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Influence of impulsive line source and non-homogeneity on the propagation of SH-wave in an isotropic medium

Rajneesh Kakar*

DIPS Polytechnic College, Hoshiarpur-146001, India

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Abstract. In this paper, the effect of impulsive line on the propagation of shear waves in non-homogeneous elastic layer is investigated. The rigidity and density in the intermediate layer is assumed to vary quadratic as functions of depth. The dispersion equation is obtained by using the Fourier transform and Green's function technique. The study ends with the mathematical calculations for transmitted wave in the layer. These equations are in complete agreement with the classical results when the non-homogeneity parameters are neglected. Various curves are plotted to show the effects of non-homogeneities on shear waves in the intermediate layer.

Keywords: non-homogeneity; Green's function; Dirac-delta function; isotropic; shear waves

1. Introduction

The Dirac delta function is also known as the unit impulse function. It is a mathematical abstraction and is often used to approximate some physical phenomenon. An idealized line source of wave can be described using the delta function. Of course, real points of wave will have finite width, but if the point is narrow enough, approximating it with a delta function can be very useful. Further, Green's function technique is very handy to solve inhomogeneous differential equations subject to certain boundary conditions. That is why; the author used this technique to solve the problem of wave propagation. The Green's function is a strong mathematical tool to carry out asymptotic approximations of solutions of differential equations.

In solid materials, during rest position, the volume elements retain their relative positions and orientations alike. On the other hand, when the solid is under the action of external forces such as elastic stresses and strains, the volume elements get departed from the original position. Therefore, the subject of elasticity of crystals has its own importance in several circumstances. In recent years, the elastic behaviour of polycrystalline solids, used as the materials in engineering construction, are of great practical importance.

Many authors have studied the problems of wave propagation in elastic and viscoelastic medium in the field of mechanics of solids, applied physics, applied mathematics, mechanical engineering and materials science by using various mathematically technique. Rommel (1990) presented a formulation for anisotropic medium, heterogeneity on the propagation of SH-waves with point

*Corresponding author, Ph.D., E-mail: rkakar_163@rediffmail.com

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with point source was studied by Chattopadhyay *et al*. (2010) and also Chattopadhyay *et al*. (2011) discussed non-homogeneity, effect of point source on various shear waves in monoclinic medium. Kumar and Gupta (2010) studied wave motion in micropolar transversely isotropic thermoelastic half space without energy dissipation. Kakar and Kakar (2012) studied propagation of Love waves in a non-homogeneous elastic media. Kakar and Gupta (2012) also discussed propagation of Love waves in a non-homogeneous orthotropic layer under 'P' overlying semi-infinite medium. Ponnusamy and Selvamani (2012) discussed wave propagation in a generalized thermo elastic plate embedded in elastic medium. Recently, Kakar and Gupta (2013) presented a note on torsional surface waves in a non-homogeneous isotropic layer over viscoelastic half-space.

Some papers on wave propagation using Green's function techniques are Vrettos (1991), and Robert (2002, 2005), Popov (2002), George and Christos (2003), Vaclav and Kiyoshi (1996), Jing *et al*. (2011), Kirpichnikova (2012). Matsuda and Glorieux (2007) used Green's function technique for dispersion relation of surface acoustic waves for functionally graded materials. Green function is widely used in investigating wave propagation in homogeneous medium without exception in inhomogeneous ones. With a view to initial and radiation condition, Li (1994) derived wave equation with Green function in inhomogeneous medium by Fourier transformation so that wave equation could be converted to Schrodinger equation and the method can be applied to resolve acoustic propagation in range-dependent inhomogeneous medium. Li *et al*. (2010) extended the method to settle Helmholtz equation with complex refractive index. Shaw (1997) used conformal mapping to obtain Green's function for two dimensional heterogeneous Helmholtz equations. It followed that the solutions can serve as a basis for developing Green's functions for use as kernels in boundary element methods used for numerical solution of complex physical problems. Daros (2013) derived a Green's function which can be used to model transient SH-wave in inhomogeneous and anisotropic media, with a power function velocity variation in one direction of Cartesian coordinate system.

In this work, the problem of propagation of SH-wave in a non-homogeneous layer of variable rigidity and density which is lying in between the two homogeneous and isotropic half spaces has been studied. The SH-waves are excited in the layer due to the presence of an impulsive line source at the interface of the intermediate layer and the lower half-space. We have taken the quadratic variation in rigidity and density. The Dirac-delta function is taken as the source of impulse in the wave propagation. The dispersion equations are obtained for this generated line source. These equations are in complete agreement with the classical results when the non-homogeneity parameters are neglected. In the end of this study, the transmitted wave in the medium is also calculated. The effect of nonhomogeneity on the generated SH-wave is also shown graphically for the various values of material parameters (earth).

2. Formulation of the problem

We have assumed that the harmonic shear wave is travelling along x-axis and z-axis is taken vertically downwards. The impulsive line source of disturbance P is taken at the line of intersection of the interface and z - axis (Fig. 1). Let H be the thickness of the inhomogeneous isotropic intermediate layer. Let μ_1, ρ_1 be the rigidity and density of the first half-space layer. Let μ_3, ρ_3 be the rigidity and density of the lower half-space layer. The variations in rigidity and density (inhomogeneous parameters) in the intermediate layer are

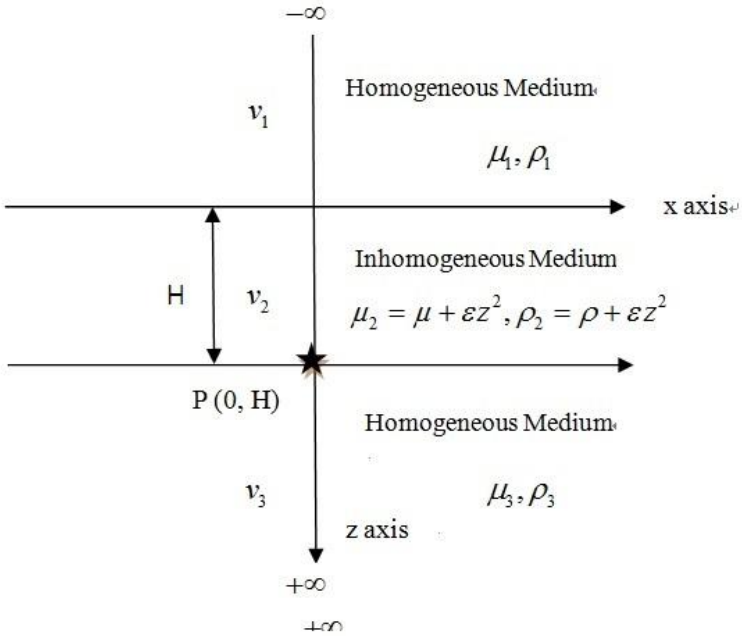

Fig. 1 Geometry of the problem

$$\mu_2 = \mu + \varepsilon z^2 \tag{1a}$$

$$\rho_2 = \rho + \varepsilon z^2 \tag{1b}$$

where ε is small positive real constant.

The equation of motion for line source can be written as

$$\tau_{ij} = F_i + \rho \ddot{u}_j \tag{2}$$

where τ_{ij} are components of the divergence of stress tensor, ρ is the density of the medium and F_i are body forces.

For shear wave propagation along the x-axis, we have

$$u = 0,\ w = 0,\ v = v(x, z, t) \tag{3}$$

where, u, v, w are the respective displacement components at time t.

Therefore, the equation of motion for upper homogeneous isotropic medium is $(-\infty < x, y < \infty, -\infty < z \le 0)$

$$\frac{\partial}{\partial x}\left(\mu_1 \frac{\partial v_1}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_1 \frac{\partial v_1}{\partial z}\right) - \rho \frac{\partial^2 v_1}{\partial t^2} = 0 \tag{4}$$

Since rigidity ‘ μ_1 ’ and density ‘ ρ_1 ’ are constant, thus (5) is a consequence of (4) for the case of constant density.

$$\mu_1\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}\right)-\rho_1\frac{\partial^2 v_1}{\partial t^2}=0 \tag{5}$$

Assuming the source is time harmonic and taking the time dependence $e^{i\omega t}$ to be understood throughout i.e., $v_1(x,z,t)=v_1(x,z)e^{i\omega t}$. Therefore, Eq. (5) reduces to

$$\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}+\frac{\rho_1}{\mu_1}\omega^2 v_1=0 \tag{6}$$

or

$$\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial z^2}+k_1^2 v_1=0 \tag{7}$$

where, $k_1^1=\frac{\rho_1}{\mu_1}\omega^2$, $\omega=kc$ is the angular frequency, *k* the wave number and *c* is the phase velocity.

Similarly, the equation of motion for lower homogeneous isotropic medium is

$$\frac{\partial}{\partial x}+\left(\mu_3\frac{\partial v_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu_3\frac{\partial v_3}{\partial z}\right)-\rho_3\frac{\partial^2 v_3}{\partial t^2}=0 \tag{8}$$

Let $v_3(x,z,t)=v_3(x,z)e^{i\omega t}$. Therefore, Eq. (8) reduces to

$$\frac{\partial^2 v_3}{\partial x^2}+\frac{\partial^2 v_3}{\partial z^2}+\frac{\rho_3}{\mu_3}\omega^2 v_3=0$$

or

$$\frac{\partial^2 v_3}{\partial x^2}+\frac{\partial^2 v_3}{\partial z^2}+k_3^2 v_3=0 \quad , \tag{9}$$

where, $k_3^1=\frac{\rho_3}{\mu_3}\omega^2$, $\omega=kc$ is the angular frequency, *k* the wave number and c is the phase velocity.

Assuming the source is time harmonic and taking the time dependence $e^{i\omega t}$ to be understood throughout, such that the equation of motion for intermediate inhomogeneous isotropic medium is

$$\frac{\partial}{\partial x}\left(\mu_2\frac{\partial v_2}{\partial x}\right)+\frac{\partial}{\partial z}\left(\mu_2\frac{\partial v_2}{\partial z}\right)-\rho_2\frac{\partial^2 v_2}{\partial t^2}=4\pi\sigma(r,t) \tag{10}$$

Here ‘*r*’ is the distance from the origin, where the force is applied to a point of coordinates,

'$\sigma(r,t)$' is the disturbances produced by the impulsive force at P and t is time.

As per our assumption $v_2(x,z,t) = v_2(x,z)e^{i\omega t}$ and $\sigma(r,t) = \sigma(r)\,\mathrm{e}^{i\omega t}$, Eq. (12) reduces to

$$\frac{\partial}{\partial x}\left(\mu_2 \frac{\partial v_2}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_2 \frac{\partial v_2}{\partial z}\right) + \rho_2 \omega^2 v_2 = 4\pi\sigma(r) \tag{11}$$

The disturbances caused by the impulsive force '$\sigma(r)$' can be written in terms of Dirac-delta function at the source point as

$$\sigma(r) = \delta(x)\delta(z-H) \tag{12}$$

Therefore, Eq. (11) reduces to

$$\frac{\partial}{\partial x}\left(\mu_2 \frac{\partial v_2}{\partial x}\right) + \frac{\partial}{\partial z}\left(\mu_2 \frac{\partial v_2}{\partial z}\right) + \rho_2 \omega^2 v_2 = 4\pi\delta(x)\delta(z-H) \tag{13}$$

Put Eq. (1) in Eq. (13), then we get

$$\mu \frac{\partial^2 v_2}{\partial x^2} + \varepsilon z^2 \frac{\partial^2 v^2}{\partial x^2} + \mu \frac{\partial^2 v_2}{\partial z^2} + \varepsilon z^2 \frac{\partial^2 v_2}{\partial z^2} + (\mu + 2\varepsilon z)\frac{\partial v^2}{\partial z} + (\rho + \varepsilon z^2)\omega^2 v_2 = 4\pi\delta(x)\delta(z = H) \tag{14}$$

Dividing Eq. (14) throughout by μ and rearranging, we get

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial z^2} + k^2 v_2 = \frac{4\pi\delta(x)}{\mu}\delta(z-H) - \frac{\varepsilon z^2}{\mu}\frac{\partial^2 v_2}{\partial z^2} - \frac{\varepsilon z^2}{\mu}\frac{\partial^2 v_2}{\partial z^2} - \frac{(\mu + 2\varepsilon z)}{\mu}\frac{\partial v_2}{\partial z} \tag{15}$$

where, $k_2^2 = \frac{\rho + \varepsilon z^2}{\mu}\omega^2$, $\omega = kc$ is the angular frequency, k the wave number and c is the phase velocity.

3. Boundary conditions

The geometry of the problem leads to the following conditions

$$v_1 = v_2, \quad \mu_1 \frac{\partial v_1}{\partial z} = \mu_2 \frac{\partial v_2}{\partial z}. \quad (at \quad z = 0, \quad -\infty < x < \infty) \tag{16a}$$

$$v_2 = v_3, \quad \left(\mu + \varepsilon H^2\right)\frac{\partial v_2}{\partial z} = \mu_3 \frac{\partial v_3}{\partial z}. \quad (at \quad z = H, \quad -\infty < x < \infty) \tag{16b}$$

$$v_1 \to 0 \quad as \quad z \to -\infty. \tag{16c}$$

$$v_1 \to 0 \quad as \quad z \to -\infty. \tag{16d}$$

4. Solution of the problem

The following transforms are used to solve Eq. (7), Eq. (9) and Eq. (15)

$$V_r(\xi, z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} v_r(x, z)\, e^{i\xi x}\, dx \tag{17a}$$

Then the inverse Fourier transform is given as

$$v_r(x, z) = \int_{-\infty}^{+\infty} V_r(\xi, z)\, e^{-i\xi x}\, d\xi \tag{17b}$$

Now using the above defining Fourier transforms for Eq. (7) and Eq. (9), we get

$$\frac{d^2V_1}{dz^2} - \alpha^2 V_1 = 0, \tag{18}$$

where, $\alpha^2 = \xi^2 - k_1^2$

$$\frac{d^2V_3}{dz^2} - \gamma^2 V_3 = 0, \tag{19}$$

where, $\gamma^2 = \xi^2 - k_3^2$

Also, Eq. (15) can be written in terms of Fourier transforms as

$$\frac{d^2V_2}{dz^2} - \beta^2 V_2 = 4\pi\sigma(z), \tag{20a}$$

where,

$$\beta^2 = \xi^2 - k_2^2 \qquad and \qquad 4\pi\sigma(z) = \left(\frac{2}{\mu}\right)\delta(z-H) - \frac{\varepsilon}{\mu}\left\{ z^2\frac{d^2V_2}{dz^2} + 2(z-H)\frac{dV_2}{dz} - z^2\xi^2 V_2 \right\}$$

In order to solve Eqs. (18)-(20) under the prescribed boundary conditions in Eqs. (16a), (16b), (16c) and (16d), we introduce the Green's Function technique. First of all take the intermediate inhomogeneous layer and it is solved with the help of Green's function $G_2(z/z_0)$ (Stakgold, 1979). The Eq. (20) will satisfy $G_2(z/z_0)$ as

$$\frac{d^2G_2(z/z_0)}{dz^2} - \beta^2 G_2(z/z_0) = \delta(z - z_0), \tag{21}$$

together with the homogeneous boundary conditions

$$\frac{dG_2(z/z_0)}{dz} = 0. \quad \text{at } z = 0, H \tag{22}$$

Here z_0 is arbitrary line in the medium 2. Multiplying Eq. (20b) by $G_2(z/z_0)$, Eq. (21) by $V_2(\xi, z)$, subtracting and integrating over $0 \le z \le H$, we get

$$G_2(H/z_0)\left[\frac{dV_2}{dz}\right]_{z=H} - G_2(0/z_0)\left[\frac{dV_2}{dz}\right]_{z=0} = \int_0^H \sigma_2(z)G_2(z/z_0)dz - V_2(z_0). \qquad (23)$$

Similarly, if $G_1(z/z_0)$ and $G_3(z/z_0)$ are Green's functions corresponding to upper and lower homogeneous media, then Eq. (18) and Eq. (19) will satisfy as

$$\frac{dG_1(z/z_0)}{dz} = 0 \quad \text{at } z = 0; \quad \frac{dG_1(z/z_0)}{dz} \to 0 \quad \text{as } z \to -\infty, \qquad (24)$$

and

$$\frac{dG_3(z/z_0)}{dz} = 0 \quad \text{at } z = H; \quad \frac{dG_3(z/z_0)}{dz} \to 0 \quad \text{as } z \to \infty \qquad (25)$$

Here z_0 is arbitrary point in the medium 1. Multiplying Eq. (18) by $G_1(z/z_0)$, Eq. (24) by $V_1(\xi, z)$, subtracting and integrating, we get

$$G_1(0/z_0)\left[\frac{dV_1}{dz}\right]_{z=0} = -V_1(z_0), \qquad (26)$$

Multiplying Eq. (19) by $G_3(z/z_0)$, Eq. (25) by $V_3(\xi, z)$, subtracting and integrating, we get

$$G_3(H/z_0)\left[\frac{dV_3}{dz}\right]_{z=H} = -V_3(z_0). \qquad (27)$$

Replacing z by z_0 and using symmetry of Green's function, Eq. (23), Eq. (26) and Eq. (27) become

$$V_2(z) = G_2(z/0)\left[\frac{dV_2}{dz}\right]_{z=0} - G_2(z/H)\left[\frac{dV_2}{dz}\right]_{z=H} + \int_0^H \sigma_2(z_0)G_2(z/z_0)dz_0, \qquad (28)$$

$$V_1(z) = -G_1(z/0)\left[\frac{dV_1}{dz}\right]_{z=0}, \qquad (29)$$

$$V_3(z) = G_3(z/H)\left[\frac{dV_3}{dz}\right]_{z=H}. \qquad (30)$$

Using boundary condition (16a) in Eq. (28), we get

$$\left[\frac{dV_2}{dz}\right]_{z=0} = \frac{1}{A}\left\{G_2(0/H)\left[\frac{dV_2}{dz}\right]_{z=H} - \int_0^H \sigma_2(z_0)G_2(0/z_0)dz_0\right\}, \qquad (31)$$

where, $A = G_2(0/0) + \frac{\mu}{\mu_1} G_1(0/0)$

Similarly, using boundary condition (16b) in Eq. (28), we get

$$\left[\frac{dV_2}{dz}\right]_{z=H} = \frac{1}{\left\{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3} AG_3(H/H)\right\}} \left\{\begin{array}{l} -G_2(H/0)\int_0^H \sigma_2(z_0) G_2(0/z_0)\,dz_0 \\ +A\int_0^H \sigma_2(z_0) G_2(H/z_0)\,dz_0 \end{array}\right\}, \quad (32)$$

where, $B = G_2(H/H) + \frac{\mu}{\mu_3} G_3(H/H)$.

Using Eq. (31) and Eq. (32) in Eq. (28), substituting back the values of $\sigma_2(z_0)$ and using the property of delta function, we get

$$V_2(z) = \frac{2(\mu + \varepsilon H^2)}{\mu\mu_3} \left\{\frac{G_2(z/H)C - G_2(z/0)D}{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3} AG_3(H/H)}\right\}$$

$$-\left\{\frac{G_2(z/H)G_2(H/0) - \left\{B + \frac{\varepsilon H^2}{\mu_3} G_3(H/H)\right\} G_2(z/0)}{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3} AG_3(H/H)}\right\} \frac{\varepsilon}{\mu} \int_0^H \left\{z_0^2 \frac{d^2V_2(z_0)}{dz_0^2} + 2z_0 \frac{dV_2(z_0)}{dz_0} - z_0^2 \xi^2 V_2(z_0)\right\} G_2(0/z_0)\,dz_0$$

$$-\left\{\frac{G_2(z/0)G_2(H/0) - G_2(z/H)A}{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3} AG_3(H/H)}\right\} \frac{\varepsilon}{\mu} \int_0^H \left\{z_0^2 \frac{d^2V_2(z_0)}{dz_0^2} + 2z_0 \frac{dV_2(z_0)}{dz_0} - z_0^2 \xi^2 V_2(z_0)\right\} G_2(H/z_0)\,dz_0$$

$$-\frac{\varepsilon}{\mu} \int_0^H \left\{z_0^2 \frac{d^2V_2(z_0)}{dz_0^2} + 2z_0 \frac{dV_2(z_0)}{dz_0} - z_0^2 \xi^2 V_2(z_0)\right\} G_2(z/z_0)\,dz_0, \quad (33)$$

where, $C = G_3(H/H)A$, $\quad D = G_3(H/H)G_2(H/0)$.

Eq. (33) is an integral equation and $V_2(z)$ can be found from this equation by using approximations. First of all we neglect the terms having ε, we get

$$V_2(z) = \frac{2}{\mu_3} \left\{\frac{G_2(z/H)C - G_2(z/0)D}{AB - G_2^2(H/0)}\right\}. \quad (34)$$

Now put Eq. (34) back in the right hand side of Eq. (33), we get

$$V_2(z) = \frac{2(\mu+\varepsilon H^2)}{\mu\mu_3}\left\{\frac{G_2(z/H)C - G_2(z/0)D}{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3}AG_3(H/H)}\right\}$$

$$-\frac{2\varepsilon}{\mu\mu_3}\left\{\frac{G_2(z/H)G_2(H/0) - \left\{B + \frac{\varepsilon H^2}{\mu_3}G_3(H/H)\right\}G_2(z/0)}{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3}AG_3(H/H)}\right\}\int_0^H\left\{z_0^2\frac{d^2\Theta(z_0)}{dz_0^2} + 2z_0\frac{d\Theta(z_0)}{dz_0} - z_0^2\xi^2\Theta(z_0)\right\}G_2(0/z_0)dz_0$$

$$-\frac{2\varepsilon}{\mu\mu_3}\left\{\frac{G_2(z/0)G_2(H/0) - G_2(z/H)A}{AB - G_2^2(H/0) + \frac{\varepsilon H^2}{\mu_3}AG_3(H/H)}\right\}\int_0^H\left\{z_0^2\frac{d^2\Theta(z_0)}{dz_0^2} + 2z_0\frac{d\Theta(z_0)}{dz_0} - z_0^2\xi^2\Theta(z_0)\right\}G_2(H/z_0)dz_0 \tag{35}$$

$$-\frac{2\varepsilon}{\mu\mu_3}\int_0^H\left\{z_0^2\frac{d^2\Theta(z_0)}{dz_0^2} + 2z_0\frac{d\Theta(z_0)}{dz_0} - z_0^2\xi^2\Theta(z_0)\right\}G_2(z/z_0)dz_0,$$

where,

$$\Theta(z_0) = \left\{\frac{G_2(z_0/H)C - G_2(z_0/0)D}{AB - G_2^2(H/0)}\right\}$$

We note that Eq. (35) completely represents the elastic displacements. These elastic displacements are due to a unit impulsive force in space and time. Also, the solution of Eq. (35) is incomplete because G_1, G_2 and G_3 are not known. We adopt the following method to find the unknown Green's function.

We have considered $G_1(z / z_0)$ as a solution of Eq. (18).

A solution of Eq. (18) can also be found as

$$\frac{d^2L}{dz^2} - \alpha^2 L = 0 \tag{36}$$

The two independent solutions of Eq. (36) will vanish at $z = -\infty$ and $z = \infty$ are

$$L_1(z) = e^{\alpha z} \qquad and \qquad L_2(z) = e^{-\alpha z} \tag{37}$$

Hence, the solution of Eq. (36) for an infinite medium is

$$\begin{aligned} &\frac{L_1(z)L_2(z_0)}{M} \qquad for \qquad z < z_0, \\ &\frac{L_1(z_0)L_2(z)}{M} \qquad for \qquad z > z_0. \end{aligned} \tag{38}$$

where, $M = L_1(z)L'_2(z) - L_2(z)L'_1(z) = -2\alpha$.

So we can write the solution of Eq. (18) as

$$-\frac{e^{-\alpha|z-z_0|}}{2\alpha}. \tag{39}$$

Since $G_1(z / z_0)$ is to satisfy the condition (Eq. (24))

$$\frac{dG_1(z/z_0)}{dz}=0 \quad \text{at } z=0; \quad \frac{dG_1(z/z_0)}{dz}\to 0 \quad \text{as } z\to-\infty, \tag{40}$$

Therefore, we assume that

$$G_1(z/z_0)=-\frac{e^{-\alpha|z-z_0|}}{2\alpha}+Ae^{\alpha z}+Be^{-\alpha z}. \tag{41}$$

The conditions as mentioned in Eq. (40) give

$$G_1(z/z_0)=-\frac{1}{2\alpha}\left[e^{-\alpha|z-z_0|}+e^{\alpha(z+z_0)}\right], \tag{42}$$

Similarly

$$G_3(z/z_0)=-\frac{1}{2\gamma}\left[e^{-\gamma|z-z_0|}+e^{\gamma(z+z_0-2H)}\right], \tag{43}$$

For the intermediate inhomogeneous layer, Green's function $G_1(z / z_0)$ can be obtained in the similar manner as above by using the boundary conditions Eq. (16a) and Eq. (16b).

$$G_2(z/z_0)=-\frac{1}{2\beta}\left[e^{-\beta|z-z_0|}+e^{\beta z}\left\{\frac{e^{-\beta(z_0+H)}+e^{\beta(H-z_0)}}{e^{\beta H}-e^{-\beta H}}\right\}+e^{-\beta z}\left\{\frac{e^{\beta(H-z_0)}+e^{\beta(H-z_0)}}{e^{\beta H}-e^{-\beta H}}\right\}\right]. \tag{44}$$

Substitute the value of Eq. (42), Eq. (43) and Eq. (44) in Eq. (35), simplifying and neglecting square and higher powers of ε , we get

$$V_2(z)=\frac{-2(\mu\beta\cosh\beta z+\mu_1\alpha\sinh\beta z)}{\mu_1\mu_3\alpha\beta^2\gamma\left\{AB-G_2^2(H/0)\right\}E(\varepsilon)\sinh\beta H}, \tag{45}$$

Where

$$E(\varepsilon)=1+\frac{\varepsilon}{4\left\{AB-G_2^2(H/0)\right\}}\left[\begin{array}{l}\dfrac{(\mu_3\gamma-\mu_1\alpha)}{\mu_1\mu_3\alpha\beta^2\gamma}\\ -\dfrac{H^2\left\{(5\mu_1\mu_3\alpha\gamma+3\mu^2\beta^2)+(\mu_1\mu_3\alpha\gamma+\mu^2\beta^2)\coth\beta H\right\}}{\mu_1\mu\mu_3\alpha\beta^2\gamma}\\ -\dfrac{H^3\left\{\mu\beta(\mu_1\alpha+\mu_3\gamma)+(\mu_1\mu_3\alpha\gamma+\mu^2\beta^2)\coth\beta H\right\}}{\mu_1\mu\mu_3\alpha\beta\gamma}\\ -\dfrac{H^2\xi^2\left\{(\mu_1\mu_3\alpha\gamma-\mu^2\beta^2)+\mu\beta(\mu_3\gamma-\mu_1\alpha)\coth\beta H\right\}}{\mu_1\mu\mu_3\alpha\beta^4\gamma}\\ +\dfrac{H^3\xi^2\left\{\mu\beta(\mu_1\alpha+\mu_3\gamma)+(\mu_1\mu_3\alpha\gamma+\mu^2\beta^2)\coth\beta H\right\}}{\mu_1\mu\mu_3\alpha\beta^3\gamma}\\ +\dfrac{\xi^2(\mu_3\gamma-\mu_1\alpha)}{\mu_1\mu_3\alpha\beta^4\gamma}\end{array}\right]. \tag{46}$$

5. Transmitted waves

Taking inverse Fourier transform of Eq. (45), the displacement in the intermediate inhomogeneous layer is

$$v_2(x,z) = -\int_{ic-\infty}^{ic+\infty} \frac{2(\mu\beta\cosh\beta z + \mu_1\alpha\sinh\beta z)e^{-i\xi x}}{\mu_1\mu_3\alpha\beta^2\gamma\left\{AB - G_2^2(H/0)\right\}E(\varepsilon)\sinh\beta H}\,d\xi, \tag{47}$$

or

$$v_2(x,z) = -\int_{ic-\infty}^{ic+\infty} \frac{2(\mu\beta\cosh\beta z + \mu_1\alpha\sinh\beta z)e^{-i\xi x}}{J(\xi,H)}\,d\xi, \tag{48}$$

where, $J(\xi,H) = \mu_1\mu_3\alpha\beta^2\gamma\left\{AB - G_2^2(H/0)\right\}E(\varepsilon)\sinh\beta H$

Eq. (48) is obtained by performing contour integration. The poles of the integrands are obtained byputting the denominator to zero. The resultant relation will give the dispersion of SH-wave in non-homogeneous elastic media subjected to impulsive line source.

Replacing β by $i\widehat{\beta}$, the dispersion relation will become

$$\tan\widehat{\beta}H = \frac{\mu\widehat{\beta}(\mu\alpha+\mu_3\gamma)}{\mu^2\widehat{\beta}^2 - \mu_1\mu_3\alpha\gamma} + \frac{\varepsilon}{4\left\{\mu^2\widehat{\beta}^2 - \mu_1\mu_3\alpha\gamma\right\}}\left[\begin{array}{l} (\mu_3\gamma-\mu_1\alpha)\tan\widehat{\beta}H \\ -\dfrac{H^2}{\mu}\left\{(5\mu_1\mu_3\alpha\gamma - 3\mu^2\widehat{\beta}^2)\tan\widehat{\beta}H + (3\mu_1\alpha+5\mu_3\gamma)\right\} \\ -\dfrac{H^3\widehat{\beta}}{\mu}\left\{(\mu_1\mu_3\alpha\gamma - \mu^2\widehat{\beta}^2) + \mu\widehat{\beta}(\mu_1\alpha+\mu_3\gamma)\tan\widehat{\beta}H\right\} \\ +\dfrac{H^2\xi^2}{\mu\widehat{\beta}^2}\left\{(\mu_1\mu_3\alpha\gamma + \mu^2\widehat{\beta}^2)\tan\widehat{\beta}H + \mu\widehat{\beta}(\mu_3\gamma-\mu_1\alpha)\right\} \\ -\dfrac{H^3\xi^2}{\mu\widehat{\beta}^2}\left\{(\mu_1\mu_3\alpha\gamma - \mu^2\widehat{\beta}^2) - \mu\widehat{\beta}(\mu_1\alpha+\mu_3\gamma)\tan\widehat{\beta}H\right\} \\ -\dfrac{\xi^2}{\widehat{\beta}^2}\left\{(\mu_3\gamma-\mu_1\alpha)\tan\widehat{\beta}H\right\} \end{array}\right] \tag{49}$$

In the absence of non-homogeneity i.e., $\varepsilon = 0$, the relation (49) reduces to

$$\tan\widehat{\beta}H = \frac{\mu\widehat{\beta}(\mu\alpha+\mu_3\gamma)}{\mu^2\widehat{\beta}^2 - \mu_1\mu_3\alpha\gamma} \tag{50}$$

The Eq. (48) is the dispersion relation for love waves in homogeneous media given by Ewing *et al*. (1957).

In order to calculate transmitted waves we have to calculate Eq. (48), for that we note that the poles of the integrand are roots $P_{2,n}$ (n = 1, 2, 3...) of

$J(\xi,H) = \mu_1\mu_3\alpha\beta^2\gamma\left\{AB - G_2^2(H/0)\right\}E(\varepsilon)\sinh\beta H$

Calculate the pole contribution at these poles, we get

$$v_2(x,z) = 2\pi\sum_{n=1}^{\infty}\frac{e^{-ip_{2,n}x}\left\{\mu k_{2,n}\cos\widehat{\beta}_{2,n}z + \mu_1\alpha_{2,n}\sin\widehat{\beta}_{2,n}z\right\}}{\left.\dfrac{dJ(\xi,H)}{d\xi}\right|_{\xi=p_{2,n}}} \tag{51}$$

where, $\hat{\beta}\big|_{\xi=p_{2,n}} = \hat{\beta}_{2,n}, \quad \alpha\big|_{\xi=p_{2,n}} = \alpha_{2,n}.$

Eq. (51) is the expression for SH-wave travelling in the x-axis. Beside the poles, we have branch points which give rise to the body waves, which is not important in the present study.

6. Numerical analysis

The effects of non-homogeneity in the intermediate layer are studied numerically by taking following parameters (Gubbins 1990).

The various curves in Fig. 2 are plotted between $\hat{\beta}H$ v/s α at various values of non-homogeneity parameter $\varepsilon^{/} = \dfrac{\varepsilon}{4\{\mu^2 k^2\}}$ by taking values of $\varepsilon^{/}$ = 0.0 to 0.4. Another graph is drawn between $\hat{\beta}H$ v/s α at various values of another non-homogeneity factor $\varepsilon^{//} = \dfrac{\varepsilon H}{4\{\mu^2 k^2\}}$ by taking $\varepsilon^{//}$ = 0.0 to 0.4 (Fig. 3). It is clear from diagrams that the phase velocity of SH-waves is affected by non-homogeneity parameters.

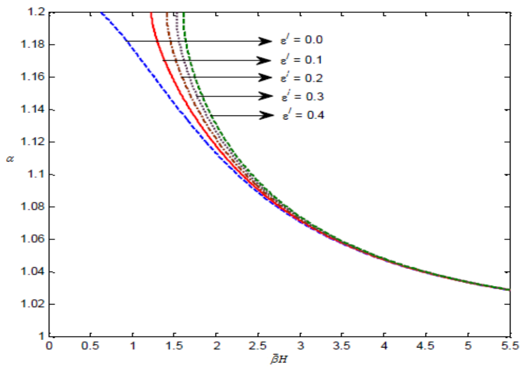

Fig. 2 Dispersion of SH-wave for $\varepsilon^{/}$

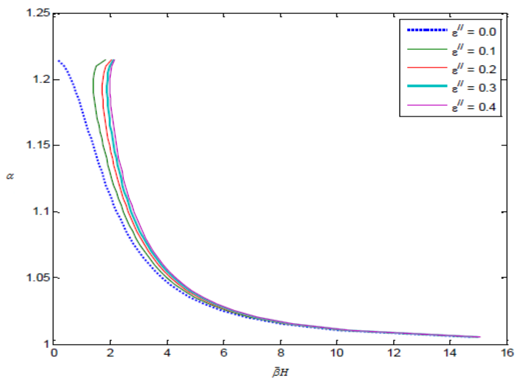

Fig. 3 Dispersion of SH-wave for $\varepsilon^{//}$

Table 1 Material Parameters

Layer	Rigidity	Density
Upper Homogeneous Layer	$\mu_1 = 11.77 \times 10^{10}$ N/m^2	$\rho_1 = 3409$ Kg / m^3
Intermediate Inhomogeneous Layer	$\mu = 11.77 \times 10^{10}$ N/m^2	$\rho = 4148$ Kg / m^3
Lower Homogeneous Layer	$\mu_3 = 11.77 \times 10^{10}$ N/m^2	$P_3 = 3944$ Kg / m^3

7. Conclusions

In this problem we assume the upper layer and lower layer is homogeneous, isotropic and semi infinite, whereas the intermediate layer is taken non-homogeneous isotropic with quadratic variation in rigidity and density. We have employed Green's function method to find the frequency equation due to a line source. Displacement in the intermediate layer is derived in closed form and the dispersion curves are drawn for various values of inhomogeneity parameters. Eq. (49) gives the dispersion of SH-wave in non-homogeneous elastic media subjected to impulsive line source. It is observed from Eq. (49), the inhomogeneity parameters affect the phase velocity of SH-wave and however their effects are negligible after a certain value of dimensionless wave number.

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