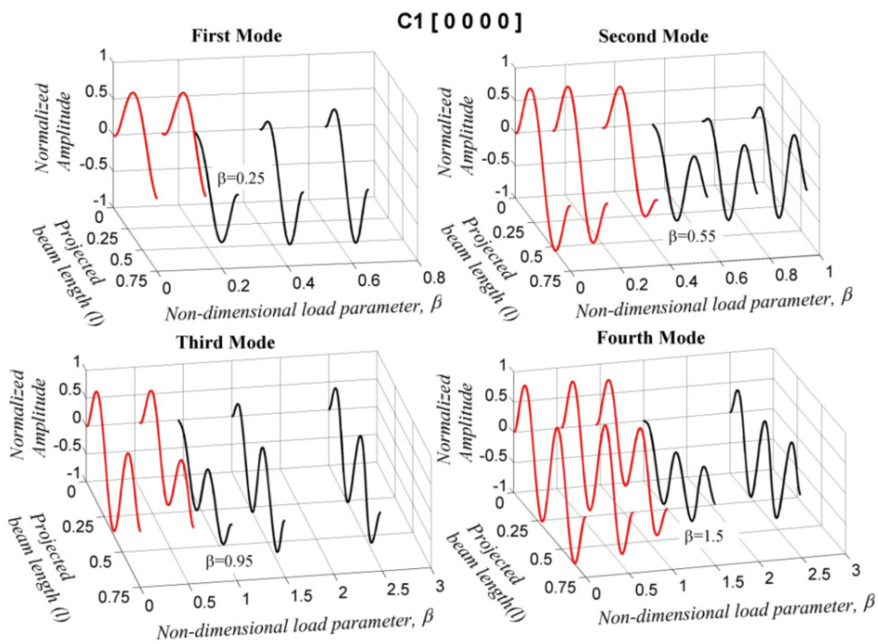


Fig. 14 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C1) in transverse vibration. (The figure is zoomed in β axes)

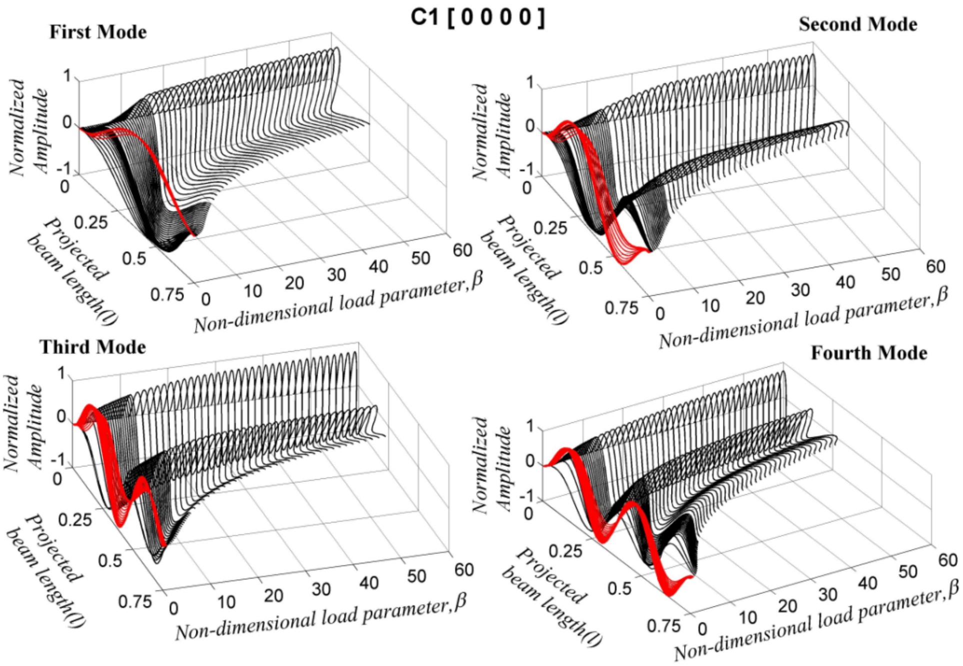


Fig. 15 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C1) in transverse vibration

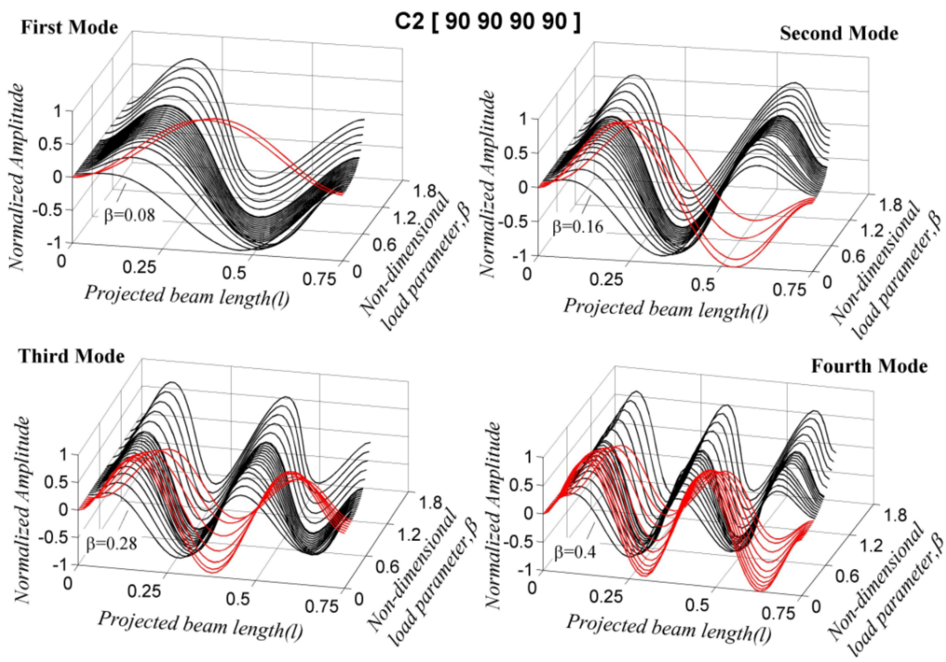


Fig. 16 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C2) in transverse vibration

C2, C3, C4, C5 and C6. All of the amplitudes of the mode shapes for all load parameters are normalized with respect to the maximum amplitude. When the load parameter increases, the mode shapes change from the mode shape of a straight beam to the mode shape of a curved beam after certain load parameter values. As seen in Figs. 14 and 15, it is noticed that the first mode shape (the lines shown in red) of C1 has no node until the value of $\beta = 0.25$, as from this value, one node begins to occur until the value of β_{max} and this node in the first mode shape becomes more obvious and its intersection point with the horizontal axes moves. Beyond this load parameter value, noteworthy variation for the first mode shapes is not seen as shown in Fig. 14. However, the other mode shapes do not show any modal variation between β = 0 and β = 0.25 values. For the laminated curved beam C1, a similar phenomenon occurs for the natural frequency parameters of $\lambda_{2,}$ λ_3 and λ_4 for different β values as seen in Figs. 14 and 15, which are 0.55, 0.95 and 1.5, respectively.

As seen in Figs. 16-20, a similar change in the mode shapes of C1 also occur for the other laminated curved beams, C2, C3, C4 , C5 and C6. This change in the first four mode shapes is seen when the non-dimensional load parameter β is "0.08, 0.16, 0.28 and 0.4 for C2", "0.213, 0.497, 0.782 and 1.209 for C3", "0.08, 0.16, 0.28 and 0.44 for C4", "0.21, 0.49, 0.77 and 1.19 for C5" and "0.24, 0.48, 0.8 and 1.2 for C6", respectively. Moreover, the values of β at the change of the mode shapes of different laminated curved beams are arranged in the ascending order as: C1, C3, C5, C6, C4 and C2.

It can be noticed from the free vibration analysis shown in Figs. 8-20 that the natural frequency parameters and mode shapes change very little after certain load parameter values. The reason for

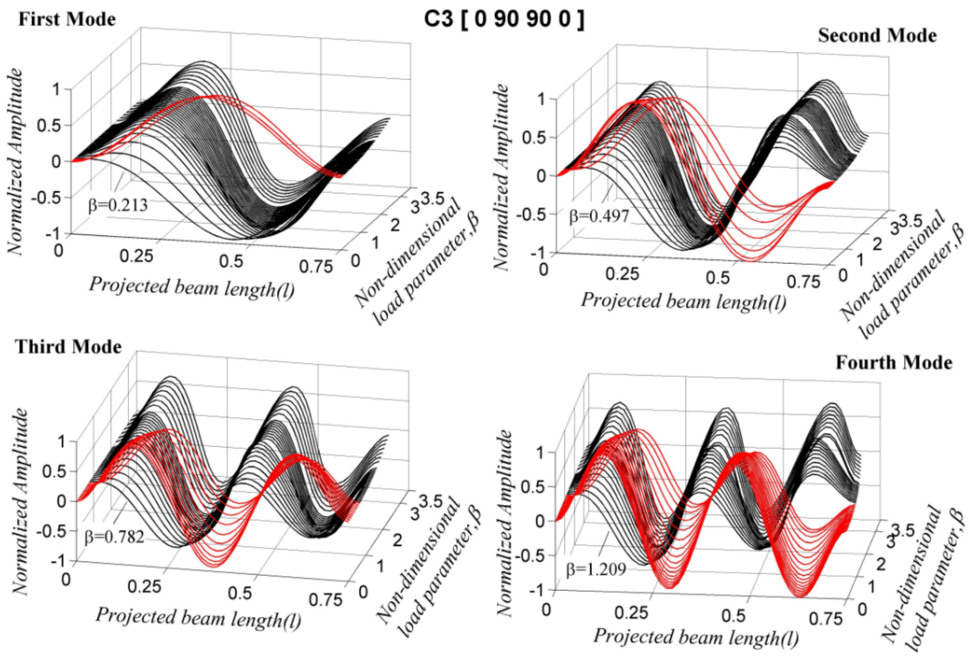


Fig. 17 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C3) in transverse vibration

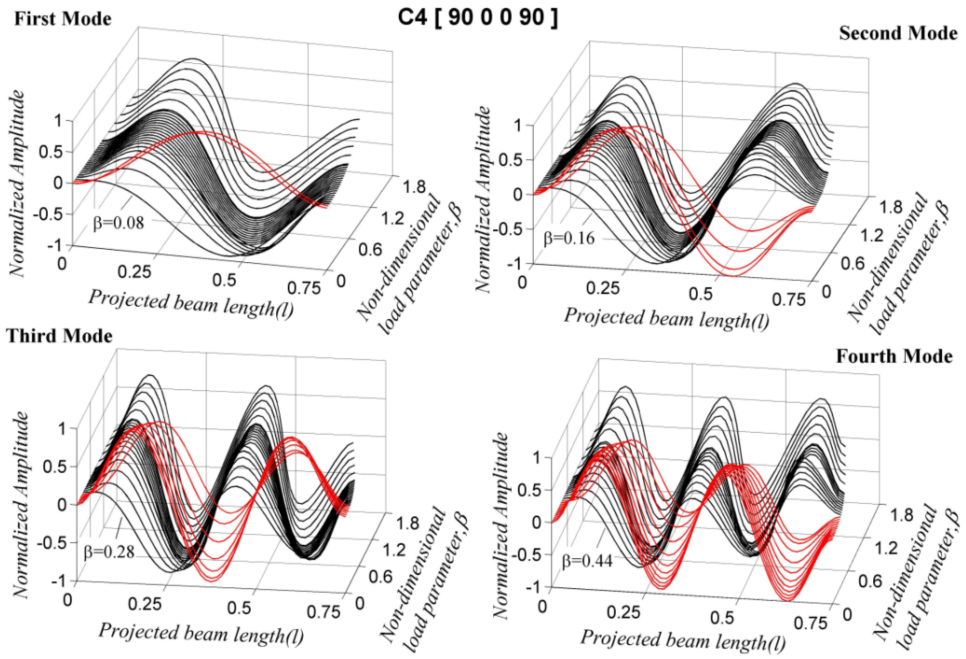


Fig. 18 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C4) in transverse vibration

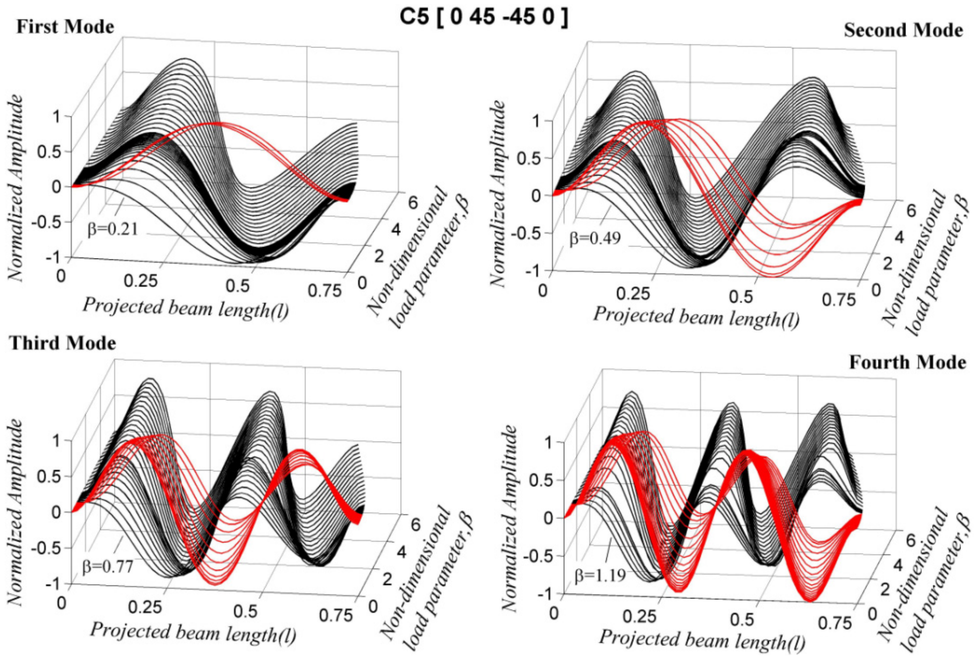


Fig. 19 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C5) in transverse vibration

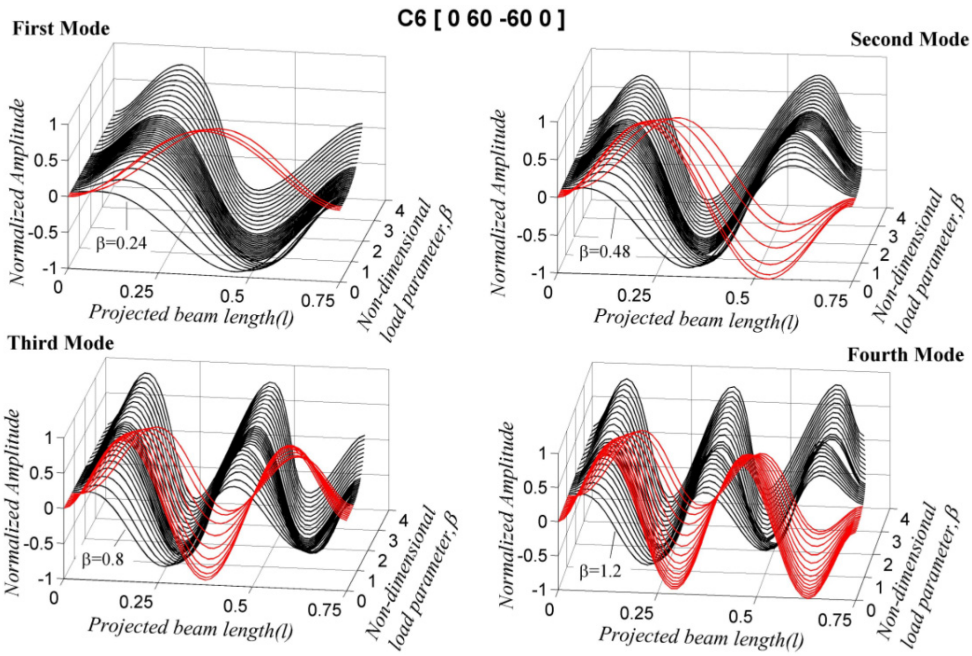


Fig. 20 First four mode shapes of a fixed-fixed pre-stressed laminated curved beam (C6) in transverse vibration

this is that mass and elastic stiffness with initial stress matrix of the laminated curved beam counteract each other due to the deformation shape of the laminated beam.

4. Conclusions

In-plane free vibration of a fixed-fixed pre-stressed laminated curved beam which is obtained from a large deflected cantilever laminated beam having six different orientation angles is presented in this study. In order to obtain the fixed-fixed curved laminated beam, an external vertical concentrated load at the free end is applied to a cantilever laminated beam and then the loading point of the deflected beam is fixed. The natural frequencies increase with respect to the effective flexural modulus which increases in the order as C1, C3, C5, C6, C4 and C2. It has been found that the effect of the non-dimensional load parameter, which forms the curved laminated beam, on the natural frequency parameter of C1, C2, C3, C4, C5 and C6 almost disappears after a certain value of the non-dimensional load parameter for each vibration mode. Furthermore, although the percentage increment in the natural frequency parameter of all laminated curved beams (C1, C2, C3, C4, C5 and C6) between β_{min} and β_{max} values decreases towards higher frequencies, the interval of the non-dimensional load parameter value which is more influential on the natural frequency parameter, shifts to higher values for high frequency mode.

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CC

Nomenclature

A	cross-sectional area
b	width of the beam
d	finite element length
E	modulus of elasticity
E_{ef}	effective flexural modulus
h	height of the beam
I	moment of inertia of the cross-sectional area of the beam about the axis of bending
L	length of the beam
l	projected beam length
$M(s)$	deflecting moment occurring at a point along the beam
M_x, M_y	bending moments per unit length
M_{xy}	twisting moment per unit length
P	vertical load at $x = L$
s	distance measured along the curve
S	initial axial force
u_1	axial displacement at the left hand side of the finite element in the $\bar{x}$ direction
u_2	axial displacement at the right hand side of the finite element in the $\bar{x}$ direction
v_1	bending displacement at the left hand side of the finite element in the $\bar{z}$ direction
v_2	bending displacement at the right hand side of the finite element in the $\bar{z}$ direction
v'_1	bending slope at the left hand side of the finite element
v'_2	bending slope at the right hand side of the finite element
x, z	global reference coordinates
$\bar{x}$, $\bar{z}$	local reference coordinates
$\mathbf{D}$	bending stiffness matrix
$\bar{\mathbf{k}}_e$	elastic stiffness matrix with respect to local reference coordinates
$\mathbf{k}_e$	elastic stiffness matrix with respect to global reference coordinates
$\mathbf{K_e}$	global elastic stiffness matrix
$\bar{\mathbf{m}}_e$	mass matrix with respect to local reference coordinates
$\mathbf{m}_e$	mass matrix with respect to global reference coordinates
$\mathbf{M_e}$	global mass matrix
$\mathbf{s_e}$	initial stiffness matrix with respect to global reference coordinates
$\bar{\mathbf{s}}_e$	initial stiffness matrix with respect to local reference coordinates
$\mathbf{S_e}$	global initial stiffness matrix
$\mathbf{q_e}$	generalized displacement vector with respect to global reference coordinates
$\bar{\mathbf{q}}_e$	generalized displacement vector with respect to local reference coordinates
$\mathbf{T}$	transformation matrix
θ	angle of rotation of the beam deflection curve

ρ	curvature of the beam
ϕ	fiber orientation angle
β	load parameter
λ	natural frequency parameter
θ_0	angle of rotation of the beam deflection curve at $x = l$
δ_x, δ_z	displacement components in the x and z directions, respectively

Appendix

$$c = \cos\phi, \quad s = \sin\phi \tag{A1}$$

$$\overline{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{33})s^2c^2 + Q_{22}s^4$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{33})s^2c^2 + Q_{12}(s^4 + c^4)$$

$$\overline{Q}_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{33})s^2c^2 + Q_{22}c^4$$

$$\overline{Q}_{13} = (Q_{11} - Q_{12} - 2Q_{33})sc^3 + (Q_{12} - Q_{22} + 2Q_{33})s^3c$$

$$\overline{Q}_{23} = (Q_{11} - Q_{12} - 2Q_{33})s^3c + (Q_{12} - Q_{22} + 2Q_{33})sc^3$$

$$\overline{Q}_{33} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{33})s^2c^2 + Q_{33}(s^4 + c^4);$$

$$Q_{11} = \frac{E_1}{1 - v_{12}v_{21}}, \quad Q_{12} = \frac{v_{12}E_2}{1 - v_{12}v_{21}}, \quad Q_{22} = \frac{E_2}{1 - v_{12}v_{21}}, \quad Q_{66} = G_{12}, \quad v_{21} = v_{12}\frac{E_2}{E_1}$$

$$\mathbf{T} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{A2}$$