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# Nonlinear vibration of thin circular sector cylinder: An analytical approach

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**Abstract.** In this paper, we try to prepare an accurate analytical solution for solving nonlinear vibration of thin circular sector cylinder. A new approximate solution called variational approach is presented and correctly applied to the governing equation of thin circular sector cylinder. The effect of important parameters on the response of the problem is considered. Some comparisons have been presented between the numerical solution and the present approach. The results show an excellent agreement between these methods. It has been illustrated that the variational approach can be a useful method to solve nonlinear problems by considering the effects of important parameters.

**Keywords:** thin circular sector cylinder; nonlinear vibration; variational approach

## 1. Introduction

Nonlinear vibrations are an important issue in mechanical and civil engineering. It is very important to find analytical solution for the nonlinear equations. In some cases with high nonlinear term, it is really difficult to find an exact solution for them. Perturbation technique is one of the well-known methods but it has its own limitations. To overcome the limitations of the traditional methods some new approximate analytical solutions have been proposed such as: Homotopy Perturbation Method (Shaban *et al*. 2010, Bayat *et al*. 2013), Hamiltonian Approach (Bayat and Pakar 2011a, 2012, 2013a, Bayat *et al*. 2013, 2014a, b) , energy balance method (He 2002, Bayat and Pakar 2011b, Pakar and Bayat 2011, 2012, Mehdipour *et al*. 2010), variational iteration method (Dehghan 2010, Pakar *et al*. 2012), amplitude frequency formulation (Bayat *et al*. 2011, 2012, Pakar and Bayat 2013a, He 2008), max-min approach (Shen and Mo 2009, Zeng and Lee 2009), variational approach (He 2007, Bayat and Pakar 2013b, Bayat *et al*. 2014c, Pakar and Bayat 2012), and the other analytical and numerical (Xu and Zhang 2009, Alicia *et al*. 2010, Kuo and Lo 2009, Wu 2011, Odibat *et al*. 2008).

The paper has been organized as follows:

Governing equation of the problem is in Section 2. Basic idea of variational approach has been described in Section 3, the complete applications of variational approach have been studied in

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Section 4, to verify the results and accuracy of the method, some comparisons between analytical and numerical solutions have done in Section 5 and the last section contains the most significant findings of the paper.

## 2. Thin circular sector cylinder formulation

In this condition a thin circular sector cylinder is considered as Shown in Fig. 1.

Thin circular sector cylinder rolls in an oscillatory motion back and forth on a flat stationary support, with no sliding effect. Governing equation of the oscillation is as follow (Shaban *et al.* 2010)

$$\left(R^2 - R\bar{y}\cos(\theta)\right)\left(2\ddot{\theta}\right) + R\left(\bar{y}\sin(\theta)\right)\theta^2 + \left(g\bar{y}\right)\sin(\theta) = 0$$
$$\theta(0) = A, \quad \dot{\theta}(0) = 0, \tag{1}$$

Where the geometrical parameters are shown in Fig. 1. The height of mass center obtained as below

$$\bar{y} = \frac{R\sin(\alpha)}{\alpha} \tag{2}$$

Introducing the dimensionless time variable

$$\bar{t} = \sqrt{\frac{1}{\bar{y}}}t = \left(\sqrt{\frac{R\sin(\alpha)}{\alpha}}\right)^{-1} t. \tag{3}$$

Eq. (1) becomes

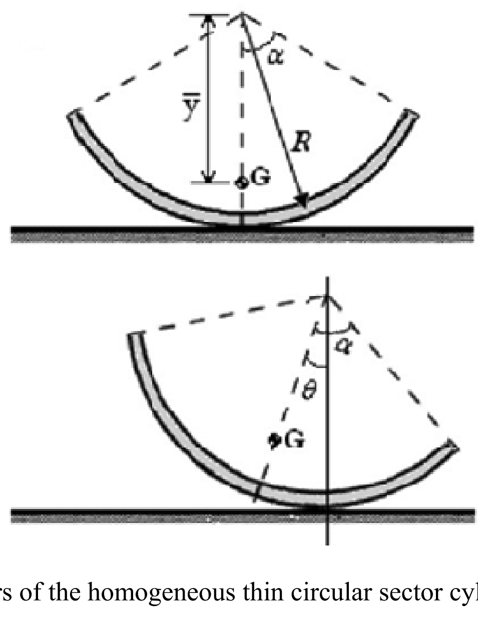



Fig. 1 Geometric parameters of the homogeneous thin circular sector cylinder (Shaban *et al*. 2010)

$$\left(\frac{R}{\bar{y}}\cos(\theta)\right)\left(2\ddot{\theta}\right)+\sin(\theta)\dot{\theta}^2+\frac{g}{R}\sin(\theta)=0$$
$$\theta(0)=A, \quad \dot{\theta}(0)=0. \tag{4}$$

And by introducing the dimensionless geometrical parameter

$$\lambda=\frac{\bar{y}}{R}=\frac{\sin(\alpha)}{\alpha} \tag{5}$$

Eq. (4) becomes

$$\left(\frac{1}{\lambda}-\cos(\theta)\right)\left(2\ddot{\theta}\right)+\sin(\theta)\dot{\theta}^2+\frac{g}{R}\sin(\theta)=0$$
$$\theta(0)=A, \quad \dot{\theta}(0)=0. \tag{6}$$

## 3. Basic concept of variational approach method

He (2007) suggested a variational approach which is different from the known variational methods in open literature. Hereby we give a brief introduction of the method

$$\ddot{\theta}+f(\theta)=0 \tag{7}$$

Its variational principle can be easily established utilizing the semi-inverse method (He 2007)

$$J(\theta)=\int_0^{T/4}\left(-\frac{1}{2}\dot{\theta}^2+F(\theta)\right)dt \tag{8}$$

Where $T$ is period of the nonlinear oscillator, $\partial F/\partial\theta=f$. Assume that its solution can be expressed as

$$\theta(t)=A\cos(\omega t) \tag{9}$$

Where $A$ and $\omega$ are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (9) into Eq. (8) results in

$$\begin{aligned} J(A,\omega)&=\int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt \\ &=\frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2 t+F(A\cos t)\right)dt \\ &=-\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\,dt+\frac{1}{\omega}\int_0^{\pi/2}F\left(A\cos t\right)dt \end{aligned} \tag{10}$$

Applying the Ritz method, require

$$\frac{\partial J}{\partial A}=0 \tag{11}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{12}$$

But with a careful inspection, for most cases He found that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2}A^2\int_0^{\pi/2}\sin^2 t\, dt - \frac{1}{\omega^2}\int_0^{\pi/2}F\left(A\cos t\right)dt < 0 \tag{13}$$

Thus, He modify conditions Eqs. (11) and (12) into a simpler form

$$\frac{\partial J}{\partial \omega} = 0 \tag{14}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

## 4. Application of variational approach

By using the Taylor's series expansion for cos($\theta(t)$), sin($\theta(t)$) we can re-write Eq. (6) in the following form

$$\left(\frac{1}{\lambda} - \left(1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4\right)\right)\left(2\ddot{\theta}\right) + \left(\theta - \frac{1}{6}\theta^3\right)\dot{\theta}^2 + \frac{g}{R}\left(\theta - \frac{1}{6}\theta^3\right) = 0 \tag{15}$$

Its variational formulation of Eq. (15) can be readily obtain as follows

$$J(\theta) = \int_0^t\left(\frac{\ddot{\theta}^2}{\lambda} - \ddot{\theta}^2 + \frac{1}{2}\ddot{\theta}^2\theta^2 - \frac{1}{24}\ddot{\theta}^2\theta^4 + \frac{g}{24R}\theta^2 + \frac{g}{24R}\theta^4\right)dt. \tag{16}$$

Choosing the trial function $\theta(t) = A = \cos(\omega t)$ into Eq. (16) we obtain

$$J(A) = \int_0^{T/4}\left(\begin{array}{l}\dfrac{A^2\omega^2\sin^2(\omega t)}{\lambda} - A^2\omega^2\sin^2(\omega t) + \dfrac{1}{2}A^4\omega^2\sin^2(\omega t)\cos^2(\omega t) \\ -\dfrac{1}{24}A^6\omega^2\sin^2(\omega t)\cos^4(\omega t) + \dfrac{g}{2R}A^2\cos^2(\omega t) - \dfrac{g}{24R}A^4\cos^4(\omega t)\end{array}\right)dt \tag{17}$$

The stationary condition with respect to $A$ leads to

$$\frac{\partial J}{\partial A} = \int_0^{T/4}\left(\begin{array}{l}\dfrac{2A\omega^2\sin^2(\omega t)}{\lambda} - 2A\omega^2\sin^2(\omega t) + 2A^3\omega^2\sin^2(\omega t)\cos^2(\omega t) \\ -\dfrac{1}{4}A^5\omega^2\sin^2(\omega t)\cos^4(\omega t) + \dfrac{g}{R}A\cos^2(\omega t) - \dfrac{g}{6R}A^3\cos^4(\omega t)\end{array}\right)dt = 0 \tag{18}$$

or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\left(\begin{array}{l}\dfrac{2A\omega^2\sin^2 t}{\lambda}-2A\omega^2\sin^2 t+2A^3\omega^2\sin^2 t\cos^2 t\\ -\dfrac{1}{4}A^5\omega^2\sin^2 t\cos^4 t+\dfrac{g}{R}A\cos^2 t-\dfrac{g}{6R}A^3\cos^4 t\end{array}\right)dt=0 \tag{19}$$

Solving Eq. (19), according to $\omega$, we have

$$\omega^2=\frac{\int_0^{\pi/2}\dfrac{g}{R}A\cos^2 t-\dfrac{g}{6R}A^3\cos^4 t}{\int_0^{\pi/2}\dfrac{2A\sin^2 t}{\lambda}-2A\sin^2 t+2A^3\sin^2 t\cos^2 t-\dfrac{1}{4}A^5\sin^2 t\cos^4 t+} \tag{20}$$

Then we have

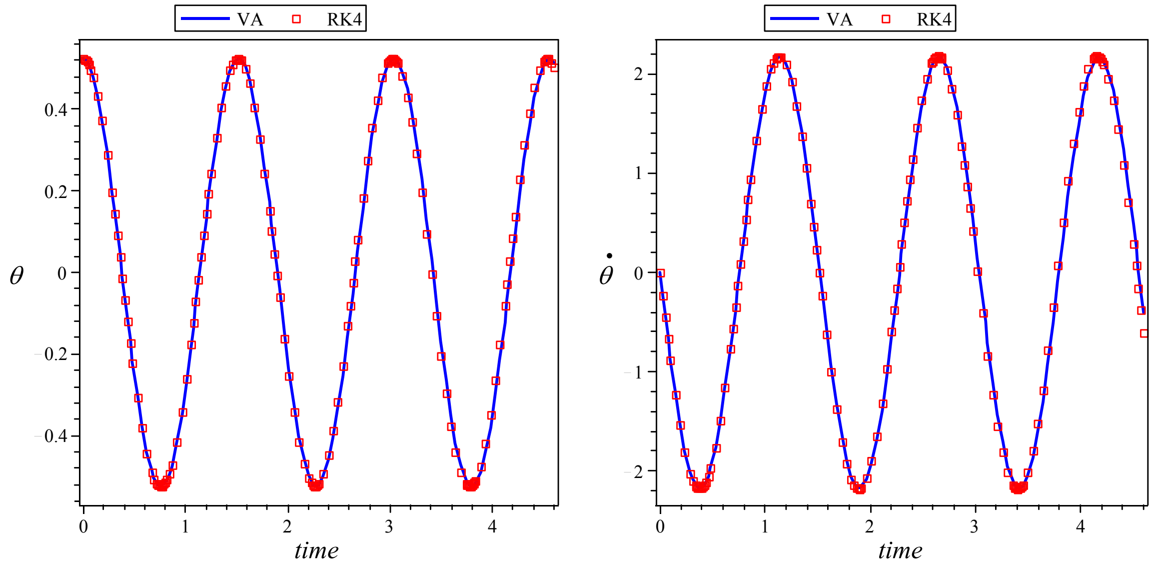
$$\omega=\sqrt{\frac{4g\lambda(-8+A^2)}{R(\lambda A^4-16\lambda A^2+64\lambda-64)}} \tag{21}$$

According to Eqs. (9) and (21), we can obtain the following approximate solution

$$\theta(t)=A\cos\left(\sqrt{\frac{4g\lambda(-8+A^2)}{R(\lambda A^4-16\lambda A^2+64\lambda-64)}}t\right) \tag{22}$$

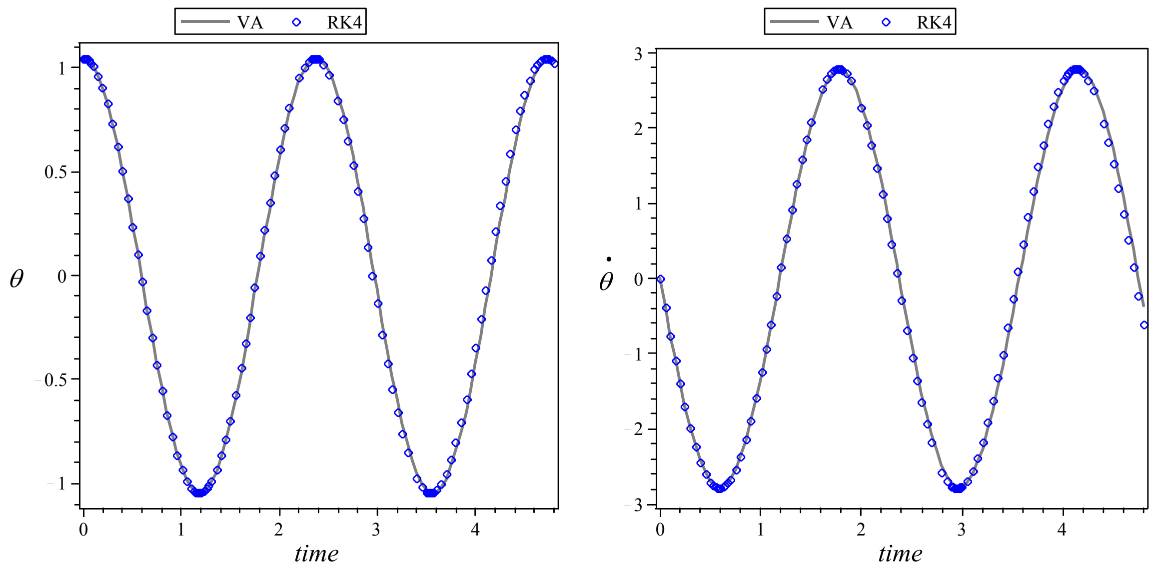
Table 1 Comparison of time history response of variational approach with Runje-Kutta

| Time | Case 1 | | | Case 2 | | |
|---|---|---|---|---|---|---|
| | VA | RK4 | Error | VA | RK4 | Error |
| 0 | 0.5236 | 0.5236 | 0 | 1.0472 | 1.0472 | 0 |
| 0.2 | 0.3216 | 0.3200 | 0.0051 | 0.7676 | 0.7648 | 0.0037 |
| 0.4 | -0.1285 | -0.1326 | 0.0316 | 0.0782 | 0.0698 | 0.1069 |
| 0.6 | -0.4795 | -0.4820 | 0.0052 | -0.6530 | -0.6628 | 0.0150 |
| 0.8 | -0.4605 | -0.4565 | 0.0088 | -1.0355 | -1.0379 | 0.0023 |
| 1 | -0.0863 | -0.0759 | 0.1199 | -0.8651 | -0.8531 | 0.0138 |
| 1.2 | 0.3545 | 0.3637 | 0.0258 | -0.2327 | -0.2082 | 0.1056 |
| 1.4 | 0.5218 | 0.5204 | 0.0027 | 0.5239 | 0.5491 | 0.0481 |
| 1.6 | 0.2865 | 0.2723 | 0.0495 | 1.0008 | 1.0101 | 0.0093 |
| 1.8 | -0.1699 | -0.1876 | 0.1043 | 0.9433 | 0.9263 | 0.0180 |
| 2 | -0.4952 | -0.5016 | 0.0129 | 0.3821 | 0.3428 | 0.1028 |
| 2.2 | -0.4385 | -0.4254 | 0.0297 | -0.3831 | -0.4256 | 0.1109 |
| 2.4 | -0.0435 | -0.0184 | 0.5772 | -0.9437 | -0.9644 | 0.0219 |
| 2.6 | 0.3851 | 0.4030 | 0.0465 | -1.0005 | -0.9831 | 0.0174 |
| 2.8 | 0.5165 | 0.5109 | 0.0109 | -0.5230 | -0.4714 | 0.0987 |
| 3 | 0.2495 | 0.2214 | 0.1125 | 0.2337 | 0.2945 | 0.2601 |




(a) (b)

Fig. 2 Comparison of analytical solution with the numerical solution for: (a) time history response of displacement; (b) time history response of velocity for $\alpha = \pi/3$, $A = \pi/6$, $g = 9.81$, $R = 1$




(a) (b)

Fig. 3 Comparison of analytical solution with the numerical solution for: (a) time history response of displacement; (b) time history response of velocity for $\alpha = \pi/6$, $A = \pi/3$, $g = 9.81$, $R = 2$

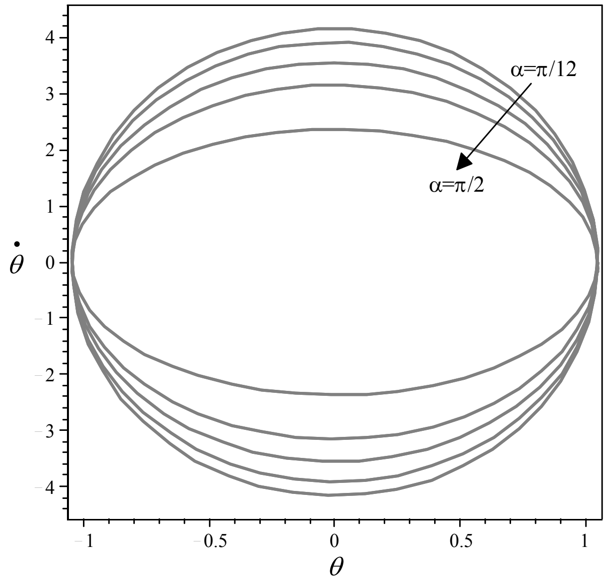



Fig. 4 Influence of $\alpha$ on phase plan, for $A = \pi/3$, $R = 2$, $g = 9.81$

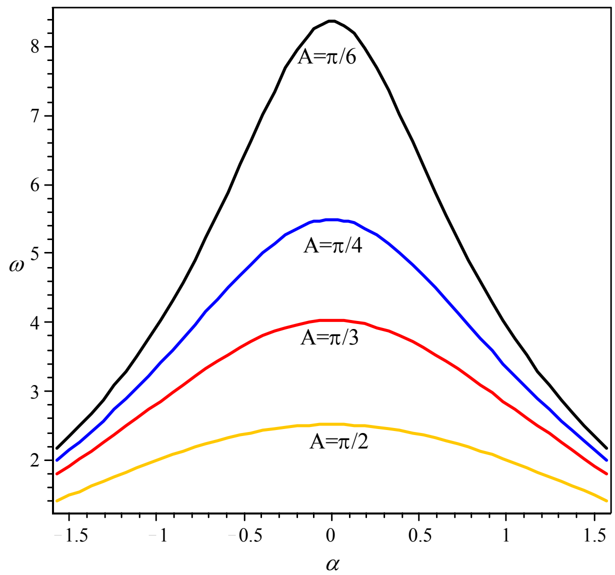



Fig. 5 Influence of amplitude and angle on nonlinear frequency

## 5. Results and discussions

In this part, the results of variational approach method and numerical solutions using Runge-kutta's algorithm (Appendix A) are compared in some figures. Table 1 is comparison of

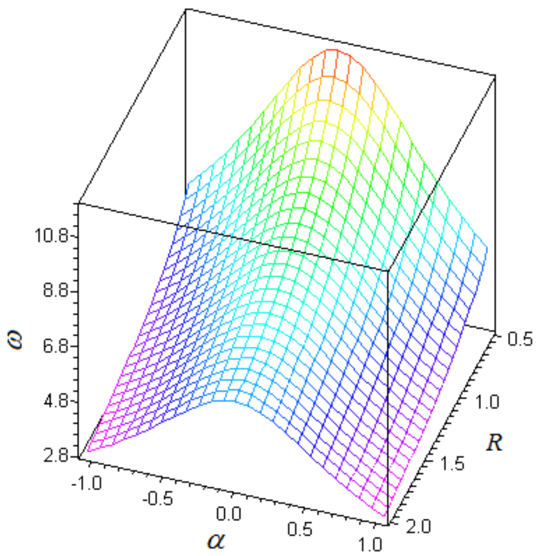



Fig. 6 Sensitivity analysis of various parameter of system on nonlinear frequency

some point values between analytical and numerical solution.

Figs. 2 and 3 are the displacement time history and velocity time history to show an excellent convergence of analytical and numerical solution. It can be obtain from the figures that the motion of the problem is periodic and it is function of amplitude. One of the most advantages of analytical methods respect to numerical ones is to see the effect of important parameters on the response of the problem easily. In Fig. 3 we consider the effects of angle on the phase plan of the problem. The effects of $\alpha$ and $A$ on the frequency of the problem is in Fig. 4. A sensitive analyze has been done on nonlinear frequency of the system by considering the effect of $\alpha$ and $R$.

It is evident that variational approach shows an excellent agreement with the numerical solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the variational approach can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

## 6. Conclusions

Nonlinear vibration of thin circular cylinder has been studied in this paper. Variational approach has been completely applied to the governing equation of the problem. Some figures and patterns have been presented to show the accuracy of this approach. It has been proved that the variational approach is very effective and doest need any linearization or small perturbation. By applying this approach we can converge to an accurate solution with only one iteration. It has been illustrated that the variational approach is extremely speedy, and easy to apply to conservative nonlinear oscillators. Variational approach provides an easy and direct procedure for determining approximations of periodic solutions.

## References

Alicia, C., Hueso, J.L., Martínez, E. and Torregros, J.R. (2010), "Iterative methods for use with nonlinear discrete algebraic models", *Math. Comput. Model.*, **52**(7-8), 1251-1257.

Bayat, M. and Pakar, I. (2011a), "Nonlinear free vibration analysis of tapered beams by Hamiltonian Approach", *J. Vibroeng.*, **13**(4), 654-661.

Bayat, M. and Pakar, I. (2011b), "Application of He's energy balance method for nonlinear vibration of thin circular sector cylinder", *Int. J. Phy. Sci.*, **6**(23), 5564-5570.

Bayat, M. and Pakar, I. (2012), "Accurate analytical solution for nonlinear free vibration of beams", *Struct. Eng. Mech., Int. J.*, **43**(3), 337-347.

Bayat, M. and Pakar, I. (2013a), "Nonlinear dynamics of two degree of freedom systems with linear and nonlinear stiffnesses", *Earthq. Eng. Eng. Vib.*, **12**(3), 411-420.

Bayat, M. and Pakar, I. (2013b), "On the approximate analytical solution to non-linear oscillation systems", *Shock Vib.*, **20**(1), 43-52.

Bayat, M., Pakar, I. and Shahidi, M. (2011), "Analysis of nonlinear vibration of coupled systems with cubic nonlinearity", *Mechanika*, **17**(6), 620-629.

Bayat, M., Pakar, I. and Domaiirry, G. (2012), "Recent developments of Some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review", *Latin Am. J. Solid. Struct.*, **9**(2), 145-234.

Bayat, M., Pakar, I. and Bayat, M. (2013), "Analytical solution for nonlinear vibration of an eccentrically reinforced cylindrical shell", *Steel Compos. Struct., Int. J.*, **14**(5), 511-521.

Bayat, M., Pakar, I. and Cveticanin, L. (2014a), "Nonlinear free vibration of systems with inertia and static type cubic nonlinearities: An analytical approach", *Mech. Mach. Theory*, **77**, 50-58.

Bayat, M., Pakar, I. and Cveticanin, L. (2014b), "Nonlinear vibration of stringer shell by means of extended Hamiltonian Approach", *Arch. Appl. Mech.*, **84**(1), 43-50.

Bayat, M., Bayat, M. and Pakar, I. (2014c), "Nonlinear vibration of an electrostatically actuated microbeam", *Latin Am. J. Solid. Struct.*, **11**(3), 534-544.

Dehghan, M. and Tatari, M. (2008), "Identifying an unknown function in a parabolic equation with over specified data via He's variational iteration method", *Chaos Solitons Fractals*, **36**(1), 157-166.

He, J.H. (2002), "Preliminary report on the energy balance for nonlinear oscillators", *Mech. Res. Communications*, **29**(2), 107-111.

He, J.H. (2007), "Variational approach for nonlinear oscillators", *Chaos Solitons Fractals*, **34**(5), 1430-1439.

He, J.H. (2008), "An improved amplitude-frequency formulation for nonlinear oscillators", *Int. J. Nonlinear Sci. Numer. Simulation*, **9**(2), 211-212.

Kuo, B.L. and Lo, C.Y. (2009), "Application of the differential transformation method to the solution of a damped system with high nonlinearity", *Nonlinear Anal.*, **70**(4), 1732-1737.

Mehdipour, I., Ganji, D.D. and Mozaffari, M. (2010), "Application of the energy balance method to nonlinear vibrating equations", *Current Appl. Phys.*, **10**(1), 104-112.

Odibat, Z., Momani, S. and Erturk, V.S. (2008), "Generalized differential transform method: application to differential equations of fractional order", *Appl. Math. Comput.*, **197**(2) , 467-477.

Pakar, I. and Bayat, M. (2011), "Analytical solution for strongly nonlinear oscillation systems using energy balance method", *Int. J. Phy. Sci.*, **6**(22), 5166-5170.

Pakar, I. and Bayat, M. (2012), "Analytical study on the non-linear vibration of Euler-Bernoulli beams", *J. Vibroeng.*, **14**(1), 216-224.

Pakar, I., Bayat, M. and Bayat, M. (2012), "On the approximate analytical solution for parametrically excited nonlinear oscillators", *J. Vibroeng.*, **14**(1), 423-429.

Pakar, I. and Bayat, M. (2013a), "An analytical study of nonlinear vibrations of buckled Euler-Bernoulli beams", *Acta Physica Polonica A*, **123**(1), 48-52.

Pakar, I. and Bayat, M. (2013b), "Vibration analysis of high nonlinear oscillators using accurate approximate methods", *Struct. Eng. Mech., Int. J.*, **46**(1), 137-151.

Shaban, M., Ganji, D.D. and Alipour, A.A. (2010), "Nonlinear fluctuation, frequency and stability analyses in free vibration of circular sector oscillation systems", *Current Appl. Phys.*, **10**(5), 1267-1285.

Shen, Y.Y. and Mo, L.F. (2009), "The max–min approach to a relativistic equation", *Comput. Math. Appl.* **58**(11), 2131-2133.

Wu, G. (2011), "Adomian decomposition method for non-smooth initial value problems", *Math. Comput. Model.*, **54**(9-10), 2104-2108.

Xu, L. (2008), "Variational approach to solution of nonlinear dispersive $K(m, n)$ equation", *Chaos Solitons Fractals*, **37**(1), 137-143.

Xu, N. and Zhang, A. (2009), "Variational approachnext term to analyzing catalytic reactions in short monoliths", *Comput. Math. Appl.*, **58**(11-12), 2460-2463.

Zeng, D.Q. and Lee, Y.Y. (2009), "Analysis of strongly nonlinear oscillator using the max–min approach", *Int. J. Nonlinear Sci. Numer. Simul.*, **10**(10), 1361-1368.

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## Appendix A

The most often used method of the Runge-Kutta family is the Fourth-Order one, which extends the idea of the mid-point method, by jumping 1/4th of the way first, then going half-way, a la the mid-point method, then going 3/4th of the way and finally jumping all the way.

Consider an initial value problem be specified as follows

$$\dot{\theta} = f(t, \theta), \quad \theta(t_0) = \theta_0 \tag{A.1}$$

$\theta$ is an unknown function of time $t$ which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{A.2}$$

for $n$ = 0, 1, 2, 3, . . . , using

$$\begin{aligned} k_1 &= f(t_n, \theta_n), \\ k_2 &= f\left(t_n + \frac{1}{2}h, \theta_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \frac{1}{2}h, \theta_n + \frac{1}{2}hk_3\right), \\ k_4 &= f(t_n + h, \theta_n + hk_3). \end{aligned} \tag{A.3}$$

Where $\theta_{n+1}$ is the RK4 approximation of $\theta(t_{n+1})$. and the next value ($\theta_{n+1}$) is determined by the present value ($\theta_n$) plus the weighted average of four increments, where each increment is the product of the size of the interval, $h$, and an estimated slope specified by function $f$ on the right-hand side of the differential equation.

- $k_1$ is the increment based on the slope at the beginning of the interval, using $\dot{\theta}$,
- $k_2$ is the increment based on the slope at the midpoint of the interval, using $\dot{\theta}$ + 1/2 $hk_1$;
- $k_3$ is again the increment based on the slope at the midpoint, but now using $\dot{\theta}$ + 1/2 $hk_2$;
- $k_4$ is the increment based on the slope at the end of the interval, using $\dot{\theta}$ + $hk_3$.