Geomechanics and Engineering, *Vol. 21, No. 1 (2020) 85-93*
DOI: https://doi.org/10.12989/gae.2020.21.1.085

Fractional order thermoelastic wave assessment in a two-dimension medium with voids

Aatef D. Hobiny[1] and Ibrahim A. Abbas*[1,2]

[1]*Nonlinear Analysis and Applied Mathematics Research Group (NAAM), Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia*
[2]*Department of mathematics, Faculty of Science, Sohag University, Sohag, Egypt*

(Received November 10, 2019, Revised February 29, 2020, Accepted March 12, 2020)

Abstract. In this article, the generalized thermoelastic theory with fractional derivative is presented to estimate the variation of temperature, the components of stress, the components of displacement and the changes in volume fraction field in two-dimensional porous media. Easily, the exact solutions in the Laplace domain are obtained. By using Laplace and Fourier transformations with the eigenvalues method, the physical quantities are obtained analytically. The numerical results for all the physical quantities considered are implemented and presented graphically. The results display that the present model with the fractional derivative is reduced to the Lord and Shulman (LS) and the classical dynamical coupled (CT) theories when the fractional parameter is equivalent to one and the delay time is equal to zero and respectively.

Keywords: fractional derivative; Laplace-Fourier transforms; porous medium; eigenvalues approach

1. Introduction

The elastic mediums with voids are basically a porous medium whose skeleton (matrix) is an elastic solid and the interstices are voids (small pores) containing nothing of energetic or physical significance. The theory of linear elastic mediums with voids is a generalization of the classical theory of elasticity. The classical theory of elasticity is not appropriate for the study of various types of geological and biological materials. The theory of linear elastic materials with voids deals with the distribution of small pores or voids in the elastic materials, where the volume of the void is included among the kinematics variables. The theory reduces to the classical theory in the limiting case for which the volume of void tending to zero. The problems of porous fluid-saturated mediums have been investigated for numerous years. To overcome the first shortcoming in the classical uncoupled thermoelastic theory, Biot (1956) established the coupled thermoelastic theory that it provides two phenomena are not consistent with the physical observations. By postulating a new law of heat conduction to replace the classical Fourier law, the generalized thermoelastic theory with one relaxation time has been established by Lord and Shulman (1967). Fractional computing has been used successfully to modify many existing physical process models. In the second half of the nineteenth century, it can be said that the whole theory of fractional derivatives and integrals has been determined. The generalization of the concepts of derivatives and integrals to a non-integer order has been the subject of several methods and various alternative definitions of fractional derivatives have emerged. In the context of generalized thermoelastic theories, Youssef (2010), Youssef and Al-Lehaibi (2010) have presented the generalized fractional-order thermoelastic of low and high thermal conductivities. Based on a Taylor expansion of time-fractional order, Ezzat (2011, Ezzat and El Karamany (2011), Ezzat and El-Karamany (2011) have given other model for a fractional order generalized thermo-elasticity. A new model is presented by using the heat conduction law as Sherief *et al.* (2010). As an important branch of mechanical properties of solid, the literature Abbas and Alzahrani (2016), Abbas and Kumar (2016, Abd-Elaziz *et al.* (2019), Alzahrani and Abbas (2016, Deswal and Kalkal (2014), Ezzat and El-Bary (2017), Hussein (2015), Kakar and Kakar (2014), Lata and Kaur (2019), Othman *et al.* (2013, Othman and Abd-Elaziz (2019, Sarkar and Lahiri (2013, Sur and Kanoria (2014, Wang *et al.* (2015), Youssef (2012) have considered different problems by numerical and analytical methods. Singh Singh (2007) studied the wave propagation in material with voids under the generalized thermoelastic theory. Marin (2009), Marin *et al.* (2017), Marin and Nicaise (2016), Marin *et al.* (2019) have presented some problems to study the elastic dipolar bodies. Karageorghis *et al.* (2014) have investigated the moving pseudo-boundary MFS in two-dimension thermoelastic materials with voids. Many authors Abbas (2007), Abbas (2009), Abbas (2014), Abbas (2014), Abbas *et al.* (2009, Hobiny and Abbas (2018), Kaur and Lata (2020, Lata and Kaur (2019), Lata and Zakhmi (2019), Mohamed *et al.* (2009) have presented the solutions of several problems under generalized thermo-elastic models. Ezzat *et al.* (2016) presented a modeling in generalized thermoelastic theory based on memory-dependent derivative. Sarkar (2017) used

*Corresponding author, Professor
E-mail: aabbas5@kau.edu.sa, ibrabbas7@gmail.com

http://www.techno-press.org/?journal=gae&subpage=7 ISSN: 2005-307X (Print), 2092-6219 (Online)

the time-fractional order two-temperature model to study the wave propagation in an initially stressed elastic half-space solids under magneto-thermoelastic theory. Ellahi *et al.* (2019), Sheikholeslami *et al.* (2019) have present the solutions for various porous medium problems under several boundary conditions. By using the fractional calculus, several actual models of physical procedures have been improved successfully. It can be safely said that all integrations models and fractional derivatives were formed in the last four decades. Otherwise, various methods and definitions of fractional derivatives have become the main object of several studies. In the domain of Laplace, the eigenvalue approaches gave the analytical solutions without any supposed restriction on the factual physical quantities.

The aim of this paper is to study the effects of fractional order derivative in a two-dimension porous medium. By using the Fourier-Laplace transforms and eigenvalues method based on an numerical and analytical methods, the governing equations are processed. For the considered physical quantities, the numerical outcomes are obtained and presented graphically.

2. Mathematical model

For an isotropic, elastic and homogeneous two-dimensional medium contains voids, the governing relations under generalized thermoelastic theory Lord and Shulman (1967), based on Cattaneo (1958) and Ezzat *et al.* (2014) and according to Singh (2007) in the absence of body forces and the heat source can be introduced by:

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}+b\varphi_{,i}-\gamma_t\Theta_{,i}=\rho\frac{\partial^2 u_i}{\partial t^2}, \tag{1}$$

$$(\lambda+\mu)u_{j,ij}+\mu u_{i,jj}+b\varphi_{,i}-\gamma_t\Theta_{,i}=\rho\frac{\partial^2 u_i}{\partial t^2}, \tag{2}$$

$$K\Theta_{,jj}=\left(1+\frac{\tau_o^\beta}{\Gamma(\beta+1)}\frac{\partial^\beta}{\partial t^\beta}\right)\left(\rho c_e\frac{\partial\Theta}{\partial t}+mT_o\frac{\partial\varphi}{\partial t}+\gamma_t T_o\frac{\partial u_{j,j}}{\partial t}\right),\quad 0<\beta\le 1 \tag{3}$$

The stress-displacement equations can be given by

$$\sigma_{ij}=\mu\left(u_{i,j}+u_{j,i}\right)+\left(\lambda u_{k,k}+b\varphi-\gamma_t\Theta\right)\delta_{ij} \tag{4}$$

By taking into consideration the above definition can be expressed by

$$\frac{\partial^\beta g(\boldsymbol{r},t)}{\partial t^\beta}=\begin{cases} g(\boldsymbol{r},t)-g(\boldsymbol{r},0), & \beta\to 0,\\ I^{\alpha-1}\dfrac{\partial g(\boldsymbol{r},t)}{\partial t}, & 0<\beta<1,\\ \dfrac{\partial g(\boldsymbol{r},t)}{\partial t}, & \beta=1,\end{cases} \tag{5}$$

where I^β is the Riemann-Liouville integral fraction introduced as a natural generalization of the well-known integral $I^\beta g(\boldsymbol{r},t)$ that can be written as a convolution type.

$$I^\beta\beta(\boldsymbol{r},t)=\int_0^t\frac{(t-s)^\beta}{\Gamma(\beta)}g(\boldsymbol{r},s)ds\,,\beta>0 \tag{6}$$

where $g(\boldsymbol{r},t)$ is a Lebesgue's integral function and $\Gamma(\beta)$ is the Gamma function. In the case where $g(\boldsymbol{r},t)$ is definitely continuous, then it is possible to write

$$\lim_{\beta\to1}\frac{\partial^\beta g(\boldsymbol{r},t)}{\partial t^\beta}=\frac{\partial g(\boldsymbol{r},t)}{\partial t} \tag{7}$$

This model can be reduced to:

i) (CT) point to the classical coupled model $\tau_o=\beta=0$.

ii) (LS) point to Lord and Shulman's model $\tau_o>0,\beta=1$, where the different values of fractional parameter $0<\beta\le 1$ cover two types of conductivity, $\beta=1$ for normal conductivity and $0<\beta<1$ for low conductivity, $\omega_o,\alpha,b,m,\psi,\zeta_1$ are the material constants due to presence of voids τ_o is the thermal relaxation time, σ_{ij} are the components of stress, ρ is the density of material, $\Theta=T-T_o$, T_o is the reference temperature, u_i are the components of displacement, K is the coefficient of thermal conductivity, λ,μ are the Lame's constants, c_e is the specific heat at constant strain, $\gamma_t=(3\lambda+2\mu)\alpha_t$, α_t is the linear thermal expansion coefficient, t is the time and $i,j,k=1,2,3$. We consider plane waves in the xy-plane therefore in the two-dimensional porous medium, we have

$$\boldsymbol{u}=(u,v,0),u=u(x,y,t),v=v(x,y,t),\Theta=\Theta(x,y,t),\quad \varphi=\varphi(x,y,t). \tag{8}$$

Subsequently, the Eqs. (1)- (4) can be given by:

$$\rho\frac{\partial^2 u}{\partial t^2}=(\lambda+2\mu)\frac{\partial^2 u}{\partial x^2}+(\lambda+\mu)\frac{\partial^2 v}{\partial x\partial y}+\mu\frac{\partial^2 u}{\partial y^2}+b\frac{\partial\varphi}{\partial x}-\gamma_t\frac{\partial\Theta}{\partial x} \tag{9}$$

$$\rho\frac{\partial^2 v}{\partial t^2}=(\lambda+2\mu)\frac{\partial^2 v}{\partial y^2}+(\lambda+\mu)\frac{\partial^2 u}{\partial x\partial y}+\mu\frac{\partial^2 v}{\partial x^2}+b\frac{\partial\varphi}{\partial y}-\gamma_t\frac{\partial\Theta}{\partial y} \tag{10}$$

$$\rho\psi\frac{\partial^2\varphi}{\partial t^2}=\alpha\left(\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}\right)-\omega_o\frac{\partial\varphi}{\partial t}-\zeta_1\varphi-b\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+m\Theta \tag{11}$$

$$K\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}\right)=\left(1+\frac{\tau_o^\beta}{\Gamma(\beta+1)}\frac{\partial^\beta}{\partial t^\beta}\right)\left(\rho c_e\frac{\partial\Theta}{\partial t}+mT_o\frac{\partial\varphi}{\partial t}+\gamma_t T_o\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right) \tag{12}$$

$$\sigma_{xx}=\lambda\frac{\partial v}{\partial y}+(\lambda+2\mu)\frac{\partial u}{\partial x}+b\varphi-\gamma_t\Theta,\sigma_{xy}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \tag{13}$$

3. Application

The problem initial conditions can be taken by

$$\Theta(x,y,0)=\frac{\partial\Theta(x,y,0)}{\partial t}=0,\varphi(x,y,0)=\frac{\partial\varphi(x,y,0)}{\partial t}=0,u(x,y,0)=\frac{\partial u(x,y,0)}{\partial t}=0,v(x,y,0)=\frac{\partial v(x,y,0)}{\partial t}=0. \tag{14}$$

While, the appropriate boundary conditions can be expressed as

$$\sigma_{xx}(0,y,t)=\sigma_{xy}(0,y,t)=0.0,-K\frac{\partial\Theta(0,y,t)}{\partial x}=q_o\frac{t^2e^{-\frac{t}{t_p}}}{16t_p^2}H(a-|y|),\frac{\partial\varphi(0,y,t)}{\partial x}=0 \tag{15}$$

where q_o is a constant H is the unit Heaviside function and t_p is the pulse heat flux characteristic time. For appropriateness, the dimensionless physical quantities can be defined by

$$\Theta' = \frac{\Theta}{T_o}, (\sigma'_{xx}, \sigma'_{xy}) = \frac{(\sigma_{xx}, \sigma_{xy})}{(\lambda+2\mu)}, (t', \tau'_o) = \eta c^2 (t, \tau_o), (x', y', u', v') = \eta c (x, y, u, v), \varphi' = \psi \eta^2 c^2 \varphi \tag{16}$$

where $\eta = \frac{\rho c_e}{k}$ and $c = \sqrt{\frac{\lambda+2\mu}{\rho}}$.

In these dimensionless parameters terms of the in Eq. (16), the above Eqs. (9)-(15) can be given by (the dashes have been neglected for convenience)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + (1-s_1)\frac{\partial^2 v}{\partial x \partial y} + s_1 \frac{\partial^2 u}{\partial y^2} + s_2 \frac{\partial \varphi}{\partial x} - s_3 \frac{\partial \Theta}{\partial x} \tag{17}$$

$$\frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial y^2} + (1-s_1)\frac{\partial^2 u}{\partial x \partial y} + s_1 \frac{\partial^2 v}{\partial x^2} + s_2 \frac{\partial \varphi}{\partial y} - s_3 \frac{\partial \Theta}{\partial y} \tag{18}$$

$$s_4 \frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} - s_5 \frac{\partial \varphi}{\partial t} - s_6 \varphi - s_7 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + s_8 \Theta \tag{19}$$

$$\frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2} = \left(1 + \frac{\tau_o^\beta}{\Gamma(\beta+1)} \frac{\partial^\beta}{\partial t^\beta}\right)\left(\frac{\partial \Theta}{\partial t} + s_9 \frac{\partial \varphi}{\partial t} + s_{10} \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right) \tag{20}$$

$$\sigma_{xx} = (1-2s_1)\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} + s_2 \varphi - s_3 \Theta, \sigma_{xy} = s_1 \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{21}$$

$$\sigma_{xx}(0,y,t) = \sigma_{xy}(0,y,t) = 0.0, \frac{\partial \Theta(0,y,t)}{\partial x} = -q_o \frac{t^2 e^{-\frac{t}{t_p}}}{16 t_p^2} H(a - |y|), \frac{\partial \varphi(0,y,t)}{\partial x} = 0 \tag{22}$$

where

$$s_1 = \frac{\mu}{\rho c^2}, \quad s_2 = \frac{b}{\rho \psi \eta^2 c^4}, \quad s_3 = \frac{\gamma_t T_o}{\rho c^2}, s_4 = \frac{\rho c^2 \psi}{\alpha}, s_5 = \frac{\omega_o}{\alpha \eta}, s_6 = \frac{\zeta_1}{\alpha \eta^2 c^2}, s_7 = \frac{b\psi}{\alpha}, s_8 = \frac{m\psi T_o}{\alpha}, s_9 = \frac{m}{\psi \eta^3 c^2 k}, s_{10} = \frac{\gamma_t}{\rho c_e}.$$

Now, the Laplace transforms for any function $M(x,y,t)$, are expressed as in Debnath and Bhatta (2014)

$$\bar{M}(x,y,s) = \int_0^\infty M(x,y,t) e^{-st} dt, s > 0, \tag{23}$$

while, the Fourier transforms for any function $\bar{M}(x,y,s)$ are given as

$$\bar{M}^*(x,q,s) = \int_{-\infty}^\infty \bar{M}(x,y,s) e^{-iqy} dx. \tag{24}$$

Thus, the governing equations under initial conditions and using the boundary conditions are given to get the system of ordinary differential equations by

$$\frac{d^2 \bar{u}^*}{dx^2} = (s^2 + s_1 q^2)\bar{u}^* - iq(1-s_1)\frac{d\bar{v}^*}{dx} - s_2 \frac{d\bar{\varphi}^*}{dx} + s_3 \frac{d\bar{\Theta}^*}{dx} \tag{25}$$

$$\frac{d^2 \bar{v}^*}{dx^2} = \frac{(s^2+q^2)}{s_1}\bar{v}^* - \frac{s_2 iq}{s_1}\bar{\varphi}^* + \frac{s_3 iq}{s_1}\bar{\Theta}^* - \frac{iq(1-s_1)}{s_1}\frac{d\bar{u}^*}{dx}, \tag{26}$$

$$\frac{d^2 \bar{\varphi}^*}{dx^2} = s_7 iq \bar{v}^* + (s^2 s_4 + q^2 + s_6 + s_5 s)\bar{\varphi}^* - s_8 \bar{\Theta}^* + s_7 \frac{d\bar{u}^*}{dx} \tag{27}$$

$$\frac{d^2 \bar{\Theta}^*}{dx^2} = s_{10} iqs\gamma \bar{v}^* + s_9 s\gamma \bar{\varphi}^* + (q^2 + s\gamma)\bar{\Theta}^* + s\gamma s_{10} \frac{d\bar{u}^*}{dx} \tag{28}$$

$$\bar{\sigma}_{xx}^* = iq(1-2s_1)\bar{v}^* + \frac{d\bar{u}^*}{dx} + s_2 \bar{\varphi}^* - s_3 \bar{\Theta}^* \tag{29}$$

$$\bar{\sigma}_{xy}^* = s_1 \left(\frac{d\bar{v}^*}{dx} + iq\bar{u}^*\right), \tag{30}$$

$$\bar{\sigma}_{xx}^* = \bar{\sigma}_{xy}^* = 0, \frac{d\bar{\varphi}^*}{dx} = 0, \frac{d\ \bar{\Theta}^*}{dx} = -\frac{q_o t_p}{8(st_p+1)^3}\sqrt{\frac{2}{\pi}}\frac{\sin(qa)}{q} \tag{31}$$

where $\gamma = 1 + \frac{s^\beta \tau_o^\beta}{\Gamma(\beta+1)}$. The differential equations vector-matrix of Eqs. (25-28) can be written by

$$\frac{d\mathrm{N}}{dx} = A\mathrm{N} \tag{32}$$

where $\mathrm{N} = \left[\bar{u}^* \quad \bar{v}^* \quad \bar{\varphi}^* \quad \bar{\Theta}^* \quad \frac{d\bar{u}^*}{dx} \quad \frac{d\bar{v}^*}{dx} \quad \frac{d\bar{\varphi}^*}{dx} \quad \frac{d\bar{\Theta}^*}{dx}\right]^T$ and the matrix $A = [a_{ij}] = 0, i,j = 1 \dots 8$.

Excepting $a_{15} = a_{26} = a_{37} = a_{48} = 1$,

$a_{51} = s^2 + s_1 q^2, a_{56} = -iq(1-s_1), a_{57} = -s_2, a_{58} = s_3, a_{62} = \frac{(s^2+q^2)}{s_1}, a_{63} = -\frac{s_2 iq}{s_1}$,

$a_{64} = \frac{s_3 iq}{s_1}, a_{65} = -\frac{iq(1-s_1)}{s_1}$, $a_{72} = s_7 iq, a_{73} = s^2 s_4 + q^2 + s_6 + s_5 s, a_{74} = -s_8, a_{75} = s_7$,

$a_{82} = s_{10} iqs\gamma, a_{83} = s_9 s\gamma,\ a_{84} = q^2 + s\gamma, a_{85} = s\gamma s_{10}$.

By using the eigenvalues approach which proposition by Abbas (2015), Abbas *et al.* (2019), Das *et al.* (1997), the analytical solutions of Eq. (32) can be given. thereafter, the matrix A has the characteristic equation can be written as:

$$\xi^8 - f_4 \xi^6 + f_3 \xi^4 + f_2 \xi^2 + f_1 = 0 \tag{33}$$

where f_1, f_2, f_3 and f_4 can be defined as in appendix A. To get the solutions of Eq. (29), the eigenvalue and its eigenvector of matrix A should be calculated. In the cases $\pm\xi_1, \pm\xi_2, \pm\xi_3, \pm\xi_4$ are the eigenvalue, the corresponding eigenvectors of eigenvalues ξ can be considered as in appendix A. Hence, the exact solutions of Eq. (29) can be expressed as:

$$N(x,q,s) = \sum_{i=1}^{4} B_i Y_i e^{-\xi_i x} \tag{34}$$

Thus, the general solutions of the field variables can be given as functions of q, s and t as:

$$\bar{\Theta}^*(x,q,p) = \sum_{i=1}^{4} B_i T_i e^{-\xi_i x} \tag{35}$$

$$\bar{u}^*(x,q,p) = \sum_{i=1}^{4} B_i u_i e^{-\xi_i x} \tag{36}$$

$$\bar{v}^*(x,q,p) = \sum_{i=1}^{4} B_i v_i e^{-\xi_i x} \tag{37}$$

$$\bar{\varphi}^*(x,q,p) = \sum_{i=1}^{4} B_i \varphi_i e^{-\xi_i x} \tag{38}$$

$$\bar{\sigma}_{xx}^{*}(x,q,p)=\sum_{i=1}^{4}B_i(-\xi_i u_i+iq(1-2s_1)v_i+s_2\varphi_i-s_3T_i)e^{-\xi_i x} \tag{39}$$

$$\bar{\sigma}_{xy}^{*}(x,q,p)=\sum_{i=1}^{4}B_i s_1(-\xi_i v_i+iqu_i)e^{-\xi_i x} \tag{40}$$

where B_1, B_2, B_3 and B_4 are constants which can be calculated by using the problem boundary conditions such that the terms containing exponentials of increasing nature in the spatial variable x have been discarded due to the regularity conditions of the solutions at infinity, while u_i, v_i, T_i and φ_i are the components of corresponding eigenvector. Now, for any function $\bar{M}^{*}(x,q,s)$, Fourier transforms inverse can be written as

$$\bar{M}(x,y,s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{M}^{*}(x,q,s)e^{iqy}dq \tag{41}$$

Finally, to get the general solutions of the displacements components, temperature increment, the stress components and the change in volume fraction field of voids distributions along the distances x, y at any time t, Stehfest Stehfest (1970) numerical inversion approach has been chosen. In this approach, the Laplace transforms inverse for $\bar{\mathrm{M}}(x,y,s)$ are approximated by

$$M(x,y,t)=\frac{ln(2)}{t}\sum_{n=1}^{N}V_n\bar{M}\left(x,y,n\frac{ln(2)}{t}\right) \tag{42}$$

where

$$V_n=(-1)^{\left(\frac{N}{2}+1\right)}\sum_{p=\frac{n+1}{2}}^{min\left(n,\frac{N}{2}\right)}\frac{(2p)!\,p^{\left(\frac{N}{2}+1\right)}}{p!\,(n-p)!\left(\frac{N}{2}-p\right)!\,(2n-1)!} \tag{43}$$

where N is the term numbers.

4. Results and discussion

For numerical computations, magnesium medium was chosen for purposes of numerical calculations. The values of parameters can be takin as in Othman and Marin (2017)

$$\zeta_1=1.475\times10^{10}(N)(m^{-2}),\alpha=3.688\times10^{-5}(N),\omega_o=0.0787\times10^{-3}(N)(m^{-2})(s^{-1}),$$

$$\alpha_t=1.98\times10^{-6}(k^{-1}),K=1.7\ (W)(m^{-1})(k^{-1}),\beta=2.68\times10^{6}(N)(m^{-2})(k^{-1}),$$

$$T_o=298(k),c_e=1040\ (J)(kg^{-1})(k^{-1}),\psi=1.753\times10^{-15}(m^{2}),a=0.25,$$

$$\mu=3.278\times10^{10}(N)(m^{-2}),\lambda=2.17\times10^{10}(N)(m^{-2}),\rho=1740(kg)(m^{-3}),$$

$$m=2\times10^{6}(N)(m^{-2})(k^{-1}),b=1.13840\times10^{10}(N)(m^{-2}),t=0.6.$$

The above data have been applied to study the fractional order thermoelastic wave assessment in a two-dimensional porous material by the eigenvalue approach. The change in volume fraction field of voids distribution φ, the displacement components u, v, the stress components σ_{xx}, σ_{xy} and the variations of temperature Θ, are studied. The material is considered to be an isotropic and homogeneous

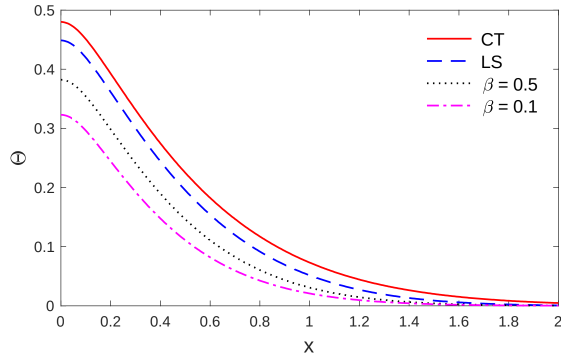

Fig. 1 The variations of temperature Θ versus x when $y = 0.5$ for different values of the fractional-order parameter β

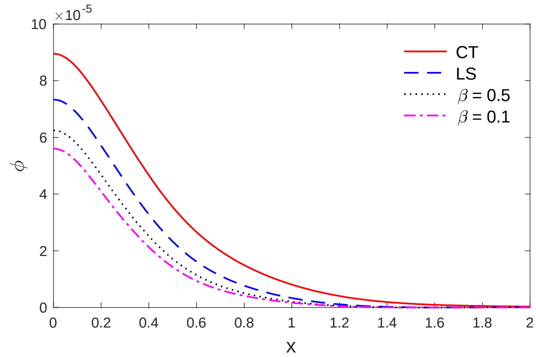

Fig. 2 The changes in volume fraction field of voids distributions φ versus x when $y = 0.5$ for different values of the fractional-order parameter β

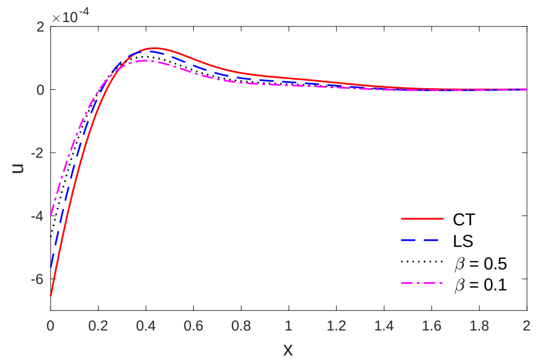

Fig. 3 The variations of horizontal displacement u versus x when $y = 0.5$ for different values of the fractional-order parameter β

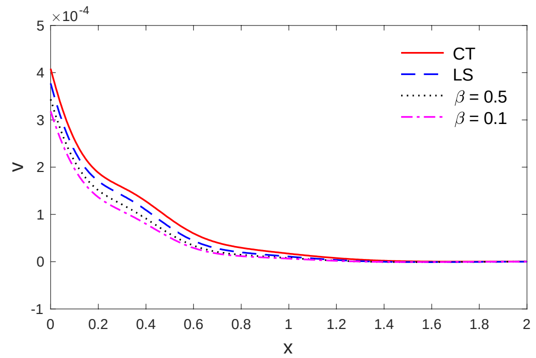

Fig. 4 The variations of vertical displacement v versus x when $y = 0.5$ for different values of the fractional-order parameter β

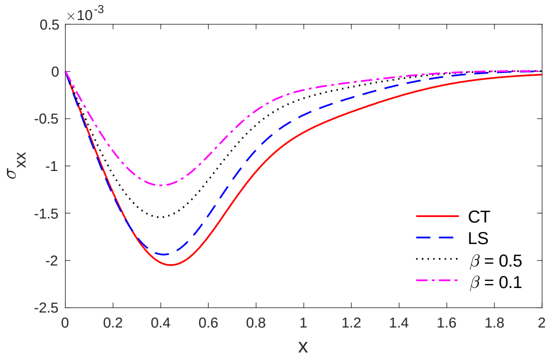

Fig. 5 The variations of stress σ_{xx} versus x when y= 0.5 for different values of the fractional-order parameter β

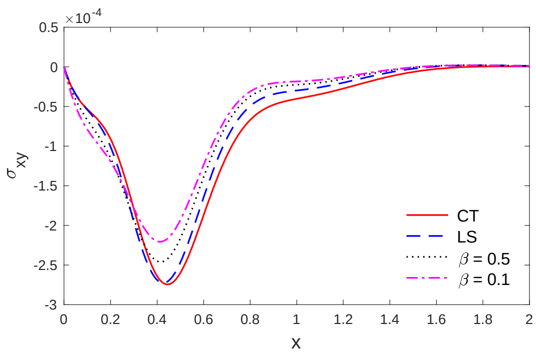

Fig. 6 The variations of stress σ_{xy} versus x when y= 0.5 for different values of the fractional-order parameter β

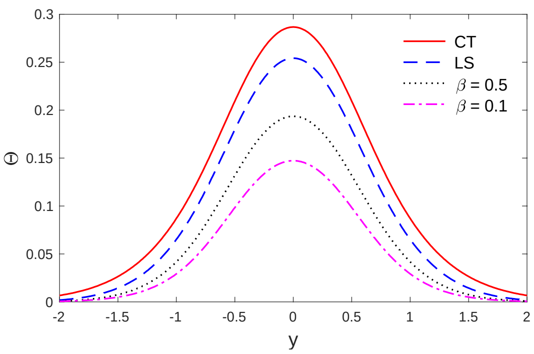

Fig. 7 The variations of temperature Θ versus y when x = 0.5 for different values of the fractional-order parameter β

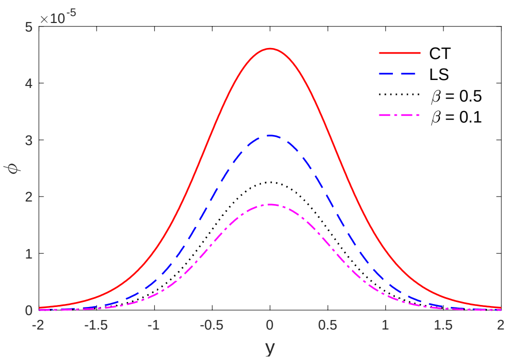

Fig. 8 The changes in volume fraction field of voids distributions φ versus y when x = 0.5 for different values of the fractional-order parameter β

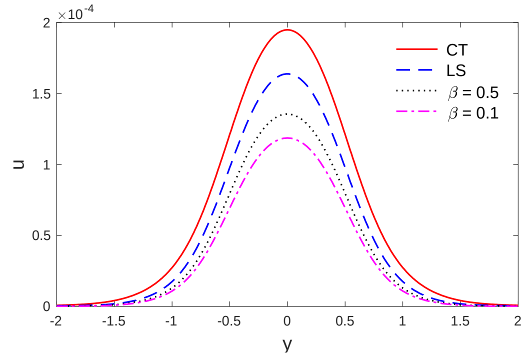

Fig. 9 The variations of horizontal displacement u versus y when x = 0.5 for different values of the fractional-order parameter β

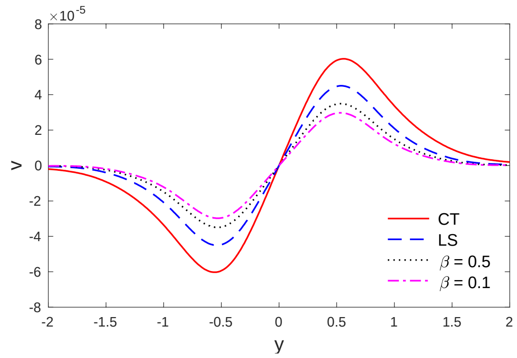

Fig. 10 The variations of vertical displacement v versus y when x= 0.5 for different values of the fractional-order parameter β

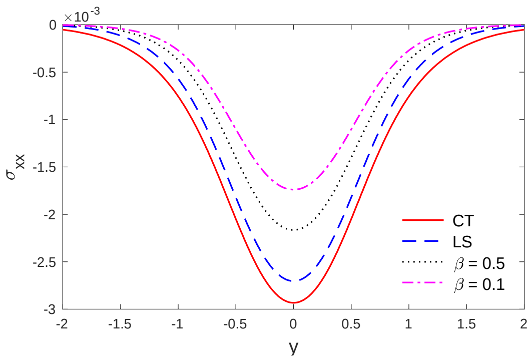

Fig. 11 The variations of stress σ_{xx} versus y when x= 0.5 for different values of the fractional-order parameter β

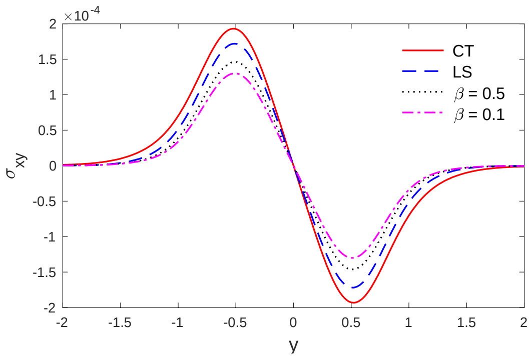

Fig. 12 The variations of stress σ_{xy} versus y when x= 0.5 for different values of the fractional-order parameter β

two-dimensional medium. Fig. 1 shows the variations of temperature with respect to the distance x. It is observed that it begins from maximum value according to the application of boundary condition and decreases with the increasing x to reach to zero. Fig. 2 depicts the change in volume fraction field of voids distributions φ with respect to the distances x. It is noticed that it decreases with the increasing x till attaining zero. Fig. 3 predicts the variations of horizontal displacement u versus x. It is observed that it attains maximum negative values and gradually increases until it attains peak values and then decreases to close to zero. Fig. 4 displays the variations of vertical displacement versus x which have maximums values on $x = 0$ and decreases with the increasing x. Figs. 5 and 6 predict the stress components variations σ_{xx} and σ_{xy} along x. It is observed that the stress magnitudes, always started from zero which satisfied the boundary conditions. Figs. 7 and 8 show the variation of temperature Θ and the variation of change in volume fraction field of voids φ versus the distance y when $x = 0.5$. It is observed that they have maximum values at the length of thermal surface $\left((|y| \leq 0.5)\right)$ then start to reduce just near the edge $\left((|y| \leq 0.5)\right)$ where they decrease and finally close to zero values. Figure 9 display the variation of horizontal displacement u versus the distance y. It is observed that it has maximum value at the length of the heating surface $(|y| \leq 0.5)$, and it starts to reduce just near the edge$(y = \pm 0.5)$, and after that decreases to zero values. Fig. 10 shows the variation of vertical displacement v versus the distance y. It indicates that the vertical displacement starts raising at the beginning and ending of the thermal surface $(|y| \leq 0.5)$, and has small values at the middle of the thermal surface, then it starts increasing and come to maximum values just near the edge $(y = \pm 0.5)$, after that it decreases to reach to zero. The components of stress σ_{xx} and σ_{xy} versus y are shown in Figs. 11 and 12.

Finally, Figs. 2 to 12 show the variations of all physical quantities with respect to the distance $x$and the distance$y$ at $t = 0.5$. These figures display the predict curves under various theories of thermo-elasticity. In these figures, the classical dynamically coupled theory (CT) appeared in solid lines, the Lord and Shulman (LS) theory appeared in dashed lines while the other curves that refer to the generalized thermoelastic theory under fractional time derivative $(\beta = 0.5, \beta = 0.1)$. As expected, it can be found that the fractional derivative parameter β have the major impact on the values of all the studied fields.

5. Conclusions

Based on the generalized thermoelastic theory with fractional derivative, the variations of temperature, the components of displacement, the components of stress and the changes in volume fraction field in a two-dimension porous medium are studied. The non-dimensional resulting has been solved employing the and Fourier and Laplace transforms technique and has been solved using the eigenvalue method. The great effects of the fractional derivative parameter are discussed for all physical quantities.

Acknowledgments

This work was supported by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under grant No (DF-182-130-1441). The authors, therefore, gratefully acknowledge the DSR technical and financial support.

References

Abbas, I.A. (2007), “Finite element analysis of the thermoelastic interactions in an unbounded body with a cavity”, *Forschung im Ingenieurwesen*, **71**(3-4), 215-222. https://doi.org/10.1007/s10010-007-0060-x.

Abbas, I.A. (2009), “Generalized magneto-thermoelasticity in a nonhomogeneous isotropic hollow cylinder using the finite element method”, *Arch. Appl. Mech.*, **79**(1), 41-50. https://doi.org/10.1007/s00419-008-0206-9.

Abbas, I.A. (2014), “Nonlinear transient thermal stress analysis of thick-walled FGM cylinder with temperature-dependent material properties”, *Meccanica*, **49**(7), 1697-1708.

Abbas, I.A. (2014), “The effects of relaxation times and a moving heat source on a two-temperature generalized thermoelastic thin slim strip”, *Can. J. Phys.*, **93**(5), 585-590. https://doi.org/10.1139/cjp-2014-0387.

Abbas, I.A. (2015), “The effects of relaxation times and a moving heat source on a two-temperature generalized thermoelastic thin slim strip”, *Can. J. Phys.*, **93**(5), 585-590. https://doi.org/10.1139/cjp-2014-0387.

Abbas, I.A. and Alzahrani, F.S. (2016), “Analytical solution of a two-dimensional thermoelastic problem subjected to laser pulse”, *Steel Compos. Struct.*, **21**(4), 791-803. https://doi.org/10.12989/scs.2016.21.4.791.

Abbas, I.A. and Kumar, R. (2016), “2D deformation in initially stressed thermoelastic half-space with voids”, *Steel Compos. Struct.*, **20**(5), 1103-1117. https://doi.org/10.12989/scs.2016.20.5.1103.

Abbas, I.A., Alzahrani, F.S. and Elaiw, A. (2019), “A DPL model of photothermal interaction in a semiconductor material”, *Waves Random Complex Media*, **29**(2), 328-343. https://doi.org/10.1080/17455030.2018.1433901.

Abbas, I.A., El-Amin, M. and Salama, A. (2009), “Effect of thermal dispersion on free convection in a fluid saturated porous medium”, *Int. J. Heat Fluid Flow*, **30**(2), 229-236. https://doi.org/10.1016/j.ijheatfluidflow.2009.01.004.

Abd-Elaziz, E.M., Marin, M. and Othman, M.I. (2019), “On the effect of Thomson and initial stress in a thermo-porous elastic solid under GN electromagnetic theory”, *Symmetry*, **11**(3), 413. https://doi.org/10.3390/sym11030413.

Alzahrani, F.S. and Abbas, I.A. (2016), “The effect of magnetic field on a thermoelastic fiber-reinforced material under GN-III theory”, *Steel Compos. Struct.*, **22**(2), 369-386. https://doi.org/10.12989/scs.2016.22.2.369.

Biot, M.A. (1956), “Thermoelasticity and irreversible thermodynamics”, *J. Appl. Phys.*, **27**(3), 240-253. https://doi.org/10.1063/1.1722351.

Cattaneo, C. (1958), “A form of heat conduction equation which eliminates the paradox of instantaneous propagation”, *Compte Rendus*, **247**(4), 431-433.

Das, N.C., Lahiri, A. and Giri, R.R. (1997), “Eigenvalue approach to generalized thermoelasticity”, *Indian J. Pure Appl. Math.*,

28(12), 1573-1594.
Debnath, L. and Bhatta, D. (2014), *Integral Transforms and Their Applications*, Chapman and Hall, CRC.
Deswal, S. and Kalkal, K.K. (2014), "Plane waves in a fractional order micropolar magneto-thermoelastic half-space", *Wave Motion*, **51**(1), 100-113. https://doi.org/10.1016/j.wavemoti.2013.06.009.
Ellahi, R., Sait, S.M., Shehzad, N. and Ayaz, Z. (2019), "A hybrid investigation on numerical and analytical solutions of electro-magnetohydrodynamics flow of nanofluid through porous media with entropy generation", *Int. J. Numer. Meth. Heat Fluid Flow*, **30**(2), 834-854. https://doi.org/10.1108/HFF-06-2019-0506.
Ezzat, M., El-Karamany, A. and El-Bary, A. (2016), "Modeling of memory-dependent derivative in generalized thermoelasticity", *Eur. Phys. J. Plus*, **131**(10), 372. https://doi.org/10.1140/epjp/i2016-16372-3.
Ezzat, M.A. (2011), "Theory of fractional order in generalized thermoelectric MHD", *Appl. Math. Modell*., **35**(10), 4965-4978. https://doi.org/10.1016/j.apm.2011.04.004.
Ezzat, M.A. and El-Bary, A.A. (2017), "A functionally graded magneto-thermoelastic half space with memory-dependent derivatives heat transfer", *Steel Compos. Struct*., **25**(2), 177-186. https://doi.org/10.12989/scs.2017.25.2.177.
Ezzat, M.A. and El-Karamany, A.S. (2011), "Fractional order theory of a perfect conducting thermoelastic medium", *Can. J. Phys*., **89**(3), 311-318. https://doi.org/10.1139/P11-022.
Ezzat, M.A. and El Karamany, A.S. (2011), "Theory of fractional order in electro-thermoelasticity", *Eur. J. Mech. A/Solids*, **30**(4), 491-500. https://doi.org/10.1016/j.euromechsol.2011.02.004.
Ezzat, M.A., AlSowayan, N.S., Al-Muhiameed, Z.I. and Ezzat, S.M. (2014), "Fractional modelling of Pennes' bioheat transfer equation", *Heat Mass Transfer*, **50**(7), 907-914. https://doi.org/10.1007/s00231-014-1300-x.
Hobiny, A.D. and Abbas, I.A. (2018), "Fractional order photo-thermo-elastic waves in a two-dimensional semiconductor plate", *Eur. Phys. J. Plus*, **133**(6), 232. https://doi.org/10.1140/epjp/i2018-12054-6.
Hussein, E.M. (2015), "Fractional order thermoelastic problem for an infinitely long solid circular cylinder", *J. Therm. Stresses*, **38**(2), 133-145. https://doi.org/10.1080/01495739.2014.936253.
Kakar, R. and Kakar, S. (2014), "Electro-magneto-thermoelastic surface waves in non-homogeneous orthotropic granular half space", *Geomech. Eng*., **7**(1), 1-36. https://doi.org/10.12989/gae.2014.7.1.001.
Karageorghis, A., Lesnic, D. and Marin, L. (2014), "A moving pseudo-boundary MFS for void detection in two-dimensional thermoelasticity", *Int. J. Mech. Sci*., **88**, 276-288. https://doi.org/10.1016/j.ijmecsci.2014.05.015.
Kaur, I. and Lata, P. (2020), "Stoneley wave propagation in transversely isotropic thermoelastic medium with two temperature and rotation", *GEM-Int. J. Geomath*., **11**(1), 4. https://doi.org/10.1007/s13137-020-0140-8.
Lata, P. and Kaur, H. (2019), "Deformation in transversely isotropic thermoelastic medium using new modified couple stress theory in frequency domain", *Geomech. Eng*., **19**(5), 369-381. https://doi.org/10.12989/gae.2019.19.5.369.
Lata, P. and Kaur, I. (2019), "Effect of time harmonic sources on transversely isotropic thermoelastic thin circular plate", *Geomech. Eng*., **19**(1), 29-36. https://doi.org/10.12989/gae.2019.19.1.029.
Lata, P. and Zakhmi, H. (2019), "Fractional order generalized thermoelastic study in orthotropic medium of type GN-III", *Geomech. Eng*., **19**(4), 295-305. https://doi.org/10.12989/gae.2019.19.4.295.
Lord, H.W. and Shulman, Y. (1967), "A generalized dynamical theory of thermoelasticity", *J. Mech. Phys. Solids*, **15**(5), 299-309. https://doi.org/10.1016/0022-5096(67)90024-5.
Marin, M. (2009), "On the minimum principle for dipolar materials with stretch", *Nonlin. Anal. Real World Appl*., **10**(3), 1572-1578. https://doi.org/10.1016/j.nonrwa.2008.02.001.
Marin, M. and Nicaise, S. (2016), "Existence and stability results for thermoelastic dipolar bodies with double porosity", *Contin. Mech. Thermodyn*., **28**(6), 1645-1657. https://doi.org/10.1007/s00161-016-0503-4.
Marin, M., Ellahi, R. and Chirilă, A. (2017), "On solutions of Saint-Venant's problem for elastic dipolar bodies with voids", *Carpathian J. Math*., **33**(2), 219-232.
Marin, M., Vlase, S., Ellahi, R. and Bhatti, M. (2019), "On the partition of energies for the backward in time problem of thermoelastic materials with a dipolar structure", *Symmetry*, **11**(7), 863. https://doi.org/10.3390/sym11070863.
Mohamed, R., Abbas, I.A. and Abo-Dahab, S. (2009), "Finite element analysis of hydromagnetic flow and heat transfer of a heat generation fluid over a surface embedded in a non-Darcian porous medium in the presence of chemical reaction", *Commun. Nonlin. Sci. Numer. Simul*., **14**(4), 1385-1395. https://doi.org/10.1016/j.cnsns.2008.04.006.
Othman, M., Sarkar, N. and Atwa, S.Y. (2013), "Effect of fractional parameter on plane waves of generalized magneto-thermoelastic diffusion with reference temperature-dependent elastic medium", *Comput. Math. Appl*., **65**(7), 1103-1118. https://doi.org/10.1016/j.camwa.2013.01.047.
Othman, M.I. and Abd-Elaziz, E.M. (2019), "Influence of gravity and micro-temperatures on the thermoelastic porous medium under three theories", *Int. J. Numer. Meth. Heat Fluid Flow*, **29**(9), 3242-3262. https://doi.org/10.1108/HFF-12-2018-0763.
Othman, M.I. and Marin, M. (2017), "Effect of thermal loading due to laser pulse on thermoelastic porous medium under GN theory", *Results Phys*., **7**, 3863-3872. https://doi.org/10.1016/j.rinp.2017.10.012.
Sarkar, N. (2017), "Wave propagation in an initially stressed elastic half-space solids under time-fractional order two-temperature magneto-thermoelasticity", *Eur. Phys. J. Plus*, **132**(4), 154. https://doi.org/10.1140/epjp/i2017-11426-8.
Sarkar, N. and Lahiri, A. (2013), "The effect of fractional parameter on a perfect conducting elastic half-space in generalized magneto-thermoelasticity", *Meccanica*, **48**(1), 231-245. https://doi.org/10.1007/s11012-012-9597-3.
Sheikholeslami, M., Ellahi, R., Shafee, A. and Li, Z. (2019), "Numerical investigation for second law analysis of ferrofluid inside a porous semi annulus: An application of entropy generation and exergy loss", *Int. J. Numer. Meth. Heat Fluid Flow*, **29**(3), 1079-1102. https://doi.org/10.1108/HFF-10-2018-0606.
Sherief, H.H., El-Sayed, A.M.A. and Abd El-Latief, A.M. (2010), "Fractional order theory of thermoelasticity", *Int. J. Solids Struct*., **47**(2), 269-275. https://doi.org/10.1016/j.ijsolstr.2009.09.034.
Singh, B. (2007), "Wave propagation in a generalized thermoelastic material with voids", *Appl. Math. Comput*., **189**(1), 698-709. https://doi.org/10.1016/j.amc.2006.11.123.
Stehfest, H. (1970), "Algorithm 368: Numerical inversion of Laplace transforms [D5]", *Commun. ACM*, **13**(1), 47-49. https://doi.org/10.1145/361953.361969.
Sur, A. and Kanoria, M. (2014), "Fractional order generalized thermoelastic functionally graded solid with variable material properties", *J. Solid Mech*., **6**(1), 54-69.
Wang, Y., Liu, D. and Wang, Q. (2015), "Effect of fractional order parameter on thermoelastic behaviors in infinite elastic medium with a cylindrical cavity", *Acta Mechanica Solida Sinica*, **28**(3), 285-293. https://doi.org/10.1016/S0894-9166(15)30015-X.
Youssef, H.M. (2010), "Theory of fractional order generalized

thermoelasticity", *J. Heat Transfer*, **132**(6), 1-7. https://doi.org/10.1115/1.4000705.

Youssef, H.M. (2012), "Two-dimensional thermal shock problem of fractional order generalized thermoelasticity", *Acta Mechanica*, **223**(6), 1219-1231. https://doi.org/10.1007/s00707-012-0627-y.

Youssef, H.M. and Al-Lehaibi, E.A. (2010), "Variational principle of fractional order generalized thermoelasticity", *Appl. Math. Lett.*, **23**(10), 1183-1187. https://doi.org/10.1016/j.aml.2010.05.008.

IC

Appendix A

$$f_1 = a_{51}a_{64}a_{72}a_{83} + a_{51}a_{62}a_{73}a_{84} - a_{51}a_{64}a_{73}a_{82} + a_{51}a_{63}a_{74}a_{82} - a_{51}a_{62}a_{74}a_{83} - a_{51}a_{63}a_{72}a_{84}.$$

$$f_2 = a_{56}a_{64}a_{73}a_{85} + a_{56}a_{65}a_{74}a_{83} + a_{57}a_{62}a_{74}a_{85} - a_{56}a_{63}a_{74}a_{85} + a_{51}a_{63}a_{72} - a_{51}a_{73}a_{84} - a_{62}a_{73}a_{84} - a_{56}a_{65}a_{73}a_{84} - a_{63}a_{74}a_{82} - a_{57}a_{62}a_{75}a_{84} + a_{58}a_{62}a_{75}a_{83} - a_{56}a_{64}a_{75}a_{83} - a_{51}a_{62}a_{84} + a_{63}a_{72}a_{84} - a_{51}a_{62}a_{73} - a_{58}a_{63}a_{75}a_{82} + a_{57}a_{64}a_{75}a_{82} - a_{58}a_{65}a_{72}a_{83} + a_{51}a_{64}a_{82} + a_{64}a_{73}a_{82} + a_{58}a_{65}a_{73}a_{82} + a_{51}a_{74}a_{83} + a_{62}a_{74}a_{83} - a_{57}a_{65}a_{74}a_{82} - a_{64}a_{72}a_{83} + a_{57}a_{65}a_{72}a_{84} + a_{56}a_{63}a_{75}a_{84} + a_{58}a_{63}a_{72}a_{85} - a_{57}a_{64}a_{72}a_{85} - a_{58}a_{62}a_{73}a_{85},$$

$$f_3 = a_{51}a_{73} - a_{64}a_{82} - a_{58}a_{65}a_{82} - a_{74}a_{83} - a_{58}a_{75}a_{83} + a_{58}a_{62}a_{85} - a_{56}a_{64}a_{85} + a_{58}a_{73}a_{85} - a_{57}a_{74}a_{85} + a_{51}a_{84} + a_{62}a_{84} + a_{56}a_{65}a_{84} + a_{73}a_{84} + a_{57}a_{75}a_{84} + a_{62}a_{73} + a_{56}a_{65}a_{73} + a_{57}a_{62}a_{75} - a_{56}a_{63}a_{75} + a_{51}a_{62} - a_{63}a_{72} - a_{57}a_{65}a_{72},$$

$$f_4 = a_{51} + a_{62} + a_{65}a_{56} + a_{75}a_{57} + a_{84} + a_{85}a_{58} + a_{73}.$$

Appendix B

$$Y_5 = Y_1\xi, Y_6 = Y_2\xi, Y_7 = Y_3\xi, Y_8 = Y_4\xi,$$

$$Y_1 = -(\xi a_{58}(a_{62} - \xi^2) - \xi a_{56}a_{64})(\xi(a_{73} - \xi^2)a_{58} - a_{57}a_{74}\xi) + \xi^2(a_{58}a_{63} - a_{57}a_{64})(a_{58}a_{72} - a_{56}a_{74}),$$

$$Y_2 = -\xi(a_{63}a_{58} - a_{64}a_{57})(a_{74}(\xi^2 - a_{51}) + \xi^2 a_{75}a_{58}) - \xi(a_{64}(\xi^2 - a_{51}) + a_{65}a_{58}\xi^2)\big((\xi^2 - a_{73})a_{58} + a_{74}a_{57}\big),$$

$$Y_3 = a_{64}(\xi^2 a_{56}a_{75} + a_{72}(\xi^2 - a_{51})) + \xi^2 a_{58}(a_{72}a_{65} + a_{75}(\xi^2 - a_{62}))) - a_{58}\xi(((\xi^2 - a_{51})(\xi^2 - a_{62}) - a_{65}a_{56}\xi^2)a_{74},$$

$$Y_4 = -a_{58}\xi\left(\xi^6 + a_{51}\big((\xi^2 - a_{73})a_{62} - \xi^4 + a_{72}a_{63} + a_{73}\xi^2\big) - \xi^2\big((a_{65}a_{57} + a_{63})a_{72} + (a_{73} + a_{75}a_{57})\xi^2 + a_{56}(a_{65}(\xi^2 - a_{73}) + a_{63}a_{75}) + a_{62}(\xi^2 - a_{73} - a_{57}a_{75})\big)\right).$$