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Buckling of non-homogeneous orthotropic conical shells subjected to combined load

A.H. Sofiyev *[1] and N. Kuruoglu [2]

[1] *Department of Civil Engineering, Engineering Faculty, Suleyman Demirel University, Isparta, Turkey*
[2] *Department of Engineering Mathematics, Faculty of Engineering and Natural Sciences, Bahcesehir University, Istanbul, Turkey*

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Abstract. The buckling analysis is presented for non-homogeneous (NH) orthotropic truncated conical shells subjected to combined loading of axial compression and external pressure. The governing equations have been obtained for the non-homogeneous orthotropic truncated conical shell, the material properties of which vary continuously in the thickness direction. By applying Superposition and Galerkin methods to the governing equations, the expressions for critical loads (axial, lateral, hydrostatic and combined) of non-homogeneous orthotropic truncated conical shells with simply supported boundary conditions are obtained. The results are verified by comparing the obtained values with those in the existing literature. Finally, the effects of non-homogeneity, material orthotropy, cone semi-vertex angle and other geometrical parameters on the values of the critical combined load have been studied.

Keywords: buckling; stability; functionally graded; material properties; axial compression; lateral-torsional-buckling; numerical analysis; composite structures

1. Introduction

Conical shells are one of the necessary structural components and commonly found in a variety of engineering applications, such as the military, turbo-machinery and marine industries, e.g., piles for holding jackets when driven into the sea bed, transition elements between two cylindrical shells of different diameter, and the legs of off-shore drilling rigs. When used as piles for jackets holding, they are, subjected to axial compression. However, when used as transition components, they are also subjected to external pressure. Hence in the case of off-shore drilling rigs, they are under combined loading (Ifayefunmi and Błachut 2013). This extensive application of conical shells in marine engineering calls for efficient tools to analyze the mechanical behavior of these structures. Stability of conical shells under combined loads is one of the most important failure modes of these structures. Research on the buckling of homogeneous isotropic conical shells under combined load has a long history. References to some of the earlier works can be found in the studies of Sachenkov (1964), Weingarten and Seide (1965), Karpov and Karpova (1981), Struk (1984) and Tani (1985).

*Corresponding author, Professor, E-mail: abdullahavey@sdu.edu.tr

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In recent years, some studies on the buckling of homogeneous isotropic and orthotropic conical shells under combined loads are published. For instance, Blachut (2011) presented the buckling of short, and relatively thick, mild steel conical shells subjected to the combined action of external pressure and axial compression. Ajdari *et al*. (2012) presented the solution of the buckling of composite truncated conical shells under combined external pressure and axial compression. Blachut (2012) presented interactive plastic buckling of cones subjected to axial compression and external pressure. Shadmehri *et al*. (2012) proposed to obtain the linear buckling response of conical composite shells using a semi-analytical approach. Naderi *et al*. (2014) reported the influence of fiber paths on the buckling load of tailored conical shells. Sofiyev (2014) presented the buckling of homogeneous composite conical shells resting on elastic foundations under a combined load.

Since 2010, new researches performed on the non-homogeneous or functionally graded (FG) isotropic cylindrical and conical shells have mostly been focused on the buckling of shells subjected to combined loads. Khazaeinejad *et al*. (2010) presented the buckling of functionally graded cylindrical shells under combined external pressure and axial compression. Sofiyev (2010) investigated the buckling of FG isotropic truncated conical shells subjected to combined axial tension and hydrostatic pressure. Sofiyev *et al*. (2012) presented the stability of FG isotropic shells subjected to combined loads with different edge conditions and resting on elastic foundations. Van-Dung *et al*. (2013) presented instability of eccentrically stiffened FG isotropic truncated conical shells under mechanical loads. Mohammadzadeh *et al*. (2013) studied the buckling of 2D-FG cylindrical shells under combined external pressure and axial compression. Wu *et al*. (2013) investigated the buckling analysis of functionally graded material circular hollow cylinders under combined axial compression and external pressure.

However, to the best of our knowledge, the buckling of non-homogeneous orthotropic conical shells under combined loading has not been examined theoretically yet. The non-homogeneity of the materials stems from the effects of humidity, surface and thermal polishing processes and methods of production, which causes the mechanical properties of the materials to vary from point to point (random, piecewise continuous or continuous functions of coordinates) (Babich and Khoroshun 2001, Awrejcewicz and Krysko 2008, Sofiyev *et al*. 2009, Grigorenko and Grigorenko 2013). Furthermore, certain parts of structural elements have to operate under radiation and elevated temperatures and which cause non-homogeneity in the material, i.e., the constants of the material become functions of space variables (Lal and Kumar 2012). In addition, the FGMs is a subgroup of non-homogeneous materials, also. They are non-homogeneous with regard to mechanical and strength properties. Depending on the processing technique, they may exhibit either isotropic or anisotropic material properties. For example, in studying the mechanics of the former class of materials produced by spark plasma sintering (SPS), a non-homogeneous isotropic model may be appropriate; and for the latter class of materials fabricated by plasma spraying or physical vapor deposition (PVD), the non-homogeneous orthotropic model may suffice as a first approximation (Kim and Paulino 2002, Ootao and Tanigawa 2007). A study that includes a non-homogeneity would be interesting as it would shed light on the effects of non-homogeneity on the buckling phenomenon of conical shells under combined loads. Thus in the present paper, the stability analyses of non-homogeneous orthotropic shells under combined loads are carried out. This is one of the dominant loading conditions for aerospace and marine structures. The content of this paper is arranged as follows: first, the equations of stability of a non-homogeneous orthotropic truncated conical shell subjected to the combined loading are derived based upon the modified Donnell type thin shell theory. Young's moduli and shear modulus of orthotropic materials vary as linear, quadratic and exponential functions in the thickness direction. Then solving the governing equations using the Galerkin's method, we obtain an expression for the critical combined load of non-homogeneous orthotropic truncated conical and cylindrical shells. The results are compared with corresponding studies presented by other authors. The effects of non-homogeneity, material orthotropy, cone semi-vertex angle and other geometrical parameters on the values of the critical combined load examined in detail.

2. Theoretical development

The configuration of a thin non-homogeneous orthotropic truncated conical shell and the coordinate system are taken as shown in Fig. 1 that R_1 and R_2 indicate the radii of the cone at its small and large ends, respectively, γ denotes the semi-vertex angle of the cone, L is the truncated cone length along its generator and S_1 is the distance of the smaller end of the truncated conical shell from the vertex. We introduce the $S\theta\zeta$ curvilinear coordinate system; S coincides with generator, θ is circumferential coordinate and ζ is perpendicular to the S-θ plane and its direction is inwards normal of the truncated conical shell. w is the displacement of the reference surface in the normal direction, positive towards the axis of the cone and assumed to be much smaller than the thickness.

It is assumed that the conical shell material is a non-homogeneous orthotropic. The non-homogeneity of orthotropic materials of the conical shell is assumed to arise due to the variation of Young's moduli and shear modulus along the thickness direction ζ (Sofiyev *et al.* 2009)

$$E_S = E_{0S}\varphi(\bar{\zeta}), \quad E_\theta = E_{0\theta}\varphi(\bar{\zeta}), \quad G = G_0\varphi(\bar{\zeta}) \tag{1}$$

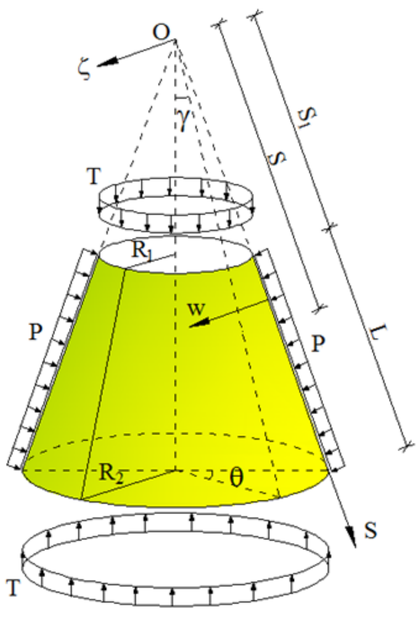

Fig. 1 Geometry of composite truncated conical shell subjected to combined load

where E_{0S} and $E_{0\theta}$ are the Young's moduli in S and θ directions, respectively, and G_0 is the shear modulus of the homogeneous orthotropic materials. Also, it is assumed that the Poisson's ratios $\nu_{S\theta}$ and $\nu_{\theta S}$ are constant and satisfying $\nu_{\theta S}E_{0S} = \nu_{S\theta}E_{0\theta}$. Here $\varphi(\bar{\zeta})$ is function of the non-homogeneity defining the variations of the Young's moduli and shear modulus, respectively.

In this study, two different variation laws for the non-homogeneity of orthotropic materials are considered: (a) simple power-law; and (b) exponential distributions through the thickness of the shells.

(a) The orthotropic material properties of the non-homogeneous shells are assumed to vary through their thickness direction according to the simple power-law distribution (Babich and Khoroshun 2001, Awrejcewicz and Krysko 2008, Sofiyev *et al.* 2009, Grigorenko and Grigorenko 2013, Lal and Kumar 2012)

$$\varphi(\bar{\zeta}) = 1 + \mu\bar{\zeta}^k \tag{2}$$

where μ is a non-homogeneity parameter, satisfying $0 \le \mu \le 1$ and $k = 1, 2,...$ is the power-law index.

(b) The orthotropic material properties of the non-homogeneous shells are assumed to vary through their thickness direction according to the exponential-law distribution, also

$$\varphi(\bar{\zeta}) = e^{\eta(\bar{\zeta}-0.5)} \tag{3}$$

where η is a exponential factor and is a real number (Kim and Paulino 2002, Ootao and Tanigawa 2007).

The truncated conical shell is subjected to simultaneous action of the axially compressive load T and external normal pressure P, as shown in Fig. 1. Under this loading the membrane stress resultant, at the critical state, may be expressed as (Agamirov 1990, Sofiyev 2014)

$$N_S^0 = -T, \quad N_\theta^0 = -PS\tan\gamma; \quad N_{S\theta}^0 = 0 \tag{4}$$

where N_s^0, N_θ^0 and $N_{S\theta}^0$ are the membrane forces for the condition with zero initial moments.

These equations based on the membrane theory of shells degenerate to their more familiar forms for cylindrical shells, when γ is set equal to zero.

For thin non-homogeneous orthotropic shells, the stresses σ_S, σ_θ and $\sigma_{S\theta}$ are related to the corresponding strains $\varepsilon_S, \varepsilon_\theta$ and $\varepsilon_{S\theta}$ by the stress-strain relationship (Reddy 2004, Sofiyev *et al.* 2009)

$$\begin{bmatrix} \sigma_S \\ \sigma_\theta \\ \sigma_{S\theta} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_S \\ \varepsilon_\theta \\ \varepsilon_{S\theta} \end{bmatrix} \tag{5}$$

Where Q_{ij}, $(i, j = 1, 2, 6)$ are quantities of non-homogeneous orthotropic materials and are

$$Q_{11} = \frac{E_{0S}\varphi(\bar{\zeta})}{1-\nu_{S\theta}\nu_{\theta S}}, \quad Q_{22} = \frac{E_{0\theta}\varphi(\bar{\zeta})}{1-\nu_{S\theta}\nu_{\theta S}}, \quad Q_{12} = Q_{21} = \nu_{\theta S}Q_{11} = \nu_{S\theta}Q_{22}, \quad Q_{66} = 2G_0\varphi(\bar{\zeta}) \tag{6}$$

The strains are defined as linear functions of the thickness coordinate ζ

$$\begin{bmatrix} \varepsilon_S \\ \varepsilon_\theta \\ \varepsilon_{S\theta} \end{bmatrix} = \begin{bmatrix} e_S \\ e_\theta \\ e_{S\theta} \end{bmatrix} - \zeta \begin{bmatrix} \dfrac{\partial^2 w}{\partial S^2} \\ \dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial \theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S} \\ \dfrac{1}{S}\dfrac{\partial^2 w}{\partial S\,\partial \theta_1} - \dfrac{1}{S^2}\dfrac{\partial w}{\partial \theta_1} \end{bmatrix} \tag{7}$$

where $\theta_1 = \theta \sin\gamma$ and e_S, e_θ, $e_{S\theta}$ are the strains on the reference surface.
The definitions of the force and moment resultants are given as (Reddy 2004)

$$\left[(N_S, N_\theta, N_{S\theta}), (M_S, M_\theta, M_{S\theta})\right] = \int_{-h/2}^{h/2} (\sigma_S, \sigma_\theta, \sigma_{S\theta})[1, \zeta\,]\,\mathrm{d}\zeta \tag{8}$$

Let Ψ (S, θ) be the stress function for the stress resultants defined by (Agamirov 1990)

$$N_S = \frac{1}{S^2}\frac{\partial^2\Psi}{\partial\theta_1^2} + \frac{1}{S}\frac{\partial\Psi}{\partial S}, \quad N_\theta = \frac{\partial^2\Psi}{\partial S^2}, \quad N_{S\theta} = -\frac{1}{S}\frac{\partial^2\Psi}{\partial S\partial\theta_1} + \frac{1}{S^2}\frac{\partial\Psi}{\partial\theta_1} \tag{9}$$

In view of Eqs. (5), (7) and (9), and when solutions (8) are used, the moments and stresses may be given in the following final explicit forms

$$\begin{bmatrix} M_S \\ M_\theta \\ M_{S\theta} \end{bmatrix} = \begin{bmatrix} c_{11}\left(\dfrac{1}{S^2}\dfrac{\partial^2\Psi}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial\Psi}{\partial S}\right) + c_{12}\dfrac{\partial^2\Psi}{\partial S^2} - c_{13}\dfrac{\partial^2 w}{\partial S^2} - c_{14}\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ c_{21}\left(\dfrac{1}{S^2}\dfrac{\partial^2\Psi}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial\Psi}{\partial S}\right) + c_{22}\dfrac{\partial^2\Psi}{\partial S^2} - c_{23}\dfrac{\partial^2 w}{\partial S^2} - c_{24}\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ c_{31}\left(\dfrac{1}{S^2}\dfrac{\partial\Psi}{\partial\theta_1} - \dfrac{1}{S}\dfrac{\partial^2\Psi}{\partial S\partial\theta_1}\right) + c_{32}\left(\dfrac{1}{S^2}\dfrac{\partial w}{\partial\theta_1} - \dfrac{1}{S}\dfrac{\partial^2 w}{\partial S\,\partial\theta_1}\right) \end{bmatrix} \tag{10}$$

$$\begin{bmatrix} e_S \\ e_\theta \\ e_{S\theta} \end{bmatrix} = \begin{bmatrix} b_{11}\left(\dfrac{1}{S^2}\dfrac{\partial^2\Psi}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial\Psi}{\partial S}\right) + b_{12}\dfrac{\partial^2\Psi}{\partial S^2} - b_{13}\dfrac{\partial^2 w}{\partial S^2} - b_{14}\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ b_{21}\left(\dfrac{1}{S^2}\dfrac{\partial^2\Psi}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial\Psi}{\partial S}\right) + b_{22}\dfrac{\partial^2\Psi}{\partial S^2} - b_{23}\dfrac{\partial^2 w}{\partial S^2} - b_{24}\left(\dfrac{1}{S^2}\dfrac{\partial^2 w}{\partial\theta_1^2} + \dfrac{1}{S}\dfrac{\partial w}{\partial S}\right) \\ b_{31}\left(\dfrac{1}{S^2}\dfrac{\partial\Psi}{\partial\theta_1} - \dfrac{1}{S}\dfrac{\partial^2\Psi}{\partial S\partial\theta_1}\right) + b_{32}\left(\dfrac{1}{S^2}\dfrac{\partial w}{\partial\theta_1} - \dfrac{1}{S}\dfrac{\partial^2 w}{\partial S\,\partial\theta_1}\right) \end{bmatrix} \tag{11}$$

where the following definitions apply

$$
\begin{aligned}
&c_{11}=a_{11}^{1}b_{11}+a_{12}^{1}b_{21}, c_{12}=a_{11}^{1}b_{12}+a_{12}^{1}b_{22},\ c_{13}=a_{11}^{1}b_{13}+a_{12}^{1}b_{23}+a_{11}^{2},\\
&c_{14}=a_{11}^{1}b_{14}+a_{12}^{1}b_{24}+a_{12}^{2},\ c_{21}=a_{21}^{1}b_{11}+a_{22}^{1}b_{21},\ c_{22}=a_{21}^{1}b_{12}+a_{22}^{1}b_{22},\\
&c_{23}=a_{21}^{1}b_{13}+a_{22}^{1}b_{14}+a_{21}^{2},\ \ c_{24}=a_{21}^{1}b_{14}+a_{22}^{1}b_{13}+a_{22}^{2},\ c_{31}=a_{66}^{1}b_{31},\\
&c_{32}=a_{66}^{1}b_{32}+a_{66}^{2}, b_{11}=a_{22}^{0}/L_0,\ b_{12}=-a_{12}^{0}/L_0,\ b_{13}=(a_{12}^{0}a_{21}^{1}-a_{11}^{1}a_{22}^{0})/L_0,\\
&b_{14}=(a_{12}^{0}a_{22}^{1}-a_{12}^{1}a_{22}^{0})/L_0, b_{21}=-a_{21}^{0}/L_0,\ \ b_{22}=a_{11}^{0}/L_0,\ b_{23}=(a_{21}^{0}a_{11}^{1}-a_{21}^{1}a_{11}^{0})/L_0,\\
&b_{24}=(a_{21}^{0}a_{12}^{1}-a_{22}^{1}a_{11}^{0})/L_0, b_{31}=1/a_{66}^{0},\ \ b_{32}=-a_{66}^{1}/a_{66}^{0},\ L_0=a_{11}^{0}a_{22}^{0}-a_{12}^{0}a_{21}^{0}
\end{aligned}
\tag{12}
$$

in which

$$
\begin{aligned}
&a_{11}^{k}=\frac{E_{0S}h^{k+1}}{1-\nu_{S\theta}\nu_{\theta S}}\int_{-1/2}^{1/2}\bar{\zeta}^{k}\varphi\left(\bar{\zeta}\right)\mathrm{d}\bar{\zeta},\quad a_{12}^{k}=\nu_{\theta S}a_{11}^{k}=a_{21}^{k}=\nu_{S\theta}a_{22}^{k},\\
&a_{22}^{k}=\frac{E_{0\theta}h^{k+1}}{1-\nu_{S\theta}\nu_{\theta S}}\int_{-1/2}^{1/2}\bar{\zeta}^{k}\varphi\left(\bar{\zeta}\right)\mathrm{d}\bar{\zeta},\quad a_{66}^{k}=2G_0h^{k+1}\int_{-1/2}^{1/2}\bar{\zeta}^{k}\varphi\left(\bar{\zeta}\right)\mathrm{d}\bar{\zeta},\quad k=0,1,2.
\end{aligned}
\tag{13}
$$

The modified Donnell type stability and strain compatibility equations of a non-homogeneous orthotropic truncated conical shell under external pressures can be written as (Agamirov 1990)

$$
\begin{aligned}
&\frac{\partial^2 M_S}{\partial S^2}+\frac{2}{S}\frac{\partial M_S}{\partial S}+\frac{2}{S}\frac{\partial^2 M_{S\theta}}{\partial S\partial\theta_1}-\frac{1}{S}\frac{\partial M_\theta}{\partial S}+\frac{2}{S^2}\frac{\partial M_{S\theta}}{\partial\theta_1}+\frac{1}{S^2}\frac{\partial^2 M_\theta}{\partial\theta_1{}^2}+\frac{\cot\gamma}{S}N_\theta\\
&+N_S^0\frac{\partial^2 w}{\partial S^2}+\frac{N_\theta^0}{S}\left(\frac{1}{S}\frac{\partial^2 w}{\partial\theta_1{}^2}+\frac{\partial w}{\partial S}\right)+2N_{S\theta}^0\left(\frac{1}{S}\frac{\partial^2 w}{\partial S\partial\theta_1}-\frac{1}{S^2}\frac{\partial w}{\partial\theta_1}\right)=0
\end{aligned}
\tag{14}
$$

$$
\frac{\cot\gamma}{\mathrm{S}}\frac{\partial^2 w}{\partial S^2}-\frac{2}{S}\frac{\partial^2 e_{S\theta}}{\partial S\partial\theta_1}-\frac{2}{S^2}\frac{\partial e_{S\theta}}{\partial\theta_1}+\frac{\partial^2 e_\theta}{\partial S^2}+\frac{1}{S^2}\frac{\partial^2 e_S}{\partial\theta_1{}^2}+\frac{2}{S}\frac{\partial e_\theta}{\partial S}-\frac{1}{S}\frac{\partial e_S}{\partial S}=0
\tag{15}
$$

Expanding the force and moment resultants and substituting into Eqs. (14) and (15), and together with (4) and (9) yields

$$
\begin{aligned}
&L(\Psi_1,w)\\
&\equiv\left(k_1\frac{\partial^4\Psi_1}{\partial z^4}+k_2\frac{\partial^3\Psi_1}{\partial z^3}+k_3\frac{\partial^2\Psi_1}{\partial z^2}+k_4\frac{\partial\Psi_1}{\partial z}+k_5\frac{\partial^4\Psi_1}{\partial\theta_1^4}+k_6\frac{\partial^4\Psi_1}{\partial z^2\partial\theta_1^2}+k_7\frac{\partial^3\Psi_1}{\partial z\partial\theta_1^2}+k_8\frac{\partial^2\Psi_1}{\partial\theta_1^2}\right)e^{2z}\\
&-k_9\frac{\partial^4 w}{\partial\theta_1^4}-k_{10}\frac{\partial^4 w}{\partial z^2\partial\theta_1^2}+k_{11}\frac{\partial^3 w}{\partial z\partial\theta_1^2}-k_{12}\frac{\partial^2 w}{\partial\theta_1^2}-k_{13}\frac{\partial^4 w}{\partial z^4}+k_{14}\frac{\partial^3 w}{\partial z^3}-k_{15}\frac{\partial^2 w}{\partial z^2}+k_{16}\frac{\partial w}{\partial z}\\
&+\left(\frac{\partial^2\Psi_1}{\partial z^2}+3\frac{\partial\Psi_1}{\partial z}+2\Psi_1\right)S_1e^{3z}\cot\gamma-e^{2z}S_1^2T\left(\frac{\partial^2 w}{\partial z^2}-\frac{\partial w}{\partial z}\right)-S_1^3e^{3z}P\tan\gamma\left(\frac{\partial^2 w}{\partial\varphi^2}+\frac{\partial w}{\partial z}\right)=0
\end{aligned}
\tag{16}
$$

$$\begin{aligned}&\left(q_1\frac{\partial^4\Psi_1}{\partial z^4}+q_2\frac{\partial^3\Psi_1}{\partial z^3}+q_3\frac{\partial^2\Psi_1}{\partial z^2}+q_4\frac{\partial\Psi_1}{\partial z}+q_5\frac{\partial^4\Psi_1}{\partial z^2\partial\theta_1^2}+q_6\frac{\partial^3\Psi_1}{\partial z\partial\theta_1^2}+q_7\frac{\partial^2\Psi_1}{\partial\theta_1^2}+q_8\frac{\partial^4\Psi_1}{\partial\theta_1^4}\right)\\&e^{2z}-q_9\frac{\partial^4 w}{\partial\theta_1{}^4}+q_{10}\frac{\partial^4 w}{\partial z^2\partial\theta_1^2}+q_{11}\frac{\partial^3 w}{\partial z\partial\theta_1^2}+q_{12}\frac{\partial^2 w}{\partial\theta_1^2}-q_{13}\frac{\partial^4 w}{\partial z^4}+q_{14}\frac{\partial^3 w}{\partial z^3}+q_{15}\frac{\partial^2 w}{\partial z^2}+q_{16}\frac{\partial w}{\partial z}\\&+\left(\frac{\partial^2 w}{\partial z^2}-\frac{\partial w}{\partial z}\right)S_1e^z\cot\gamma=0\end{aligned}\tag{17}$$

where $z = \ln(S/S_1)$ and $\Psi_1 = \Psi e^{-2z}$, and the following definitions apply

$$\begin{aligned}&k_1=c_{12},\ k_2=c_{11}-c_{22}-4c_{12},\ k_3=5c_{12}+3c_{11}-3c_{22}-c_{21},\ k_4=2(c_{11}-c_{22}+c_{12}-c_{21}),\\&k_5=c_{21},\ k_6=c_{11}-2c_{31}+c_{22},\ k_7=c_{11}-4c_{31}+3c_{22},\ k_8=2(c_{22}-c_{31}+c_{21}),\ \ k_9=c_{24},\\&k_{10}=c_{14}+c_{23}+2c_{32},\ k_{11}=c_{13};\ k_{12}=2(c_{14}+c_{32}+c_{24}),\ k_{13}=c_{13},\ k_{14}=4c_{13}+c_{23}-c_{14},\\&k_{15}=5c_{13}+3c_{23}-3c_{14}-c_{24},\ k_{16}=2(c_{13}+c_{23}-c_{14}-c_{24}),\\&q_1=b_{22},\ q_2=b_{21}-b_{12}-4b_{22},\ q_3=-3b_{21}-b_{11}+5b_{22}+3b_{12},\ q_4=2(b_{11}+b_{21}-b_{12}-b_{22}),\\&q_5=2b_{31}+b_{21}+b_{12},\ q_6=-4b_{31}-3b_{12}-b_{21},\ q_7=2(b_{31}+b_{21}+b_{11}),\ q_8=b_{11},\ q_9=b_{14},\\&q_{10}=2b_{32}-b_{13}-b_{24},\ q_{11}=3b_{24}-4b_{32}+b_{13},\ q_{12}=2(b_{32}-b_{24}-b_{14}),\ q_{13}=b_{23},\\&q_{14}=b_{13}-b_{24}+4b_{23},\ q_{15}=-3b_{13}+3b_{24}-5b_{23}+b_{14},\ q_{16}=2b_{13}+2b_{23}-2b_{14}-2b_{24}.\end{aligned}\tag{18}$$

The Eqs. (16) and (17) are the basic equations describing the required buckling response of NH orthotropic conical shells subjected to combined loading of external pressure and axial compression.

3. Solution of basic equations

It is assumed that the non-homogeneous orthotropic conical shell is subject to the simply-supported boundary conditions. According to study of Agamirov (1990), the deflection of axially and laterally combine-loaded conical shells can be expressed as follows

$$w = fe^z\sin(\beta_1 z)\sin(\beta_2\theta_1)\tag{19}$$

where f is the unknown amplitude, $\beta_1 = m\pi/z_0$ and $\beta_2 = n/\sin\gamma$, in which $z_0 = \ln(1 + L/S_1)$ and m and n are number of half-waves along a generatrix and an integer representing the circumferential wave number of the buckled conical shell, respectively.

Taking into account (19), the compatibility Eq. (17) is solved exactly to yield the stress function Ψ_1, satisfying the boundary conditions, as

$$\Psi_1 = f\left[A_1e^{-z}\sin(\beta_1 z)+A_2e^{-z}\cos(\beta_1 z)+A_3\cos(\beta_1 z)+A_4\sin(\beta_1 z)\right]\sin(\beta_2\theta_1)\tag{20}$$

where the following definitions apply

$$A_1 = \frac{z_1 x_1 + y_1 z_2}{x_1^2 + y_1^2}, \qquad A_2 = \frac{x_1 z_2 - z_1 y_1}{x_1^2 + y_1^2},$$
$$A_3 = \frac{\beta_1(\beta_1 y_0 - x_0)S_1 \cot\gamma}{x_0^2 + y_0^2}, \quad A_4 = \frac{\beta_1(\beta_1 x_0 + y_0)S_1 \cot\gamma}{x_0^2 + y_0^2} \tag{21}$$

in which

$$x_0 = -q_3\beta_1^2 + q_5\beta_1^2\beta_2^2 - q_7\beta_2^2 + q_1\beta_1^4 + q_8\beta_2^4, \; y_0 = q_2\beta_1^3 - q_4\beta_1 + q_6\beta_1\beta_2^2,$$
$$x_1 = q_5\beta_1^2\beta_2^2 + (3q_2 - 6q_1 - q_3)\beta_1^2 + (q_6 - q_7 - q_5)\beta_2^2 + q_1\beta_1^4 + q_8\beta_2^4 + q_1 - q_2 + q_3 - q_4,$$
$$y_1 = (q_6 - 2q_5)\beta_1\beta_2^2 + (2q_3 - q_4 + 4q_1 - 3q_2)\beta_1 + (q_2 - 4q_1)\beta_1^3, \tag{22}$$
$$z_1 = q_9\left[(\beta_2^2 - 1)^2 + \beta_1^2\right] + \beta_1^2\left[q_{13}(\beta_1^2 + 1) - q_{10}\beta_2^2\right], \quad z_2 = (4q_{13} - q_{14})\beta_1(\beta_2^2 - \beta_1^2 - 1).$$

Introduction of Eqs. (19) and (20) into Eq. (16) and applying the Galerkin method, yields

$$\int_0^{z_0}\int_0^{2\pi\sin\gamma} L(\Psi_1, w)e^z \sin(\beta_1 z)\sin(\beta_2\theta_1)dz d\theta_1 = 0, \tag{23}$$

(a) The truncated conical shell is subjected to an axial compressive load only, i.e., $N_S^0 = -T$, $N_\theta^0 = N_{S\theta}^0 = 0$. In this case, after integrating Eq. (23), for the dimensionless critical axial compressive load of the non-homogeneous orthotropic conical shell, yields

$$T_{1cr}^{ax} = \frac{8(\beta_1^2 + 4)}{S_1^2\beta_1^2(\beta_1^2 + 2)(e^{4z_0} - 1)}\frac{Q}{E_{0S}h} \tag{24}$$

where Q is parameter and defined as

$$Q = \begin{bmatrix} A_4\left(\beta_1^3 k_2 + \beta_1^2 k_3 - 2\beta_1 k_4 - \beta_1\beta_2^2 k_7\right) + A_3\left(k_1\beta_1^4 + \beta_2^4 k_5 + \beta_1^2\beta_2^2 k_6 - 2\beta_2^2 k_8\right) \\ + \left(A_1\beta_1 - A_2\beta_1^2\right)S_1\cot\gamma - \dfrac{2\beta_1 A_4}{3}\left(k_1\beta_1^4 - \beta_1^2 k_3 + \beta_2^4 k_5 + \beta_1^2\beta_2^2 k_6 - 2\beta_2^2 k_8\right) \\ - \dfrac{2\beta_1 A_3}{3}\left(\beta_1^3 k_2 + \beta_1\beta_2^2 k_7 - 2\beta_1 k_4\right) + \dfrac{2\beta_1}{3}\left(A_1\beta_1^2 + A_2\beta_1\right)S_1\cot\gamma \end{bmatrix}\frac{\beta_1\left(e^{3z_0} - 1\right)}{4\beta_1^2 + 9}$$
$$+ \begin{Bmatrix} A_4\left(\beta_1^3 k_2 + \beta_1^2 k_3 - 2\beta_1 k_4 - \beta_1\beta_2^2 k_7\right) + A_3\left(k_1\beta_1^4 + \beta_2^4 k_5 + \beta_1^2\beta_2^2 k_6 - 2\beta_2^2 k_8\right) \\ + \left(A_1\beta_1 - A_2\beta_1^2\right)S_1\cot\gamma - A_2\left(\beta_1^3 + \beta_1 - \beta_1\beta_2^2\right)\left(k_4 - k_1 + k_5\right)\beta_1 + k_{11}\beta_1^5 \\ - A_1\left[\left(\beta_1^4 + \beta_1^2\right)k_1\beta_1 + \left(\beta_1^2 + 1 + \beta_2^4 - 2\beta_2^2\right)k_5\beta_1 + \beta_1^3\beta_2^2 k_6\right] \\ + \left(k_{11} + k_9\right)\beta_1^3 + k_9\beta_1 + k_9\beta_2^4\beta_1 + \beta_1^3\beta_2^2 k_{10} - 2k_9\beta_2^2\beta_1 \end{Bmatrix}\frac{\beta_1\left(e^{2z_0} - 1\right)}{4(\beta_1^2 + 1)} \tag{25}$$
$$+ \left(A_3\beta_1^2 + A_4\beta_1^3 + 4A_3 + 4A_4\beta_1\right)\frac{\beta_1\left(e^{4z_0} - 1\right)}{8(\beta_1^2 + 4)}S_1\cot\gamma$$

(b) The truncated conical shell is subjected to uniform lateral pressure only, i.e., $N_S^0 = 0$, $N_\theta^0 = -PS\tan\gamma$; $N_{S\theta}^0 = 0$. In this case, after integrating Eq. (23), for the dimensionless critical lateral pressure of the non-homogeneous orthotropic conical shell, the following equation is obtained

$$P_{1cr}^L = \frac{5(25+4\beta_1^2)\cot\gamma}{S_1^3\beta_1^2(2\beta_2^2+3)(e^{5z_0}-1)}\frac{Q}{E_{0S}} \tag{26}$$

(c) The truncated conical shell subjected to uniform hydrostatic pressure only, i.e., $N_S^0 = -0.5PS\tan\gamma$, $N_\theta^0 = -PS\tan\gamma$; $N_{S\theta}^0 = 0$. In this case, after integrating Eq. (23), for the dimensionless critical hydrostatic pressure of the non-homogeneous orthotropic conical shell, the following equation is obtained

$$P_{1cr}^H = \frac{10(25+4\beta_1^2)\cot\gamma}{S_1^3\beta_1^2(2\beta_1^2+4\beta_2^2+11)(e^{5z_0}-1)}\frac{Q}{E_{0S}} \tag{27}$$

(d) The following equation is used for the critical combined axial compressive load and lateral pressure of the non-homogeneous orthotropic conical shell

$$\frac{T_1^{ax}}{T_{1cr}^{ax}} + \frac{P_1^L}{P_{1cr}^L} = 1 \tag{28}$$

where T_1^{ax} and P_1^L are dimensionless axial compressive load and dimensionless lateral pressure, respectively and the following definitions apply

$$T_1^{ax} = \frac{T}{E_{0S}h},\quad P_1^L = \frac{P_L}{E_{0S}} \tag{29}$$

Case (1) high values of axial compression combined with relatively low lateral pressure (Shen 2001, Sofiyev 2014), i.e., $T_1^{ax} = B_1P_1^L$. If $T_1^{ax} = B_1P_1^L$ is considering in Eq. (28), the expression for dimensionless critical axial compressive load and lateral pressure of the non-homogeneous orthotropic truncated conical shell is rewritten as

$$P_{1cr}^{cb} = \frac{T_{1cr}^{ax}P_{1cr}^L}{B_1P_{1cr}^L + T_{1cr}^{ax}} \tag{30}$$

where $B_1 = T_1^{ax}/P_1^L$ is the load-proportional parameter and is a positive number.

Case (2) high values of external pressure combined with relatively low axial load (Shen 2001), i.e., $P_1^L = B_2T_1^{ax}$. If $P_2^L = B_2T_1^{ax}$ is considering in Eq. (28), the expression for dimensionless critical combined axial compressive load and lateral pressure of the non-homogeneous orthotropic truncated conical shell is rewritten as

$$T_{1cr}^{cb} = \frac{T_{1cr}^{ax}P_{1cr}^L}{P_{1cr}^L + B_2T_{1cr}^{ax}} \tag{31}$$

where $B_2 = P_1^L / T_1^{ax}$ is the load-proportional parameter and is a positive number.

As $\gamma \to 0$, the truncated conical shell is transformed into a cylindrical shell, i.e.,

$$\gamma = \pi / 180000 \to 0, \quad S_1 \to \infty, \quad S_1 \sin\gamma = R, \quad \beta_1 \sin\gamma = \frac{m\pi R}{L} = m_2, \\ z_0 = \ln\left(1 + \frac{L}{S_1}\right) \approx \frac{L}{S_1}, \quad e^{-az_0} \approx 1 - a\frac{L}{S_1}, \quad a > 0, \quad L = L_1, \quad R_1 = R_2 = R \tag{32}$$

If (32) is taken into account in Eqs. (24), (26), (27), (30) and (31) corresponding expressions for the non-homogeneous orthotropic cylindrical shell are found, as a special case.

As $\mu = 0$ (or $\eta = 0$); $E_{0S} = E_{0\theta} = E_0$; $\nu_{S\theta} = \nu_{\theta S} = \nu_0$, the appropriate expressions for the homogeneous isotropic conical shell are found, as a special case.

As $\mu = 0$ (or $\eta = 0$), the appropriate expressions for homogeneous orthotropic conical shells are found, as a special case.

4. Numerical analysis and discussions

In this section, numerical results are presented and compared with existing data.

4.1 Comparisons

In order to test the validity of this research is carried out two comparisons. In the first example, the values of the dimensionless critical hydrostatic pressure of homogeneous isotropic cylindrical and truncated conical shells are compared in Table 1 with the results of Baruch *et al*. (1967) for

Table 1 Comparison the values of the critical hydrostatic pressure of homogeneous isotropic cylindrical and truncated conical shells with those of Baruch *et al*. (1967)

		$P_{1cr}^H \times 10^6$; (n_{cr})	
γ	L/R_1	Baruch *et al*. (1967)	Present study
0°	0.5	21.06(11)	21.238(11)
10°		19.40(11)	19.373(11)
30°		14.55(11)	14.397(11)
50°		8.813(11)	8.6403(11)
0°	1	9.838(8)	9.77997(8)
10°		8.569(9)	8.5339(9)
30°		5.843(9)	5.6651(9)
50°		3.285(9)	3.0976(9)
0°	2	4.744(6)	4.7461(6)
10°		3.740(7)	3.6878(7)
30°		2.237(8)	2.0770(8)
50°		1.164(8)	1.0105(8)

Table 2 Comparison of dimensionless critical loads homogeneous orthotropic cylindrical shells under combined lateral pressure and axial compression

$E_{0S}\,/\,E_{0\theta}$	B_1	$P_{1cr} \times 10^3$	
		Shen (2001)	Present study
5	0	0.1465 (3)	0.1473(3)
	0.1	0.1462 (3)	0.1472(3)
10	0	0.1644 (3)	0.1699(3)
	0.1	0.1641 (3)	0.1698(3)

different semi-vertex angle γ and $L\,/\,R_1$. The other computing data are $E_0 = 2 \times 10^{11}$Pa, $\nu_0 = 0.3$, $h = 0.01$ m and $R_1 = 1$ m. For the first comparison, the expression (27) is used. By taking $\mu = 0$ (or $\eta = 0$), $E_{0S} = E_{0\theta} = E_0$ and $\nu_{S\theta} = \nu_{\theta S} = \nu_0$, into the expression (27), the appropriate expression a dimensionless hydrostatic pressure for the homogeneous isotropic truncated conical shell is found, as a special case. In brackets indicate the circumferential wave numbers (n_{cr}) corresponding to the minimum values of a critical hydrostatic pressure and $\gamma \to 0°$ corresponds to the cylindrical shell. It can be seen that the present results are in good agreement with results of Baruch *et al*. (1967)

In addition, the dimensionless critical loads P_{1cr}^{L} and P_{1cr}^{Cb} for homogeneous orthotropic cylindrical shells under combined loading case (1) are compared in Table 2 with results of Shen (2001), for different values of stiffness ratio $E_{0S}\,/\,E_{0\theta}$ shown. In the second comparison, the expression (30) is used. By taking $\gamma \to 0°$ and $\mu = 0$ (or $\eta = 0$) into the expressions (26) and (30), the appropriate expressions for the dimensionless critical lateral and critical combined loads of the homogeneous orthotropic cylindrical shell is found, as a special case. $B_1 = 0$ indicates the loading case of uniform lateral pressure. The computing data adopted here are: $E_{0S} = 206.844$ GPa, $G_0 = 0.6\,E_{0\theta}$, $\nu_{S\theta} = 0.25$ and $h = 0.01$ m, $R\,/\,h = 20$, $L_1\,/\,R = 5$ (see, Shen 2001). Values in parentheses are the wave numbers (n_{cr}) corresponding to the critical loads. It can also be seen that the present results agree well with the results of Shen (2001).

4.2 Critical combined loads of NH orthotropic cylindrical and truncated conical shells

The buckling analysis has been presented for non-homogenous orthotropic truncated conical shells subjected to combined loading of external pressure and axial compression. Numerous examples were solved to illustrate their application to the performance of NH orthotropic cylindrical and truncated conical shells. Numerical computations for the critical loads (lateral, hydrostatic and combined loads) of homogeneous (H) and non-homogenous (NH) orthotropic cylindrical and truncated conical shells have been carried out using expressions (24), (26), (27), (30) and (31) and the results are presented in Figs. 2-5 and Table 3. The calculations were performed for the following types of orthotropic materials (except of Table 3). The homogeneous orthotropic material properties are taken to be (Glass/epoxy): $E_{0S} = 5.37791 \times 10^{10}$ Pa, $E_{0\theta} = 1.79264 \times 10^{10}$ Pa, $G_0 = 8.9632 \times 10^9$ Pa, $\nu_{S\theta} = 0.25$ and (Graphite/epoxy): $E_{0S} = 1.724 \times 10^{11}$ Pa, $E_{0\theta} = 7.79 \times 10^9$ Pa, $\nu_{S\theta} = 0.35$. In all subsequent calculations, the Young's muduli and shear modulus of orthotropic materials vary as linear, quadratic or exponential functions. The variation coefficients are taken into account as $\mu = 1$ or $\eta = 1$. The material properties of the shells are homogenous, as $\mu = 0$ or $\eta = 0$. The following expression is used for percents: $[(\mathrm{NH} - \mathrm{H})\,/\,\mathrm{H}] \times 100\%$. The negative sign in front of the percents show that the values of critical loads in non-homogeneity case are smaller than in homogeneity case.

Fig. 2 shows the variation of critical combined loads for H and NH graphite/epoxy and glass/epoxy truncated conical shells against the load-proportional parameter B_1, i.e., under combined loading case (1). Then Fig. 3 shows the variation of the values of critical combined loads for H and NH graphite/epoxy and glass/epoxy truncated conical shells versus the load-proportional parameter B_2, i.e., under combined loading case (2). The truncated conical shell characteristics are taken to be $R_1 / h = 100$; $L / R_1 = 2$, $\gamma = 45°$. It is seen that from Figs. 2 and 3, the

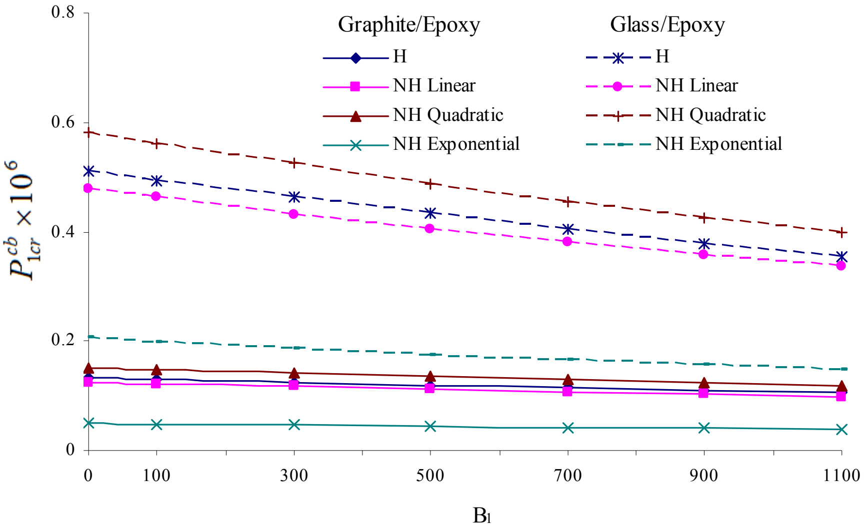

Fig. 2 Variation of the values of critical combined loads for H and NH graphite/epoxy and glass/epoxy truncated conical shells versus the load-proportional parameter B_1

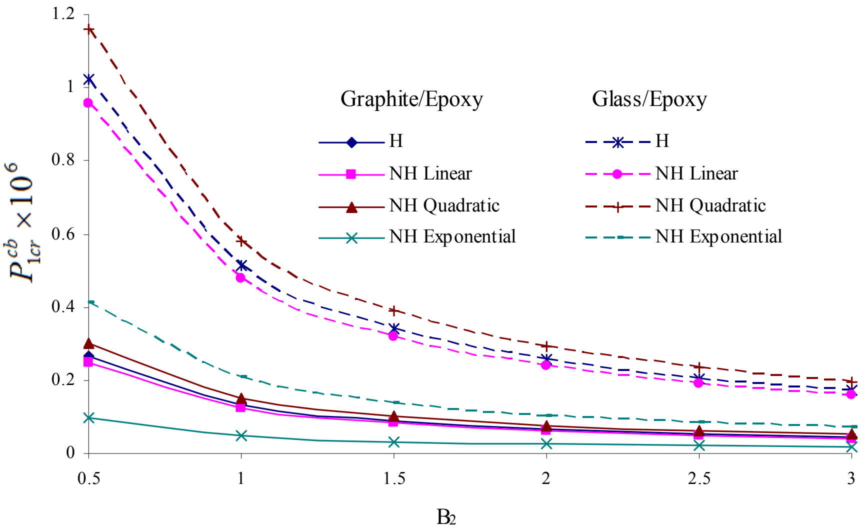

Fig. 3 Variation of the values of critical combined loads for H and NH graphite/epoxy and glass/epoxy truncated conical shells versus the load-proportional parameter B_2

values of critical combined loads for H and NH graphite/epoxy and glass/epoxy truncated conical shells decrease, as load-proportional parameters B_1 or B_2 increase. The influence of the variation of B_1 on the values of critical combined loads for the glass/epoxy conical shell is higher than the graphite/epoxy conical shell. The effect of changes of B_2, on the values of critical combined loads for both orthotropic conical shells is almost the same. The values of critical combined loads for H and NH glass/epoxy conical shells are higher than the H and NH graphite/epoxy conical shells for different load proportional parameters B_1 and B_2. The influences of load-proportional parameters on the values of the critical combined loads are important in the large values of B_1 and B_2.

The distribution of the values of critical loads for H and NH graphite/epoxy and glass/epoxy conical shells versus the stiffness ratio $E_{0S}/E_{0\theta}$ with the linear, quadratic and exponential profiles are tabulated in Table 3. The conical shell characteristics and material properties adopted here are $R_1/h = 150$; $L/R_1 = 2$ and $E_{0S} = 2 \times 10^{11}$ (Pa), $E_{0\theta} = E_{0S}/i$, $i = 10$; 25; 40, $\nu_{S\theta} = 0.2$; $\rho_0 = 7800$ kg/m^3. The truncated conical shell under combined loading case (1), with load-proportional parameter $B_1 = 500$. The number in brackets (n_{cr}) indicate the circumferential wave numbers corresponding to minimum values of critical loads. The values of critical lateral and hydrostatic pressures and combined load (axial compression and lateral pressure) for H and NH orthotropic conical shells decrease, while corresponding circumferential wave numbers increase, as the stiffness ratio $E_{0S}/E_{0\theta}$ increases. The values of critical combined loads for H and NH orthotropic conical shells are lower than the critical lateral or hydrostatic pressures. The effect of non-homogeneity on the values of critical combined loads for orthotropic conical shells changed irregularly, as the stiffness ratio $E_{0S}/E_{0\theta}$ increases from 10 to 40 by step 15. For example, the effect of heterogeneity on the values of critical combined loads for orthotropic conical shells with linear quadratic and exponential profiles are (6.49%, 5.0%, 6.9%) (-12.99% -15.0% -13.79%) and (63.64%, 62.5%, 62.07%), respectively, as the stiffness ratio $E_{0S}/E_{0\theta} = 10$, 25 and 40, respectively.

The distribution of the values of critical combined loads for H and NH graphite/epoxy and glass/epoxy conical shells versus the semi-vertex angle γ with the linear, quadratic and exponential profiles are given in Fig. 4. Here, $\gamma \to 0°$ corresponds to a cylindrical shell. The conical shell has the following geometric parameters: $R_1/h = 150$ and $L/R_1 = 2$. The shells under combined loading case (1), with load-proportional parameter $B_1 = 500$. The values of critical combined loads for H and NH graphite/epoxy and glass/epoxy conical shells decrease, as the semi-vertex angle γ increases. As the values of critical combined loads for NH orthotropic shells with linear,

Table 3 Variation of critical loads for H and NH graphite/epoxy and glass/epoxy conical shells versus the $E_{0S}/E_{0\theta}$ with the linear, quadratic and exponential profiles

	$P_{1cr}^{H}\times10^6$	$P_{1cr}^{L}\times10^6$	$P_{1cr}^{cb}\times10^6$	$P_{1cr}^{H}\times10^6$	$P_{1cr}^{L}\times10^6$	$P_{1cr}^{cb}\times10^6$
$E_{0S}/E_{0\theta}$	Homogeneous			NH linear		
10	0.081(12)	0.083(12)	0.077(12)	0.075(12)	0.077(12)	0.072(12)
25	0.042(13)	0.043(13)	0.04(13)	0.039(13)	0.040(13)	0.038(13)
40	0.030(13)	0.031(14)	0.029(13)	0.028(14)	0.029(14)	0.027(14)
$E_{0S}/E_{0\theta}$	NH quadratic			NH exponential		
10	0.092(12)	0.094(12)	0.087(11)	0.030(12)	0.030(12)	0.028(12)
25	0.048(13)	0.049(13)	0.046(13)	0.015(13)	0.016(13)	0.015(13)
40	0.034(13)	0.035(13)	0.033(13)	0.011(14)	0.011(14)	0.011(13)

quadratic and exponential profiles are compared with each other, the largest influence is observed in an exponential case. As the values of the critical combined loads for NH orthotropic cylindrical shells ($\gamma \to 0$) with linear, quadratic and exponential profiles are compared with homogeneous orthotropic cylindrical shells, the influence of non-homogeneity on the values of critical combined loads for graphite/epoxy (or glass/epoxy) cylindrical shells are 6.92%, (-13.84%), 62.89% (or 5.72%, -12.56%, 61.21%), respectively. As the values of critical combined loads for NH orthotropic conical shells with linear, quadratic and exponential profiles are compared with the homogeneous orthotropic conical shells, the effect of non-homogeneity on the values of critical combined loads for graphite/epoxy (or glass/epoxy) conical shells is significant and slightly change, as γ increases from 15° to 60° by step 15°. It is apparent from the Fig. 4 that the values of critical combined loads for H and NH orthotropic cylindrical shells are higher than the corresponding values of critical combined loads for H and NH orthotropic conical shells. The values of critical combined loads of H and NH glass/epoxy conical shells are higher than the H and NH graphite/epoxy conical shells, whereas, the influence of non-homogeneity on the values of critical combined load of graphite/epoxy shell is higher than the corresponding effect for the glass/epoxy shell.

Fig. 5 shows the effect of shell geometric parameter (L / R_1) on the values of critical combined loads of graphite/epoxy and glass/epoxy truncated conical shells under combined loading case (1), with the load-proportional parameter $B_1 = 500$. In computations, the following conical shell parameters are used: $R_1 / h = 150$ and $\gamma = 45°$. With increasing of the ratio L / R_1, the values of critical combined loads for graphite / epoxy and glass / epoxy truncated conical shells decrease. It is observed that the effect of non-homogeneity is significant, while this effect is changed with increasing of L / R_1. Comparing the values of critical combined loads of NH orthotropic conical shell with those of homogeneous orthotropic conical shell: (a) the effects of linear, quadratic and exponential profiles on the values of critical combined load for the graphite/epoxy conical shell are (6.52%; -13.77%, 63.04%), (4.76%; -14.29%, 61.9%) and (12.5%; -12.5, 62.5%), whereas, for

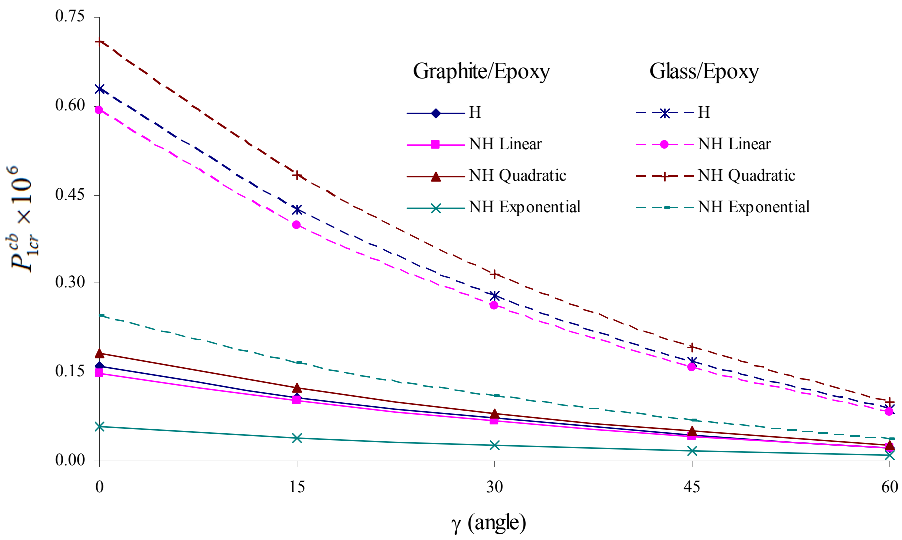

Fig. 4 Variation of critical combined loads for H and NH graphite/epoxy and glass/epoxy conical shells versus the semi-vertex angle γ with different non-homogeneity profiles

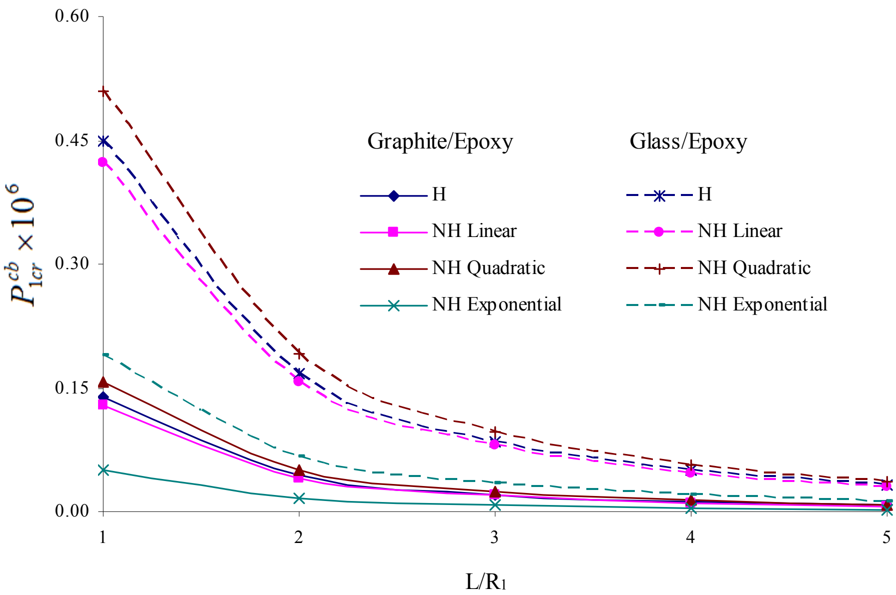

Fig. 5 Variation of the values of critical combined loads for H and NH graphite/epoxy and glass/epoxy conical shells versus L/R_1 with different non-homogeneity profiles

the glass/epoxy conical shell are (6%; -13.33%, 58%), (5.88%; -12.94%, 60%) and (6.25%; -15.63, 59.38%), respectively for L/R_1 = 1, 3 and 5.

5. Conclusions

In this study, the buckling of non-homogeneous orthotropic truncated conical shells under combined axial compression and lateral pressure is investigated. The governing equations have been obtained for the non-homogeneous orthotropic truncated conical shell, the material properties of which vary continuously in the thickness direction. By applying the Galerkin's method to the governing equations, the expressions for critical loads (axial, lateral, hydrostatic and combined) of non-homogeneous orthotropic truncated conical shells are obtained.

The numerical results support the following conclusions:

(a) The values of critical combined loads for H and NH graphite/epoxy and glass/epoxy truncated conical shells decrease, as load-proportional parameters B_1 or B_2 increase.

(b) The influence of the variation of B_1 on the values of critical combined loads for H and NH glass/epoxy conical shells is higher than H and NH graphite/epoxy conical shells, respectively, while the effect of changes of B_2 is almost the same.

(c) The values of critical combined loads for H and NH glass/epoxy conical shells are higher than the H and NH graphite/epoxy conical shells for different load proportional parameters B_1 and B_2.

(d) The values of critical lateral and hydrostatic pressures and combined load (axial compression and lateral pressure) for H and NH orthotropic conical shells decrease, while corresponding circumferential wave numbers increase, as $E_{0S}/E_{0\theta}$ increases.

(e) The effect of non-homogeneity on the values of critical combined loads for orthotropic conical shells changed irregularly, as the stiffness ratio $E_{0S} / E_{0\theta}$ increases .
(f) The values of critical combined loads for H and NH graphite/epoxy and glass/epoxy conical shells decrease, as the semi-vertex angle γ and L / R_1 increase.
(g) The effect of non-homogeneity on the values of critical combined loads for graphite/epoxy (or glass/epoxy) conical shells is significant and slightly change, as γ increases, while this effect is significant and changed with increasing of L / R_1.
(h) The values of critical combined loads of H and NH glass/epoxy conical shells are higher than the H and NH graphite/epoxy conical shells, whereas, the influence of non-homogeneity on the values of critical combined load of graphite/epoxy shell is higher than the corresponding effect for the glass/epoxy shell.
(i) The largest influence of non-homogeneity on the critical combined load is observed in an exponential case.
(j) The values of critical combined loads for H and NH orthotropic conical shells are lower than the critical lateral or hydrostatic pressures.

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Nomenclature

A_i (i = 1, 2, 3, 4)	:	Parameters
a_{ij}, c_{ij}, b_{ij} ($i.j$ = 1, 2, 6)	:	Coefficients depending on the material properties and shell characteristics
B_1, B_2	:	Load-proportional parameters
E_0	:	Young's moduli of homogeneous isotropic material
E_{0S}, $E_{0\theta}$	:	Young's moduli of homogeneous orthotropic material in S and θ directions, respectively
E_S, E_θ	:	Young's moduli of non-homogeneous orthotropic material in S and θ directions, respectively
e_S, e_θ, $e_{S\theta}$	:	Strains on the reference surface
f	:	Unknown amplitude
G	:	Shear modulus of the non-homogeneous orthotropic materials
G_0	:	Shear modulus of the homogeneous orthotropic materials
H	:	Shortening of the "homogeneous"
k = 1, 2,...	:	Power-law index
k_j, q_j (j = 1, 2,..., 16)	:	Coefficients depending on the material properties and shell characteristics
L	:	Length of truncated cone
L (Ψ_1, w)	:	Differential operator
M_s, M_θ, $M_{S\theta}$	:	Moment resultants
m	:	Number of half-waves along a generatrix
NH	:	Shortening of the "non-homogeneous"
N_S, N_θ, $N_{S\theta}$	:	Force resultants
N_s^0, N_θ^0, $N_{S\theta}^0$	:	Membrane forces for the condition with zero initial moments
n	:	Circumferential wave number
P	:	External normal pressure
P_1^L	:	Dimensionless lateral pressure
P_1^H	:	Dimensionless hydrostatic pressure
P_{1cr}^{cb}	:	Dimensionless critical combined load for low lateral pressure
P_{1cr}^H	:	Dimensionless critical hydrostatic pressure
P_{1cr}^L	:	Dimensionless critical lateral pressure

Q_{ij} $(i, j = 1, 2, 6)$	:	Quantities of non-homogeneous orthotropic materials
R_1, R_2	:	Radii of the truncated cone at its small and large ends
S	:	Axis in the direction of the generator of the cone
$S\theta\zeta$	:	Curvilinear coordinate system
S_1	:	Distance of the smaller end of the truncated conical shell from the vertex
T	:	Axial compressive load
T_1^{ax}	:	Dimensionless axial compressive load
T_{1cr}^{ax}	:	Dimensionless critical axial compressive load
T_{1cr}^{cb}	:	Dimensionless critical combined load for low axial load
w	:	Displacement of the reference surface in the normal direction
z	:	Independent variable
z_0	:	Parameter depending on shell characteristics
β_1, β_2	:	Parameters
γ	:	Semi-vertex angle of the cone
ε_S, ε_θ, $\varepsilon_{S\theta}$	:	Strains
η	:	Exponential factor
θ	:	Axis in the circumferential direction
θ_1	:	Variable depending on θ
μ	:	Non-homogeneity parameter
ν_0	:	Poisson's ratio of homogeneous isotropic material
$\nu_{S\theta}$, $\nu_{\theta S}$	:	Poisson's ratios of homogeneous orthotropic material
σ_S, σ_θ, $\sigma_{S\theta}$	:	Stresses
$\varphi(\bar{\zeta})$	:	Non-homogeneity function
ζ	:	Axis perpendicular to the S-θ plane
x_i, y_i, z_i $(i = 0, 1, 2)$	:	Parameters depending on the material properties and shell characteristics
Q	:	Parameter depending on the material properties and shell characteristics
Ψ (S, θ)	:	Stress function
Ψ_1	:	Function depending on Airy stress function