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A note on Hvorslev's shape factor for a flush bottom piezometer in uniform soil

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Abstract. This note presents an analytical solution for the determination of the shape factor of a flush bottom piezometer in a uniform, isotropic, and incompressible deep soil deposit. The deduced shape factor is compared to published values obtained by approximate methods. Depending on the selected value, the difference may reach 11%.

Keywords: hydraulic conductivity; shape factor; flush bottom piezometer; analytical solution; comparisons.

1. Introduction

Lother (1978) drew attention to a difficulty encountered in the determination of hydraulic conductivity values from data obtained from constant-head pumping tests in piezometers. He noted that the formulae given by Hvorslev (1951) for the case of a piezometer point-filter are, in the limit of zero length of filter, different from those given for a flush bottom piezometer in uniform soil, although they should be the same. Youngs (1980) indicated that the reason stemmed from the approximate nature of the formulae used by Hvorslev (1951).

In the case of the intake formulae which have the general form

$$Q = FkH \tag{1}$$

where Q is the rate of flow, k the hydraulic conductivity, H the fixed excess head, and F a shape factor, the value of which depends on the geometry of the cavity, the boundary conditions, and the anisotropy of the soil, Hvorslev (1951) noted that simplifications were made on the shape of the cavity in order to obtain shape factors applicable in practice. Hvorslev (1951) also indicated that the formulae were all derived on the assumption that the soil stratum in which the well point or piezometer is placed is of infinite thickness and that the inflow or outflow is so small that it does not cause any appreciable change in the ground-water level or pressure.

The shape factor is generally a characteristic of an axisymmetric flow net, since the porous element is nearly always symmetric in shape (Brand and Premchitt 1980). A flush bottom piezometer

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of diameter d similar to that shown in Fig. 1 is considered in the present paper. The piezometer is located below the water table in a uniform, isotropic, and incompressible soil medium. For the problem at hand, the thickness of the soil is considered to be infinite. This means that the relative distances d_w/d and s/d are greater than about 10, respectively, on the basis of results obtained by Youngs (1968, 1980) using an electric analogue model. While d_w represents the depth of the intake area below the water table, s is the vertical distance between the intake area and an underlying impervious layer, as shown in Fig. 1. For the geometry of Fig. 1, Hvorslev (1951) gives $F/d = 2.75$, where d is the diameter of the porous opening at the bottom of the cylindrical casing. He mentions that the value of 2.75 is based on F/d values ranging between 2.4 and 2.8 found by Harza (1935) using an electric analogue model, and a value of 2.85 obtained by Taylor (1948) by means of a simple graphical flow net. The intake area in the flow net studied by Taylor (1948) was located at $d_w = 3.5d$ below the water table and $s = 1.5d$ above an impervious stratum. It should be also noted that Luthin and Kirkham (1949) obtained $F/d = 2.5$. A few years later, Brand and Premchitt (1980), using both a finite difference method and an electric analogue model, found $F/d = 2.63$ for a flat, circular disc piezometer. The electrolytic tank used by Brand and Premchitt (1980) measured 900 mm in diameter and 500 mm in depth. The piezometer model was placed at mid-depth in the center of the tank for the simulation of infinite soil thickness. The ratio between the diameter of the tank and that of the piezometer was equal to 300. Youngs (1968, 1980) obtained $F/d = 2.8$ from an electric analogue model for $d_w/d \geq 6$ and $s/d \geq 4$. More recently, Ratnam *et al.* (2001), on the basis of a finite element method, indicated that F/d tended towards a value of 3.11 when the vertical intake area of the cylindrical piezometer with a pervious bottom approached zero. The domain analyzed by Ratnam *et al.* (2001) measured 12 m in diameter and 7 m in depth. Once again, the

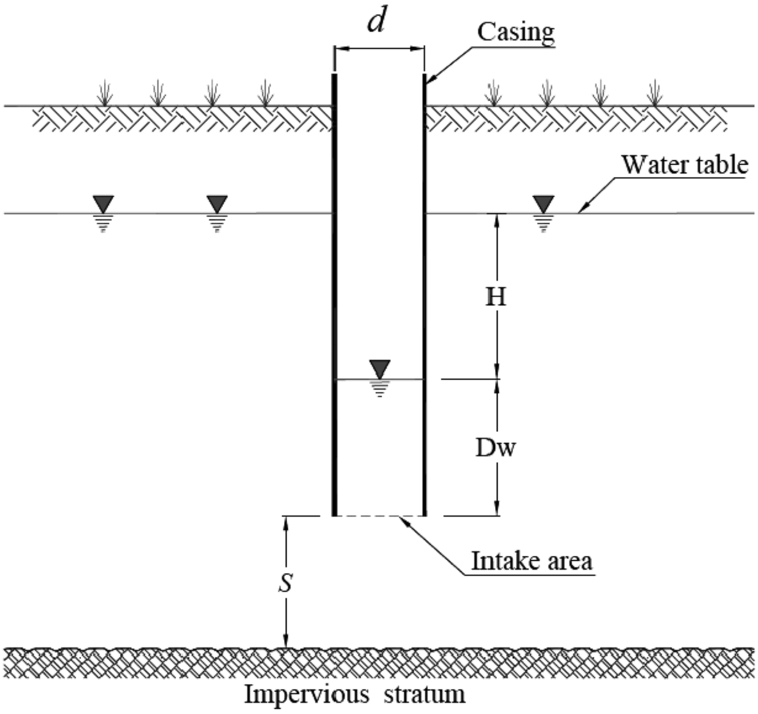

Fig. 1 Geometry of piezometer below the water table

piezometer was placed at mid-depth in the center of the domain for the simulation of infinite soil thickness.

This paper presents, for the first time, an analytical solution for the shape factor of a flush bottom piezometer in uniform, isotropic, and incompressible soil of infinite thickness. Comparisons are made with various published results.

2. Theoretical analysis

Laplace equation which governs the steady state flow to or from the piezometer in a constant-head test is given by the following expression

$$\nabla^2 h = 0 \tag{2}$$

where h is the excess head. Although the solution of this equation may be attempted using cartesian coordinates, it will be necessary to rewrite this equation in terms of some other suitable coordinates before a final solution to the problem at hand can be obtained. The solution is rendered easier by the use of the orthogonal curvilinear coordinates u, v, θ shown in Fig. 2. These are related to the cartesian coordinates through the following expressions (Appendix)

$$\begin{aligned} x &= \frac{2}{2\pi}(2u + \sin 2u \cosh 2v)\cos\theta \\ y &= \frac{2}{2\pi}(2u + \sin 2u \cosh 2v)\sin\theta \\ z &= \frac{2}{2\pi}(2v + \cos 2u \sinh 2v) \end{aligned} \tag{3}$$

Examination of the various curves drawn in Fig. 2 shows that the intake area of the cylindrical piezometer corresponds to $v = 0$, $-\pi/2 < u < \pi/2$; the impervious cylindrical surface of the casing to $v > 0$, $u = \pm\,\pi/2$; and the positive z-axis to $v > 0$, $u = 0$.

In view of the axisymmetric nature of the flow, Laplace equation reduces to (Cassan 1980)

$$\frac{\partial}{\partial v}\left(\frac{e_u e_\theta}{e_v}\frac{\partial h}{\partial v}\right) = 0 \tag{4}$$

where h is the excess head, and e_u, e_v, e_θ are metric coefficients or scale factors (See, for example, Moon and Spencer 1961). These are obtained from

$$\begin{aligned} e_u^2 &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\ e_v^2 &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\ e_\theta^2 &= \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \end{aligned} \tag{5}$$

Integration of Laplace equation gives

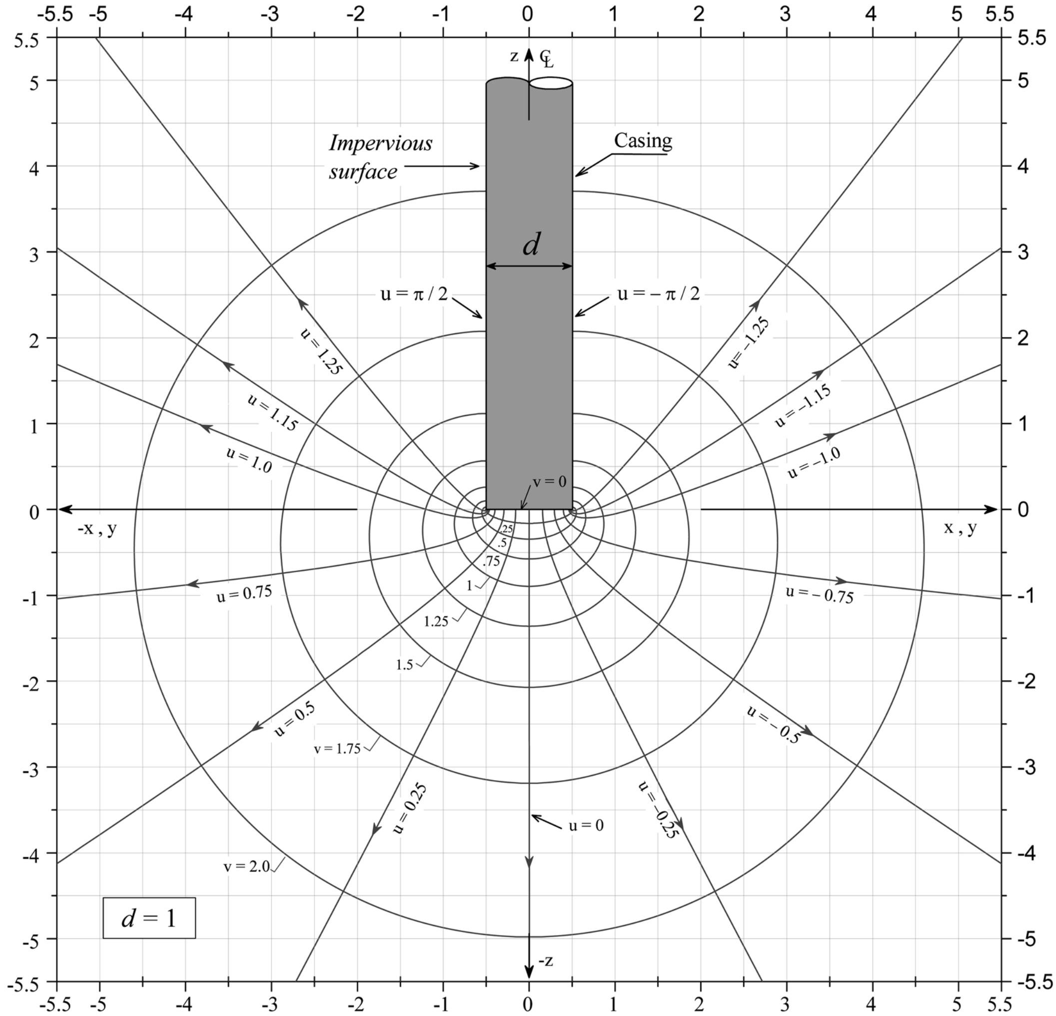

Fig. 2 Flush bottom piezometer in uniform soil of infinite thickness

$$\frac{e_u e_\theta}{e_v}\frac{\partial h}{\partial v} = A(u,\ \theta) \tag{6}$$

The function $A(u,\ \theta)$ is equal to (Cassan 1980)

$$A(u,\ \theta) = \frac{H}{\int_{v=\infty}^{v=0} \dfrac{e_v}{e_u e_\theta} dv} \tag{7}$$

Because of the symmetry of the flow, the rate of flow Q is given by

$$Q = -2k \int_{u=0}^{u=\pi/2} \int_{\theta=0}^{\theta=2\pi} A(u,\ \theta) d\theta du \tag{8}$$

On the basis of Eqs. (3) and (5), the metric coefficients become

$$e_u = e_v = \frac{d}{2\pi}(2v + \cos 2u \sinh 2v) \tag{9}$$

and

$$e_\theta = \frac{d}{2\pi}(2u + \sin 2u \cosh 2v) \tag{10}$$

As a result, the function $A(u, \theta)$ is given by

$$A(u, \theta) = -\frac{Hd}{2\pi \int_{v=\infty}^{v=0} \frac{dv}{(2u + \sin 2u \cosh 2v)}} \tag{11}$$

Integration of the denominator in Eq. (11) allows finding the function $A(u, \theta)$. It is given by

$$A(u, \theta) = -\frac{H}{2\pi} \frac{\sqrt{4u^2 - \sin^2 2u}}{\ln\left(\frac{2u + \sin 2u + \sqrt{4u^2 - \sin^2 2u}}{2u + \sin 2u - \sqrt{4u^2 - \sin^2 2u}}\right)} \tag{12}$$

The integral in Eq. (11) is tabulated in the treatise by Gradshteyn and Ryzhik (1980). Substitution of Eq. (12) into Eq. (8) gives the rate of flow

$$Q = \frac{kHd}{\pi} \int_{u=0}^{u=\pi/2} \int_{\theta=0}^{\theta=2\pi} \frac{\sqrt{4u^2 - \sin^2 2u}}{\ln\left(\frac{2u + \sin 2u + \sqrt{4u^2 - \sin^2 2u}}{2u + \sin 2u - \sqrt{4u^2 - \sin^2 2u}}\right)} d\theta du \tag{13}$$

Since $A(u, \theta)$ in Eq. (12) is independent of θ, Eq. (13) reduces to

$$Q = 2kHd \int_{u=0}^{u=\pi/2} \frac{\sqrt{4u^2 - \sin^2 2u}}{\ln\left(\frac{2u + \sin 2u + \sqrt{4u^2 - \sin^2 2u}}{2u + \sin 2u - \sqrt{4u^2 - \sin^2 2u}}\right)} du \tag{14a}$$

or

$$Q = 2kHd \int_{u=0}^{u=\pi/2} \frac{\sqrt{4u^2 - \sin^2 2u}}{\ln\left(\frac{2u + \sqrt{4u^2 - \sin^2 2u}}{\sin 2u}\right)} du \tag{14b}$$

or, even,

$$Q = kHd \int_{u=0}^{u=\pi/2} \frac{\sqrt{4u^2 - \sin^2 2u}}{\tanh^{-1}\left(\frac{\sqrt{4u^2 - \sin^2 2u}}{2u + \sin 2u}\right)} du \tag{14c}$$

By writing Eq. (14b) in the form of $Q = FkH$, the ratio F/d is given by

$$\frac{F}{d} = 2\int_{u=0}^{u=\pi/2} \frac{\sqrt{4u^2 - \sin^2 2u}}{\ln\left(\dfrac{2u + \sqrt{4u^2 - \sin^2 2u}}{\sin 2u}\right)} du \tag{15}$$

Since the integrand in Eq. (15) is rather complex, the integral was evaluated numerically, resulting in $F/d = 2.804$.

3. Comparison

Comparison with the various shape factor ratios, F/d, presented previously indicates that while the value found by Youngs (1980) coincides with the theoretical value obtained in this study, the maximum difference is 11% in the case of Luthin and Kirkham (1949), and Ratnam *et al.* (2001). As for the value of 2.75 proposed by Hvorslev (1951), it is very close to the theoretical value of 2.80.

4. Conclusions

The present paper presents an analytical solution for the determination of the shape factor of a flush bottom piezometer in uniform soil. Comparison with published values indicates that the largest difference reaches 11%.

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Appendix

Conformal mapping is important in engineering mathematics, since it is a standard method for solving boundary value problems in two-dimensional potential theory by transforming a given complicated region into a simpler one. In the context of this paper, conformal mapping is used to obtain curvilinear coordinates which facilitate the solution of Laplace equation.

In order to find the orthogonal curvilinear coordinates u, v, θ that will allow the determination of the shape factor, it is necessary to employ a double conformal mapping. First, the actual geometry in the t plane, where $t = r + iz$, with $i^2 = -1$, in Fig. A1(a) is mapped onto the upper half-plane $n \geq 0$ of Fig. A1(b), where $p = m + in$, by means of the following transformation (See, for example, Kober 1957)

$$t = r + iz = -\frac{di}{\pi}(p\sqrt{p^2 - 1} - \cosh^{-1}p) + \frac{d}{2} \tag{A1}$$

The points A, B, C, D in Fig. A1(a) correspond to the points A', B', C', D' in Fig. A1(b).

The upper half-plane $n \geq 0$ in Fig. A1(b) is next mapped onto the semi-infinite strip of width π in the w-plane, with $w = u + iv$ shown in Fig. A1(c), by means of the transformation

$$p = \sin w \tag{A2}$$

The points A", B", C", D" in Fig. A1(c) correspond to the points A', B', C', D' in Fig. A1(b).

Substitution of Eq. (A2) into Eq. (A1) yields

$$t = -\frac{di}{\pi}[\sin w\sqrt{\sin^2 w - 1} - \cosh^{-1}(\sin w)] + \frac{d}{2} \tag{A3}$$

However, since $\sqrt{\sin^2 w - 1} = i\cos w$, and $\cosh^{-1}(\sin w) = \ln(\sin w + \sqrt{\sin^2 w - 1})$, Eq. (A3) becomes

$$t = \frac{di}{\pi}[i\sin w\cos w - \ln(\sin w + i\cos w)] + \frac{d}{2} \tag{A4}$$

In addition, since $\sin w = \sin(u + iv) = \sin u \cosh v + i\cos u \cosh v$, $\cos w = \cos(u + iv) = \cos u \cosh v - i\sin u \sinh v$, $\sin w \cos w = \sin 2w/2 = (\sin 2u \cosh 2v + i\cos 2u \sinh 2v)/2$, and $\ln(\sin w + i\cos w) = v + i(-u + \pi/2)$, Eq. (A4) transforms into

$$t = r + iz = \frac{d}{2\pi}[(2u + \sin 2u \cosh 2v) + i(2v + \cos 2u \sinh 2v)] \tag{A5}$$

On the basis of Eq. (A5), the r-axis is given by

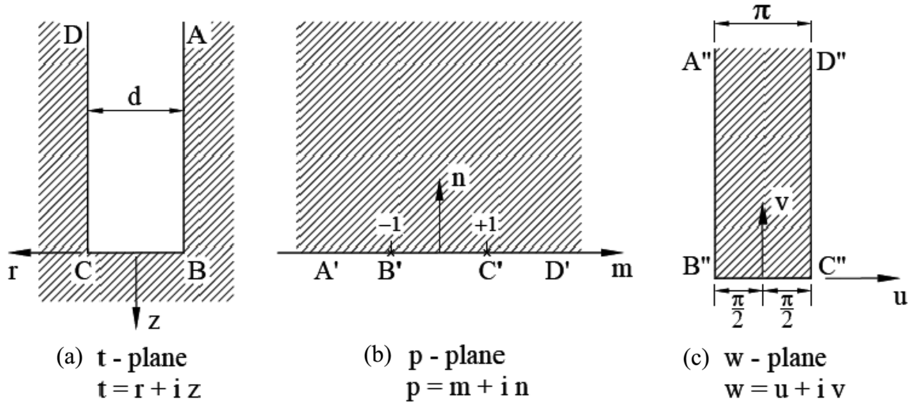

Fig. A1 Double conformal mapping

$$r = \frac{d}{2\pi}[(2u + \sin 2u \cosh 2v)] \tag{A6}$$

Consequently, as the x-and y coordinates in Fig. 2 are equal, respectively, to $r\cos\theta$ and $r\sin\theta$, they become

$$x = \frac{d}{2\pi}(2u + \sin 2u \cosh 2v)\cos\theta \tag{A7}$$

and

$$y = \frac{d}{2\pi}(2u + \sin 2u \cosh 2v)\sin\theta \tag{A8}$$

Finally, Eq. (A5) gives the z-axis as

$$z = \frac{d}{2\pi}(2v + \cos 2u \sinh 2v) \tag{A9}$$

It should be noted that while $u =$ constant curves correspond to streamlines, those represented by $v =$ constant define equipotential lines in Fig. 2.