Geomechanics and Engineering, *Vol. 21, No. 3 (2020) 247-258*
DOI: https://doi.org/10.12989/gae.2020.21.3.247

Calculation model for the shear strength of unsaturated soil under nonlinear strength theory

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(Received January 17, 2020, Revised March 6, 2020, Accepted March 16, 2020)

Abstract. The shear strength of unsaturated soils, a research hotspot in geotechnical engineering, has great guiding significance for geotechnical engineering design. Although kinds of calculation models for the shear strength of unsaturated soil have been put forward by predecessors, there is still need for new models to extensively consider the nonlinear variation of shear strength, particularly for the nonlinear effect of the net normal stress on the shear strength of unsaturated soil. Here, the shear strength of unsaturated soils is explored to study the nonlinear effects of net normal stress with the introduction of a general nonlinear Mohr-Coulomb (M-C) strength criterion, and the relationship between the matric suction (or suction stress) and degree of saturation (DOS) constructed by the soil-water characteristics curve (SWCC) of van Genuchten is also applied for unsaturated soil. Then, two calculation models (i.e., an envelope shell model and an effective stress model) are established for the shear strength of unsaturated soils under the nonlinear strength theory. In these two models, the curve of the shear strength of unsaturated soils versus the net normal stress exhibits a tendency to gently. Moreover, the proposed formulas have flexibility and convenience with five parameters (for the effective stress model) or six parameters (for the envelope shell model), which are from the M-C strength parameters of the saturated soil and fitting parameters of SWCC of van Genuchten. Thereafter, by comparison with the classical theory of the shear strength of unsaturated soils from some actual cases, the rationality and accuracy of the present models were verified.

Keywords: shear strength of unsaturated soil; soil-water characteristic curve (SWCC); suction stress; envelope shell model; effective stress model

1. Introduction

Different from saturated soil composed of liquid and solid phases, unsaturated soil consists of liquid, solid and gas phases. Due to the special properties of the gas phase and the interface of the water, gas, and soil skeleton, the strength property of unsaturated soil is more complicated than that of saturated soil. In actual engineering, the assumption of a completely saturated condition is utilized in design of engineering. However, this practice is too conservative, resulting in the unnecessary waste of resources because most soils are in an unsaturated state, and the saturated state is just a special case. Meanwhile, the shear strength of the soil is related to its degree of saturation (DOS). With the variation in the DOS, the shear strength of the soil also changes, thereby threatening engineering security. For example, the Welipenna landslide happened in Sri Lanka on November 2, 2012. After the gradual penetration and accumulation of rainwater, the unsaturated zone increased in this landslide, and the shear strength of the soil decreased with the increase in its DOS. As a result (Kankanamge *et al.* 2018), the slope collapsed when the rainfall stopped. Therefore, there is great engineering significance in studying the shear strength of unsaturated soil.

As a basic property, the shear strength of the soil is usually used to solve various engineering problems, including the slope stability, foundation bearing capacity, etc. (Konrad and Lebeau 2015, Deng and Li 2019a, b, Deng *et al.* 2019a, b, c). Many factors have an effect on the shear strength of unsaturated soil, and matric suction is an important factor. Studies from Zhang *et al.* (2014) and Aqtash and Bandini (2015) show that the shear strength of unsaturated soil has a nonlinear relationship with matric suction, which explains why it is nonlinear for the envelope of the shear strength of unsaturated soil. Therefore, it is necessary to know the matric suction for solving geotechnical problems, such as the stability of an unsaturated slope or the bearing capacity of an unsaturated foundation (Chiorean *et al.* 2017, Estabrag and Javadi 2012). Due to the complex, time-consuming, and expensive nature of suction testing, Fredlund *et al.* (1995) applied the soil-water characteristics curve (SWCC) to research regarding unsaturated soil. The SWCC manifests the relationship between the water content (or the DOS) and suction. Practices from Roopnarine *et al.* (2014), Al-Mahbashi *et al.* (2015), Johari *et al.* (2018), and Lin *et al.* (2018) proved that good results regarding the shear strength of unsaturated soil, slope stability, pavement design, etc., could be achieved by applying the SWCC.

At present, many scholars have put forward a variety of

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ISSN: 2005-307X (Print), 2092-6219 (Online)

formulas for calculating the shear strength of unsaturated soil, but the shear strength theory of unsaturated soil is still in the process of development and improvement. These proposed formulas can be roughly divided into two categories. One (Khalil *et al.* 1998, Tarantino 2007) is the formula based on the theoretical framework of effective stress from Bishop *et al.* (1963). The other (Xu and Cao 2015, Patil *et al.* 2017) is the formula based on the theoretical framework of double stress state variables from Fredlund *et al.* (1978). For most of these existing formulas for calculating the shear strength of the unsaturated soil, few factors with the introduction of few numbers of parameters are taken into account in the established model, thereby limiting their applicability. In other words, it is difficult to simulate the shear behavior of different kinds of unsaturated soils for these established models. For example, some models could not consider the shear behavior of the unsaturated soil when the matric suction is greater than the residual matric suction. In addition, the existing shear strength formulas of unsaturated soil are mainly established on the basis of the linear Mohr-Coulomb (M-C) strength criterion, and the nonlinear M-C strength criterion is rarely introduced into the shear strength of unsaturated soil to consider the nonlinear effect of the net normal stress.

Here, based on previous research, the general nonlinear M-C strength criterion is introduced into the shear strength of unsaturated soil to consider the nonlinear effects of net normal stress. Moreover, the relationship constructed by the SWCC of van Genuchten (1980) between the matric suction (or suction stress) of the unsaturated soil and the DOS is applied. Then, an envelope shell model and an effective stress model for the shear strength of unsaturated soil under nonlinear strength theory are established. The proposed formulas have flexibility and convenience with five parameters (for the effective stress model) or six parameters (for the envelope shell model), which are from the M-C strength parameters of saturated soil and fitting parameters of SWCC of van Genuchten (1980). Thereafter, by comparison with the classical theory of the shear strength of unsaturated soils from some actual cases, the rationality and accuracy of the present models are verified. On basic of the reliable prediction for the shear strength of unsaturated soil from the present models, the ultimate bearing capacity of unsaturated foundations and the stability of unsaturated slopes could be reasonably evaluated and calculated under the nonlinear strength criterion. Therefore, the present models would have good practicability in the design of unsaturated slope and unsaturated foundation.

2. Calculation model for the shear strength of unsaturated soil under nonlinear strength theory

2.1 Envelope shell model for the shear strength of unsaturated soil under nonlinear strength theory

Based on the linear M-C strength criterion, Fredlund *et al.* (1978) considered the effect of matric suction on the shear strength of unsaturated soil and then proposed a formula of the shear strength of the unsaturated soil under a double stress state variable (i.e., net normal stress ($\sigma - u_a$) and matric suction ($u_a - u_w$)), which is called the envelop shell model for the shear strength of unsaturated soil under linear strength theory. The formula of the shear strength of unsaturated soil obtained by Fredlund *et al.* (1978) is:

$$\tau_f = c' + (\sigma - u_a)\tan\varphi' + (u_a - u_w)\tan\varphi_b \tag{1}$$

where τ_f is the shear strength, σ is the total normal stress, u_a is the pore air pressure of the soil, u_w is the pore water pressure of the soil, ($\sigma - u_a$) is the net normal stress, ($u_a - u_w$) is the matric suction, c' is the effective cohesion of the saturated soil, φ' is the effective internal friction angle of the saturated soil, and φ_b is the suction friction angle.

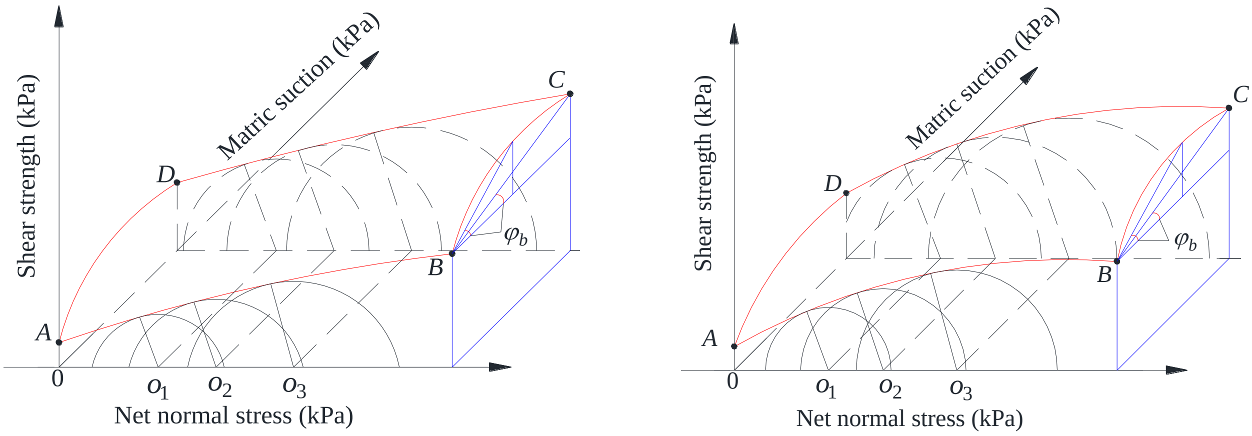
In Eq. (1), $\tan\varphi_b$ is the parameter describing the contribution of matric suction on the shear strength of the unsaturated soil. Originally, Fredlund *et al.* (1978) thought that $\tan\varphi_b$ was a constant. Therefore, the original equation of the shear strength of the unsaturated soil established by Fredlund *et al.* (1978) is expressed as the plane on the stress space (i.e., the surface *ABCD* in Fig. 1(a) is a plane), which is formed by the spatial expansion of the Mohr failure envelope (straight line). Meanwhile, the inclination of the plane for the shear strength of the unsaturated soil is $\tan\varphi_b$ (constant), and the trace of the Mohr failure envelope represents the shear strength of unsaturated soil when the matric suction ($u_a - u_w$) = 0 (i.e., the saturated state).

With further research, many experiments show that $\tan\varphi_b$ is not a constant but a variable related to the suction, and the shear strength has a nonlinear relationship with the matric suction, as shown in Fig. 1(a). Thus, the results of the shear strength of unsaturated soil show a curved surface in the stress space. In other words, the surface *ABCD* in Fig. 1(a) is a curved surface. For $\tan\varphi_b$, many scholars give different nonlinear calculation formulas based on their experiments, theories, and experiences, such as Vanapalli *et al.* (1996) and Vilar (2006). Among these formulas, Vanapalli *et al.* (1996) established the relationship between $\tan\varphi_b$ with the effective internal friction angle and effective DOS of the soil. Meanwhile, Vanapalli *et al.* (1996) compared the model with the experimental results of glacial soil and used the SWCC to consider the contribution of matric suction on the shear strength, thereby further verifying the good correlation between the predicted value and the measured value of the shear strength of the unsaturated soil. The formula of the suction friction angle established by Vanapalli *et al.* (1996) is simple, and it is convenient to explore the influence of the matric suction on the shear strength of unsaturated soil in combination with the SWCC to get prediction results that are more consistent with the actual situation. Therefore, Vanapalli's model has been widely used since it was proposed. In Vanapalli's model, the calculation formula of $\tan\varphi_b$ is:

$$\tan\varphi_b = S_e \tan\varphi' \tag{2}$$

where S_e is the effective DOS, and $S_e = (S - S_r) / (1 - S_r)$ or $S_e = (\theta - \theta_r) / (\theta_s - \theta_r)$, S is the DOS, S_r is the residual DOS, θ is the volume water content, θ_s is the saturated volume water content, and θ_r is the residual volume water content.

In Eq. (2), Schnellmann *et al.* (2014) indicated that the effective DOS (S_e) is the control parameter of the soil characteristic function of unsaturated soil. The effective DOS can be solved by the model of the SWCC. To date, many models of the SWCC have been proposed by scholars



(a) Envelope shell model for the shear strength of unsaturated soil under the linear M-C strength criterion (i.e., $m = 1$)

(b) Envelope shell model for the shear strength of unsaturated soil under the nonlinear M-C strength criterion (i.e., $m > 1$)

Fig. 1 Envelope shell model for the shear strength of unsaturated soil

(van Genuchteetal 1980, Fredland and Xing 1994, Pham and Fredland 2005) in the development of unsaturated soil mechanics. Among these models of the SWCC, the SWCC of van Genuchten (1980) is the most widely used (Wang *et al.* 2017, Oh and Vanapalli 2018, Bao *et al.* 2018). Based on the SWCC of van Genuchten (1980), the formula of the soil effective DOS (S_e) can be obtained as:

$$S_e = \frac{1}{\{1+[\alpha(u_a-u_w)]^n\}^{m_v}} \tag{3}$$

where α, n, and m_v are all the fitting parameters of the SWCC, $\alpha = 0 \sim 0.5$ kPa^{-1} and $n = 1.1 \sim 8.5$, in general (Lu and Likos 2004).

In Eq. (3), m_v can be calculated in the following ways: (1) m_v = arbitrary value (Bi *et al.* 2018); (2) $m_v = (n-1)/n$ (Mualem 1976); and (3) $m_v = (n-2)/n$ (Burdine 1953).

Substituting Eq. (3) into Eq. (2), the formula of $\tan\varphi_b$ can be converted into:

$$\tan\varphi_b = \frac{1}{\{1+[\alpha(u_a-u_w)^n]\}^{m_v}}\tan\varphi' \tag{4}$$

Eq. (1) is the formula of the shear strength of the unsaturated soils established by Fredlund *et al.* (1978) under the linear M-C strength criterion. However, most experimental results show that, only when the geotechnical body bears a small normal stress is there a linear correlation between the shear strength of the geotechnical body with the normal stress. Furthermore, the curve of the shear strength versus the normal stress is likely a straight line. If the normal stress on the geotechnical body is large, the shear strength of the geotechnical body nonlinearly increases with the increase in the normal stress, and the curve of the shear strength gradually flattens out. Then, using the linear strength criterion will overestimate the shear strength of the geotechnical body. Hence, the shear strength of the geotechnical body usually shows a nonlinear characteristic, and the linear strength criterion is only a special case of the nonlinear strength criterion (Baker 2004, Deng *et al.* 2015). In other words, using the linear M-C strength criterion to describe the mechanism of shear failure on the geotechnical body is a simplified measure. Thus, based on the general nonlinear M-C strength criterion, this work improves the linear envelope shell model for the shear strength of unsaturated soil and obtains an envelope shell model under nonlinear strength theory shown in Fig. 1(b). In Fig. 1(b), the present envelope shell model takes the nonlinear effects of net normal stress on shear strength and the nonlinear effects of matric suction on the suction friction angle into consideration.

If Eq. (4) is used to calculate the suction friction angle, the formula of the shear strength of the unsaturated soil under the nonlinear M-C strength criterion can be obtained as:

$$\tau_f = c_0\left(1+\frac{\sigma-u_a}{\sigma_t}\right)^{\frac{1}{m}} + \frac{u_a-u_w}{\{1+[\alpha(u_a-u_w)]^n\}^{m_v}}\tan\varphi' \tag{5}$$

where c_0, σ_t, and m are the soil strength parameters and $m \geq 1$.

If Eq. (2) is used to calculate the suction friction angle and Eq. (3) is used to calculate the matric suction by the effective DOS, the formula of the shear strength of the unsaturated soil under nonlinear M-C strength theory can be obtained as:

$$\tau_f = c_0\left(1+\frac{\sigma-u_a}{\sigma_t}\right)^{\frac{1}{m}} + \frac{S_e}{\alpha}\left[S_e^{(-1/m_v)}-1\right]^{\frac{1}{n}}\tan\varphi' \tag{6}$$

For the nonlinear M-C strength criterion, φ' in Eqs. (5) and (6) is the instantaneous effective internal friction angle of the saturated soil. From Eqs. (5) and (6), the formula of $\tan\varphi'$ is given as:

$$\tan\varphi' = \frac{c_0}{m\sigma_t}\left(1+\frac{\sigma-u_a}{\sigma_t}\right)^{\frac{1-m}{m}} \tag{7}$$

Eqs. (5) and (6) represent the envelope shell model for the shear strength of unsaturated soil under the nonlinear strength criterion, which is abbreviated as the envelope shell model. Eq. (5) shows the relationship between the shear strength of the unsaturated soil and the matric suction under the nonlinear M-C strength criterion. Eq. (6) shows the relationship between the shear strength of the unsaturated soil with the effective DOS under the nonlinear M-C strength criterion. In other words, if the matric suction ($u_a - u_w$) or the effective DOS (S_e) of unsaturated soil under certain normal stress (σ) is known, the envelope shell model can be used to calculate the shear strength of the unsaturated soil.

Consistent with the above, when the matric suction ($u_a - u_w$) = 0 (i.e., $S_e = 1$), the strength parameters c_0, σ_t, and m in Eqs. (5) and (6) are the nonlinear M-C strength parameters of saturated soil. Furthermore, the strength parameters c_0, σ_t, and m can be obtained by fitting the experimental data of the shear strength of saturated soil based on the nonlinear M-C strength criterion.

When $m = 1$ (i.e., the linear M-C strength criterion), Eqs. (5) and (6) can be simplified into:

$$\tau_f = c' + (\sigma - u_a)\tan\varphi' + \frac{u_a - u_w}{\{1+[\alpha(u_a - u_w)]^n\}^{m_v}}\tan\varphi' \tag{8}$$

$$\tau_f = c' + (\sigma - u_a)\tan\varphi' + \frac{S_e}{\alpha}[S_e^{(-1/m_v)} - 1]^{\frac{1}{n}}\tan\varphi' \tag{9}$$

where $c' = c_0$ and $\tan\varphi' = c_0 / \sigma_t$.

Eqs. (8) and (9) are the same as Eq. (1), and both are envelope shell models for the shear strength of unsaturated soil under linear strength theory. Despite the envelope shell model for the shear strength of unsaturated soil under nonlinear strength theory being established also based on the envelope shell model of Fredlund *et al.* (1978) with a double stress state variable, the present model considers the nonlinear effect of normal stress on the shear strength, which is different from the envelope shell model for the shear strength of unsaturated soil under linear strength theory. In addition, the envelope shell model for the shear strength of unsaturated soil under nonlinear strength theory calculates the effect of the normal stress and matric suction on the shear strength by simple superposition, and the coupling effects of the normal stress and matric suction on shear strength are not considered. Theoretically, the stress state of the soil has a certain influence on the SWCC (Schnellmann *et al.* 2013, Zhou *et al.* 2016). Zhai *et al.* (2019) found that the soil suction will generate the additional net normal stress on the soil skeleton, and it has an additional adhesion contribution on the soil particles. Therefore, when Eqs. (5) and (6) are used to calculate the shear strength of unsaturated soil under nonlinear strength theory, all of the parameters of the SWCC (α, n, and m_v) should account for the coupling effect of the normal stress and matric suction on the shear strength. In other words, for the formula with the envelope shell model for the shear strength of unsaturated soil under nonlinear strength theory (i.e., Eqs (5) and (6)), all parameters of the SWCC (α, n, and m_v) should be obtained by measuring the shear strength of unsaturated soil instead of the results from the SWCC.

2.2 Effective stress model for the shear strength of unsaturated soil under nonlinear strength theory

As previously mentioned, the envelope shell model is not theoretically rigorous because the coupling effect of the normal stress and matric suction on shear strength is not considered. Meanwhile, to consider the coupling effect of the normal stress and matric suction on the shear strength in the envelope shell model, the parameters (α, n, and m_v) from the SWCC cannot be directly applied to the formula of the shear strength. Thus, based on the general nonlinear M-C strength criterion and the effective stress model of unsaturated soil (Lu and Likos 2004), the effective stress model for the shear strength of unsaturated soil under nonlinear strength theory is established.

The principle of effective stress is an important theory that distinguishes soil mechanics from other mechanics. Terzaghi came up with the basic concept of the effective stress principle in 1923. Then, Terzaghi (1943) recorded the complete principle of effective stress and clarified the great difference between granular materials and continuous solid materials in the stress-strain relationship. Thereby, soil mechanics became an independent subject. According to the characteristics of the saturation and dryness of the soil, Bishop (1959) modified the classic expression of Terzaghi (1943) with a parameter χ and then proposed the concept of the effective stress of unsaturated soil. Thereafter, Lu and Likos (2004 and 2006) introduced the suction concept to further expand the effective stress formula of unsaturated soil established by Bishop (1959), which is consistent with Terzaghi's effective stress expression. Lu and Likos (2006) proposed an effective stress formula for unsaturated soil as:

$$\sigma' = \sigma - u_a - \sigma_s \tag{10}$$

where σ' is the effective normal stress, σ_s is the suction stress, and the physical meanings of σ and u_a are the same as in Eq. (1).

The envelope of the shear strength of the soil indicates that, for any soil (whether saturated or unsaturated), its shear failure state can be described by the general nonlinear M-C strength criterion. However, according to the effective stress principle of Terzaghi (1943), the shear failure behavior of the soil is directly related to the effective normal stress on the soil. In other words, the shear strength of the soil is dominated by the effective normal stress rather than the total normal stress. Therefore, when the effective stress model of Lu and Likos (2006) is introduced into the general nonlinear M-C criterion, this work could obtain the effective stress formula of the shear strength of unsaturated soil under nonlinear strength theory as:

$$\tau_f = c_0[1 + (\sigma - u_a - \sigma_s)/\sigma_t]^{1/m} \tag{11}$$

where the physical meanings of c_0, σ_t, and m are the same as in Eq. (5).

For the suction stress σ_s in Eq. (11), Lu *et al.* (2010) gave its calculation formula as:

$$\sigma_s = -(u_a - u_w)S_e \tag{12}$$

According to the SWCC of van Genuchten (i.e., Eq. (3)) and the fitting parameter $m_v = (n - 1) / n$ (which reduces the flexibility of the van Genuchten model but significantly resulting in greater stability during parameter optimization and permitting closed-form solution of the hydraulic conductivity function), Lu *et al.* (2010) obtained the formula of suction stress (σ_s) as:

$$\sigma_s = -\frac{(u_a - u_w)}{\{1+[\alpha(u_a - u_w)]^n\}^{(n-1)/n}} \tag{13}$$

where the physical meanings of α and n are the same as in Eq. (3), and both are fitting parameters of the SWCC of van Genuchten (1980).

In the suction stress model of Lu *et al.* (2010), if the formula of suction stress is established with the effective DOS (S_e) as the independent variable, it can be determined that:

$$\sigma_s = -\frac{S_e}{\alpha}[S_e^{n/(n-1)} - 1]^{1/n} \tag{14}$$

Eq. (13) is the expression of suction stress in the whole range of matric suction, while Eq. (14) is the expression of suction stress in the whole range of the effective DOS.

If Eq. (13) is substituted into Eq. (11), the formula of the shear strength of the unsaturated soil in the whole range of matric suction under the effective stress model can be obtained as:

$$\tau_f = c_0\left\{1+\frac{\sigma-u_a}{\sigma_t}+\frac{(u_a-u_w)}{\sigma_t\{1+[\alpha(u_a-u_w)]^n\}^{(n-1)/n}}\right\}^{\frac{1}{m}} \tag{15}$$

If Eq. (14) is substituted into Eq. (11), the formulas of the shear strength of unsaturated soil in the whole range of saturation under the effective stress model can be calculated as:

$$\tau_f = c_0\left\{1+\frac{\sigma-u_a}{\sigma_t}+\frac{S_e}{\alpha\sigma_t}[S_e^{(\frac{n}{1-n})}-1]^{1/n}\right\}^{\frac{1}{m}} \tag{16}$$

Eqs. (15) and (16) represent the effective stress model for the shear strength of unsaturated soil under nonlinear strength theory, which is abbreviated as the effective stress model.

When m = 1 (i.e., the linear M-C strength criterion), Eqs. (15) and (16) can be simplified into:

$$\tau_f = c'+(\sigma-u_a)\tan\varphi' + \frac{1}{\{1+[\alpha(u_a-u_w)]^n\}^{(n-1)/n}}(u_a-u_w)\tan\varphi' \tag{17}$$

$$\tau_f = c'+(\sigma-u_a)\tan\varphi'+\frac{S_e}{\alpha}[S_e^{n/(1-n)}-1]^{\frac{1}{n}}\tan\varphi' \tag{18}$$

where the physical meanings of c' and φ' are the same as in Eq. (1), while $c' = c_0$ and $\tan\varphi' = c_0 / \sigma_t$.

Compared with Eqs. (8) and (9), Eqs. (17) and (18) are the simplified formulas of Eqs. (8) and (9) when $m_v = (n - 1) / n$. In other words, under linear M-C strength theory (i.e., m = 1), the formula of the shear strength of unsaturated soil established by the effective stress model still belongs to the category of the envelope shell model. When $m > 1$ (i.e., the nonlinear M-C strength criterion), the effective stress formula (i.e., Eqs. (15) and (16)) is derived based on the general nonlinear M-C strength theory and the effective stress model of Lu and Likos (2006). Hence, the effective stress model is more rigorous in theory than the envelope shell model under nonlinear strength theory. Meanwhile, in the effective stress model, not only can the strength parameters c_0, σ_t, and m be taken as the nonlinear M-C shear strength parameters of saturated soil but also the values of parameters α and n can be taken directly from the fitting results of the SWCC.

3. Comparison and analysis

3.1 Comparison with the classical formulas of the shear strength of unsaturated soil

For the shear strength of unsaturated soil, many scholars have carried out a large number of the studies from different perspectives (Bishop *et al.* 1963, Fredlund *et al.* 1978, Jokar and Mirasi 2018, Pedrotti and Tarantino 2018, Naghadeh and Toker 2019). Among them, the strength theory of unsaturated soil with the single stress state variable from Bishop *et al.* (1963) and the strength theory of unsaturated soil with the double stress state variable from Fredlund *et al.* (1978) are more classical and widely used. The above two theories are based on the linear M-C strength criterion, corresponding to the case in which the nonlinear strength parameter m = 1 in the envelope shell model and effective stress model.

The classical strength theory of the unsaturated soil from Bishop *et al.* (1963) is established by extending the effective stress equation of Terzaghi (1943) for the saturated soil and combining it with the linear M-C strength criterion. The shear strength formula of unsaturated soil from Bishop *et al.* (1963) is:

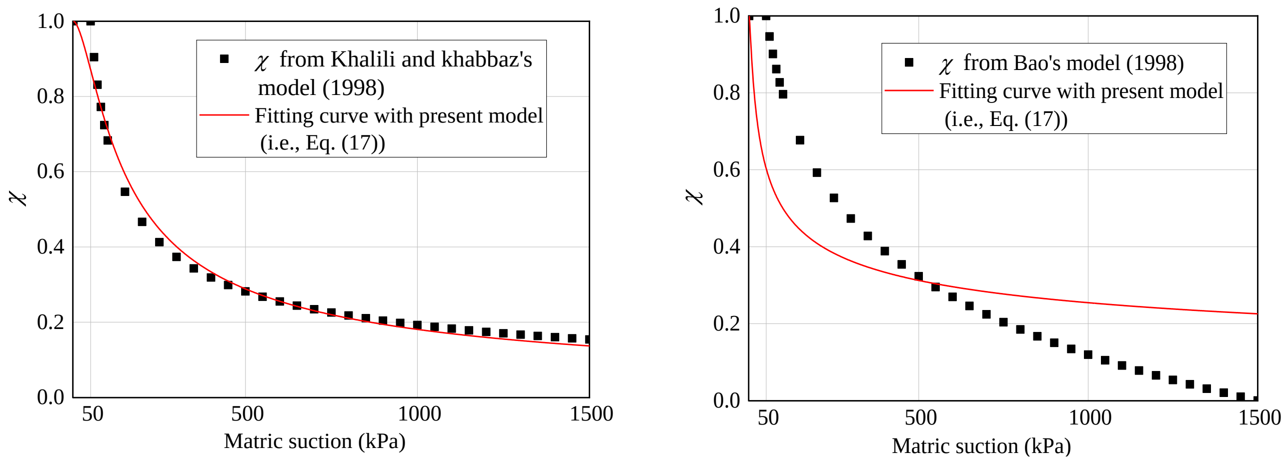
$$\tau_f = c' + (\sigma - u_a)\tan\varphi' + \chi(u_a - u_w)\tan\varphi' \tag{19}$$

where the physical meanings of σ, u_a, u_w, c', and φ' are the same as Eq. (1), χ is a parameter related to saturation; $\chi = 1$ when $S = 0$, and $\chi = 0$ when $S = 1$.

Since Bishop *et al.* (1963) determined the formula of the shear strength of unsaturated soil, many scholars studied the parameter χ according to experiments, theoretical derivation, and experience, reporting different formulas for χ. The double independent variable strength formula (i.e., Eq (1)) proposed by Fredlund *et al.* (1978) can be written as an expression containing χ, $\tan\varphi_b = \chi\tan\varphi'$. Table 1 shows the different formulas of the parameter χ in the classical criterion of the shear strength of unsaturated soil. In Table 1, Vanapalli *et al.* (1996) thought that χ was equal to the effective DOS (S_e) of the soil, Khalil and Khabbaz (1998) established the power function relationship of χ with matric suction and the air-entry value of the SWCC, and Bao *et al.* (1998) adopted χ as the logarithmic function of the residual matric suction and the air-entry value of the SWCC. Meanwhile, when the nonlinear strength parameter m = 1 (i.e., the linear M-C strength criterion), the envelope shell model (i.e., Eq. (8)) and the effective stress model (i.e., Eq. (17)) can also establish the same formula as Eq. (19) for the

Table 1 Different formulas for parameter χ in the classical shear strength criterion of unsaturated soil

Methods		Formulas of parameter χ
Vanapalli *et al.*(1996)		$\chi = S_e$
Khalili and Khabbaz (1998)		$\chi = 1.0$ when $(u_a - u_w) \le (u_a - u_w)_{AE}$, where $(u_a - u_w)_{AE}$ is air-entry value
		$\chi = \left[\dfrac{u_a - u_w}{(u_a - u_w)_{AE}}\right]^{-0.55}$ when $(u_a - u_w) > (u_a - u_w)_{AE}$
Bao *et al.* (1998)		$\chi = 1.0$ when $(u_a - u_w) \le (u_a - u_w)_{AE}$
		$\chi = \dfrac{\log(u_a - u_w)_r - \log(u_a - u_w)}{\log(u_a - u_w)_r - \log(u_a - u_w)_{AE}}$ when $(u_a - u_w) \ge (u_a - u_w)_{AE}$ and $(u_a - u_w) < (u_a - u_w)_r$, where $(u_a - u_w)_r$ is the residual matric suction
		$\chi = 0.0$ when $(u_a - u_w) \ge (u_a - u_w)_r$
Present method	Envelope shell model	$\chi = \dfrac{1}{\{1 + [\alpha(u_a - u_w)]^n\}^{m_v}}$
	Effective stress model	$\chi = \dfrac{1}{\{1 + [\alpha(u_a - u_w)]^n\}^{(n-1)/n}}$



(a) Comparison of effective stress model with the model of Khalili and Khabbaz (1998)

(b) Comparison of effective stress model with the model of Bao *et al.* (1998)

Fig. 2 Comparison of models for the shear strength of unsaturated soil under the linear M-C strength criterion

shear strength, and then the expression of parameter χ is shown in Table 1. Notably, despite the formula of parameter χ in the Vanapalli's method being different from the present method, the calculation results of χ are the same (all are the effective DOS).

Here, letting $(u_a - u_w)_{AE}$ = 50 kPa and $(u_a - u_w)_r$ = 1500 kPa from the previous research, two methods (i.e., those of Khalili and Khabbaz and Bao *et al.*) are used to obtain a series discrete points of χ when the variation range of $(u_a - u_w)$ is 0 kPa ~ 1500 kPa. Then, the present effective stress model is used to fit these discrete points with the curve. Fig. 2 shows the best results of the curve fitting. The fitting parameters are α = 0.012 and n = 1.694 in Fig. 2(a), and α = 0.100 and n = 1.297 in Fig. 2(b). Thus, α and n obtained by the curve fitting are all within the specified range.

Fig. 2(a) shows that χ from the present method is quite close to that of the method of Khalili and Khabbaz (1998), indicating that the present models have good applicability. Fig. 2(b) shows that χ from the present method is quite different from that of Bao *et al.* (1998). The main reason is that Bao *et al.* (1998) directly ignore the influence of matric suction on the shear strength when the matric suction is greater than the residual matric suction. Although no experimental data support and explain the shear behavior beyond the residual state (Vanapalli *et al.* 1996), the shear strength of unsaturated soil may increase, decrease, or remain constant when the matric suction is greater than the residual matric suction. However, ignoring this phenomenon will deviate from the actual situation. Moreover, the existing research also shows that it is necessary to properly explain the shear behavior of the unsaturated soil in the residual state (Schnellmann *et al.* 2014, Hoyos *et al.* 2014). Therefore, the present method is more reasonable than that of Bao *et al.* (1998).

3.2 Comparison and analysis between the envelope shell model and the effective stress model

Both the established envelope shell model and effective stress model in this work can be used to calculate the shear strength of unsaturated soil under the nonlinear M-C strength criterion. However, the envelope shell model and the effective stress calculation model are built using different methods. The formula in the envelope shell model is based on the model of Fredlund *et al.* (1978). It uses the nonlinear M- C strength criterion to replace the linear M-C strength criterion under saturated conditions and extends the nonlinear strength criterion to the matric suction dimension by the suction friction angle obtained from the model of Vanapalli *et al.* (1996). The envelop shell model does not consider the coupling effect of the normal stress and matric suction on the shear strength, so its formula is not theoretically rigorous. The effective stress formula is derived from the general nonlinear M-C strength theory and effective stress model of Lu and Likos (2006), and it is a rigorous theoretical solution.

Letting the nonlinear strength parameters c_0 = 2.0 kPa, σ_t = 0.3 kPa, and m = 1.5 in these two models, five groups of the SWCC according to the different combinations of parameters α and n are formed, as shown in Table 2. Meanwhile, letting the net normal stress be a constant (i.e., $(\sigma - u_a)$ = 75 kPa), the discrete points of the shear strength of unsaturated soil with the effective stress model are calculated for the matric suction varying from 0 kPa to 30 kPa. It was previously mentioned that, when $m_v = (n-1)/n$ under linear M-C strength theory, the envelope shell model has the same formula as the effective stress model. However, compared with the effective stress model, the envelope shell model does not take the coupling effect of the normal stress and matric suction on the shear strength into consideration. Therefore, to consider the coupling effect in the envelope model, the curve fitting method is utilized to fit the discrete points of the shear strength of unsaturated soil from the effective stress model, and the parameters α , n , and m_v are recalculated.

Fig. 3 shows the results of the envelope model by fitting discrete points of the shear strength of unsaturated soil from the effective stress model. Table 2 gives the original parameters (α, n, and mv) of the SWCC and the best fitting parameters in the envelope shell model. From Table 2 and Fig. 3, despite the results of the shear strength of unsaturated soil obtained by the envelope shell model and the effective stress model under the nonlinear M-C strength criterion being close, the parameters (α, n, and m_v) determined by the curve fitting show a significant

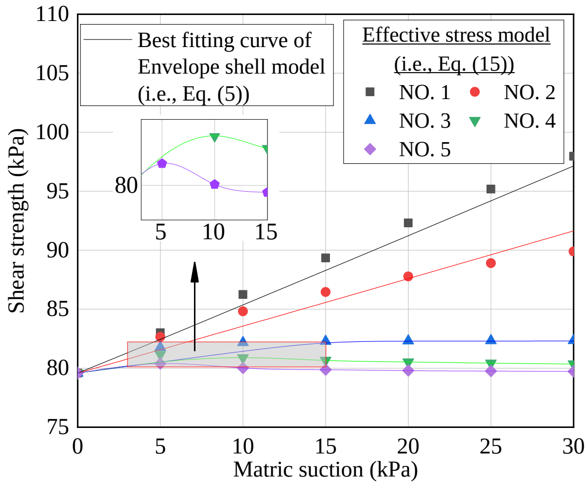


Fig. 3 Comparison between effective stress model and envelope shell model for the shear strength of unsaturated soil with different parameters of the SWCC

Table 2 Comparison between original parameters of the SWCC in the effective stress model and the best fitting parameters in the envelope shell model

Test No.	Original parameters of SWCC of van Genuchten in effective stress model (i.e., Eq. (15))			Best fitting parameters of envelope shell model (i.e., Eq. (5))		
	α	n	$m_v = (n-1)/n$	α	n	m_v
1	0.05	1.1	0.091	1.493×10^{42}	0.183	-0.062
2	0.12	1.5	0.333	9.968×10^{40}	0.170	-0.046
3	0.25	2.0	0.500	0.069	33.106	0.030
4	0.30	2.5	0.600	0.127	22.105	0.067
5	0.40	3.0	0.667	0.208	14.239	0.140

difference from the parameters of the original SWCC (α, n, and m_v). In other words, when the coupling effect of the net normal stress and matric suction on the shear strength is considered in the envelop shell model, the parameters α, n, and m_v need to be fitted with the measured shear strength of the unsaturated soil rather than completely using the fitting results of the SWCC.

Fig. 3 shows that there is a peak phenomenon in NO. 4 and NO. 5. The peak behavior is an important feature of coarse-grained materials, and the shear strength envelope of the coarse-grained materials exhibits nonlinear behavior and shows a peak near air-entry value; then, the shear strength will decrease with the increase in matric suction (Houston *et al.* 2008, Likos *et al.* 2010). Lu *et al.* (2010) pointed out that most formulas of the shear strength of unsaturated soil under the double-stress state variable frame of Fredlund *et al.* (1978) or the classic effective stress frame of Bishop *et al.* (1963) could not reflect this feature. Although Zhao *et al.* (2013) believed that the methods established under the framework of Fredlund *et al.* (1978) and Bishop *et al.* (1963) can provide a reasonable prediction for the shear strength of fine-grained soil, most of them cannot show the peak behavior of the shear strength of coarse-grained materials in unsaturated conditions. However, the present models can simulate the peak behavior of coarse-grained soil. Thus, the present models have good applicability to the shear failure behavior of fine-grained soil and coarse-grained soil under unsaturated conditions.

According to the above analysis, the main difference between the formulas of the shear strength involved in Section 2 is the function form and material parameters. These differences determine the accuracy of the model to predict the shear strength of unsaturated soil. In general, Sheng *et al.* (2011) pointed out that the models with more parameters are more flexible in fitting different data sets. In this work, there are 6 fitting parameters (c_0, σ_t, m, α, n, and m_v) in the envelope shell model and 5 fitting parameters (c_0, σ_t, m, α, and n) in the effective stress model, and all parameters are from the previous theory models. Meanwhile, the present envelope shell model and effective stress model cover the formulas for the shear strength of unsaturated soil in the full DOS range and matric suction range. Therefore, the shear strength of unsaturated soil can be predicted by measuring the matric suction or DOS of the soil sample when the present models are used. Hence, the two present models have many benefits, such as wide application range, excellent flexibility, and definite parameter values. Nevertheless, the present models need more experimental data to ensure their accuracy and reliability.

4. Verification of examples

4.1 Verification of the formula for the shear strength of unsaturated soil under linear strength theory

Example 1: Lee *et al.* (2005) researched the influence of the stress state on the shear strength of Korean residual soil in the unsaturated state by the triaxial test. When the confining pressure is 0 kPa, Lee *et al.* (2005) obtained the experimental data of the SWCC (see Fig. 4), the test results of the shear strength were fitted by linear M-C strength, and the parameters of saturated soil subject to the linear M-C strength criterion were obtained as c' = 19.3 kPa and φ' = 41.4°. Taking $m_v = (n - 1) / n$, the parameters in the SWCC model of van Genuchten (1980) are α = 0.407 kPa^{-1} and n = 1.317, which are obtained by fitting the experimental data of the SWCC from Lee *et al.* (2005). Moreover, the air entry value is 2.0 kPa in the literature of Lee *et al.* (2005). Then, substituting the parameters c', φ', α, and n into Eq. (17), this example can use the effective stress model to predict the shear strength of the unsaturated Korean residual soil, and the results are shown in Fig. 5. It should be noted that, if $m_v = (n - 1) / n$ under the liner M-C strength criterion, the effective stress model and envelope shell model would have the same shear strength results. In addition, the initial model for the shear strength of unsaturated soil from Fredlund *et al.* (1978) (i.e., the suction friction angle in Eq. (1) is considered as a constant rather than a suction related variable) is used to fit the experimental data of the shear strength of the unsaturated Korean residual soil from Lee *et al.* (2005), and the suction friction angle is obtained as φ_b = 11.1°. Meanwhile, the predicted results of the initial model for the shear strength of unsaturated soil from Fredlund *et al.* (1978) are plotted in

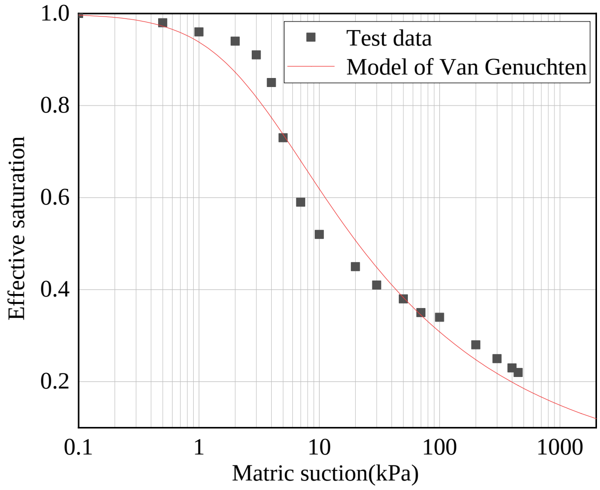


Fig. 4 SWCC in example 1

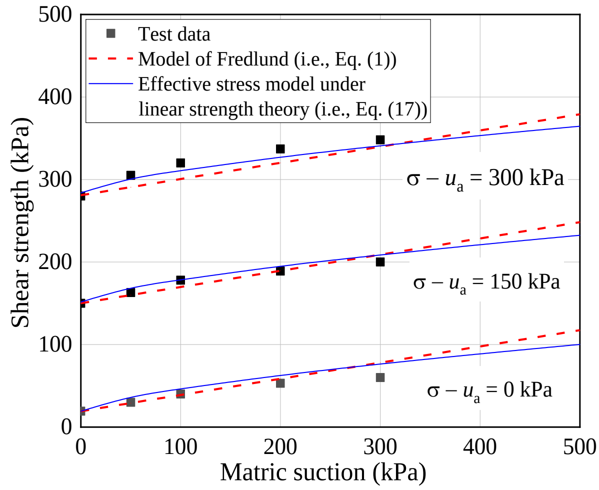


Fig. 5 Curve of shear strength vs. matric suction under different ($\sigma - u_a$) conditions in example 1

Fig. 5 to compare with the predicted results of the present models and the test data from Lee *et al.* (2005). Fig. 5 shows the test results of the shear strength of the unsaturated Korean residual soil under the net normal stress ($\sigma - u_a$) = 0 kPa, 150 kPa, and 300 kPa as well as the corresponding prediction results of the present models and the model of Fredlund *et al.* (1978).

Fig. 5 shows that when the net normal stress ($\sigma - u_a$) = 0 kPa, the results from the model of Fredlund *et al.* (1978) are closer to the test results, but the errors of the present models are smaller when the net normal stress are 150 kPa and 300 kPa. In addition, when the net normal stress ($\sigma - u_a$) is constant, the results from the model of Fredlund *et al.* (1978) show a linear increase with the increase of the matric suction ($u_a - u_w$), while the present models tend to be flat, which is closer to the engineering practice.

4.2 Verification of the formula of the shear strength of unsaturated soil under nonlinear strength theory

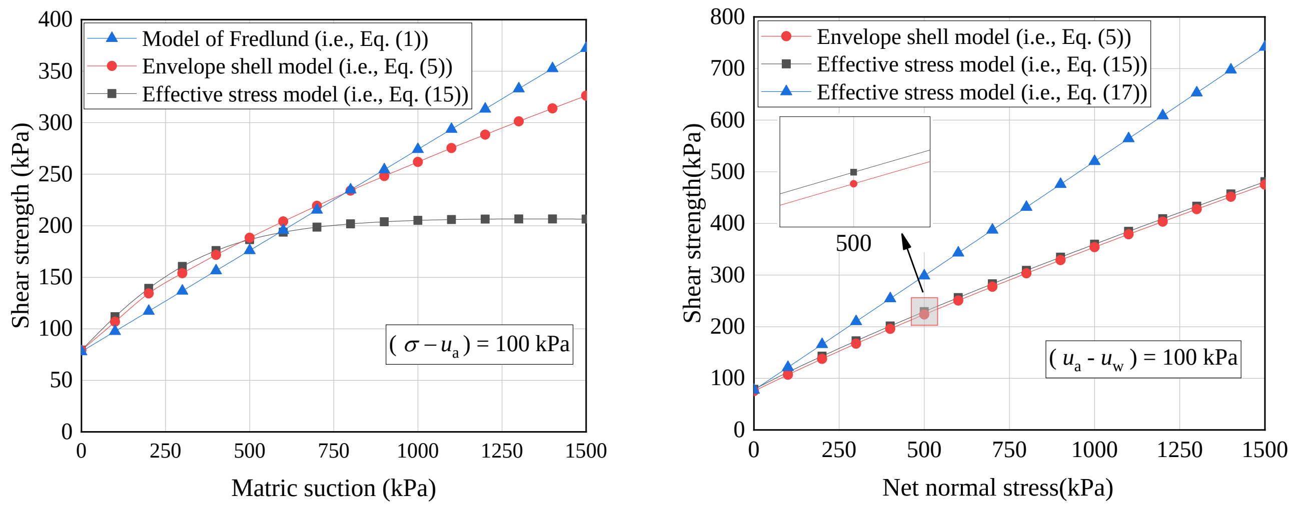
Example 2: to study the relationship between the shear strength of unsaturated soil and SWCC, Zhou *et al.* (2010) measured the shear strength of Jingmen expansive soil

Table 3 Experimental data of the SWCC in example 2

Test No.	$(u_a - u_w)$ (kPa)	Test results			Calculated results
		θ	θ_s	θ_r	S_e
1	0	0.4023	0.4023	0.1217	1.000
2	10	0.4015	0.4023	0.1217	0.997
3	50	0.3888	0.4023	0.1217	0.952
4	100	0.3832	0.4023	0.1217	0.932
5	150	0.3769	0.4023	0.1217	0.909
6	200	0.3704	0.4023	0.1217	0.886
7	250	0.3644	0.4023	0.1217	0.865
8	300	0.3576	0.4023	0.1217	0.841
9	350	0.3432	0.4023	0.1217	0.789
10	400	0.3263	0.4023	0.1217	0.729
11	450	0.3140	0.4023	0.1217	0.685
12	500	0.2995	0.4023	0.1217	0.634
13	600	0.2707	0.4023	0.1217	0.531
14	800	0.2420	0.4023	0.1217	0.429
15	900	0.2325	0.4023	0.1217	0.395

Table 4 Comparison of experimental data and the results from the theoretical model in example 2

Test No.	$(\sigma - u_a)$ (kPa)	$(u_a - u_w)$ (kPa)	Test results	Calculated results							
				Calculation model under linear strength theory				Calculation model under nonlinear strength theory			
				Model of Fredlund *et al.* (1978) (i.e., Eq. (1))		Effective stress model (i.e., Eq. (17))		Envelope shell model (i.e., Eq. (5))		Effective stress model (i.e., Eq. (15))	
			τ_f (kPa)	τ_f (kPa)	Diff (%)	τ_f (kPa)	Diff (%)	τ_f (kPa)	Diff (%)	τ_f (kPa)	Diff (%)
1	101.0	0	78.4	78.357	-0.05%	78.343	-0.07%	79.559	1.48%	79.559	1.48%
2	126.0	100	121.9	109.055	-10.54%	132.966	9.08%	114.953	-5.70%	119.969	-1.58%
3	138.3	200	153.8	134.125	-12.79%	177.416	15.36%	145.165	-5.61%	150.786	-1.96%
4	155.8	400	183.9	181.118	-1.51%	239.806	30.40%	185.594	0.92%	191.953	4.38%
5	169.6	500	205.0	206.852	0.90%	262.291	27.95%	204.733	-0.13%	206.324	0.65%
6	181.7	600	222.7	231.834	4.10%	278.837	25.21%	222.378	-0.14%	216.765	-2.67%



(a) Prediction results of linear and nonlinear strength criterions when the net normal stress is constant

(b) Prediction results of linear and nonlinear strength criterions when matric suction is constant

Fig. 6 Comparison of models for shear strength of unsaturated soil under the linear and nonlinear strength criterions under different net normal stress ($\sigma - u_a$) and matric suction ($u_a - u_w$) values. According to the experimental data of the SWCC in Zhou *et al.* (2010) (see Table 3), the curve fitting was carried out on the experimental data under $m_v = (n - 1) / n$, and the parameters in the SWCC model of van Gennchten are $\alpha = 0.002$ kPa^{-1} and $n = 1.766$. In addition, according to Zhou *et al.* (2010), the air entry value is 210 kPa and the residual suction value is 3000 kPa.

Furthermore, based on the test data of shear strength of the unsaturated Jingmen expansive soil from Zhou *et al.* (2010), the model of Fredlund *et al.* (1978) for the shear strength of unsaturated soil (i.e., Eq. (1)) under linear strength theory is used to fit these measured data, and the corresponding strength parameters are obtained. Zhou *et al.* (2010) determined an effective cohesive force $c' = 33.6$ kPa and an effective internal friction angle $\varphi' = 23.9°$ for saturated soil. Fitting the original experimental data with the model of Fredlund *et al.* (1978) for the shear strength of unsaturated soil, φ_b is obtained as 11.1°. Table 4 shows the test results and the predicted results of shear strength under the model of Fredlund *et al.* (1978) for Jingmen expansive soil and their relative error. Table 4 shows that the maximum prediction error for the model of Fredlund *et al.* (1978) is 10.54%, which is mainly because the initial model of Fredlund *et al.* (1978) considers the suction friction angle as a constant. Under linear strength theory (i.e., $m = 1$), the effective stress model for the shear strength of unsaturated soil (i.e., Eq. (17)) is also applied to analyze the accuracy of its prediction results. The effective stress model considers the nonlinear relationship between the suction friction angle and suction and can be solved using the parameters α, n, c', and φ'. Table 4 shows the prediction results of the effective stress model and their relative errors with the test results. Table 4 shows that the maximum prediction error of the effective stress model (i.e., Eq. (17)) will reach 30.40%. Obviously, it is not suitable to use the calculation model under linear strength theory in this case. Thereby, the envelope shell model and the effective stress model for the shear strength of unsaturated soil under nonlinear strength theory are further used in this case. For the effective stress model, the strength parameters obtained by the curve fitting are $c_0 = 42.830$ kPa, $\sigma_t = 90.539$ kPa, and $m = 1.210$. For the envelope shell model, the fitting strength parameters from the effective stress model can also be taken, i.e., $c_0 = 42.830$ kPa, $\sigma_t = 90.539$ kPa, and $m = 1.210$. Meanwhile, to consider the coupling effect of net normal stress and matric suction on the shear strength, the envelope shell model is used to fit the original experimental data set, and other parameters are obtained as $\alpha = 0.005$, $n = 51.417$, and $m_v = 0.005$. Table 4 shows the predicted results of the calculation model under nonlinear strength theory and their relative errors with the test results. Table 4 shows that the predicted results of the effective stress model under nonlinear strength theory are quite close to the test results, and these errors are less than 5%. Moreover, the difference between the predicted results of the envelope shell model under nonlinear strength theory and the test results is also basically less than 7%. Therefore, a better prediction result can be obtained by using the calculation model for the shear strength of unsaturated soil under nonlinear strength theory.

To study the difference between the calculation model for the shear strength of unsaturated soil between linear and nonlinear strength theory, Example 2 is used to compare and analyze the calculation model for the shear strength of unsaturated soil under linear strength theory (i.e., the initial model of Fredlund *et al.* (1978) and the effective stress model under linear strength theory for the shear strength of unsaturated soil) and under nonlinear strength theory (i.e., the effective stress model and envelope shell model). In this research, when the net normal stress ($\sigma - u_a$) = 100 kPa, the predicted results of the shear strength of unsaturated soil are obtained from the two theoretical models of the matric suction ($u_a - u_w$) in the range of 0 kPa to 1500 kPa are plotted in Fig. 6(a). Meanwhile, when the matric suction ($u_a - u_w$) = 100 kPa, the predicted results of the shear strength of unsaturated soil obtained from the two theoretical models with a net normal stress ($\sigma - u_a$) in the range of 0 kPa to 1500 kPa are plotted in Fig. 6(b).

Fig. 6(a) indicates that the model of Fredlund *et al.* (1978) shows that the shear strength of unsaturated soil increases linearly with the increase in the matric suction when the net normal stress is kept in the same level. The calculation model for the shear strength of unsaturated soil under nonlinear strength theory can stimulate the nonlinear effects of the matric suction on the shear strength of unsaturated soil. Fig. 6(b) indicates that, when the matric suction is a constant, the predicted results of the effective stress model under linear strength theory shows a linear increase with the increase in the net normal stress, Conversely, although the predicted results of the shear strength of the two calculation models under nonlinear shear strength also increase with the increase in the net normal stress, there is a clear tendency to slow down. In other words, the prediction results of the present theoretical model under nonlinear strength theory are more conservative, safer and agree with the nonlinear characteristics of the strength of the geotechnical body. This is also the advantage of the present models over the traditional models.

To summarize, the two present calculation models for the shear strength of unsaturated soil under nonlinear strength theory consider not only the nonlinear effects of the matric suction on the shear strength of unsaturated soil but also the nonlinear effects of the net normal stress on the shear strength of unsaturated soil. Moreover, the present models can degenerate into the calculation model for the shear strength of unsaturated soil under linear strength theory and consider the coupling effect of the matric suction and net normal stress on the shear strength of unsaturated soil. Therefore, compared with the traditional model, the present models have certain advantages in predicting the shear strength of unsaturated soil under complex conditions.

5. Conclusions

Based on previous achievement regarding the shear strength of unsaturated soil under the linear strength criterion, this work introduces a nonlinear relationship between the normal stress and shear strength and then establishes an envelope shell model and effective stress model for the shear strength of unsaturated soil under nonlinear strength theory. The feasibility and applicability of the present models are illustrated through the analysis and verification of examples. Meanwhile, studies show that:

• Except for considering the nonlinear effect of matric suction on the shear strength of unsaturated soil, the nonlinear effect of the net normal stress on the shear strength of unsaturated soil is also considered in the present two models. Thus, the present models can reflect that the curve of the shear strength of unsaturated soil obviously slows down with net normal stress increase.

• The present models adopt the appropriate parameters to enhance the flexibility of the calculation formula. Moreover, they take advantage of the M-C strength parameters of saturated soil and fitting parameters in the SWCC of van Genuchten to predict the shear strength of unsaturated soil. Thereby, the present models are simple and convenient to use.

• The present models can preferably simulate the shear strength behavior of fine-grained soil and manifest the peak phenomenon of the shear strength of unsaturated coarse-grained soil.

Acknowledgments

The research described in this paper was financially supported by the National Natural Science Foundation of China (No. 51608541) and the Natural Science Foundation of Hunan Province, China (Grant No. 2019JJ50772).

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