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Characteristic equation solution of nonuniform soil deposit: An energy-based mode perturbation method

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Abstract. The mode perturbation method (MPM) is suitable and efficient for solving the eigenvalue problem of a nonuniform soil deposit whose property varies with depth. However, results of the MPM do not always converge to the exact solution, when the variation of soil deposit property is discontinuous. This discontinuity is typical because soil is usually made up of sedimentary layers of different geologic materials. Based on the energy integral of the variational principle, a new mode perturbation method, the energy-based mode perturbation method (EMPM), is proposed to address the convergence of the perturbation solution on the natural frequencies and the corresponding mode shapes and is able to find solution whether the soil properties are continuous or not. First, the variational principle is used to transform the variable coefficient differential equation into an equivalent energy integral equation. Then, the natural mode shapes of the uniform shear beam with same height and boundary conditions are used as Ritz function. The EMPM transforms the energy integral equation into a set of nonlinear algebraic equations which significantly simplifies the eigenvalue solution of the soil layer with variable properties. Finally, the accuracy and convergence of this new method are illustrated with two case study examples. Numerical results show that the EMPM is more accurate and convergent than the MPM. As for the mode shapes of the uniform shear beam included in the EMPM, the additional 8 modes of vibration are sufficient in engineering applications.

Keywords: soil deposit; perturbation method; characteristic equation; variational principle; dynamic characteristics

1. Introduction

A large volume of seismic damage data shows that site conditions play a very important role in the characteristic of the ground motion (Rathje *et al.* 2000, Shiuly *et al.* 2015, Tonyali *et al.* 2019). In the 1906 San Francisco earthquake in the United States, buildings on sedimentary layers near the coastline or on the marsh fills suffered more damage than similar houses on hard or thin soil layers (Yasuhara *et al.* 1982). The peak acceleration recorded on the soft clay field in the 1985 Mechoacan earthquake in Mexico, located in the lake area of Mexico City, was four times the peak acceleration of the hard-field Tacubaya site. Caused severe damage to high-rise and 5-15 story medium-rise buildings in the lake area of Mexico City (Campillo *et al.* 1989). A lot of earthquakes also demonstrated obvious effects that varied by site condition, such as the 1989 Loma Prieta earthquake in the United States, the 2008 Wenchuan earthquake in China, the 2009 L'Aquila earthquake and the 2012 Emilia earthquakes in Italy, the 2011 Tohuku-oki earthquake in Japan, and the 2016 Gyeongju earthquakes in Korea (Borcherdt *et al.* 1992, Bergamaschi *et al.* 2011, Wu *et al.* 2012, Kim *et al.* 2013, Goda *et al.* 2015, Minghini *et al.* 2016, Kim *et al.* 2016).

Because the mechanical properties of a soil layer vary less in the transverse direction than in depth, the ground motion analysis is commonly simplified into one-dimensional soil layer to consider the change of the soil and rock mechanical properties along the depth. The effects of local site conditions on ground motion (Roy and Sahu 2012) can be analyzed by using two categories of methods. One is quantitative analysis and the other is qualitative estimation.

The quantitative analysis method establishes the computational mechanical model of soil layers to calculate the seismic response of soil layers caused by the waves propagating from the underlying bedrocks. Idriss and Seed (1968) proposed a one-dimensional equivalent linearization method in 1968, which is the most common and mature method for nonlinear seismic response in soil layers. Combined with the basic principle of equivalent linearization, the wave method or vibration method can be used to calculate the seismic response of soil layers. Shake (Schnabel *et al.* 1972) and Shake 91 (Idriss *et al.* 1992) calculated the seismic response of soil layers in the

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frequency domain based on wave theory. DEEPSOIL (Hashash *et al.* 2001, Hashash *et al.* 2017) used the vibration method to calculate the seismic response of soil layers in the frequency domain or time domain. Chen *et al.* (2017) applied the equivalent linear method to analyze the seismic responses of loess soil layers in Lanzhou region, China. Käser *et al.* (2008) applied finite element method to analyze seismic wave propagation.

The quantitative analysis method can obtain ground motion parameters such as acceleration time history and response spectrum at any depth of the soil layer, which provides scientific ground motion parameters for a seismic design of large-scale projects. However, for general engineering structures, a qualitative method is often used to estimate ground motion parameters by site class (FEMA 356. 2000). The soil site is usually classified by the shear wave velocity of the soils (Thokchom *et al.* 2017) or the fundamental frequency. The fundamental frequency has an important influence on the seismic response of the upper structure (Ruiz *et al.* 2009) and is important to recognize the characteristics of soil deposit (Mohamed Adel *et al.* 2008, Liang and Hou 2016).

Idriss and Seed (1968) assumed that soil has a uniform density, and the shear modulus of the soil increases as a power function of depth. The Bessel function is used to obtain the analytical solution of the natural frequencies of the soil layer. Xiong *et al.* (1986) used the transfer matrix approach to obtain the analytical solution of the natural frequency of the layered soil deposit. Luan *et al.* (2003) used the layered shear beam method to calculate the dynamic characteristics of the layered soil deposit. The analytical solution is very attractive for finding the expression of the function and for investigating the characteristics of the problem. However, the cases where the analytical solution can be obtained are very limited. The characteristic equation for a soil deposit whose property varies arbitrarily with depth, which is a differential equation with variable parameters, is difficult to obtain using an analytical solution. Therefore, a numerical solution or a semi-analytical solution is adopted. Even though the natural frequencies of the arbitrary soil deposit can be estimated by the finite element method (Dobry *et al.* 1976, Chopra *et al.* 1995), much research has focused on the semi-analytic solution, which is not only because it finds the function expression of the dynamic characteristics, but also it is convenient for studying the characteristics of the solution. Xiong *et al.* (1986) proposed a direction method to solve the approximation of natural frequencies. Dobry *et al.* (1976) proposed a simplified calculation method for the natural vibration period of soil layer based on the basic principle of Rayleigh method. The simplified calculation method meets the engineering accuracy requirement for soil layers with relatively uniform soil properties. However, the calculation accuracy is low when the soil properties vary greatly with depth. The Ritz method, formed by the direct variation method of the Hamilton principle, obtains a solution by transforming the eigenvalue differential equation into a matrix eigenvalue which can be found in the textbooks of Dynamics of Structures (Chopra *et al.* 1995). Lou (1997) proposed a modal perturbation method (MPM) for the dynamic characteristics of soil layers with variable properties based on the differential equations of free vibration of shear beams and the basic principles of perturbation method. MPM is essentially a Ritz method, and the main advantage of MPM is that it can transform the differential characteristic equation into a set of nonlinear algebraic equations and simplify the solution process. MPM is highly accurate when the properties of the soil layer and beams change continuously. Lou *et al.* (2005) applied MPM to solve the eigenvalues of the prismatic Timoshenko beam. Pan *et al.* (2011) established a modified modal perturbation method for solving the eigenvalues of an arbitrary variable cross-section Timoshenko beam. However, when the soil properties are discontinuous, the MPM results of the eigenvalues do not always converge to the exact solution.

Based on the energy integral of the variational principle, this paper proposes an energy-based mode perturbation method for solving the eigenvalues under arbitrary variation of soil properties to address the convergence issue of the mode perturbation method when the soil properties are discontinuous. First, the differential equations related to the soil layer's characteristics are transformed into an energy integral equation by using the variational principle. Then, the mode shapes of the uniform shear beam are used as the Ritz base function, and the integral equation is transformed into a set of nonlinear algebraic equations based on the modal perturbation principle. Finally, the accuracy and convergence of the proposed method is validated with two numerical examples and compared with the previous methods.

2. The energy-based modal perturbation method

2.1 Non-uniform soil layers

Linearly elastic soil overlying rigid bedrock with properties that vary with depth is shown in Fig. 1. Assuming that the ground surface, rock surface and the boundaries between soil layers are horizontal, the equation of ground response motion caused by SH-wave propagating vertically from the underlying bedrock is

$$\rho(y)\frac{\partial^2 u}{\partial t^2}+c(y)\frac{\partial u}{\partial t}-\frac{\partial}{\partial y}\left[G(y)\frac{\partial u}{\partial y}\right]=-\rho(y)\ddot{u}_g(t) \tag{1}$$

in which $G(y)$ is the shear modulus at depth y, $\rho(y)$ is the mass density at depth y, $c(y)$ is the viscous damping coefficient at depth y, h is the total thickness of deposited soil, $u(y,t)$ is the relative displacement at depth y at time t, and $\ddot{u}_g(t)$ is the seismic acceleration of the bedrock motion. If the layer is composed of soils that are linearly elastic, the solution of Eq. (1) may be obtained by the method of separation of variables, letting

$$u(y,t)=\bar{\phi}(y)Y(t) \tag{2}$$

in which $\bar{\phi}(y)$ and $Y(t)$ are the mode shape and modal amplitude. The mode shape can be obtained by the characteristic equation

$$\bar{\lambda}\rho(y)\bar{\phi}(y)+\frac{d}{dy}[G(y)\frac{d\bar{\phi}(y)}{dy}]=0 \tag{3}$$

in which $\bar{\lambda}$ is eigenvalue. When $\rho(y)$ and $G(y)$ are

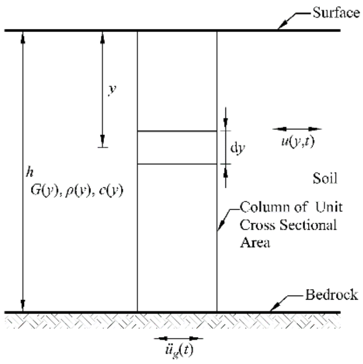

Fig. 1 Soil layer with variable properties

constant or some specific function, Eq. (3) can obtain an analytical solution. However, it is very difficult to solve Eq. (3) for arbitrary $\rho(y)$ and $G(y)$ functions. To obtain the semi-analytical solution of the Eq. (3), the energy integral corresponding to the Eq. (3) can be obtained by the variational principle:

$$\Pi = \frac{1}{2}\int_0^h G(y)\left[\bar{\phi}'(y)\right]^2 dy - \frac{1}{2}\bar{\lambda}\int_0^h \rho(y)\bar{\phi}^2(y)dy \tag{4}$$

The minimum value problem of Eq. (4) is equivalent to the differential equation of Eq. (3). The total potential Π would be the extremum for the j^{th} mode of vibration, that is,

$$\Pi_j = \frac{1}{2}\int_0^h G(y)\left[\bar{\phi}_j'(y)\right]^2 dy - \frac{1}{2}\bar{\lambda}_j\int_0^h \rho(y)\bar{\phi}_j^2(y)dy = \text{extremum} \tag{5}$$

in which $\bar{\lambda}_j = \bar{\omega}_j^2$ is the j^{th} eigenvalue, $\bar{\omega}_j$ is the j^{th} natural circular frequency, $\bar{\phi}_j(y)$ is the j^{th} mode shape.

2.2 Corresponding uniform shear beam

The corresponding uniform shear beam is defined as the member that has the same height and boundary conditions as the nonuniform soil layers with properties varying with depth. It is represented by a constant density ρ_0 and shear modulus G_0. To ensure that the uniform shear beam is a good representation of nonuniform soil layers, ρ_0 and G_0 can be determined to be the average value of soil layers as follows:

$$\rho_0 = \int_0^h \rho(y)dy / h \tag{6}$$

$$G_0 = \int_0^h G(y)dy / h \tag{7}$$

The characteristic equation of the corresponding uniform shear beam can be expressed as

$$\lambda\rho_0\phi(y) + G_0\frac{d^2\phi(y)}{dy^2} = 0 \tag{8}$$

where λ and $\phi(y)$ respectively denote the eigenvalue and mode shape of the uniform shear beam. The general solution of Eq. (8) can then be analytically obtained as

$$\lambda_j = \left[(2j-1)\pi / 2h\right]^2 G_0 / \rho_0 \tag{9}$$

$$\phi_j(y) = \cos\left[(2j-1)\pi y / 2h\right] \tag{10}$$

2.3 The perturbation solution

In the EMPM, the j^{th} eigenvalue and its associated mode of vibration for nonuniform soil layers with properties varying with depth are related to those of the corresponding uniform shear beam by

$$\bar{\lambda}_j = \lambda_j + \Delta\lambda_j \tag{11}$$

$$\bar{\phi}_j(y) = \sum_{k=1}^{n}\phi_k(y)q_k \tag{12}$$

in which λ_j and $\phi_j(y)$ are the j^{th} eigenvalue and mode shape of the uniform shear beam, respectively, $\Delta\lambda_j$ is the perturbation of the j^{th} eigenvalue, q_k is the generalized coordinates for the Ritz functions $\phi_k(y)$.

Substituting Eqs. (11)-(12) into Eq. (5) gives

$$\Pi_j = \frac{1}{2}\int_0^h G(y)\left[\sum_{k=1}^{n}\phi_k'(y)q_k\right]^2 dy - \frac{1}{2}\bar{\lambda}_j\int_0^h \rho(y)\left[\sum_{k=1}^{n}\phi_k(y)q_k\right]^2 dy \tag{13}$$

On introducing the symbols K_{ik} and M_{ik}, defined by

$$K_{ik} = \int_0^h G(y)\phi_i'(y)\phi_k'(y)dy \tag{14}$$

$$M_{ik} = \int_0^h \rho(y)\phi_i(y)\phi_k(y)dy \tag{15}$$

Eq. (13) can be rewritten as:

$$\Pi_j = \frac{1}{2}\{q\}^T[K]\{q\} - \frac{1}{2}\bar{\lambda}_j\{q\}^T[M]\{q\} \tag{16}$$

in which $\{q\} = \{q_1 \quad q_2 \quad \cdots \quad q_n\}^T$. To minimize the Π_j, the first derivative of Π_j with respect to $\{q\}$ is set to be zero:

$$\frac{\partial \Pi_j}{\partial\{q\}} = 0 \tag{17}$$

from which

$$\left([K] - \bar{\lambda}_j[M]\right)\{q\} = 0 \tag{18}$$

If all the elements of $\{q\}$ in Eq. (18) are unknown, Eq. (18) is the conventional Rayleigh-Ritz method. For the modal perturbation method, it is assumed that the soil layer with variable parameters is regarded as a new system obtained by modifying the parameters of the uniform shear beam. The main mode shape and eigenvalue of the new system can be approximately obtained with simplified perturbation analysis by using the modal characteristics of the original system. That is, q_j=1, and Eq. (12) can be rewritten as:

$$\bar{\phi}_j(y)=\phi_j(y)+\sum_{k=1,k\neq j}^{n}\phi_k(y)q_k \tag{19}$$

Substituting Eqs. (11) and (19) into Eq. (18) gives

$$\left(-\left[\tilde{K}\right]+\lambda_j\left[E\right]+\frac{\Delta\lambda_j}{\lambda_j}\lambda_j[M]\right)\{\tilde{q}\}=\{R\} \tag{20}$$

in which the square matrices $\left[\tilde{K}\right]$, $[E]$ and the vectors $\{\tilde{q}\}$, $\{R\}$ are given below:

$$\tilde{K}_{ik}=\begin{cases}K_{ik} & k\neq j\\ 0 & k=j\end{cases},\quad E_{ik}=\begin{cases}M_{ik} & k\neq j\\ 0 & k=j\end{cases},$$

$$\tilde{q}_k=\begin{cases}q_k & k\neq j\\ 1 & k=j\end{cases},\quad R_k=K_{kj}-\lambda_j M_{kj}.$$

Set $x_k=\begin{cases}q_k & k\neq j\\ \Delta\lambda_j/\lambda_j & k=j\end{cases}$, Eq. (20) can be rearranged as

$$\left(-\left[\tilde{K}\right]+\lambda_j[M]+x_j\lambda_j\left[E\right]\right)\{x\}=\{R\} \tag{21}$$

in which $\{x\}=\{x_1\quad x_2\quad \dots\quad x_n\}^T$. Eq. (21) is a nonlinear algebraic equation with n unknowns, which can be rewritten as

$$(\left[D\right]+x_j\lambda_j\left[E\right])\{x\}=\{R\} \tag{22}$$

in which $\left[D\right]=-\left[\tilde{K}\right]+\lambda_j[M]$.

Differential Eq. (3), with unknown functions and unknown eigenvalue $\bar{\lambda}_j$, has been transformed into a set of nonlinear algebraic equations in Eq. (22). After $\{x\}$ is solved, the j^{th} eigenvalue and associated mode shape of soil layers with variable parameters can be obtained from Eqs. (11) and (19). In general, solving the nonlinear algebraic equations is easier than solving the differential equation and eigen-problem equation.

Eq. (22) is the same as the MPM from the forms perspective. The difference is that the MPM is derived from the differential equation, and the proposed method is based on the variational principle. Therefore, this method is called the energy modal perturbation method (EMPM). Another significant difference is the calculation of the coefficient K_{ik} in Eq. (14). In the MPM, the coefficients K_{ik} is:

$$K_{ik}=-\int_0^h\phi_i(y)\frac{d}{dy}\left[G(y)\phi_k'(y)\right]dy \tag{23}$$

Comparing Eq. (14) and Eq. (23), it can be seen that when the soil property is a continuous function, Eq. (14) and Eq. (23) are the same after using integration by parts and boundary conditions. But when there are mutations in properties of the layered soil, Eq. (23) causes the matrix $[K]$ to become asymmetric because of the boundary conditions of the abrupt interface, which often leads to a misconvergence in the perturbation solution. From Eq. (14), it is known that the potential energy of the soil layer is reflected by K_{ik}. It is meaningless to discuss the magnitude of energy at point level for any continuous medium system. The constant term by the integration by parts in Eq. (23) is actually the potential energy of the abrupt boundary. Therefore, it is more reasonable to select K_{ik} as obtained by the variational principle.

3. Solution of nonlinear algebraic equation

For a set of nonlinear algebraic equation, Eq. (22) can be solved by the Newton-Raphson iteration method. Assume

$$\left\{f\left(\{x\}\right)\right\}=(\left[D\right]+\lambda_j x_j\left[E\right])\{x\}-\{R\} \tag{24}$$

Then the iteration solution of $\{x\}$ can be obtained by

$$\left\{x^{(l+1)}\right\}=\left\{x^{(l)}\right\}-\left[H^{(l)}\right]^{-1}\left\{f\left(\left\{x^{(l)}\right\}\right)\right\} \tag{25}$$

in which the superscript (l) denotes the l^{th} iteration. The matrix $[H^{(l)}]$ is given below:

$$H_{ik}^{(l)}=\begin{cases}D_{ik}+\lambda_j x_j^{(l)}E_{ik} & k\neq j\\ D_{ij}+\lambda_j\sum_{k=1}^{n}E_{ik}x_k^{(l)} & k=j\end{cases}.$$

The initial approximation $\{x^{(0)}\}$ is set as

$$\left\{x^{(0)}\right\}=\left[D\right]^{-1}\left\{R\right\} \tag{26}$$

The iteration termination judgment can be

$$\left|x_j^{(l+1)}-x_j^{(l)}\right|/\left|x_j^{(l+1)}\right|<e \tag{27}$$

in which e is the allowable value of the convergence error and usually is set to 1.0E-6.

4. The algorithm workflow

In order to make the procedure of the algorithm clearer, the workflow of the EMPM described above is shown in Fig. 2.

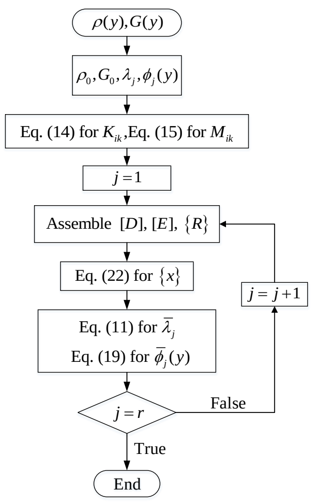

Fig. 2 Flowchart of the EMPM

The algorithm is executed via five steps:

Step 1: Given $\rho(y)$ and $G(y)$, calculate the parameters ρ_0, G_0 of the corresponding uniform shear beam from Eq. (6) and Eq. (7), and obtain the eigenvalue and mode shape λ_j, $\phi_j(y)$ from Eq. (9) and Eq. (10).

Step 2: Calculate K_{ik} from Eq. (14) and M_{ik} from Eq. (15), and set j=1.

Step 3: Assemble the matrices [D], [E] and the vector {R}.

Step 4: Obtain {x} by solve Eq. (22).

Step 5: Obtain $\bar{\lambda}_j$ from Eq. (11) and $\bar{\phi}_j(y)$ from (19), then set j=j+1, and go back to Step 3 until j=r which r is the desirable highest mode.

5. Seismic response of soil layer

In linear or equivalent linear seismic response analysis, after obtaining the frequency $\bar{\omega}_j$ and its associated mode $\bar{\phi}_j(y)$ of the nonuniform soil layer, the well-known modal superposition method can be used to calculate the seismic response, that is, the solution of the displacement can be expressed as follows

$$u(y,t)=\sum_{j=1}^{r}\bar{\phi}_j(y)Y_j(t) \tag{28}$$

in which r is the number of mode shapes of the nonuniform soil layer. The generalized coordinates $Y_j(t)$ can be derived from

$$\ddot{Y}_j(t)+2\zeta_j\bar{\omega}_j\dot{Y}_j(t)+\bar{\omega}_j^2Y_j(t)=-\bar{\gamma}_j\ddot{u}_g(t) \tag{29}$$

where $\bar{\gamma}_j$ is the modal participation factor,

$$\bar{\gamma}_j=\int_0^h\rho(y)\bar{\phi}_j(y)dy\Big/\int_0^h\rho(y)\bar{\phi}_j^2(y)dy \tag{30}$$

In the calculation process of the energy modal perturbation method used to calculate the seismic response of the soil layer with variable parameters, the calculation related to the variable parameters is only ρ_0, G_0, M_{ik}, K_{ik}. These parameters are the results of the integral operation. If the analytical integration cannot be obtained, it can be carried out by means of numerical integration. There is no special requirement for $\rho(y)$ and $G(y)$, which may or may not be an analytical function. Such soil layer parameters can directly use survey results in engineering applications.

6. Verification with numerical results

Theoretically, the method can be applied to free and forced vibration analysis of soil layers whose density and shear modulus vary arbitrarily along their depths. Because the seismic response of the soil layer is a conventional modal superposition method, the key step is to obtain highly accurate dynamic characteristics. Therefore, the following is to mainly investigate the validity and accuracy of the frequencies of the EMPM.

6.1 Soil shear modulus power exponent variation with depth

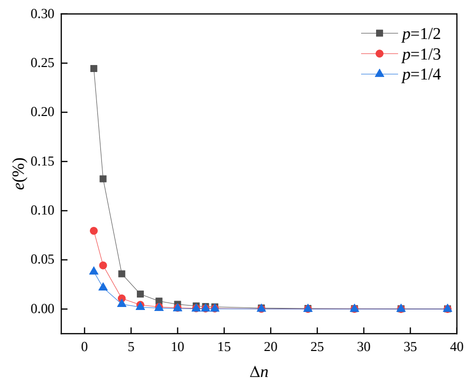

Fig. 3 Relative error of the fundamental frequency by EMPM

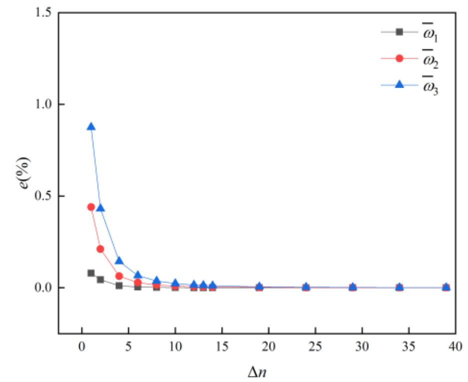

Fig. 4 Relative error of the first three frequencies by EMPM (p=1/3)

For a soil layer with uniform density ρ_0, the shear modulus varies with depth as given by:

$$G(y)=G_b\left(y/h\right)^p \tag{31}$$

in which G_b is the shear modulus at the depth h and p is a constant. For p≤1/2, Idriss and Seed (1968) derived the closed-form solutions:

$$\omega_n^*=\frac{\beta_n\sqrt{\dfrac{G_b}{h^p\rho_0}}}{\theta h^{1/\theta}},(n=1,2,...) \tag{32}$$

in which β_n is the roots of $J_{-q}(\beta_n)=0$, J_{-q} is the Bessel function of the first kind of order -q. q and θ are constants related to p by

$$\theta=-\frac{2}{p-2} \tag{33}$$

$$q=\frac{p-1}{p-2} \tag{34}$$

The EMPM would converge to the exact solutions of $\bar{\lambda}_j$ and $\bar{\phi}_j(y)$ for soil layers with variable parameters when an infinite number of modes $\phi_k(y)\,(k=1,2,...,\infty)$ are required in Eq. (19). However, relatively few modes can provide sufficiently accurate results in engineering applications. Similar to the subspace iteration method

Table 1 The first three natural frequencies of soil shear modulus power exponent variation with depth (rad/s)

p	1/4				1/3				1/2			
ω	Exact solution	EMPM	Rayleigh -Ritz	Direct solution	Exact solution	EMPM	Rayleigh -Ritz	Direct solution	Exact solution	EMPM	Rayleigh -Ritz	Direct solution
ω_1	15.684	15.684	15.684	15.762	15.381	15.381	15.381	15.519	14.755	14.755	14.755	15.061
ω_2	44.561	44.562	44.562	45.084	42.853	42.852	42.856	43.761	39.432	39.445	39.445	41.378
ω_3	73.513	73.519	73.519	74.606	70.419	70.435	70.435	72.301	64.228	64.294	64.294	68.206

Table 2 The relative errors of the first three natural frequencies (%)

p	1/4			1/3			1/2		
ω	EMPM	Rayleigh-Ritz	Direct solution	EMPM	Rayleigh-Ritz	Direct solution	EMPM	Rayleigh-Ritz	Direct solution
ω_1	0.0003	0.0003	0.4988	0.0008	0.0008	0.8967	0.0032	0.0032	2.0738
ω_2	0.0029	0.0029	1.1750	0.0033	0.0072	2.1184	0.0310	0.0310	4.9344
ω_3	0.0093	0.0093	1.4872	0.0233	0.0233	2.6726	0.1035	0.1035	6.1941

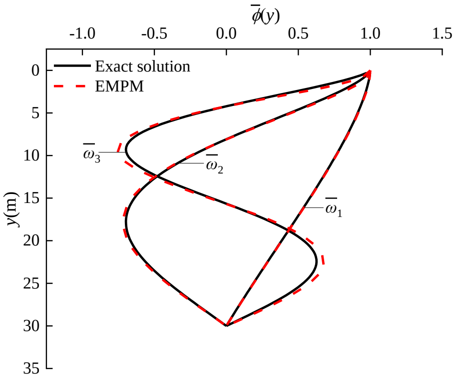

Fig. 5 The first three natural mode shapes (p=1/3)

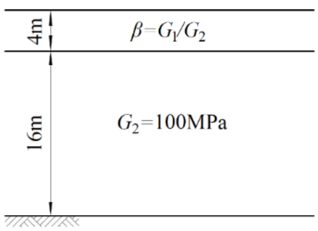

Fig. 6 Two layers site

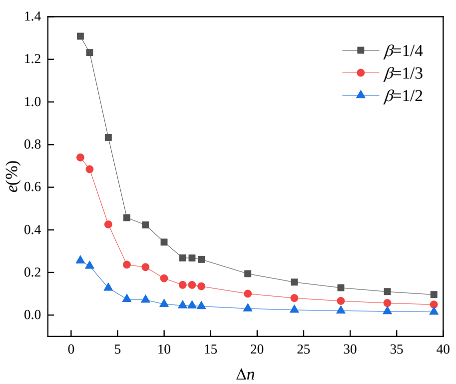

Fig. 7 Relative error of the fundamental frequency by EMPM

(Bathe 1996), to calculate the j^{th} eigenvalue of soil layers with variable parameters, the number of modes, n, of its corresponding uniform shear beam must be

$$n = j + \Delta n \tag{35}$$

As the additional number of modes, Δn, increases, the result gradually converges to the exact solution. The relative error e of the j^{th} natural frequency obtained by using n modes of vibration can be expressed as

$$e = \frac{\left|\bar{\omega}_j - \omega_j^*\right|}{\omega_j^*} \times 100\% \tag{36}$$

where $\bar{\omega}_j$ and ω_j^* are the approximate and exact frequencies, respectively.

The relative errors of the fundamental frequency by EMPM are presented in Fig. 3 for various p values. The relative errors of different natural frequencies by EMPM are shown in Fig. 4 with p=1/3 for various number of modes Δn. In the EMPM, the corresponding uniform shear beam parameter G_0 is equal to $G_b/(p+1)$. It can be observed from Fig. 3 that p significantly influences the convergence rate. The soil property is more uniform (p is much smaller), so the results are more rapidly converge. When p is between 1/2 and 1/4, the relative error of the fundamental frequency is within 0.05% when Δn is larger than 8. Fig. 4 shows that when the order of natural frequency is higher, the error is slightly greater. However, the higher frequencies are all convergent with the increase of Δn. The curves of error are almost merging into one when Δn is greater than 8. Therefore, referring to previous research results (Pan *et al.* 2011, Bathe 1996), n=j+8 in the EMPM is satisfactory for engineering applications.

Table 1 lists the first three natural frequencies of soil layers in which the shear modulus power exponent variation with depth are calculated by exact solution, EMPM, Rayleigh-Ritz method and the direct solution proposed by Xiong *et al.* (1986). For soil layer thickness h=30 m, density ρ_0=2000 kg/m^3, and bottom shear modulus G_b=200 MPa. Table 2 lists the relative errors of the first three natural frequencies by EMPM, Rayleigh-Ritz method and direct solution. Based on the previous results, 11 vibration modes of the corresponding uniform shear beam are included in the solution of Eq. (19). The same number of modes was used in the Rayleigh-Ritz method. It can be seen from Table 2 that the relative error of the first three natural frequencies obtained by EMPM is within 0.2% when Δn=8. The calculation results of EMPM and Rayleigh-Ritz method are almost same. The difference between the two methods is that the Rayleigh-Ritz method solves the characteristic equation while EMPM solves the linear algebraic equations. The EMPM is simpler than Rayleigh-Ritz method. And the EMPM is most likely more accurate than the direct solution.

Fig. 5 presents the first three natural mode shapes with p=1/3 obtained by the exact solution and the EMPM. EMPM is solved by 11 vibration modes of the corresponding uniform shear beam. The mode shapes obtained by EMPM are almost identical with the shapes obtained by exact solution. Therefore, once the natural frequencies are convergent, the associated mode shapes would also be highly precise.

6.2 Layered soil

The main difference between the EMPM and the previous MPM is the integration of K_{ik} in Eq. (14). To investigate the effect of K_{ik} for layered soil, the natural frequencies of a two-layer site, as shown in Fig. 6, are calculated by EMPM, MPM, the Rayleigh-Ritz method and direct solution, respectively. Xiong *et al.* (1986) derived the transcendental equation of natural frequencies by transfer matrix method as follows

$$\frac{\rho_1 v_1}{\rho_2 v_2}\tan\frac{\omega^* h_1}{v_1}\tan\frac{\omega^* h_2}{v_2}=1 \tag{37}$$

in which $v_i=\sqrt{G_i/\rho_i}$ (i=1,2) is the shear wave velocity of the i^{th} layer, G_i and ρ_i are the shear modulus and mass density of the i^{th} layer.

The integration of K_{ik} in Eq. (14) of a two-layers site is

$$K_{ik}=\int_0^{h_1} G_1(y)\phi_i'(y)\phi_k'(y)dy+\int_{h_1}^{h_2} G_2(y)\phi_i'(y)\phi_k'(y)dy \tag{38}$$

And the integration of K_{ik} in Eq. (23) of a two-layers site is given by

$$\begin{aligned}K_{ik}{}'=&\int_0^{h_1} G_1(y)\phi_i'(y)\phi_k'(y)dy+\int_{h_1}^{h_2} G_2(y)\phi_i'(y)\phi_k'(y)dy\\&+(G_2-G_1)\phi_i(h_1)\phi_k'(h_1)\end{aligned} \tag{39}$$

Suppose $\beta=G_1/G_2$ is the ratio of shear modulus between the first and second layer. When the first and second layer thickness of soil deposit are h_1=4 m and h_2=16 m with ρ_1=1800 kg/m^3, ρ_2=2000 kg/m^3, G_2=100 MPa, the relative error of the fundamental frequency by EMPM and MPM are shown in Figs. 7-8 for various β values. The relative errors of different natural frequencies by EMPM and MPM are shown in Figs. 9-10 for various number of modes Δn with ρ_1=1800 kg/m^3, ρ_2=2000 kg/m^3, G_2=100 MPa, β=1/2. Corresponding uniform shear beam parameters are $G_0=0.2G_1+0.8G_2$ and $\rho_0=0.2\rho_1+0.8\rho_2$.

It can be seen that the EMPM leads to monotonically

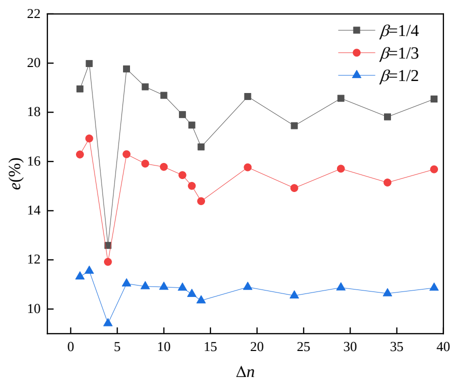

Fig. 8 Relative error of the fundamental frequency by MPM

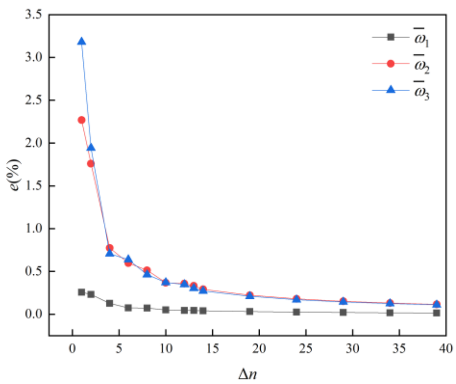

Fig. 9 Relative error of the first three frequencies by EMPM(β=1/2)

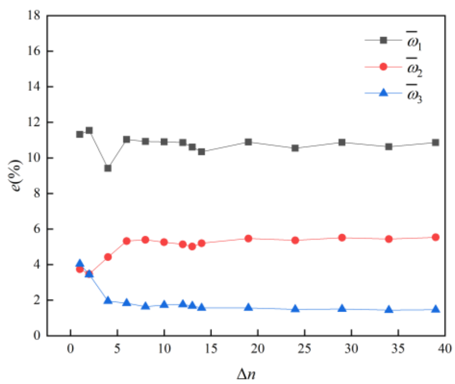

Fig. 10 Relative error of the first three frequencies by MPM(β=1/2)

convergent solutions with the increase of Δn, but errors of frequencies by MPM fluctuate and do not converge to zero. This means that the results by MPM cannot be relied upon to converge to an exact solution for the layered soil. Therefore, the EMPM is more reasonable and accurate than the MPM.

The effect of soil non-homogeneity on the error is the same as the soil shear modulus power exponent variation with depth. That is, when β is between 1/4 and 1/2, the larger the β is, the soil is more homogeneous, and the error is smaller. When the order of frequency is higher, the error is larger. But the errors of natural frequencies by EMPM for Δn=8 are less than 5%.

Table 3 The first three natural frequencies of two layered soils (rad/s)

β	ω	Exact solution	EMPM	MPM	Rayleigh-Ritz		Direct solution	
					Based on Eq. (38)	Based on Eq. (39)	Based on Eq. (38)	Based on Eq. (39)
1/4	ω_1	17.593	17.640	14.442	17.640	14.442	17.825	15.090
	ω_2	45.091	45.981	42.572	45.981	42.572	51.471	47.075
	ω_3	71.362	71.751	75.155	71.751	75.155	81.779	81.779
1/3	ω_1	17.702	17.727	14.969	17.727	14.969	17.835	15.429
	ω_2	47.945	48.483	44.978	48.483	44.978	51.700	47.831
	ω_3	74.865	75.312	77.811	75.312	77.811	82.577	82.577
1/2	ω_1	17.808	17.816	15.874	17.816	15.874	17.854	16.086
	ω_2	50.869	51.052	48.195	51.052	48.195	52.155	49.309
	ω_3	80.599	80.616	81.994	80.899	81.994	84.150	84.150

Table 4 The relative errors of the first three natural frequencies (%)

β	ω	EMPM	MPM	Rayleigh-Ritz		Direct solution	
				Based on Eq.(38)	Based on Eq.(39)	Based on Eq.(38)	Based on Eq.(39)
1/4	ω_1	0.2672	17.911	0.2661	17.908	1.3203	14.230
	ω_2	0.6461	8.0121	0.6456	8.0111	11.216	1.7178
	ω_3	0.5451	5.3152	0.5445	5.3149	14.598	14.597
1/3	ω_1	0.1412	15.439	0.1435	15.441	0.7510	12.842
	ω_2	1.1221	6.1883	1.1215	6.1894	7.8318	0.2375
	ω_3	0.5971	3.9351	0.5970	3.9347	10.301	10.301
1/2	ω_1	0.0449	10.860	0.0471	10.860	0.2599	9.6722
	ω_2	0.3598	5.2566	0.3592	5.2557	2.5282	3.0672
	ω_3	0.0211	1.7308	0.3716	1.7312	4.4055	4.4055

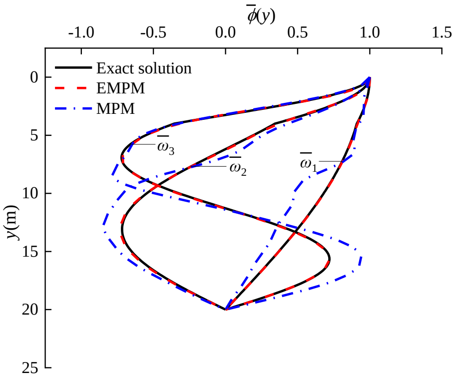

Fig. 11 The mode shapes of first three frequencies (β=1/2)

Table 3 compares the first three frequencies for various β values, which are calculated by the exact solution, EMPM, MPM, the Rayleigh-Ritz method and direct solution. Table 4 lists the relative errors of the first three natural frequencies by EMPM, MPM, the Rayleigh-Ritz method and direct solution. Eleven modes of vibration of the corresponding uniform shear beam are included in the solution of Eq. (19). The same number of modes was used in the Rayleigh-Ritz method. As can be seen, when $1/4 \le \beta \le 1/2$, the relative error calculated by EMPM is less than 2%.

In the Rayleigh-Ritz method, the terms of the stiffness matrix K_{ik}, can also be calculated with either Eq. (38) or (39) (Xiong *et al.* 1986). It can be seen from Table 3 and Table 4 that the results of both the Rayleigh-Ritz method and EMPM are the same when Eq. (38) is adopted. The results of Rayleigh-Ritz method based on Eq. (39) is almost same as MPM. The phenomena are caused by the fact that the mode perturbation method derive from Rayleigh-Ritz method, so their errors are similar. The accuracy obtained by Rayleigh-Ritz method using the K_{ik} in Eq. (38) is better than that in Eq. (39), therefore, the coefficients of stiffness matrix K_{ik} is more reasonable based on energy integration.

As for the direct solution, it can be seen from the calculation results, if the Eq. (38) which is based on energy integral is used, the direct solution is reasonable for the fundamental frequency calculation. However, when the nonlinear property of the soil layer becomes stronger, the error of the direct solution increases rapidly so it is not suitable for the calculation of higher order frequencies. The stiffness matrix K_{ik} obtained by the differential form of Eq. (39) is not recommended for use because the error is large. The results given by EMPM are probably more accurate than both MPM and direct solution.

The first three natural mode shapes obtained by the exact solution, EMPM and MPM are as shown in Fig.11 with β=1/2. It is evident that the results obtained by the EMPM are perfectly in agreement with those by exact solution. However, those mode shapes obtained by MPM could lead to significantly inconsistent results. Once again, this shows that EMPM is more accurate than MPM from a mode shape perspective.

7. Conclusions

This paper introduces an energy-based modal perturbation method (EMPM) for the evaluation of natural frequencies and mode shapes of soil deposits with properties varying with depth. The results of examples show that the proposed method is highly accurate and convergent. This method treats soil layers with variable parameters as a perturbed system of its corresponding uniform shear beam with the same boundary conditions and thickness. Based on extensive analyses and numerical results, the following conclusions can be drawn:

• The EMPM can decouple the energy integral equation for the soil deposit with properties varying with depth into a set of algebraic equations by using the vibration modes of corresponding uniform shear beam as Ritz functions. The method is particularly suitable for finding a few individual orders of frequency and mode.

• When the soil property variation with depth is continuous, the EMPM is the same as the MPM. For the discontinuous variations, the EMPM is superior to the previous MPM, no matter the accuracy and stability levels. The solution of EMPM is monotonically convergent, but the results by MPM cannot be relied upon to converge to exact solution.

• As the number of vibration modes of corresponding uniform shear beam included in EMPM analysis increases, the end results converge to the exact solutions. The difference of soil properties has a significant effect on the convergence rate. If the j^{th} natural frequency and associated mode of shape is of the interest, including ($8+j$) number of vibration modes from its corresponding uniform shear beam is sufficient. In this paper, for the first three natural frequencies, the error of the solution obtained by EMPM is less than 2%, which is the preferred method for the evaluation of natural frequencies and mode shapes of soil deposit property variation with depth.

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