Displacements of the helical stator tooth for an electromechanical integrated toroidal drive

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Abstract. The stator tooth is a key component of the electromechanical integrated toroidal drive system. The stator tooth is spiral in shape and the calculation of its displacements is difficult. In this paper, using the coordinate transformation method, the displacements of the stator tooth in the local coordinate system are expressed as the function of the variable in the drive coordinate system. Using the minimum potential energy principle, the equations of the displacements of the stator tooth under the loads are deduced. The displacement distributions within the stator tooth are investigated and the changes of the displacement distributions along with the main parameters are analyzed. This research can offer the basis for the strength and stiffness design of the drive system.

Keywords: toroidal drive; electromechanical integrated; displacement; strain energy.

1. Introduction

A toroidal drive can transmit large torques in a very small size and is suitable for applications to the technical fields such as aviation and space flight, etc (Kuehnle et al. 1981, Yao et al. 2006, Wei et al. 2008). As more and more electrical and control techniques are utilized in the mechanical engineering field, electromechanical integrated drive systems have become an advancing edge of the mechanical science.

Many types of the magnetic gear drive have been proposed and investigated recently (Mezani et al. 2006, Jorgensen et al. 2008, Wang et al. 2006, Liu et al. 2009). The magnetic gear drives are passive drive systems in which meshing occurs without physical contact. The electromagnetic harmonic drive (Delin and Huamin 1993) and piezoelectric harmonic drive (Oliver 2000) are active drive systems in which the meshing forces between the flexible and rigid gears are controlled by the electromagnetic or piezoelectric force with the drive and power integrated.

Based on the toroidal drive (Xu and Huang 2003), the authors presented a kind of electromechanical integrated drive system without contact, i.e., the electromechanical integrated toroidal drive. This drive is a new concept of the generalized composite drive. With this drive, the toroidal drive, power and control are all integrated (Xu and Huang 2005). The drive consists of four basic elements as shown in Fig. 1: (a) radically positioned planets; (b) a central worm; (c) a toroidal shaped stator; and (d) a rotor, which forms the central output shaft upon which the planets are

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mounted. The central worm is fixed and the coils are mounted along the helical grooves of its surface. The planets have permanent magnets instead of teeth. The $N$ and $S$ polar permanent magnets are mounted alternately on a planet. The stator has helical permanent magnets instead of helical teeth. In the same manner as the planet, the $N$ and $S$ polar helical permanent magnets are mounted alternately on the stator. Besides permanent magnets, steel plates can also be used for the teeth of the stator. When the teeth of the stator are produced with plate permanent magnets, the plate permanent magnet has also two poles. The pole axis of the magnet is perpendicular to the plate permanent magnet surface. One pole is near the planet and the other pole is located on the other surface of the magnet plate. As the teeth of the planet are also permanent magnets, between the planet teeth and steel plate, enough magnetic force can be generated as well. Therefore, under the condition that the torque to be transmitted is not large, the teeth of the stator can be produced with steel plates as well. Cylinder or segment permanent magnets can be used for the teeth of the planet. The cylinder or segment permanent magnets are mounted on the planet body by bonding or other joints. The central worm consists of a number of silicon steel sheets. Several helical slots are cut on the worm. The windings made of insulated wires are mounted along the helical slots.

If the relation between the planet pitch, tooth number and helical angle on the stator, and number of pole pairs and helical angle on the worm is specified, then the $N$ pole of one element will correspond to the $S$ pole of the other element all along. The attractive forces between the $N$ and $S$ poles of the different elements are the driving forces and meshing occurs without physical contact. When the alternate current is charged into the coils of the worm, a toroidal circular field is formed. It can drive several planets rotating about their own axis. And by means of magnetic forces between the teeth of the planet and stator, the rotor is driven to rotate about its own axis. Thus, a power of low speed and large torque is generated.

Compared with the toroidal drive, the new drive can be easily produced, without wear, and with no need for lubrication. It can be substituted for a servo system to simplify the structure of the existing electromechanical systems. Besides the above-mentioned fields that require compactness, the drive can be used in fields that require accurate control, such as robots, etc.

Fig. 1 The model machine for the drive system
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The authors have conducted some research for the drive system to investigate the driving torque and its affecting factors (Xu and Huang 2005), while presenting a control theory for the drive system (Xu and Fan 2006). As the loading torque becomes large, one should pay special attention to the strength and stiffness problems of the drive system. The stator tooth is the key component of the drive system, of which the strength and stiffness have important influence on the operating performance of the drive system. When the drive is in operation, speed fluctuation can occur with the output shaft. It is mainly caused by the non-uniform air-gap between the stator teeth and the planet teeth, which can be attributed to two factors. One is the manufacturing and mounting errors of the machine, another is the elastic displacements of the stator teeth. When the manufacturing and mounting errors are ignored, an elastic displacement of 0.04 mm for the stator tooth can cause a speed fluctuation of about 8% for the output shaft under the condition that the air-gap between the stator teeth and the planet teeth is about 0.5 mm. To reduce the speed fluctuation, the manufacturing and mounting accuracy should be improved. Besides this, the stiffness of the stator teeth should be chosen properly. Therefore, it becomes a prerequisite to calculate the elastic displacement analysis of the stator teeth, which serves as the basis for designing the stiffness of the stator teeth. The problem can be resolved using standard finite elements codes. However, an analytical solution has one obvious advantage over the finite element solution, because it enables us to draw some inherent rules that are favorable for the parameter design and optimum of the drive system. For the reason stated, we shall focus on derivation of an analytical solution for the stator tooth that is spiral in shape.

In this paper, the coordinate systems for the drive system and its stator tooth are established, and the transformation matrices between the different coordinate systems are presented. Based on this, the displacements of the stator tooth in the local coordinate system are expressed as a function of the variables in the drive coordinate system. Using the minimum potential energy principle and selecting proper trial function, the equations of the displacements of the stator tooth under the loads are derived. Using these equations, the displacement distributions within the stator tooth for the drive system are investigated and the changes of the displacement distributions with respect to the main parameters are analyzed. The results show that the related parameters have obvious influence on the displacement distribution of the stator tooth. In order to decrease the displacements of the stator tooth, these parameters should be selected properly. The present research offers a useful basis for the strength and stiffness design of the drive system.

2. Coordinate transformation

Fig. 2 shows a helical stator tooth and its related coordinate systems. As the stator teeth are all of helical surface, their displacement calculation is quite complicated. In order to obtain the displacement, we must use the coordinate transformation principles. To this end, we select the coordinate systems for electromechanical integrated toroidal drive as shown in Fig. 2. The coordinate system $s(x, y, z)$ is attached to the stator, $s_1(x_1, y_1, z_1)$ is attached to the planet, $s_2(x_2, y_2, z_2)$ to the normal section of the stator tooth, $s_t(x_t, y_t, z_t)$ to the rotor, and $s_{10}(x_{10}, y_{10}, z_{10})$ also to the rotor. The $z_s$- and $z_r$-axes are the rotating axes of the rotor, particularly the $z_r$-axis is the rotating axis of the planet. Here, $\phi$ is the planet rotating angle, $R$ is the planet radius, $\lambda$ is the lead angle of the stator tooth, \[\tan \lambda = \frac{z_t R}{z_1 (a + R \cos \phi)}\], with $a$ indicating the central distance between the planet and
stator, \( z_s \) is the tooth number of the stator, and \( z_1 \) is the tooth number of the planet.

The matrices \( M_{101} \), \( M_{r10} \), \( M_{r2} \) and \( M_{21} \) are the coordinates transformation matrices from the coordinate system \( s_{10} \) to the system \( s_1 \), from the system \( s_{10} \) to the system \( s_r \), from the system \( s_r \) to the system \( s \), and from the system \( s_1 \) to the system \( s_2 \), respectively. These matrices can be given as follows:

\[
M_{101} = \begin{bmatrix}
\cos \phi_1 & \sin \phi_1 & 0 \\
-\sin \phi_1 & \cos \phi_1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad (1)
\]

\[
M_{r10} = \begin{bmatrix}
1 & 0 & 0 & a \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad (2)
\]
3. The minimum potential energy principle and the displacement function

In order to obtain the displacement of the helical stator tooth, we use the minimum potential energy principle as below:

$$\delta E_i = 0$$  \hspace{1cm} (5)

where $E_i = U - W$, $U$ is the strain energy of the stator tooth, and $W$ is the work of the forces. From Eq. (5), one obtains

$$\delta U = \delta W$$  \hspace{1cm} (6)

where $\delta U$ and $\delta W$ are the virtual strain energy and virtual work, respectively.

The virtual strain energy of the stator tooth can be given as follows:

$$\delta U = \frac{1}{2} \int \left[ \int \left( \sigma_{x2} \delta \varepsilon_{x2} + \sigma_{y2} \delta \varepsilon_{y2} + \sigma_{z2} \delta \varepsilon_{z2} + \tau_{x2y2} \delta \gamma_{x2y2} + \tau_{y2z2} \delta \gamma_{y2z2} + \tau_{z2x2} \delta \gamma_{z2x2} \right) dv \right]$$

where $\sigma_{x2}, \sigma_{y2}, \sigma_{z2}$ are the stress components of the stator tooth in the coordinate axes $x_2, y_2$ and $z_2$, respectively; $\varepsilon_{x2}, \varepsilon_{y2}, \varepsilon_{z2}$ are the strain components of the stator tooth in the coordinate axes $x_2, y_2$ and $z_2$, respectively; $\tau_{x2y2}, \tau_{y2z2}, \tau_{z2x2}$ are the shear stress components of the stator tooth in three different directions, respectively; $\gamma_{x2y2}, \gamma_{y2z2}, \gamma_{z2x2}$ are the shear strain components of the stator tooth in three different directions, respectively; and $V$ is the volume of the stator tooth.

The virtual work done by the loads applied to the stator tooth can be given as follows:

$$\delta W = \int \left[ \int \left( F_{bx2} \delta u_2 + F_{by2} \delta v_2 + F_{bz2} \delta w_2 \right) dV \right]$$

where $F_{bi}$ is the load applied to the stator tooth ($i = x_2, y_2$ and $z_2$), $u_2$, $v_2$ and $w_2$ are the displacements of the stator tooth in the coordinate axes $x_2, y_2$ and $z_2$, respectively.

Eq. (7) can be changed into the following form

$$\delta U = \int \left( \delta \varepsilon_{x2} T_{x2} + \delta \varepsilon_{y2} T_{y2} + \delta \varepsilon_{z2} N_{z2} + \delta \gamma_{x2y2} F_{x2} + \delta \gamma_{y2z2} F_{y2} + \delta \gamma_{z2x2} T_{z2} \right) ds$$

where $T_{x2}$ is the bending moment about axis $x_2$, $T_{y2}$ the bending moment about axis $y_2$, $N_{z2}$ is the tension force in the $z_2$ direction, $F_{x2}$ is the shear force in the $x_2$ direction, $F_{y2}$ is the shear force in the $y_2$ direction, and $T_{z2}$ is the torsional moment about axis $z_2$. 

\[
M_{0r} = \begin{bmatrix}
\cos \phi_r & -\sin \phi_r & 0 \\
\sin \phi_r & \cos \phi_r & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
M_{2r} = \begin{bmatrix}
1 & 0 & 0 & -R \\
0 & \cos \lambda & \sin \lambda & 0 \\
0 & -\sin \lambda & \cos \lambda & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
The strains of the tooth can be expressed as functions of the displacements,

\[
\begin{align*}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \phi_x} \frac{\partial \phi_x}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \phi_y} \frac{\partial \phi_y}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z} = \frac{\partial w}{\partial \phi_z} \frac{\partial \phi_z}{\partial z} \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial \phi_x} \frac{\partial \phi_y}{\partial y} + \frac{\partial v}{\partial \phi_y} \frac{\partial \phi_x}{\partial x} \\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{\partial u}{\partial \phi_x} \frac{\partial \phi_z}{\partial z} + \frac{\partial w}{\partial \phi_z} \frac{\partial \phi_x}{\partial x} \\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \frac{\partial v}{\partial \phi_y} \frac{\partial \phi_z}{\partial z} + \frac{\partial w}{\partial \phi_z} \frac{\partial \phi_y}{\partial y}
\end{align*}
\]
In coordinate system \(s_{10}(x_{10}, y_{10}, z_{10})\), a point on the stator tooth can be expressed as

\[
\begin{align*}
  x_{10} &= R \cos \phi_i \\
  y_{10} &= -R \sin \phi_i \\
  z_{10} &= 0
\end{align*}
\]  

(12)

Using the coordinates transformation, the point can be given in the coordinate system \(s(x, y, z)\) as follows:

\[
\begin{bmatrix}
  x \\
  y \\
  z \\
  1
\end{bmatrix} = M_{0}M_{r10} \begin{bmatrix}
  x_{10} \\
  y_{10} \\
  z_{10} \\
  1
\end{bmatrix} = \begin{bmatrix}
  (R \cos \phi_i + a) \cos \phi_i \\
  (R \cos \phi_i + a) \sin \phi_i \\
  R \sin \phi_i \\
  1
\end{bmatrix}
\]  

(13)

From Eq. (13), \(\frac{\partial \phi_i}{\partial x}, \frac{\partial \phi_i}{\partial y}\) and \(\frac{\partial \phi_i}{\partial z}\) can be obtained as follows:

\[
\phi_x = \frac{\partial \phi_i}{\partial x} = \frac{1}{-R \left( \sin \phi_i \cos \frac{z_{10}}{z_s} + \frac{z_{10}}{z_s} \cos \phi_i \sin \frac{z_{10}}{z_s} + \frac{z_{10}a}{z_s} \sin \frac{z_{10}}{z_s} \right)}
\]

(14a)

\[
\phi_y = \frac{\partial \phi_i}{\partial y} = \frac{1}{-R \left( \sin \phi_i \sin \frac{z_{10}}{z_s} \cos \frac{z_{10}}{z_s} \cos \phi_i \sin \frac{z_{10}}{z_s} + \frac{z_{10}a}{z_s} \cos \frac{z_{10}}{z_s} \right)}
\]

(14b)

\[
\phi_z = \frac{\partial \phi_i}{\partial z} = \frac{1}{R \cos \phi_i}
\]

(14c)

Let the trial function of the displacements \(u, v \) and \(w\) be as follows:

\[
u(\phi) = a_1 \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right) + a_2 \sin \left( \frac{2n(\phi - \theta_i)}{\Delta \theta} \right) + \ldots + a_n \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right) + \ldots - \sum_{n=1}^{\infty} a_n \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right)
\]

(15a)

\[
v(\phi) = b_1 \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right) + b_2 \sin \left( \frac{2n(\phi - \theta_i)}{\Delta \theta} \right) + \ldots + b_n \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right) + \ldots - \sum_{n=1}^{\infty} b_n \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right)
\]

(15b)

\[
w(\phi) = c_1 \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right) + c_2 \sin \left( \frac{2n(\phi - \theta_i)}{\Delta \theta} \right) + \ldots + c_n \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right) + \ldots - \sum_{n=1}^{\infty} c_n \sin \left( \frac{n(\phi - \theta_i)}{\Delta \theta} \right)
\]

(15c)

where \(\Delta \theta = \theta_2 - \theta_1, \theta_1\), and \(\theta_2\) are the initial and end mesh angles of the planet tooth as shown in Fig. 3. From Eq. (15), \(\frac{\partial u}{\partial \phi}, \frac{\partial v}{\partial \phi}\) and \(\frac{\partial w}{\partial \phi}\) can be calculated.
In the coordinate system \(s(x, y, z)\), the arc length \(ds\) can be calculated using Eqs. (13) and (15),

\[
ds = \sqrt{(x'^2 + y'^2 + z'^2)} \, d\phi_1 = RSd\phi_1 \tag{16a}
\]

where

\[
s = \frac{\pi}{4} \left[ \frac{z}{z} \sin \frac{\phi}{z} \cos \phi + \cos \frac{\phi}{z} \sin \phi \cdot \frac{z}{R} \sin \frac{\phi}{z} \right] + \left[ \frac{z}{z} \cos \frac{\phi}{z} \cos \phi \sin \phi + \frac{z}{z} \cos \frac{\phi}{z} \sin \phi \right] + (\cos \phi) \]

Combining Eqs. (9) with (10), (11), (14), (15) and (16), the strain energy of the stator tooth can be determined.

If the magnetic forces \(F_{b2x}\) and \(F_{b2y}\) are applied to the stator tooth (see Fig. 2), with the angle of the acting point denoted by \(\phi_0\), the work done by the forces can be computed from Eq. (8) as

\[
W = F_{b2x}u_2(\phi_0) + F_{b2y}v_2(\phi_0) \tag{17a}
\]

where

\[
\begin{bmatrix}
u_2 \\ v_2 \\ w_2 \\ 0
\end{bmatrix} = B
\begin{bmatrix}
u \\ w \\ 0
\end{bmatrix}, \text{ here } B = M_21M_{110}M_{100}M_{00} = \begin{bmatrix}
B_{11} & B_{12} & B_{13} & B_{14} \\
B_{21} & B_{22} & B_{23} & B_{24} \\
B_{31} & B_{32} & B_{33} & B_{34} \\
B_{41} & B_{42} & B_{43} & B_{44}
\end{bmatrix} \tag{17b}
\]

Substituting Eq. (17) and the strain energy of the stator tooth into Eq. (6) yields

\[
\begin{align*}
\frac{\partial U}{\partial a_i} &= \frac{\partial W}{\partial a_i} \\
\frac{\partial U}{\partial b_i} &= \frac{\partial W}{\partial b_i} \quad (i = 1, 2, ..., K) \\
\frac{\partial U}{\partial c_i} &= \frac{\partial W}{\partial c_i}
\end{align*}
\tag{18}
\]
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From Eq. (18), one obtains

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\vdots \\
\alpha_k \\
\beta_1 \\
\vdots \\
\beta_k \\
\gamma_1 \\
\vdots \\
\gamma_k
\end{bmatrix}
= \begin{bmatrix}
(F_{b_{11}B_{21}} + F_{b_{12}B_{21}}) \sin \frac{\pi (\phi_0 - \theta)}{\Delta \theta} \\
\vdots \\
(F_{b_{11}B_{22}} + F_{b_{12}B_{22}}) \sin \frac{\pi (\phi_0 - \theta)}{\Delta \theta} \\
\vdots \\
(F_{b_{11}B_{23}} + F_{b_{12}B_{23}}) \sin \frac{\pi (\phi_0 - \theta)}{\Delta \theta}
\end{bmatrix}
\]

where

\[
A_{ij} = \begin{bmatrix}
\alpha_{i1} & \cdots & \alpha_{ik} \\
\alpha_{k1} & \cdots & \alpha_{kk}
\end{bmatrix}
\]

In matrix \(A_{11}\):

\[
\alpha_{ij} = 2E I_{2} \int_{\phi_i}^{\phi_j} C_{ij}(l_{1}^2 \phi_x + l_{2} m_{1} \phi_y + n_{1} l_{1} \phi_z)^2 \phi \phi S \phi \phi S + 2E I_{3} \int_{\phi_i}^{\phi_j} C_{ij}(l_{2}^2 \phi_x + l_{2} m_{2} \phi_y + n_{2} l_{2} \phi_z)^2 \phi \phi S \phi \phi S + 2k_{s2} G A \int_{\phi_i}^{\phi_j} C_{ij}(2l_{1} l_{2} \phi_x + (l_{2} m_{2} + l_{1} m_{1}) \phi_y + (n_{2} l_{2} + n_{1} l_{1}) \phi_z)^2 \phi \phi S \phi \phi S + 2k_{s2} G A \int_{\phi_i}^{\phi_j} C_{ij}(2l_{1}^2 \phi_x + (l_{1} m_{1} + l_{2} m_{2}) \phi_y + (n_{1} l_{1} + n_{2} l_{2}) \phi_z)^2 \phi \phi S \phi \phi S
\]

In matrix \(A_{12}\):

\[
\alpha_{ij} = 2E I_{2} \int_{\phi_i}^{\phi_j} C_{ij}(m_{1}^2 \phi_y + l_{1} m_{1} \phi_y + m_{1} n_{1} \phi_z)(l_{2}^2 \phi_x + l_{2} m_{2} \phi_y + n_{2} l_{2} \phi_z) S \phi \phi S + 2E I_{3} \int_{\phi_i}^{\phi_j} C_{ij}(l_{2}^2 m_{1} \phi_y + l_{2} m_{2} \phi_z + m_{2} n_{2} \phi_z)(l_{2}^2 \phi_x + l_{2} m_{2} \phi_y + n_{2} l_{2} \phi_z) S \phi \phi S + 2k_{s2} G A \int_{\phi_i}^{\phi_j} C_{ij}(2m_{1} m_{2} \phi_y + (l_{2} m_{2} + l_{1} m_{1}) \phi_y + (m_{1} n_{2} + m_{2} n_{1}) \phi_z)^2 \phi \phi S \phi \phi S
\]
\[(2l_1l_2\phi_x + (l_1m_2 + l_2m_1)\phi_y + (n_2l_1 + n_1l_2)\phi_z)Sd\phi_s + 2k_{y_2}GA\int_{\phi_1}^{\phi_2} C_{ij}(2m_2m_3\phi_y + (l_2m_3 + l_3m_2)\phi_x + (m_2n_3 + m_3n_2)\phi_z) d\phi_i \]

\[+ 2l_2l_3\phi_x + (l_2m_3 + l_1m_2)\phi_y + (n_2l_3 + n_1l_2)\phi_z)Sd\phi_s \]

\[+ 2GJ\int_{\phi_1}^{\phi_2} C_{ij}(2m_3m_1\phi_y + (l_3m_1 + l_1m_3)\phi_x + (m_3n_1 + m_1n_3)\phi_z)(2l_1l_4\phi_x + (l_4m_1 + l_1m_4)\phi_y + (n_4l_1 + n_1l_4)\phi_z) d\phi_i \]

In matrix $A_{15}$:

\[\alpha_{ij} = 2EI_{x_2}\int_{\phi_1}^{\phi_2} C_{ij}(n_1^2\phi_x + m_1n_1\phi_y + n_1l_1\phi_z)(\int_{\phi_1}^{\phi_2} \phi_x + l_1m_1\phi_y + n_1l_1\phi_z)Sd\phi_s \]

\[+ 2EI_{x_2}\int_{\phi_1}^{\phi_2} C_{ij}(n_2^2\phi_x + m_2n_2\phi_y + n_2l_2\phi_z)(\int_{\phi_1}^{\phi_2} \phi_x + l_2m_2\phi_y + n_2l_2\phi_z)Sd\phi_s \]

\[+ 2EA\int_{\phi_1}^{\phi_2} C_{ij}(n_3^2\phi_x + m_3n_3\phi_y + n_3l_3\phi_z)(\int_{\phi_1}^{\phi_2} \phi_x + l_3m_3\phi_y + n_3l_3\phi_z)Sd\phi_s \]

\[+ 2k_{y_2}GA\int_{\phi_1}^{\phi_2} C_{ij}(2n_1n_2\phi_x + (m_1n_2 + m_2n_1)\phi_y + (n_1l_2 + n_2l_1)\phi_z) d\phi_i \]

\[+ 2l_3l_2\phi_x + (l_2m_3 + l_1m_2)\phi_y + (n_2l_3 + n_1l_2)\phi_z)Sd\phi_s \]

\[+ 2GJ\int_{\phi_1}^{\phi_2} C_{ij}(2n_1n_2\phi_x + (m_1n_2 + m_2n_1)\phi_y + (n_1l_2 + n_2l_1)\phi_z)(2l_1l_4\phi_x + (l_4m_1 + l_1m_4)\phi_y + (n_4l_1 + n_1l_4)\phi_z) d\phi_i \]

In matrix $A_{22}$:

\[\alpha_{ij} = 2EI_{x_2}\int_{\phi_1}^{\phi_2} C_{ij}(l_1^2\phi_x + l_1m_1\phi_y + m_1n_1\phi_z)^2 Sd\phi_s + 2EI_{x_2}\int_{\phi_1}^{\phi_2} C_{ij}(l_2^2\phi_y + l_2m_2\phi_x + m_2n_2\phi_z)^2 Sd\phi_s \]

\[+ 2EA\int_{\phi_1}^{\phi_2} C_{ij}(l_3^2\phi_x + l_3m_3\phi_y + m_3n_3\phi_z)^2 Sd\phi_s \]

\[+ 2k_{y_2}GA\int_{\phi_1}^{\phi_2} C_{ij}(2l_1m_2\phi_y + (l_1m_2 + l_2m_1)\phi_x + (m_1n_2 + m_2n_1)\phi_z)^2 Sd\phi_s \]

\[+ 2k_{y_2}GA\int_{\phi_1}^{\phi_2} C_{ij}(2l_2m_3\phi_y + (l_2m_3 + l_3m_2)\phi_x + (m_2n_3 + m_3n_2)\phi_z)^2 Sd\phi_s \]

\[+ 2GJ\int_{\phi_1}^{\phi_2} C_{ij}(2l_3m_1\phi_x + (l_3m_1 + l_1m_3)\phi_y + (m_3n_1 + m_1n_3)\phi_z)^2 d\phi_i \]

In matrix $A_{23}$:

\[\alpha_{ij} = 2EI_{x_2}\int_{\phi_1}^{\phi_2} C_{ij}(n_1^2\phi_x + m_1n_1\phi_y + n_1l_1\phi_z)(l_1^2\phi_y + l_1m_1\phi_y + m_1n_1\phi_z)Sd\phi_s \]

\[+ 2EI_{x_2}\int_{\phi_1}^{\phi_2} C_{ij}(n_2^2\phi_x + m_2n_2\phi_y + n_2l_2\phi_z)(l_2^2\phi_y + l_2m_2\phi_y + m_2n_2\phi_z)Sd\phi_s \]

\[+ 2EA\int_{\phi_1}^{\phi_2} C_{ij}(n_3^2\phi_x + m_3n_3\phi_y + n_3l_3\phi_z)(l_3^2\phi_y + l_3m_3\phi_y + m_3n_3\phi_z)Sd\phi_s \]
+ 2k_xG A \int_{\theta_1}^{\theta_2} C_{ij}(2n_1n_2 \phi_z + (m_1n_2 + m_2n_1) \phi_y + (n_2l_1 + n_1l_2) \phi_x) d\phi_i \\
(2m_1n_2 \phi_x + (l_1m_2 + l_2m_1) \phi_x + (m_1n_2 + m_2n_1) \phi_z) S d\phi_1 \\
+ 2k_yG A \int_{\theta_1}^{\theta_2} C_{ij}(2n_2n_3 \phi_z + (m_2n_3 + m_3n_2) \phi_y + (n_2l_3 + n_1l_2) \phi_x) d\phi_i \\
(2m_2n_3 \phi_y + (l_2m_3 + l_2m_3) \phi_x + (m_2n_3 + m_3n_2) \phi_z) S d\phi_1 \\
+ 2GJ \int_{\theta_1}^{\theta_2} C_{ij}(2n_1n_3 \phi_z + (m_1n_1 + m_3n_3) \phi_y + (n_3l_1 + n_1l_3) \phi_x) d\phi_i \\
(2m_1n_3 \phi_y + (l_1m_1 + l_1m_3) \phi_x + (m_1n_1 + m_3n_3) \phi_z) S d\phi_1 \\
+ 2GJ \int_{\theta_1}^{\theta_2} C_{ij}(2n_3n_1 \phi_z + (m_3n_1 + m_1n_3) \phi_y + (n_1l_3 + n_3l_1) \phi_x) d\phi_i \\
(2m_3n_1 \phi_y + (l_3m_1 + l_3m_3) \phi_x + (m_3n_1 + m_1n_3) \phi_z) S d\phi_1 \\

In matrix A_{33}:

\begin{align*}
A_{ij} = 2EIz_x \int_{\theta_1}^{\theta_2} C_{ij}(n_1^2 \phi_z + m_1n_1 \phi_y + n_1l_1 \phi_x)^2 S d\phi_i S + 2EIz_y \int_{\theta_1}^{\theta_2} C_{ij}(n_2^2 \phi_z + m_2n_2 \phi_y + n_2l_2 \phi_x)^2 S d\phi_i \\
+ 2EIz_z \int_{\theta_1}^{\theta_2} C_{ij}(n_3^2 \phi_z + m_3n_3 \phi_y + n_3l_3 \phi_x)^2 S d\phi_i \\
+ 2k_xG A \int_{\theta_1}^{\theta_2} C_{ij}(2n_1n_2 \phi_z + (m_1n_2 + m_2n_1) \phi_y + (n_2l_1 + n_1l_2) \phi_x)^2 S d\phi_i \\
+ 2k_yG A \int_{\theta_1}^{\theta_2} C_{ij}(2n_2n_3 \phi_z + (m_2n_3 + m_3n_2) \phi_y + (n_2l_3 + n_3l_2) \phi_x)^2 S d\phi_i \\
+ 2GJ \int_{\theta_1}^{\theta_2} C_{ij}(2n_1n_3 \phi_z + (m_1n_1 + m_3n_3) \phi_y + (n_1l_3 + n_3l_1) \phi_x)^2 S d\phi_i \\
A_{21} = A_{12}, A_{31} = A_{13}, A_{32} = A_{23}, C_{ij} = \left(\frac{\pi}{\Delta \theta}\right)^2 \frac{1}{i \cos \left(\frac{\pi (\phi_i - \phi)}{\Delta \theta}\right) j \cos \left(\frac{\pi (\phi_i - \phi)}{\Delta \theta}\right) \Delta \theta}
\end{align*}

From Eq. (19), the coefficients a_i, b_j and c_j (i=1 to k) in the trial function (15) can be determined. Hence, the displacements of the stator tooth under the applied forces can be obtained.

**4. Results and discussions**

In order to obtain the displacement distribution within the stator tooth, the above equations have been implemented in a software tool developed in the Matlab environment. The parameters of the numerical example have been listed in Table 1. In operation of the drive, two magnetic forces are applied to one stator tooth. One force is applied along the radial direction of the planet and kept constant under the condition that the air-gap between the stator and planet teeth, and their material and size are determined. The force is perpendicular to the direction of the air-gap and has obvious effects on the air-gap. For the model machine developed by the authors, the load is about 10 N.

<table>
<thead>
<tr>
<th>Table 1 Parameters of the example system</th>
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<tbody>
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<td>a / R</td>
</tr>
<tr>
<td>------</td>
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<tr>
<td>2</td>
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</tbody>
</table>
Hence, the force (load) is taken as 10 N in simulation. Another force is applied along the tangential direction of the planet tooth and its amplitude is dependent on the output torque. The force is parallel to the direction of the air-gap and has very small effects on the air-gap. Hence, the force (load) is also taken as 10 N in simulation. Fig. 4 shows the displacement distribution of the stator tooth under load $F_{bx_2}$ in the $x_2$ direction ($F_{bx_2} = -10$ N $F_{by_2} = 0$ N), Fig. 5 shows the displacement distribution of the stator tooth under load $F_{by_2}$ in the $y_2$ direction ($F_{bx_2} = 0$ N $F_{by_2} = 10$ N), Fig. 6 shows the displacement distribution of the stator tooth under loads $F_{bx_2}$ and $F_{by_2}$ ($F_{bx_2} = -10$ N $F_{by_2} = 10$ N).

From Figs. 4-6, one can observe the following facts:

1. For the load applied to the stator tooth in the $x_2$ direction, the displacements $u_2$, $v_2$ and $w_2$ in three directions occur. The displacement $u_2$ in the $x_2$ direction is much larger than displacements $v_2$ and $w_2$ in the other two directions.

2. As the mesh point between the planet and the stator changes, the maximum displacement in
Displacements of the helical stator tooth for an electromechanical integrated toroidal drive

the $x_2$ direction changes as well. When the mesh point is far from two ends of the stator tooth, the position of the maximum displacement $u_{2\text{max}}$ is nearly identical to the mesh point. However, as the mesh point is near the two ends of the stator tooth, the shift between the position of the maximum displacement $u_{2\text{max}}$ and the mesh point occurs. Such a position is located between the mesh point and the center point of the tooth, but it is near to the mesh point.

(3) When the two mesh points are symmetrical to the center point of the stator tooth, the displacement distributions are approximately symmetrical to the center point. However, they are not completely symmetrical, and the displacements for the mesh point in front of the center point of the stator tooth ($\phi_0 \leq 0^0$) are a bit larger than those for the mesh point behind the center point of the stator tooth ($\phi_0 \geq 0^0$).

(4) As the stator tooth is spiral, displacement $v_2$ occurs in the $y_2$ direction when only the load in
the $x_2$ direction is applied to the stator tooth.

(5) When the mesh point is in the center point of the stator tooth ($\phi_0 = 0^0$), the displacement $v_2$ is a minimum. When the mesh point moves toward the end point of the tooth, the displacement $v_2$ first grows then drops.

(6) When the mesh point is at center point of the tooth, the displacement $v_2$ is negative between the mesh point and one end point ($\phi_0 \leq 0^0$), and positive between the mesh point and another end point ($\phi_0 \geq 0^0$).

(7) When the mesh point is within the angle range $\phi_0 \leq 0^0$, the displacement $v_2$ is negative along the whole stator tooth, and when the mesh point is within the angle range $\phi_0 \geq 0^0$, the displacement $v_2$ is positive along the whole stator tooth.

(8) When only the load in the $x_2$ direction is applied to the stator tooth, the displacement $w_2$ occurs as well. The displacement $w_2$ is larger than the displacement $v_2$.

(9) When the mesh point is in the center point of the stator tooth ($\phi_0 = 0^0$), the displacement $w_2$
is also a minimum. When the mesh point moves toward the end point of the tooth, the displacement $w_2$ first grows then drops as well.

10) Along the whole stator tooth, tensile strain zone ($w_2 \geq 0$) and compression strain zone ($w_2 \leq 0$) occur. At most mesh points, one tensile strain zone and one compression strain zone occur. At some mesh points, two tensile strain zones or two compression strain zones occur.

11) When only the load in the $y_2$ direction is applied to the stator tooth, the displacements $u_2$, $v_2$, and $w_2$ in three directions occur as well. The displacement $v_2$ in the $y_2$ direction is much larger than displacements $u_2$ and $w_2$ in the other two directions.

12) As the mesh point between the planet and the stator changes, the maximum displacement in the $y_2$ direction changes as well. The changes are similar to those of the displacement $u_2$ mentioned above.

13) When the two mesh points are symmetrical to the center point of the stator tooth, the displacement $v_2$ distributions are approximately symmetrical to the center point as well. However, the displacements for the mesh point behind the center point of the stator tooth ($\phi_0 \geq 0^\circ$) are a bit larger than those for the mesh point in front of the center point of the stator tooth ($\phi_0 \leq 0^\circ$).
(14) As the stator tooth is spiral, displacement $u_2$ occurs in the $x_2$ direction when only the load in the $y_2$ direction is applied to the stator tooth.

(15) When the mesh point is in the center point of the stator tooth ($\phi_0 = 0^\circ$), the displacement $u_2$ is a minimum as well. When the mesh point moves toward the end point of the tooth, the displacement $u_2$ first grows then drops.

(16) When only the load in the $y_2$ direction is applied to the stator tooth, the displacement $w_2$ occurs as well. The displacement $w_2$ is smaller than the displacement $u_2$. The changes of displacement $w_2$ here are similar to the case when only the load in the $x_2$ direction is applied to the stator tooth.

(17) When the loads in the $x_2$ and $y_2$ directions are applied to the stator tooth simultaneously, the displacements $u_2$, $v_2$ and $w_2$ in three directions occur as well. However, both $u_2$ and $v_2$ are large.

(18) Under the condition that the load in the $x_2$ direction is equal to the load in the $y_2$ direction, the displacement in the $y_2$ direction is larger than that in the $x_2$ direction.

(19) When the loads in the $x_2$ and $y_2$ directions are applied to the stator tooth simultaneously, the displacement $w_2$ is also larger than the case when only the load in one direction is applied.

(20) Along the whole tooth, the displacements $u_2$ are all negative and the displacements $v_2$ are all positive for any mesh points.

Fig. 7 shows the variation of the displacement distributions versus the main parameters. Here, the mesh point is at $\phi_0 = 0^\circ$, $F_{bx} = -10N$ and $F_{by} = 10N$, $i$ is the ratio of the length to width of the section of the stator tooth, $i = l/d$. From Fig. 7, the following observations can be made:

(1) For constant planet radius $R$, as the center distance $a$ increases, the displacements $u_2$, $v_2$ and $w_2$ grow, among which the increase of the displacement $v_2$ is more obvious. For constant planet radius $R$, as the center distance $a$ increases, the lead angle of the stator tooth becomes small and the length of the stator tooth increases, so the displacements grow with increasing center distance.

(2) As the stator tooth number $z_2$ increases, the displacements $u_2$ and $v_2$ drop. When the stator tooth number is small, the decrease of the displacements along with tooth number $z_2$ is obvious. When the stator tooth number is large, the decrease of the displacements along with tooth number $z_2$ is very small. As the stator tooth number $z_2$ increases, the lead angle of the stator tooth becomes large and the length of the stator tooth decreases, so the displacements drop with increasing tooth number $z_2$. When the tooth number $z_2$ is large, the length variation of the stator tooth along with the tooth number $z_2$ is quite small, so the decrease of the displacements along with the tooth number $z_2$ is not obvious.

(3) As the stator tooth number $z_2$ increases, the displacement $w_2$ first drops then grow in the opposite direction.

(4) As the planet tooth number $z_1$ increases, the displacements $u_2$, $v_2$ and $w_2$ grows. It is also because the lead angle of the stator tooth becomes small and the length of the stator tooth increases with increasing the planet tooth number.

(5) As the ratio $i$ increases, the displacements $u_2$ and $w_2$ grows, but $v_2$ drops.

(6) These results show that the related parameters have obvious influence on the displacement distribution of the stator tooth. In order to decrease the displacements of the stator tooth, these parameters should be selected properly.

Here, a FEM analysis package, ANSYS, is used to simulate the elastic displacements of the stator tooth (see Fig. 8). The differences between the FEM results and those by the strain energy method are compared in Table 2. The results show that the maximum error of the displacements is smaller than 15%, which illustrates the applicability of the results presented in the paper.
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5. Conclusions

In this paper, we use the coordinate transformation method and express the displacements of the stator tooth in the local coordinate system as a function of the variables in the drive coordinate system. By the minimum potential energy principle, the equations of the displacements of the stator tooth under the loads are derived. The displacement distributions within the stator tooth for the drive system are investigated. The results show that the related parameters have obvious influence on the displacement distribution of the stator tooth. In order to decrease the displacements of the stator tooth, these parameters should be properly selected.

<table>
<thead>
<tr>
<th></th>
<th>Energy method (m)</th>
<th>FEM (m)</th>
<th>errors</th>
</tr>
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<tbody>
<tr>
<td>$u$</td>
<td>0.0000275</td>
<td>0.0000236</td>
<td>14.18%</td>
</tr>
<tr>
<td>$v$</td>
<td>0.0000315</td>
<td>0.0000268</td>
<td>14.92%</td>
</tr>
<tr>
<td>$w$</td>
<td>0.0000330</td>
<td>0.0000305</td>
<td>4.55%</td>
</tr>
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</table>

![Fig. 8 The elastic displacements of the stator tooth from FEM](image-url)
Acknowledgments

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References