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One-dimensional consolidation with asymmetrical exponential drainage boundary

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Abstract. In this paper, asymmetric drainage boundaries modeled by exponential functions which can simulate intermediate drainage from pervious to impervious boundary is proposed for the one-dimensional consolidation problem, and the solution for the new boundary conditions was derived. The new boundary conditions satisfy the initial and the steady state conditions, and the solution for the new boundary conditions can be degraded to the conventional solution by Terzaghi. Convergence study on the infinite series solution showed that only one term in the series is needed to meet the precision requirement for larger degree of consolidation, and that more terms in the series for smaller degree of consolidation. Comparisons between the present solution with those by Terzaghi and Gray are also provided.

Keywords: Terzaghi's one-dimensional consolidation equation; exponential drainage boundary

1. Introduction

Consolidation theory has been an important topic for soil mechanics since Terzaghi (1925) developed the one dimensional consolidation theory. Subsequently, Rendulic (1936) extended the consolidation theory to two and three dimensional conditions, and Biot (1941a, b) proposed the general consolidation theory based on the principles of effective stress, continuity, and equilibrium, and provided solutions to consolidation of semi-infinite foundation under a strip loading at the surface. McNamee and Gibson (1960a, b) also developed solutions to a number of consolidation problems. In general, Terzaghi's one-dimensional consolidation theory is applied extensively in practice. A number of recent improvements were also developed regarding the characteristics of the soil system and loading condition in the one-dimensional consolidation problem (e.g., Zhu and Yin 1999, Xie *et al*. 2008a, b, Huang and Griffiths 2010). Many solutions to consolidation of multilayered soil have been developed (e.g., El-Zein 2006, Kim and Mission 2011, Walker *et al*. 2012). In addition, finite layer numerical method was also adopted by a number of researchers (e.g., Cheung and Tham 1983, Mei *et al*. 2004, Chen *et al*. 2005) to the consolidation problem.

However, only perfectly drained and undrained boundaries were considered in Terzaghi's solution of consolidation equation, and continuous variation between these two extreme boundary

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conditions were not taken into account. In reality, the boundary condition is not exactly drained or undrained, which is one of the major limitations for application of Terzaghi's solution. Gray (1944) provided a solution to the consolidation problem with drainage boundary of finite permeability. Huang (1982) also proposed the partial drained boundary, but analytical solutions are not always achievable due the complexity of the problem. In this study, an asymmetric continuous time dependent drainage boundary in form of an exponential function was proposed, and the solution to the consolidation equation was developed, which allows the application of the consolidation theory to more general conditions. Predictions based on the new solution were examined in this paper.

2. General boundary conditions

The consolidation equation was proposed by Terzaghi as follows

$$\frac{\partial u}{\partial t} = C_V \frac{\partial^2 u}{\partial z^2} \qquad (0 \le z \le 2h) \tag{1}$$

Initial condition

$$u(0,z) = p \tag{2}$$

Boundary condition

$$u(t,0) = 0 \tag{3}$$

As shown in Fig. 1, the asymmetrical exponential boundary conditions are proposed in this study as follows

$$u(t,0) = pe^{-bt} = pe^{-BT} \tag{4a}$$

$$u(t,2h) = pe^{-ct} = pe^{-CT} \tag{4b}$$

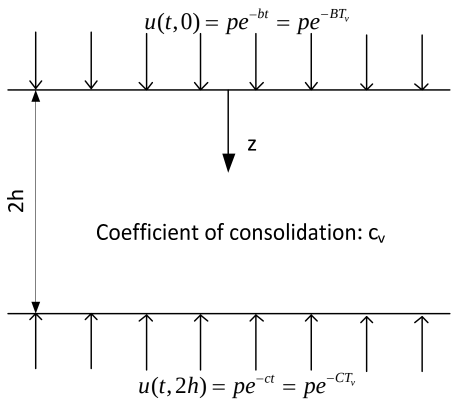


Fig. 1 One dimensional consolidation with asymmetrical exponential drainage boundary

In which b, c, B, and C are coefficients that describe the variation of pore pressure at the boundary. The values of b, c or B, C can be determined by curve fitting of experimental data. Note that the exponential boundary conditions can also be defined in terms of the real time or time factor; the latter will be used to simplify the solution during the derivation. The transformation between the real time and the time factor follows Terzaghi's definition.

$$T_v = \frac{C_v t}{h^2} \tag{5a}$$

$$B = b\frac{h^2}{C_v} \tag{5b}$$

$$C = c\frac{h^2}{C_v} \tag{5c}$$

The boundary conditions in Eqs. (4a) and (4b) are inhomogeneous but they satisfy the following

$$(u)_{z=0} = (u)_{z=2H} = p \qquad \text{for} \qquad t = 0 \tag{6a}$$

$$(u)_{z=0} = (u)_{z=2H} = 0 \qquad \text{for} \qquad t = \infty \tag{6b}$$

As shown in Eqs. (6a) and (6b) and in Fig. 2, the general boundary condition satisfies the initial condition and is stable over time. The exponential form was chosen to fit the variation of pore pressure generally observed during consolidation. As B, $C \to \infty$, $u(t,0) = 0$, which is the perfectly drained boundary in Terzaghi's solution. As B, $C \to 0$, $u(t,0) = p$, which is the perfectly undrained boundary in Terzaghi's solution. The exponential boundary condition proposed in this study

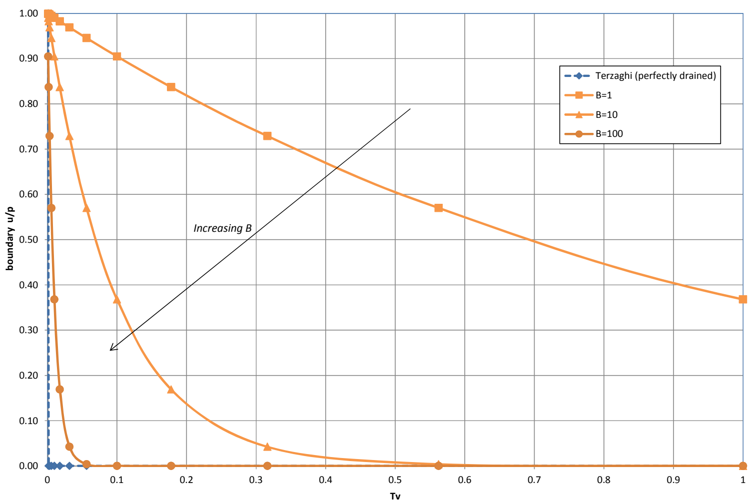


Fig. 2 Variation of pressure at the exponential boundary with time factor

actually provides simulation of the continuous variation between drained and undrained boundary conditions, while Terzaghi's original solution only considers the two extremes, perfectly drained and undrained conditions. Therefore, Eqs. (1)-(4) can be considered as the consolidation problem with general boundary conditions.

3. Solution to the consolidation problem with general boundary condition

The solution given by Terzaghi for the one-dimensional consolidation equation is as follows

$$u(t,z)=\frac{4}{\pi}p\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{k\pi z}{2h}\right)e^{\frac{k^2\pi^2}{4}T_v}\qquad (k=1,3,5...)\tag{7}$$

where T_v is the time factor defined in Eq. (5).

The solution for the general consolidation problem described by Eqs. (1)-(4) can be derived as follows (details shown in appendix)

$$u(t,z)=\frac{p}{2h}\left\{\begin{array}{l}\dfrac{4}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\left\{\begin{array}{l}\dfrac{B(2h-z)}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-B\right\}}\left(e^{-BT},-e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}\right)\\+\dfrac{Cz}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-C\right\}}\left(e^{-CT},-e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}\right)\end{array}\right\}\sin\dfrac{(2k-1)\pi z}{2h}\\+\left[(2h-z)e^{-BT_v}+ze^{-CT}\right]\end{array}\right\}\tag{8}$$

Notice that the solution for the general boundary condition is given in terms of the time factor by using the definition of boundary condition in time factor as in Eqs. (4) and (5).

If $B = C$, Eq. (8) can be reduced to

$$u(t,z)=p\left\{\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)}\left\{\frac{B}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-B\right\}}\left(e^{-BT_v},-e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}\right)\right\}\sin\frac{(2k-1)\pi z}{2h}+e^{-BT_v}\right\}\tag{9}$$

If $B = C \to \infty$, Eq. (9) can be further reduced to

$$u(t,z)=\frac{4p}{\pi}\sum_{k=1}^{\infty}\frac{\sin\frac{(2k-1)\pi z}{2h}}{(2k-1)}e^{-\frac{((2k-1)\pi)^2}{4}T_v}\tag{10}$$

which is the same as Eq. (7).

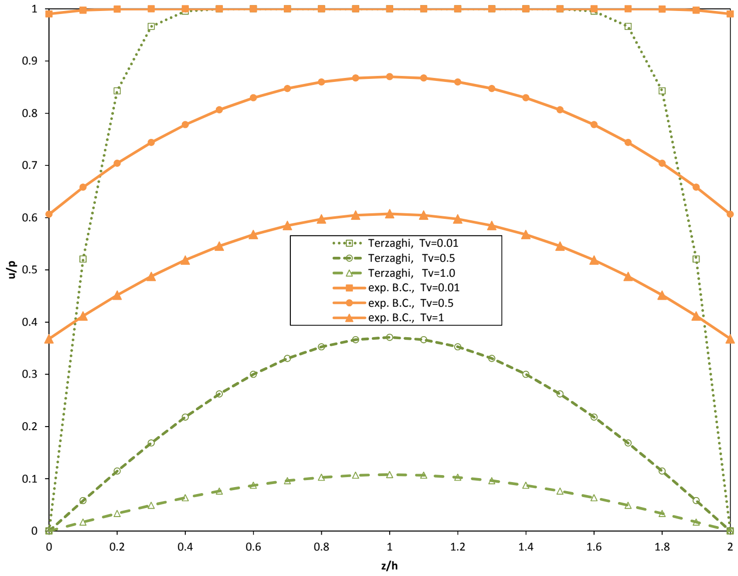


Fig. 3 Comparison between results of present solution and Terzaghi's consolidation theory ($B = C = 1.0$)

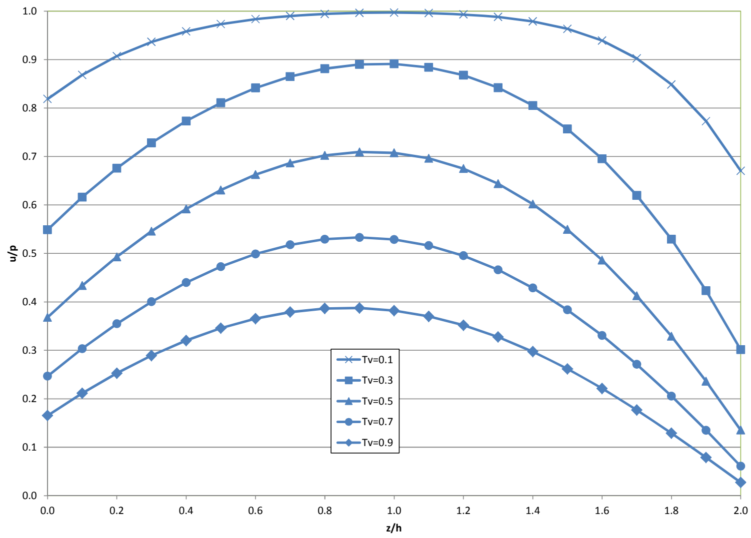


Fig. 4 Asymmetric pressure distribution simulated by the present solution for $B = 2$ and $C = 4$

4. Study of effects of modeling parameters

4.1 Effects of parameters B, C

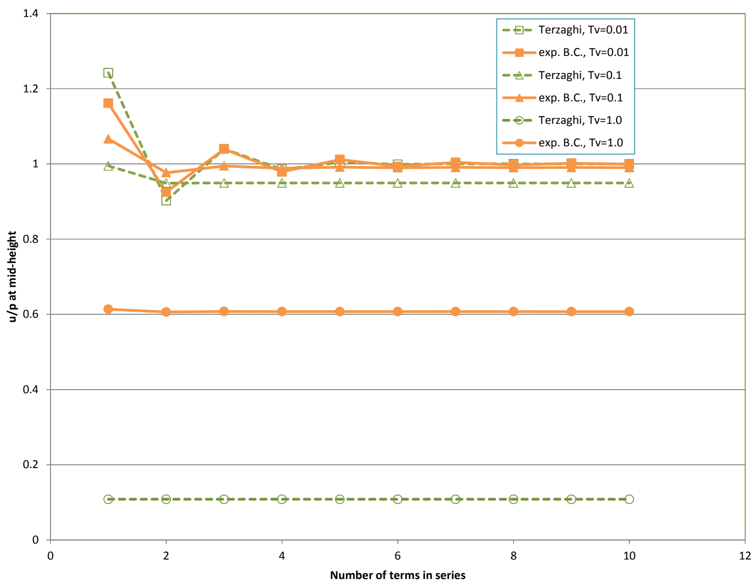
To validate the solution, calculations were made based on Eq. (8). When the parameter $B = C$, the pore pressure distribution will be symmetric about the centerline as shown in Fig. 3. As the time factor T_v increases, the pore pressure dissipates gradually at the drainage boundary and in the soil. On the contrary, pore pressure at the perfectly drained boundary is always zero from Terzaghi's solution.

Analysis was carried out with $B = 2$ and $C = 4$, and the pore pressure distribution is shown in Fig. 4. The slope of pore pressure in the lower part of the soil layer is steeper than that of the upper part since the permeability of the bottom boundary is higher with a value of C larger than B. Again, this situation of different permeability at top and bottom boundaries cannot be described by Terzaghi's solution properly with only perfectly drained and undrained boundary conditions. As illustrated by Fig. 4, more realistic modeling of the permeability of drainage boundaries can be achieved with the general boundary conditions.

As $B = C \to \infty$ and $B = C \to 0$, the exponential boundary condition is reduced to the perfectly drained and undrained boundary, respectively. As shown in Figs. 3 and 4, the exponential boundary condition provides prediction for continuous variation of permeability of drainage boundary, including the perfectly drained and undrained boundaries by Terzaghi's solution. By adjusting the parameters b and c to model different boundary permeability, unsymmetrical pore pressure distribution can be simulated. Therefore, the exponential boundary condition provides more general results while Terzaghi's solution is only a special case of the exponential boundary condition.

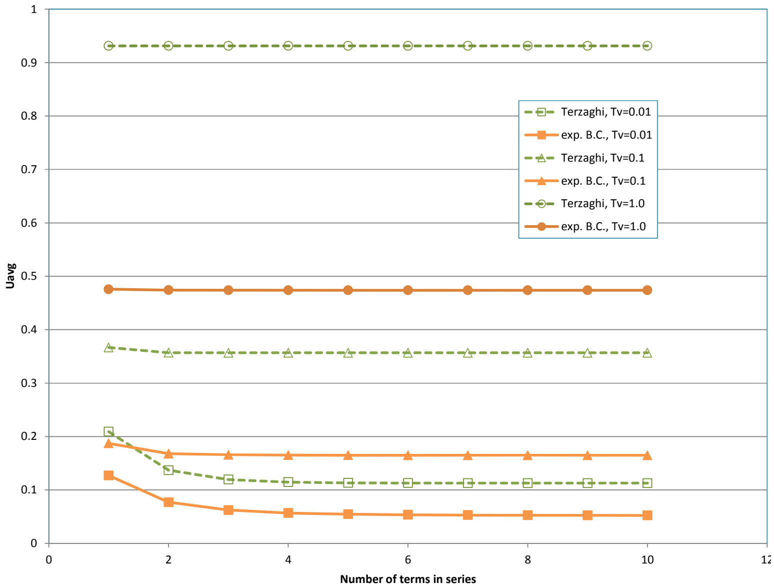
4.2 Calibration of parameters B, C

In order to calibrate the modeling parameters B and C, consolidation experiments should be carried out with measurements of pore pressure in the specimen (e.g., at the mid height). The measured pore pressure can be compared with a figure similar to Fig. 3 with certain values of B



(a)

Fig. 5 Convergence study on number of term in series ($BT = CT = 1$): (a) u/p at mid-height; (b) average degree of consolidation



(b)

Fig. 5 Continued

and C. The acceptable values of B and C can then be estimated based on the matching of the measured and calculated pore pressure.

4.3 Effects of the number of terms in series -- convergence study

Calculations were made with the time factor $BT_v = CT_v = 1.0$. As shown in Figs. 5(a) and 5(b), only one term is needed to get accurate results for large values of T_v (e.g., T_v =1.0), but more terms are needed for smaller T_v (e.g., T_v = 0.01). In general, five terms are enough for all practical purposes for calculations of both pore pressure and degree of consolidation.

5. Average degree of consolidation for the general boundary condition

5.1 Comparison with Terzaghi's solution

The average degree of consolidation given by Terzaghi is as follows

$$U = 1 - \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}e^{-\frac{k^2\pi^2}{4}T_v} \qquad (k = 1,3,5\ldots) \tag{11}$$

For the exponential boundary condition, U can also be derived as follows (details shown in appendix)

$$U=\frac{S_t}{S}=1-\frac{u_t}{p}=1-\frac{\frac{1}{2h}\int_0^{2h}udz}{p}$$

$$=1-\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left\{\begin{array}{l}\dfrac{B\left(e^{-BT_v}-e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}\right)}{\left\{\left[\dfrac{(2k-1)\pi}{2}\right]^2-B\right\}}\\[2ex]+\dfrac{C\left(e^{-CT_v}-e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}\right)}{\left\{\left[\dfrac{(2k-1)\pi}{2}\right]^2-C\right\}}\end{array}\right\}-\frac{1}{2}\left(e^{-BT_v}+e^{-CT_v}\right) \tag{12}$$

when $B = C$, Eq. (12) can be reduced to

$$U=1-\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left\{\frac{B\left(e^{-BT_v}-e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}\right)}{\left\{\left[\dfrac{(2k-1)\pi}{2}\right]^2-B\right\}}\right\}-e^{-bt} \tag{13}$$

Furthermore, as $B = C \to \infty$, Eq. (13) is reduced to

$$U=1-\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{e^{-C_v\left[\frac{(2k-1)\pi}{2h}\right]^2t}}{(2k-1)^2}=1-\frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{e^{-\left[\frac{(2k-1)\pi}{2}\right]^2T_v}}{(2k-1)^2}$$

which is the same as Eq. (11).

The average degree of consolidation calculated with Terzaghi's solution and the new solution in this study is presented in Fig. 6. The new solution is asymptotic to Terzaghi's solution with increasing values of B and C. On the other hand, the new solution becomes different from Terzaghi's solution with smaller B and C which implies lower permeability of the drainage boundary. Also, when the time factor is smaller, the difference between Terzaghi's solution and new solution is bigger, and the difference becomes negligible for large values of time factor.

5.2 Comparison with Gray's solution

The degree of consolidation for the case of impeded drainage is given by Gray (1944) as follows

$$U=1-2\sum_{k=1}^{\infty}\frac{e^{-T_v r_n^2}}{r_n}\frac{\sin^2 r_n}{r_n+\frac{1}{2}\sin 2r_n} \tag{14}$$

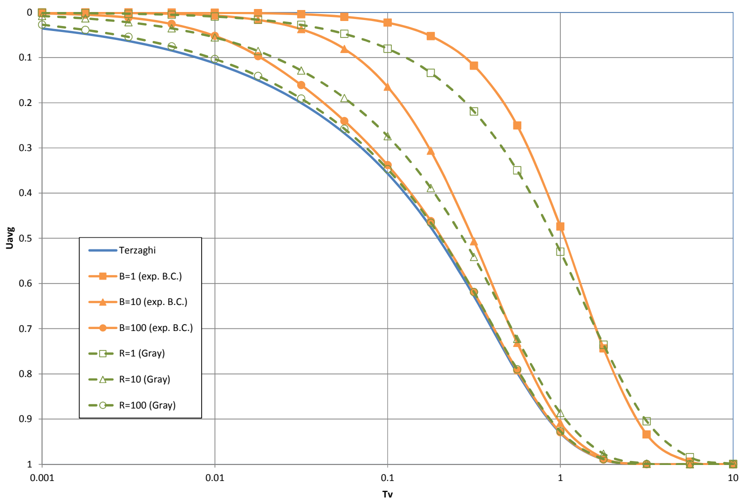


Fig. 6 Comparison between degree of consolidation calculated by Gray's method and the present solution

where r_n is the nth root of the following equation

$$r \tan r = R = \frac{H_s}{H_d}\frac{k_d}{k_s} \tag{15}$$

where H_s, k_s, H_d, k_d are the thickness, permeability of the soil and the drainage layer, respectively.

The average degree of consolidation calculated with Gray's solution and the new solution (with $B = C$) in this study is presented in Fig. 6. Although the two solutions are of different origins, interesting similarities can be observed. Both solutions are asymptotic to Terzaghi's solution for a perfectly drained boundary with increasing values of B or R. By observation from Fig. 6, approximated results were obtained for Gray's solution with the new solution by setting $R = B$. For $R = B$, Gray's solution provides large degree of consolidation than that of the new solution for small T_v but becomes very close to each other for large T_v.

6. Conclusions

The solution for the one-dimensional consolidation problem with asymmetrical exponential boundary conditions was developed and examined. The following conclusions were drawn from this study

(1) The exponential boundary condition satisfies the initial condition of Terzaghi's consolidation theory. It can also simulate the perfectly drained and undrained boundary conditions of Terzaghi's solution. With the exponential boundary condition, variation of the boundary permeability can be considered.

(2) By adjusting the model parameters, different permeability at the top and bottom drainage boundaries can be modeled, and, therefore, asymmetric pore pressure distribution can be

simulated.

(3) With the same time factor, the pore pressure from the new solution is generally smaller than that of the Terzaghi's solution which is due to the infinite permeability of the perfectly drained boundary assumed in Terzaghi's solution.

(4) From the convergence study of the infinite series solution, it was found that only one term is needed for large value of C_v. With smaller C_v, more terms are needed to obtain accurate results but it is still within reasonably practical computational efforts.

(5) The solution with exponential boundary condition is a generalization of Terzaghi's solution, which was proved to be a special case of the present solution.

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Appendix: Derivation of solution for exponential boundary condition

$$\frac{\partial u}{\partial t}=C_v\frac{\partial^2 u}{\partial z^2}$$
$$\begin{cases} u(z,0)=p \\ u(0,t)=pe^{-bt}=pe^{-BT} \\ u(2h,t)=pe^{-ct}=pe^{-CT} \end{cases} \tag{A.1}$$

Let

$$u=v+\frac{(2h-z)pe^{-bt}}{2h}+\frac{zpe^{-ct}}{2h} \tag{A.2}$$

The boundary condition can be made homogeneous as

$$\frac{\partial v}{\partial t}=C_v\frac{\partial^2 v}{\partial z^2}+\frac{bp(2h-z)e^{-bt}}{2h}+\frac{cpze^{-ct}}{2h}$$
$$\begin{cases} v(z,0)=0 \\ v(0,t)=0 \\ v(2h,t)=0 \end{cases} \tag{A.3}$$

Assuming the solution of the form

$$v(z,t)=\sum_{n=1}^{\infty}\phi_n(t)\sin\frac{n\pi z}{2h} \tag{A.4}$$

Substitute into Eq. (A.3)

$$\sum_{n=1}^{\infty}\phi_n(t)\sin\frac{n\pi z}{2h}+C_v\sum_{n=1}^{\infty}\left(\frac{n\pi}{2h}\right)^2\phi_n(t)\sin\frac{n\pi z}{2h}=p\left[\frac{b(2h-z)e^{-bt}}{2h}+\frac{cze^{-ct}}{2h}\right]\sum_{n=1}^{\infty}d_n\sin\frac{n\pi z}{2h} \tag{A.5}$$

where

$$d_n=\frac{1}{h}\int_0^{2h}\sin\frac{n\pi z}{2h}dz=\frac{2}{n\pi}\left[1-(-1)^n\right]=\begin{cases}\dfrac{4}{n\pi} & (n=2k-1) \\ 0 & (n-2k)\end{cases} \tag{A.6}$$

$$\sum_{n=1}^{\infty}\left[\phi_n(t)+C_v\left(\frac{n\pi}{2h}\right)^2\phi_n(t)-pd_n\left[\frac{b(2h-z)e^{-bt}}{2h}+\frac{cze^{-ct}}{2h}\right]\right]\sin\frac{n\pi z}{2h}=0 \tag{A.7}$$

Since the equation is valid for any value of n

$$\phi_n(t) + C_v\left(\frac{n\pi}{2h}\right)^2 \phi_n(t) = pd_n\left[\frac{b(2h-z)e^{-bt}}{2h} + \frac{cze^{-ct}}{2h}\right] \tag{A.8}$$

The solution for this nonlinear ordinary differential equation has two parts. The homogeneous solution is

$$\phi_n(t) = B_n e^{-C_v\left(\frac{n\pi}{2h}\right)^2 t} \tag{A.9}$$

And the particular solution is

$$\phi_n(t) = A_1 e^{-bt} + A_2 e^{-ct} \tag{A.10}$$

Substitute the particular solution into Eq. (A.8)

$$\begin{gathered} -bA_1e^{-bt} - cA_2e^{-ct} + C_v\left(\frac{n\pi}{2h}\right)^2 A_1e^{-bt} + C_v\left(\frac{n\pi}{2h}\right)^2 A_2e^{-ct} = pd_n\frac{b(2h-z)e^{-bt}}{2h} + pd_n\frac{cze^{-ct}}{2h} \\ e^{-bt}\left[pd_n\frac{b(2h-z)}{2h} + bA_1 - C_v\left(\frac{n\pi}{2h}\right)^2 A_1\right] + e^{-ct}\left[pd_n\frac{cz}{2h} + cA_2 - C_v\left(\frac{n\pi}{2h}\right)^2 A_2\right] = 0 \end{gathered} \tag{A.11}$$

Since the equation is valid for any t, one can obtain

$$pd_n\frac{b(2h-z)}{2h} + bA_1 - C_v\left(\frac{n\pi}{2h}\right)^2 A_1 = 0 \quad \Rightarrow \quad A_1 = \frac{pd_n b(2h-z)}{2h\left[C_v\left(\frac{n\pi}{2h}\right)^2 - b\right]}$$

$$pd_n\frac{cz}{2h} + cA_2 - C_v\left(\frac{n\pi}{2h}\right)^2 A_2 = 0 \quad \Rightarrow \quad A_2 = \frac{pd_n cz}{2h\left[C_v\left(\frac{n\pi}{2h}\right)^2 - c\right]}$$

And the general solution is

$$\begin{aligned} \phi_n(t) = {} & \frac{pd_n b(2h-z)e^{-bt}}{2h\left[C_v\left(\frac{n\pi}{2h}\right)^2 - b\right]} + \frac{pd_n cze^{-ct}}{2h\left[C_v\left(\frac{n\pi}{2h}\right)^2 - c\right]} \\ & + B_n e^{-C\left(\frac{n\pi}{2h}\right)^2 t} \end{aligned} \tag{A.12}$$

$$\downarrow$$

$\uparrow$

$$
\begin{aligned}
u(z,t) &= v(z,t) + \frac{(2h-z)pe^{-bt}}{2h} + \frac{pze^{-ct}}{2h} \\
&= \sum_{k=1}^{\infty} \frac{2bp(2h-z)e^{-bt}}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2 - b\right\}\pi} \sin\frac{(2k-1)\pi z}{2h} \\
&\quad + \sum_{k=1}^{\infty} \frac{2cpze^{-ct}}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2 - c\right\}\pi} \sin\frac{(2k-1)\pi z}{2h} \\
&\quad + \sum_{k=1}^{\infty} B_k e^{-C(\frac{k\pi}{2h})^2 t} \sin\frac{k\pi z}{2h} + \frac{(2h-z)pe^{-bt}}{2h} + \frac{pze^{-ct}}{2h}
\end{aligned}
\tag{A.12}
$$

Using the initial condition, one can obtain

$$
\begin{aligned}
u(z,0) &= \sum_{k=1}^{\infty} \frac{2bp(2h-z)}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2 - b\right\}\pi} \sin\frac{(2k-1)\pi z}{2h} \\
&\quad + \sum_{k=1}^{\infty} \frac{2cpz}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2 - b\right\}\pi} \sin\frac{(2k-1)\pi z}{2h} + \sum_{k=1}^{\infty} B_k \sin\frac{k\pi z}{2h} + p \\
&= \sum_{k=1}^{\infty} \frac{2bp(2h-z)}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2 - b\right\}\pi} \sin\frac{(2k-1)\pi z}{2h} \\
&\quad + \sum_{k=1}^{\infty} \frac{2cpz}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2 - b\right\}\pi} \sin\frac{(2k-1)\pi z}{2h} \\
&\quad + \sum_{k=1}^{\infty} B_{2k-1} \sin\frac{(2k-1)\pi z}{2h} + \sum_{k=1}^{\infty} B_{2k} \sin\frac{2k\pi z}{2h} + p
\end{aligned}
\tag{A.13}
$$

$\downarrow$

$\uparrow$

$$
= \sum_{k=1}^{\infty} \left[\frac{2bp(2h-z)}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} \atop \frac{2cpz}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} + B_{2k-1} \right] \sin \frac{(2k-1)\pi z}{2h} \tag{A.13}
$$

$$
+ \sum_{k=1}^{\infty} B_{2k} \sin \frac{2k\pi z}{2h} + p
$$

$$
= p
$$

Since the equation is valid for any z, one can obtain from each term of the series

$$
\begin{cases} B_{2k-1} = \dfrac{2bp(2h-z)}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} - \dfrac{2cpz}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} \\ B_{2k} = 0 \end{cases} \tag{A.14}
$$

Therefore

$$
\begin{aligned}
u(t,z) = & \sum_{k=1}^{\infty} \frac{2bp(2h-z)}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} \sin \frac{(2k-1)\pi z}{2h} \\
& + \sum_{k=1}^{\infty} \frac{2cpze^{-ct}}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - c \right\}\pi} \sin \frac{(2k-1)\pi z}{2h} \\
& - \sum_{k=1}^{\infty} \left\{ \frac{2bp(2h-z)}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} + \frac{2cpz}{h(2k-1)\left\{ C_v \left[\frac{(2k-1)\pi}{2h} \right]^2 - b \right\}\pi} \right\} e^{-C\left(\frac{(2k-1)\pi}{2h}\right)^2 t} \sin \frac{(2k-1)\pi z}{2h}
\end{aligned} \tag{A.15}
$$

$\downarrow$

$$\uparrow$$

$$+\frac{(2h-z)pe^{-bt}}{2h}+\frac{pze^{-ct}}{2h}$$

$$=\sum_{k=1}^{\infty}\sin\frac{(2k-1)\pi z}{2h}\left[\left(\frac{2bp(2h-z)\left(e^{-bt}-e^{-C\left(\frac{(2k-1)\pi}{2k}\right)^2 t}\right)}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2-b\right\}\pi}\right)+\left(\frac{2cpz\left(e^{-bt}-e^{-C\left(\frac{(2k-1)\pi}{2k}\right)^2 t}\right)}{h(2k-1)\left\{C_v\left[\frac{(2k-1)\pi}{2h}\right]^2-b\right\}\pi}\right)\right] \tag{A.15}$$

$$+\frac{(2h-z)pe^{-bt}}{2h}+\frac{pze^{-ct}}{2h}$$

Using the definition of boundary conditions in terms of time factor (i.e., $bt = BT_v$ and $ct = CT_v$), the solution can also be expressed in terms of time factor

$$u(t,z)=\frac{p}{2}\left\{\begin{array}{l}\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{2}{(2k-1)}\left\{\begin{array}{l}\frac{B(2-z/h)}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-B\right\}}\left(e^{-BT}-e^{-\left[\frac{(2k-1)\pi}{2k}\right]^2 T_v}\right)\\+\frac{C\frac{z}{h}}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-C\right\}}\left(e^{-CT}-e^{-\left[\frac{(2k-1)\pi}{2k}\right]^2 T_v}\right)\end{array}\right\}\sin\frac{(2k-1)\pi z}{2h}\\+\left[(2-z/h)e^{-BT_v}+z/he^{-CT_v}\right]\end{array}\right\} \tag{A.16}$$

And

$$\frac{1}{2h}\int_0^{2h} 5h\sin\frac{(2k-1)\pi z}{2h}dz = \frac{4h}{(2k-1)\pi} \quad ;$$

$$\begin{aligned}\frac{1}{2h}\int_0^{2h} z\sin\frac{(2k-1)\pi z}{2h}dz &= \frac{1}{2h}\frac{2h}{(2k-1)\pi}\int_0^{2h} z\sin\frac{(2k-1)\pi z}{2h}\, d\,\frac{(2k-1)\pi z}{2h} \\ &= \frac{1}{2h}\frac{2h}{(2k-1)\pi}\int_0^{2h} zd\left(-\cos\frac{(2k-1)\pi z}{2h}\right) \\ &= \frac{1}{2h}\frac{2h}{(2k-1)\pi}\left[z\left(-\cos\frac{(2k-1)\pi z}{2h}\right)\Bigg|_0^{2h} + \int_0^{2h}\cos\frac{(2k-1)\pi z}{2h}dz\right] \\ &= \frac{2h}{(2k-1)\pi}\end{aligned} \tag{A.17}$$

The average degree of consolidation can then be expressed as

$$\begin{aligned}U &= 1-\frac{u_t}{p} = 1-\frac{\frac{1}{2h}\int_0^{2h} udz}{p} \\ &= 1-\frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left\{\frac{B\left(e^{-BT}-e^{-\left[\frac{(2k-1)\pi}{2k}\right]^2 T_v}\right)}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-B\right\}}+\frac{C\left(e^{-CT}-e^{-\left[\frac{(2k-1)\pi}{2k}\right]^2 T_v}\right)}{\left\{\left[\frac{(2k-1)\pi}{2}\right]^2-C\right\}}\right\} \\ &\quad -\frac{1}{2}\left(e^{-BT_v}+e^{-CT_v}\right)\end{aligned} \tag{A.18}$$