Steel and Composite Structures, *Vol. 18, No. 4 (2015) 909-924*
DOI: http://dx.doi.org/10.12989/scs.2015.18.4.909

Thermoelastic interaction in functionally graded nanobeams subjected to time-dependent heat flux

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(*Received March 29, 2014, Revised August 04, 2014, Accepted October 02, 2014*)

Abstract. This paper investigates the vibration phenomenon of a nanobeam subjected to a time-dependent heat flux. Material properties of the nanobeam are assumed to be graded in the thickness direction according to a novel exponential distribution law in terms of the volume fractions of the metal and ceramic constituents. The upper surface of the functionally graded (FG) nanobeam is pure ceramic whereas the lower surface is pure metal. A nonlocal generalized thermoelasticity theory with dual-phase-lag (DPL) model is used to solve this problem. The theories of coupled thermoelasticity, generalized thermoelasticity with one relaxation time, and without energy dissipation can extracted as limited and special cases of the present model. An analytical technique based on Laplace transform is used to calculate the variation of deflection and temperature. The inverse of Laplace transforms are computed numerically using Fourier expansion techniques. The effects of the phase-lags (PLs), nonlocal parameter and the angular frequency of oscillation of the heat flux on the lateral vibration, the temperature, and the axial displacement of the nanobeam are studied.

Keywords: thermoelasticity; DPL model; FG nanobeam; nonlocal Euler-Bernoulli theory; heat flux

1. Introduction

The classical uncoupled thermoelasticity theory predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory was given without any elastic terms. Second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves. Biot (1956) introduced the coupled thermoelasticity (CTE) theory to overcome the first shortcoming. The governing equations for this theory are coupled, eliminating the first paradox of the classical theory. However, both theories share the second shortcoming since the governing equations for the coupled are of the mixed type, parabolic-hyperbolic.

Two generalizations to the coupled theory were introduced. The first is due to Lord and Shulman (LS) (1967), who postulate a new law of heat conduction to replace the classical Fourier's law in their wave-type heat equation. This wave-type heat equation automatically

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http://www.techno-press.org/?journal=scs&subpage=8 ISSN: 1229-9367 (Print), 1598-6233 (Online)

ensures finite speeds of propagation for heat and elastic waves. The remaining governing equations for LS theory, namely, the equations of motion and constitutive relations, remain the same as those for the coupled and uncoupled theories. The second generalization to the CTE theory is what is known as the theory of thermoelasticity with two relaxation times or the theory of temperature-rate-dependent thermoelasticity. Müller (1971), in a review of the thermodynamics of thermoelastic solids, proposed an entropy production inequality, with the help of which he considered restrictions on a class of constitutive equations. A generalization of this inequality was proposed by Green and Laws (1972). Green and Lindsay (1972) obtained an explicit version of the constitutive equations. Green and Naghdi (GN) (1993) proposed a new generalized thermoelasticity theory by including the thermal-displacement gradient among the independent constitutive variables. An important feature of this theory, which is not present in other thermoelasticity theories, is that it does not accommodate dissipation of thermal energy.

Tzou (1995a, 1996) proposed the dual-phase-lag (DPL) model, which describes the interactions between phonons and electrons on the microscopic level as retarding sources causing a delayed response on the macroscopic scale. For macroscopic formulation, it would be convenient to use the DPL mode for investigation of the micro-structural effect on the behavior of heat transfer. The physical meanings and the applicability of the DPL model have been supported by the experimental results (Tzou 1995b). The DPL is such a modification of the classical thermoelastic model in which Fourier law is replaced by an approximation to a modified Fourier law with two different time translations: a PL of the heat flux τ_q and a PL of the temperature gradient τ_θ. Prasad *et al*. (2010, 2011) and Mukhopadhyay *et al*. (2011) investigated the effects of phase-lags on the wave propagation problems of some deformable structures.

Micro-scale mechanical resonators have high sensitivity as well as fast response and are widely used as sensors and modulators. Recently, micro- and nano-mechanical resonators have attracted considerable attention due to their important technological applications. Accurate analysis of various effects on the characteristics of resonators, such as resonant frequencies and quality factors, is crucial for designing high-performance components. Many authors have studied the vibration and heat transfer process of beams. Fang *et al*. (2006) studied the vibration and heat transfer process of beams. Al-Huniti *et al*. (2001) investigated the thermally-induced displacements and stresses of a rod using the Laplace transformation technique. Kidawa-Kukla (2003) studied the problem of transverse vibrations of a beam induced by a mobile heat source. The analytical solution to the problem was obtained using the Green's functions method without considering the thermoelastic coupling effect.

The present nanobeam is made of a ceramic and a metal for the purpose of thermal protection against large temperature gradients. The ceramic material provides a high temperature resistance due to its low thermal conductivity, while the ductile metal constituent prevents fracture due to its greater toughness. This is a new class of composite materials, which is known as functionally graded materials (FGMs). FGM characterizes a class of materials where the microstructures are spatially graded to achieve specific thermal properties to suit the functionality of the structure. The gradient compositional variation of the constituents from one surface to the other provides an elegant solution to the problem of high transverse shear stresses that are induced when two dissimilar materials with large difference in material properties are bonded. Gradually varying the material properties can prevent from interface cracking, delamination and residual stresses and thus maintain structural integrity to a desirable level.

Ching and Yen (2006) presented numerical solutions obtained by the meshless local Petrov-Galerkin method for transient thermoelastic deformations of FG beams. Malekzadeh and

Shojaee (2014) investigated the dynamic response of FG beams under a moving heat source. The material properties are assumed to be temperature-dependent and graded in the thickness direction. Mareishi *et al*. (2013) developed the thermo-mechanical vibrations of FG beams. Governing equations are obtained based on higher-order variation of transverse shear strain through the depth of the beam. Recently, Abbas and Zenkour (2013) presented the electro-magneto-thermoelastic analysis problem of an infinite FGM hollow cylinder based upon LS theory.

The nonlocal elasticity theory initiated by Eringen (1972, 1983) and Eringen and Edelen (1972) is widely used. The local theories assume that the stress at a point is a function of strain at that point. However, the nonlocal elasticity theory assumes that the stress at a point is a function of strains at all points in the continuum. In this paper, the vibration phenomenon of a nanobeam subjected to a time-dependent heat flux is studied (see also, Zenkour and Abouelregal 2014a, b, c). Material properties of the nanobeam are assumed to be graded in the thickness direction according to a novel exponential distribution in terms of the volume fractions of the metal and ceramic constituents. The solution for the generalized thermoelastic vibration of the nanobeam induced by a time-dependent heat flux is developed. The Laplace transform method is used to determine the lateral vibration, the temperature and the displacement of the nanobeam. The effect due to some parameters, especially the nonlocal one, on thermal oscillation quantities is studied and represented graphically.

2. Basic equations of generalized thermoelasticity theory

The governing equations of the linear theory of thermoelasticity with DPLs are:

Equations of motion:

$$\sigma_{ji,j} + F_i = \rho \ddot{u}_i \tag{1}$$

where σ_{ij} are the components of the stress tensor, F_i are the components of body force vector, ρ is the material density, and u_i are the components of the displacement vector.

Constitutive equations:

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e_{kk}\delta_{ij} - \gamma\theta\delta_{ij}, \tag{2}$$

where λ and μ are Lame's coefficients, e_{ij} is the strain tensor, e_{kk} is the strain dilatation, $\gamma = (3\lambda + 2\mu)\,\alpha_t$ is the coupling parameter, in which α_t being the coefficient of linear thermal expansion, $\theta = T - T_0$ denotes the thermodynamical temperature, in which T is the temperature and T_0 is the reference temperature assumed to be such that $|\theta / T_0| << 1$, and δ_{ij} is the Kronecker's delta.

Equation of entropy (Energy equation):

$$\rho\theta_0\dot{\eta}_i = -q_{i,i} + Q \tag{3}$$

where η is the entropy per unit volume measured from the entropy of the reference state, q_i are the components of the heat flows vector, Q is the heat supplied per unit volume from the external world, and θ_0 is the reference thermodynamical temperature.

The modified Fourier's law:

$$\left(1 + \tau_q \frac{\partial}{\partial t}\right) q_i = -K\left(1 + \tau_\theta \frac{\partial}{\partial t}\right)\theta_{,i} \tag{4}$$

where K is the thermal conductivity, τ_θ and τ_q denote the finite times required for thermal equilibrium to be obtained and for effective collisions to take place between the electrons and the phonons, respectively. The delay time τ_θ is said to be the PL of the temperature gradient and the other delay time τ_q, the PL of the heat flux, which will ensure that the heat conduction equation will predict finite speeds of heat propagation.

The entropy-strain-temperature relation:

$$\rho\eta = \gamma e_{kk} + \frac{\rho C_E}{\theta_0}\theta \tag{5}$$

where C_E is the specific heat at constant strain.

The strain-displacement relations:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{6}$$

In all of the above equations, the comma followed by a suffix denotes partial derivation with respect to the space variables and the superposed dot denotes the derivation with respect to the time t. Eqs. (3)-(5) give the heat equation of the generalized theory of thermoelasticity with DPLs as (Tzou 1995b):

$$\left(1+\tau_\theta \frac{\partial}{\partial t}\right)\nabla\cdot(K\nabla\varphi) = \rho C_E\left(1+\tau_q \frac{\partial}{\partial t}\right)\left[\frac{\partial\theta}{\partial t} + \gamma T_0 \frac{\partial e_{kk}}{\partial t} - Q\right] \tag{7}$$

In what follows, Lame's coefficients λ and μ will be given in terms of the engineering coefficients: Young's modulus E and Poisson's ratio v.

3. Formulation of the problem

Let us consider a FG thermoelastic solid nanobeam in Cartesian coordinate systems $Oxyz$. The x axis is drawn along the axial direction of the beam and the y and z axes correspond to the width and thickness, respectively (see Fig. 1). In equilibrium, the beam is unstrained, unstressed and at temperature T_0 everywhere. The small flexural deflections of the nanobeam with dimensions of length L ($0 \le x \le L$), width b ($-b/2 \le y \le +b/2$) and thickness h ($-h/2 \le z \le +h/2$) are considered. The basic governing equations of motion, balance of equilibrated force and heat conduction in the context of generalized (non-Fourier) thermoelasticity for displacements u_i in the absence of body forces, external loads, extrinsic equilibrated body force and heat sources are also considered.

A new model of FGMs is presented to treat the governing equations of the thermoelastic nanobeam that subjected to a sinusoidal pulse heating. Based on this model, the effective material property $P(z)$ gradation through the thickness direction is presented by (Zenkour 2006, 2014)

$$P(z) = P_m e^{n_P(2z-h)/h}, \quad n_P = \ln\sqrt{P_m/P_c} \tag{8}$$

where P_M and P_c represent the metal and ceramic properties, respectively. This study assumes that Young's modulus E, material density ρ, thermal conductivity coefficient K and the stress-temperature modulus γ of the FGM change continuously through the thickness direction of the

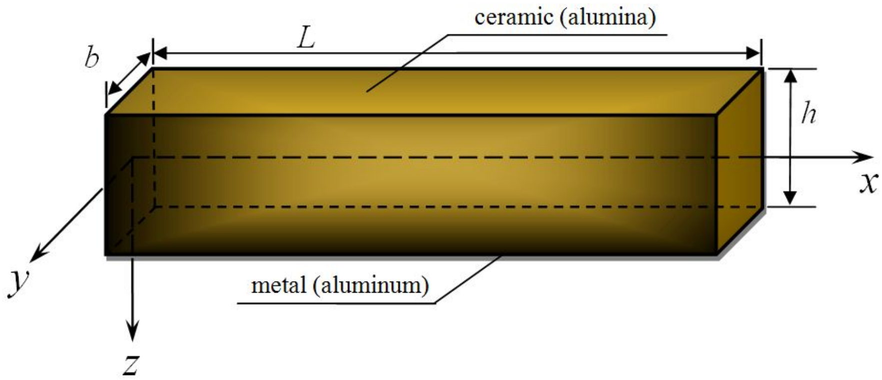

Fig. 1 Schematic diagram for the FG nanobeam

beam according to the gradation relation given in Eq. (8). It is to be noted that the material properties of the considered beam are metal-rich (fully metal) at the lower surface ($z = +h/2$) and ceramic-rich (fully ceramic) at the upper surface ($z = -h/2$) of the beam.

The beam undergoes bending vibrations of small amplitude about the x-axis such that the deflection is consistent with the linear Euler-Bernoulli beam theory. That is, any plane cross-section initially perpendicular to axis of beam remains, plane and perpendicular to the neutral surface during bending. Thus, the displacements $u_i \equiv (u, v, w)$ are given by

$$u = -z\frac{\partial w}{\partial x}, \quad v = 0, \quad w(x, y, z, t) = w(x, t) \tag{9}$$

where w is the lateral deflection. Substituting this Euler-Bernoulli assumption into Eq. (7), with the aid of Eq. (8), gives the thermal conduction equation for the beam without the heat source (Q = 0), as

$$\begin{aligned} & K_m\left(1+\tau_\theta\frac{\partial}{\partial t}\right)e^{n_K(2z-h)/h}\left(\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}+\frac{2n_K}{h}\frac{\partial\theta}{\partial z}\right) \\ & =\left(1+\tau_q\frac{\partial}{\partial t}\right)\left[\rho_m C_m^e e^{n_{\rho C^e}(2z-h)/h}\frac{\partial\theta}{\partial t}-z\gamma_m e^{n_\gamma(2z-h)/h}T_0\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right] \end{aligned} \tag{10}$$

where K_m, ρ_m, γ_m and C_m^e are, respectively, thermal conductivity coefficient, the material density, thermal modulus, and the specific heat per unit mass at constant strain of the metal material. Note that the parameters n_K, n_γ, n_ρ and $n_{\rho C^e}$ are given according to Eq. (8) in terms the properties of ceramic and metal materials, and

$$\gamma_m = \frac{E_m\alpha_m}{1-2v_m}, \quad \rho_m C_m^e = \frac{K_m}{\chi_m} \tag{11}$$

in which α_m, E_m, v_m and χ_m are the thermal expansion coefficient, Young's modulus, Poisson's ratio and the thermal diffusivity of the metal material, respectively.

There is no heat flow across the upper and lower surfaces of the beam (thermally insulated), so that $\partial\theta/\partial z$ should be vanish at the upper and lower surfaces of the beam $z = \pm h/2$. For a very thin beam (nanobeam), assuming that the temperature increment varies sinusoidally along the thickness direction. That is

$$\theta(x,z,t)=\theta_1(x,t)\sin(pz) \tag{12}$$

where $p = \pi/h$. Now, substituting Eq. (12) into Eq. (10) and integrating the resulting equation with respect to z through the beam thickness from $-h/2$ to $h/2$, yields

$$\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\partial^2\theta_1}{\partial x^2}=\left(1+\tau_q\frac{\partial}{\partial t}\right)\left[\bar{\mu}_{\rho C^e}\eta\frac{\partial\theta_1}{\partial t}-\frac{\bar{\mu}_\gamma T_0 h\gamma_{\mathrm{m}}}{K_{\mathrm{m}}}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right] \tag{13}$$

where $\eta = \rho_m C_m^e/K_m$, $\bar{\mu}_{\rho C^e}=\mu_{\rho C^e}/\mu_K$ and $\bar{\mu}_\gamma=\mu_\gamma/\mu_K$ in which.

$$\mu_K=\frac{2n_K(1+\mathrm{e}^{-2n_K})}{\pi^2+4n_K^2},\quad \mu_{\rho C^e}=\frac{2n_{\rho C^e}\left(1+\mathrm{e}^{-2n_{\rho C^e}}\right)}{\pi^2+4n_{\rho C^e}^2},\quad \mu_\gamma=\frac{n_\gamma(1+\mathrm{e}^{-2n_\gamma})-1+\mathrm{e}^{-2n_\gamma}}{4n_\gamma^2} \tag{14}$$

It is to be noted that, the nonlocal theory assumes that stress at a point depends not only on the strain at that point but also on strains at all other points of the body. So, the one-dimensional constitutive equation gives the uniaxial tensile stress only, according to the differential form of the nonlocal constitutive relation proposed by Eringen (1972, 1983) and Eringen and Edelen (1972), as

$$\sigma_x-\xi\frac{\partial^2\sigma_x}{\partial x^2}=-E_{\mathrm{m}}\left(\mathrm{e}^{n_E(2z-h)/h}\,z\frac{\partial^2 w}{\partial x^2}+\alpha_{\mathrm{m}}\,\mathrm{e}^{n_{E\alpha}(2z-h)/h}\,\theta\right) \tag{15}$$

where $n_{E\alpha}=\ln\sqrt{E_m\alpha_m/E_c\alpha_c}$ in which α_c and E_c are the thermal expansion coefficient and Young's modulus of the ceramic material, respectively. Note that $\xi = (e_0L)^2$ is the nonlocal parameter in which e_0 is a constant appropriate to each material and L is the internal characteristic length. In general, a conservative estimate of the nonlocal parameter is $e_0L < 2.0$ nm for a single wall carbon nanotube (Wang and Wang 2007).

The flexure moment of the cross-section is given, with the aid of Eq. (12), by

$$M(x,t)-\xi\frac{\partial^2 M}{\partial x^2}=-bh^2E_m\left(h\mu_E\frac{\partial^2 w}{\partial x^2}+\alpha_m\mu_{E\alpha}\theta_1\right) \tag{16}$$

where

$$\begin{aligned}\mu_E&=\frac{(n_E^2+2)(1-\mathrm{e}^{-2n_E})-2n_E(1+\mathrm{e}^{-2n_E})}{8n_E^3}\\ \mu_{E\alpha}&=\frac{n_{E\alpha}(\pi^2+4n_{E\alpha}^2)(1-\mathrm{e}^{-2n_{E\alpha}})+(\pi^2-4n_{E\alpha}^2)(1+\mathrm{e}^{-2n_{E\alpha}})}{(\pi^2+4n_{E\alpha}^2)^2}\end{aligned} \tag{17}$$

The equation of transverse motion for the present beam is

$$\frac{\partial^2 M}{\partial x^2}=\frac{1-e^{-2n_\rho}}{2n_\rho}\rho_m A\frac{\partial^2 w}{\partial t^2} \tag{18}$$

where $A = bh$ is the cross-section area. Substituting Eq. (16) into Eq. (18), one can get the motion equation of the beam as

$$\frac{\partial^4 w}{\partial x^4}+\frac{1-\mathrm{e}^{-2n_\rho}}{2n_\rho \mu_E \varepsilon^2 h^2}\left(\frac{\partial^2 w}{\partial t^2}-\xi\frac{\partial^4 w}{\partial x^2 \partial t^2}\right)+\frac{\alpha_m \bar{\mu}_{E\alpha}}{h}\frac{\partial^2 \theta_1}{\partial x^2}=0 \tag{19}$$

where $\bar{\mu}_{E\alpha}=\mu_{E\alpha}/\mu_E$ and $\varepsilon=\sqrt{E_m/\rho_m}$.

The preceding governing equations can be put in non-dimensional forms using the following dimensionless parameters

$$\begin{aligned}&(x',L',u',w',z',h')=\eta\varepsilon(x,L,u,w,z,h),\\&(t',\tau_0',t_0')=\eta\varepsilon^2(t,\tau_0,t_0),\quad \xi=\eta^2 c^2\xi,\quad \theta_1'=\frac{\theta_1}{T_0}\end{aligned} \tag{20}$$

So, the governing equations, and the bending and constitutive equations in non-dimensional forms are simplified as (dropping the primes for convenience)

$$\begin{aligned}&\frac{\partial^4 w}{\partial x^4}+A_1\left(\frac{\partial^2 w}{\partial t^2}-\xi\frac{\partial^4 w}{\partial x^2 \partial t^2}\right)+A_2\frac{\partial^2 \theta_1}{\partial x^2}=0\\&\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\frac{\partial^2 \theta_1}{\partial x^2}=\left(1+\tau_q\frac{\partial}{\partial t}\right)\left[A_3\frac{\partial \theta_1}{\partial t}-A_4\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right]\end{aligned} \tag{21}$$

where

$$A_1=\frac{1-e^{-2n_\rho}}{2n_\rho \mu_E h^2},\quad A_2=\frac{\alpha_m \bar{\mu}_{E\alpha} T_0}{h},\quad A_3=\bar{\mu}_{\rho C^e},\quad A_4=\frac{\bar{\mu}_\gamma h \gamma_m}{\eta K_m} \tag{22}$$

4. Initial and boundary conditions

The initial and boundary conditions should be considered to solve the present problem. The initial conditions of the problem are taken as

$$w(x,t)\big|_{t=0}=\left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0}=0,\quad \theta_1(x,t)\big|_{t=0}=\left.\frac{\partial \theta_1(x,t)}{\partial t}\right|_{t=0}=0 \tag{23}$$

These conditions are supplemented by considering the two ends of the nanobeam satisfy the boundary conditions

$$w(x,t)\big|_{x=0,L}=0,\qquad \left.\frac{\partial^2 w(x,t)}{\partial x^2}\right|_{x=0,L}=0 \tag{24}$$

A dimensionless time dependent heat flux $q(z, t)$ of constant intensity q_0 is applied on the boundary $x = 0$

$$\frac{\partial\theta}{\partial x}=q(z,t)=q_0\sin(pz)q_1(t)\ \text{ on }\ x=0 \tag{25}$$

Let us consider the heat flux $q_1(t)$ is varying harmonically with time described mathematically as follow

$$q_1(t)=\cos(\omega t),\quad \omega>0\quad \text{on}\ \ x=0 \tag{26}$$

where ω is the angular frequency of thermal ocillation ($\omega = 0$ for a constant heat flux). Using Eqs. (12) and (25), then one gets

$$\frac{\partial\theta_1}{\partial x}=q_0\cos(\omega t)\ \text{ on }\ x=0 \tag{27}$$

Asumming that the boundary $x = L$ is thermally insulated, this means that the following relation will be satisfied

$$\frac{\partial\theta_1}{\partial x}=0\quad \text{on}\ \ x=L \tag{28}$$

5. Solution procedure

The closed form solution of the governing and constitutive equations may be possible by adapting the Laplace transformation method. Applying the Laplace transform to Eqs. (21), one gets the field equations as

$$\begin{aligned}&\frac{d^4\overline{w}}{dx^4}-A_1s^2\left(\xi\frac{d^2\overline{w}}{dx^2}-\overline{w}\right)+A_2\frac{d^2\overline{\theta}_1}{dx^2}=0\\&\frac{d^2\overline{\theta}_1}{dx^2}=q\left(A_3\overline{\theta}_1-A_4\frac{d^2\overline{w}}{dx^2}\right)\end{aligned} \tag{29}$$

where an over bar symbol denotes its Laplace transform, s denotes the Laplace transform parameter and

$$q=\frac{s(1+\tau_q s)}{1+\tau_\theta s} \tag{30}$$

Elimination $\overline{\theta}_1$ or $\overline{w}$ from Eqs. (29) gives the following differential equation for either $\overline{w}$ or $\overline{\theta}_1$

$$\left[\frac{d^6}{dx^6}-A\frac{d^4}{dx^4}+B\frac{d^2}{dx^2}-C\right]\{\overline{w},\overline{\theta}_1\}=0, \tag{31}$$

where the coefficients A, B and C are given by

$$A = q(A_3 + A_2 A_4) + A_1 s^2 \xi, \quad B = A_1 s^2 (1 + A_3 q \xi), \quad C = A_1 A_3 s^2 q \tag{32}$$

Introducing m_i (i = 1, 2, 3) into Eq. (31), one gets

$$[(D^2 - m_1^2)(D^2 - m_2^2)(D^2 - m_3^2)]\{\overline{w}, \overline{\theta}_1\} = 0 \tag{33}$$

where D = d/dx and m_1^2, m_2^2 and m_3^2 are the roots of the characteristic equation

$$m^6 - Am^4 + Bm^2 - C = 0 \tag{34}$$

These roots are given by

$$\begin{aligned} m_1^2 &= \tfrac{1}{3}[2p_0 \sin(q_0) + A], \\ m_2^2 &= -\tfrac{1}{3} p_0[\sqrt{3}\cos(q_0) + \sin(q_0)] + \tfrac{1}{3} A \\ m_3^2 &= \tfrac{1}{3} p_0[\sqrt{3}\cos(q_0) - \sin(q_0)] + \tfrac{1}{3} A \end{aligned} \tag{24}$$

where

$$p_0 = \sqrt{A^2 - 3B}, \quad q_0 = \tfrac{1}{3}\sin^{-1}\left(-\frac{2A^3 - 9AB + 27C}{2p_0^3}\right) \tag{36}$$

The solution of Eqs. (33) in the Laplace transformation domain can be represented as

$$\{\overline{w}, \overline{\theta}_1\} = \sum_{i=1}^{3} \left(\{C_i, F_i\} e^{-m_i x} + \{C_{i+3}, F_{i+3}\} e^{m_i x}\right) \tag{37}$$

where C_i and F_i are parameters depending on s. The compatibility between these two equations and Eqs. (29), gives

$$F_i = \beta_i C_i, \quad F_{i+3} = \beta_i C_{i+3}, \quad \beta_i = -\frac{m_i^4 + A_1 s^2 - A_1 s^2 \xi m_i^2}{A_2 m_i^2} \tag{38}$$

So

$$\{\overline{\theta}_1, \overline{w}\} = \sum_{i=1}^{3} \{\beta_i, 1\}\left(C_i e^{-m_i x} + C_{i+3} e^{m_i x}\right) \tag{39}$$

Then, the axial displacement after using the above equation takes the form

$$\overline{u} = -z \frac{d\overline{w}}{dx} = z \sum_{i=1}^{3} m_i \left(C_i e^{-m_i x} - C_{i+3} e^{m_i x}\right) \tag{40}$$

In addition, the strain will be

$$\bar{e} = \frac{d\bar{u}}{dx} = -z\sum_{i=1}^{3} m_i^2\left(C_i e^{-m_i x} + C_{i+3}e^{m_i x}\right) \tag{41}$$

After using Laplace transform, the boundary conditions take the forms

$$\begin{aligned} &\bar{w}(x,s)\big|_{x=0,L} = 0, \qquad \left.\frac{\mathrm{d}^2\,\bar{w}(x,s)}{\mathrm{d}x^2}\right|_{x=0,L} = 0, \\ &\left.\frac{\partial\bar{\theta}_1(x,s)}{\partial x}\right|_{x=0} = \frac{q_0 s}{s^2+\omega^2} = \bar{G}(s), \qquad \left.\frac{\mathrm{d}\bar{\theta}_1(x,s)}{\mathrm{d}x}\right|_{x=L} = 0 \end{aligned} \tag{42}$$

Substituting Eqs. (39) into the above boundary conditions, one obtains six linear equations in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{-m_1 L} & e^{-m_2 L} & e^{-m_3 L} & e^{m_1 L} & e^{m_2 L} & e^{m_3 L} \\ m_1^2 & m_2^2 & m_3^2 & m_1^2 & m_2^2 & m_3^2 \\ m_1^2 e^{-m_1 L} & m_2^2 e^{-m_2 L} & m_3^2 e^{-m_3 L} & m_1^2 e^{m_1 L} & m_2^2 e^{m_2 L} & m_3^2 e^{m_3 L} \\ -m_1\beta_1 & -m_2\beta_2 & -m_3\beta_3 & m_1\beta_1 & m_2\beta_2 & m_3\beta_3 \\ -m_1\beta_1 e^{-m_1 L} & -m_2\beta_2 e^{-m_2 L} & -m_3\beta_3 e^{-m_3 L} & m_1\beta_1 e^{m_1 L} & m_2\beta_2 e^{m_2 L} & m_3\beta_3 e^{m_3 L} \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \bar{G}(s) \\ 0 \end{Bmatrix} \tag{43}$$

The solution of the above system of linear equations gives the unknown parameters C_i and C_{i+3}. This completes the solution of the problem in the Laplace transform domain.

6. Inversion of the Laplace transforms

In order to determine the conductive and thermal temperature, displacement, and stress distributions in the time domain, the Riemann-sum approximation method is used to obtain the numerical results. In this method, any function in the Laplace domain can be inverted to the time domain as

$$f(t) = \frac{e^{\zeta t}}{t}\left[\tfrac{1}{2}\mathrm{Re}[\bar{F}(\zeta)] + \mathrm{Re}\sum_{n=0}^{N}\left(\bar{F}\left(\zeta + \frac{in\pi}{t}\right)(-1)^n\right)\right] \tag{44}$$

where Re is the real part and i is the imaginary number unit. For faster convergence, numerical experiments have shown that the value that satisfies the above relation is $\zeta \approx 47/t$ (Tzou 1996).

7. Numerical results

In terms of the Riemann-sum approximation defined in Eq. (44), numerical Laplace inversion is performed to obtain the dimensionless field variables of the nanobeam. The aluminum as lower metal surface and alumina as upper ceramic surface are used for the present nanobeam. The material properties are assumed to be *metal* (*aluminum*):

$$E_m = 70\,\text{GPa}, \quad \nu_m = 0.35, \quad \rho_m = 2700\,\text{Kg}\,\text{m}^{-3}$$
$$\alpha_m = 23.1\times10^{-6}\,\text{K}^{-1}, \ \chi_m = 84.18\times10^{-6}\,\text{m}^2/\text{s}, \ K_m = 237\,\text{W}\,\text{m}^{-1}\,\text{K}^{-1} \tag{45}$$

ceramic (*alumina*):

$$E_c = 116\,\text{GPa}, \quad \nu_c = 0.33, \quad \rho_c = 3000\,\text{Kg}\,\text{m}^{-3}$$
$$\alpha_c = 8.7\times10^{-6}\,\text{K}^{-1}, \ \ \chi_c = 1.06\times10^{-6}\,\text{m}^2/\text{s}, \ \ K_c = 1.78\,\text{W}\,\text{m}^{-1}\,\text{K}^{-1} \tag{46}$$

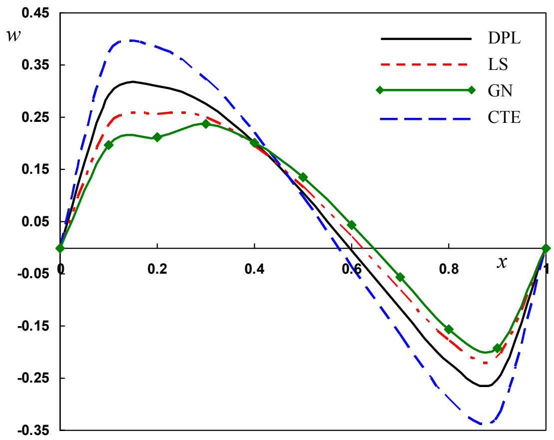

Fig. 2 The transverse deflection due to different models of thermoelasticity ($\bar{\xi} = 2, \omega = 5, z = h/6$)

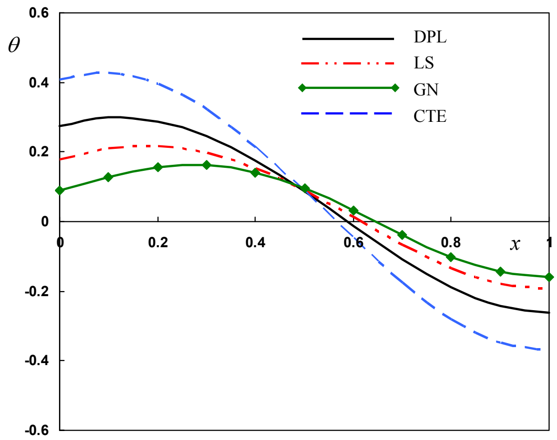

Fig. 3 The temperature due to different models of thermoelasticity ($\bar{\xi} = 2,\ \omega = 5,\ z = h/6$)

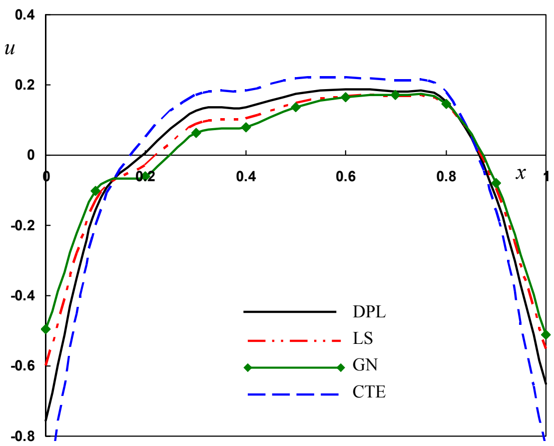

Fig. 4 The axial displacement due to different models of thermoelasticity ($\bar{\xi} = 2,\ \ \omega = 5,\ z = h/6$)

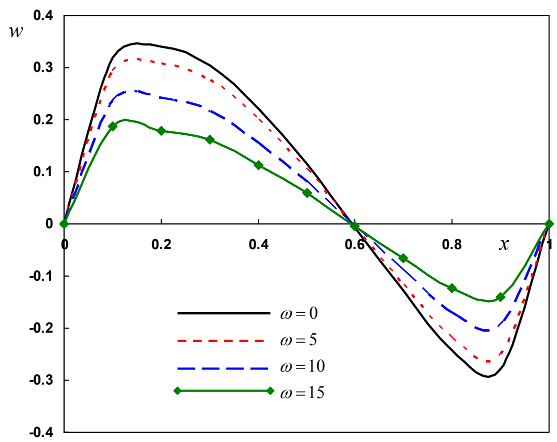

Fig. 5 Dependence of lateral vibration on different values of the angular frequency of the heat flux ω ($\bar{\xi} = 2,\ \tau_q = 0.02, \tau_\theta = 0.01,\ \ z = h/6$)

The reference temperature of the nanobeam is $T_0 = 293$ K. The length-to-thickness ratio of the beam is fixed as $L / h = 10$. The plots are prepared by using the dimensionless variables for a wide range of beam length when, except otherwise stated, $L = 1$, $z = h / 3$ and $t = 0.12$. In what follows, the nonlocal parameter $\bar{\xi}$ ($\bar{\xi} = 10^6\,\xi$) is used. It should be less than 4(μm^2). The local theory of the beam is given when $\bar{\xi} = 0$. Some plots consider the present quantities through the length of the beam and others take into account both the length and thickness directions. Numerical calculations are carried out to investigate the dimensionless lateral vibration w, thermodynamic temperature θ and axial displacement u. Figs. 2-13 represent the curves predicted for these quantities.

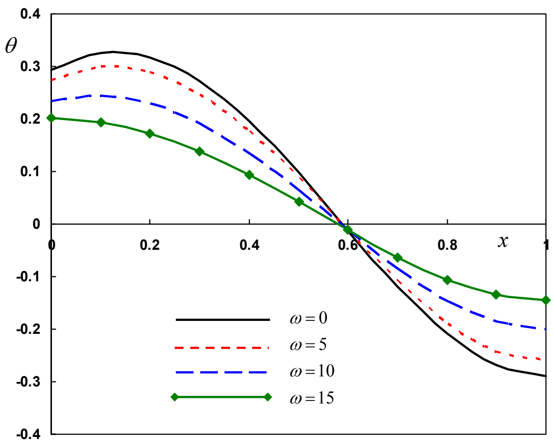

Fig. 6 Dependence of temperature on different values of the angular frequency of the heat flux ω ($\bar{\xi} = 2$, $\tau_q = 0.02$, $\tau_\theta = 0.01$, $z = h/6$)

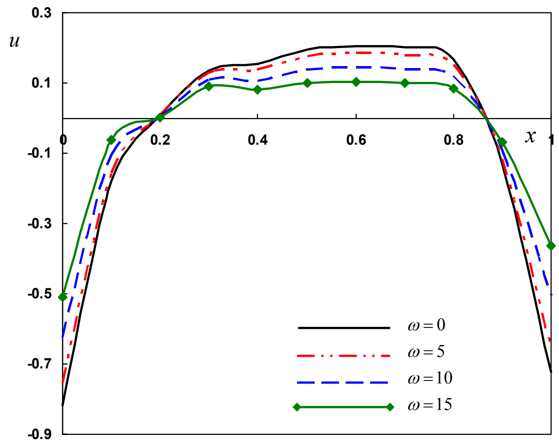

Fig. 7 Dependence of axial displacement on different values of the angular frequency of the heat flux ω ($\bar{\xi} = 2$, $\tau_q = 0.02$, $\tau_\theta = 0.01$, $z = h/6$)

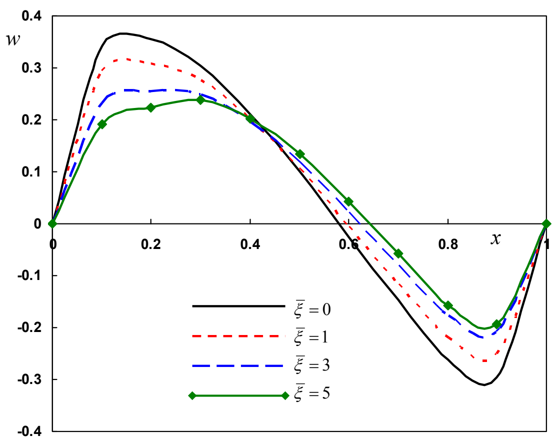

Fig. 8 Dependence of lateral vibration on different values of nonlocal parameter $\bar{\xi}$ ($\omega = 5$, $\tau_q = 0.02$, $\tau_\theta = 0.01$, $z = h/6$)

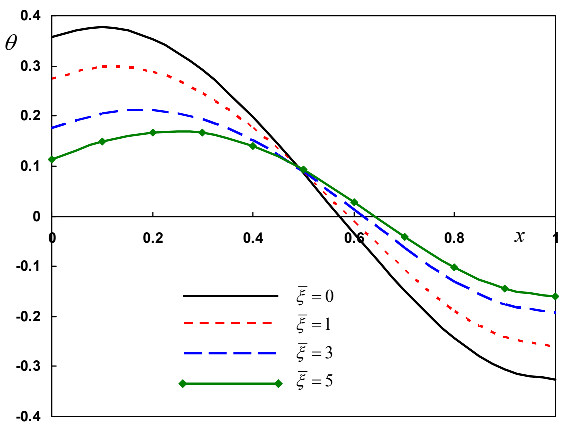

Fig. 9 Dependence of temperature on different values of nonlocal parameter $\bar{\xi}$ ($\omega = 5$, $\tau_q = 0.02$, $\tau_\theta = 0.01$, $z = h/6$)

From application point of view, the graphs have been divided into four categories. The first category (Figs. 2-4) depicts the dimensionless lateral vibration, temperature and axial displacement in the context of different theories of thermoelasticity. The DPL theory, LS model, GN model and the CTE theory are used for this purpose when $\bar{\xi}=2$, ω = 5 and $z = h/6$. These figures show the difference between the local generalized thermoelasticity and the nonlocal generalized thermoelasticity models.

Fig. 2 shows the distribution of the lateral vibration w through the length of the beam. It always begins at zero value and non-uniformly vibrates through the beam length to vanish once again at the end of the beam. This satisfies the conditions at beam boundaries. The behavior of DPL model may be similar than those of the other models. The wave amplitude of the CTE theory is the largest one while the wave amplitude of the GN model is the smallest one. Fig. 3 shows that the temperature θ is no longer increasing and get its maximum near the first edge of the nanobeam. Then, it is decreasing as the axial distance x increases to move in the direction of wave propagation. The temperatures of CTE theory may be larger than those of other theories in $0 \le x < 0.5$ and smaller than them in $0.5 < x \le 1$. Fig. 4 illustrates that the axial displacement u moves directly in the direction of wave propagation. Once again, the behavior of generalized theories of thermoelasticity may be similar than that of CTE model. The different curves change their behaviors twice at $x \cong 0.15$ and $x \cong 0.85$.

The effect of the angular frequency of thermal oscillation of the heat flux ω has been carried out in the second category (Figs. 5-7) at constant values of τ_q = 0.02, τ_θ 0.01, $z = h/6$ and $\bar{\xi}=2$. In this category, all the field quantities have been examined for four different values of the parameter ω (ω = 5, 10, 15 for time-dependent heat flux) and (ω = 0 for constant heat flux). It has been observed that both the wave amplitude of lateral vibration and temperature decrease as ω increases. Also, the axial displacement increases as ω increases only above the x-axis.

In the third category, the dimensionless lateral vibration, temperature and axial displacement are investigated in Figs. 8-10 for various values of the nonlocal parameter $\bar{\xi}$. In this case the angular frequency of thermal oscillation of the heat flux remains constant (ω = 5) as well as

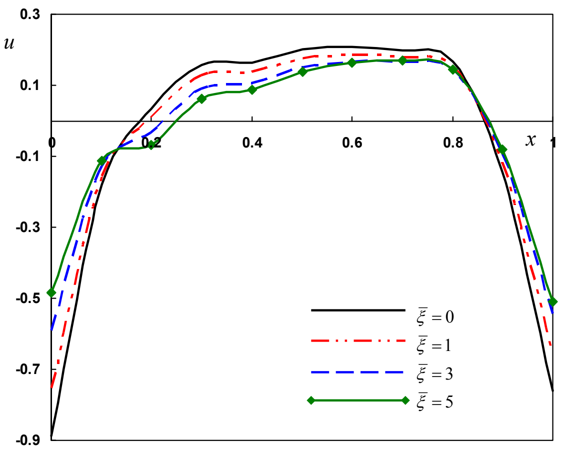

Fig. 10 Dependence of axial displacement on different values of nonlocal parameter $\bar{\xi}$ (ω = 5, τ_q = 0.02, τ_θ = 0.01, $z = h/6$)

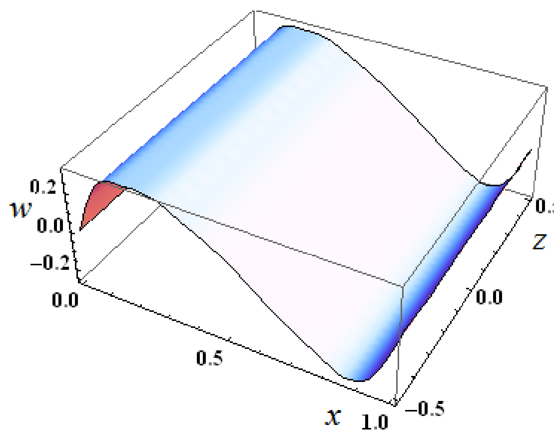

Fig. 11 The transverse deflection distribution versus the axial and thickness directions ($\bar{\xi}$ = 2, ω = 5, τ_q = 0.02, τ_θ = 0.01)

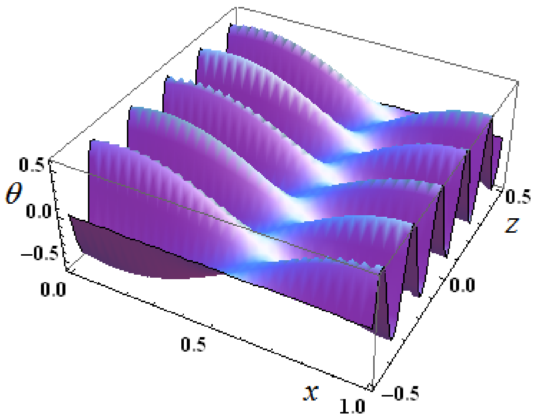

Fig. 12 The temperature distribution versus the axial and thickness directions ($\bar{\xi}=2$, $\omega=5$, $\tau_q=0.02$, $\tau_\theta=0.01$)

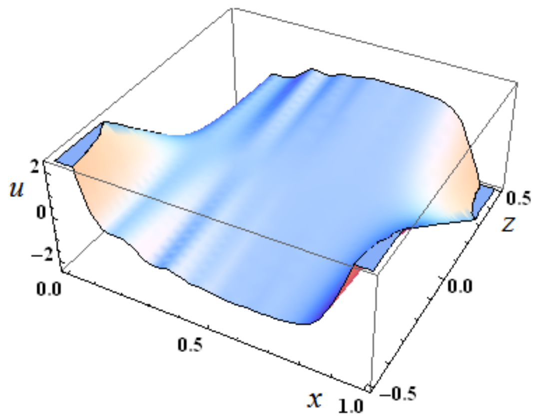

Fig. 13 The axial displacement distribution versus the axial and thickness directions ($\bar{\xi}=2$, $\omega=5$, $\tau_q=0.02$, $\tau_\theta=0.01$)

$\tau_q=0.02$, τ_θ 0.01 and $z=h/6$. It can be seen that the deflection, temperature and displacement are verey sensitive to the variation of nonlocal parameter. In general, the field variables are decrease with the increasing value of the nonlocal parameter.

The last category presented the three-dimensions of the field quantities of the nanobeam at constant values of $\tau_q=0.02$, $\tau_\theta=0.01$, $\omega=5$ and $\bar{\xi}=2$ in a wide range of thickness $-1/2 \le z/h \le 1/2$. In Figs. 11-13, one can see the behaviour of these quantities through-the-thickness of the nanobeam. The plots of temperature (Fig. 12) and axial displacement (Fig. 13) are very sensitive to the variation of the thickness direction. It is observed from Fig. 11, as it is expected, that the lateral vibration w is uniform through the thickness direction. The plot of θ in Fig. 12 is independent of the thickness distance z only when $x=0.5$ and around its neighborhood. Otherwise, the temperature θ is strongly depending on the variation of z, especially, at the beam boundaries. Fig. 13 shows that the axial displacement is vanished, as it is expected, at the center of the beam. Otherwise, u is strongly depending on the thickness distance z.

8. Conclusions

In this paper, a new model of nonlocal generalized thermoelasticity with dual-phase-lags for the Euler-Bernoulli nanobeam is constructed. The vibration characteristics of the transverse deflection, thermodynamic temperature, and axial displacement of nanobeam subjected to time-dependent heat flux are investigated. The effects of the nonlocal parameter $\bar{\xi}$ and the angular frequency of thermal oscillation of the heat flux ω on the field variables are investigated. Numerical technique based on the Laplace transformation has been used. The effects of the thickness, nonlocal parameter and the ω parameter on all the studied field quantities have been shown and presented graphically. According to the obtained results, it is found that the nonlocal parameter $\bar{\xi}$ has significant effects on all the studied fields. On the other hand the thermoelastic displacements and temperature have a strong dependency on the angular frequency of thermal oscillation of the heat flux parameter. The method which used in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity. It is also observed that the theories of coupled thermoelasticity and generalized thermoelasticity with one relaxation time can be obtained as limited cases from our discussion problem.

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