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# Nonlinear frequency analysis of beams resting on elastic foundation using max-min approach

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**Abstract.** In this paper, nonlinear vibration of Euler-Bernoulli beams resting on linear elastic foundation is studied. It has been tried to prepare a semi-analytical solution for whole domain of vibration. Only one iteration lead us to high accurate solution. The effects of linear elastic foundation on the response of the beam vibration are considered and studied. The effects of important parameters on the ratio of nonlinear to linear frequency of the system are studied. The results are compared with numerical solution using Runge-Kutta 4$^{th}$ technique. It has been shown that the Max-Min approach can be easily extended in nonlinear partial differential equations.

**Keywords:** elastic foundation; max-min approach; analytical method; Runge-Kutta 4$^{th}$

## 1. Introduction

Generally, Mathematical modeling of natural phenomena usually express with linear and nonlinear partial differential equations.

In large deflection problems, preparing analytical solution for these problems ate vey limit. Therefore analyzing of the beam vibration problems uses numerical techniques such as: finite difference, spectral methods, finite elements and differential quadrature methods.

One of the interesting areas of scientific is to analysis the structures on elastic foundations, which are widely, use in engineering area, such as foundation, pavement and railroad, pipeline, and some aerospace structures applications. Many simplified engineering problems related to the Soil-Structure Interaction (SSI) are modeled as Winkler model to consider the interaction as a beam resting on elastic foundation.

It has been so long time that many researchers have been working on nonlinear vibration problems in soil structure interactions (SSI). One of the well-known problems is Euler Bernoulli beams resting on linear on nonlinear foundations. There are different approaches that use for modeling the soil such as Winkler, Pasternak or Vlasov, Flonenko - Borodich foundations.

Winkler approach is a linear algebraic relationship is introduces between the normal displacement of the structure and the contact pressure (Gorbunov-Posadov *et al.* 1973). A set of mutually parallel independent spring elements are used in the Winkler model to represent the soil medium (Al-Hosani *et al.* 1995). In this way, considering the nonlinear behavior of the problems going to be more easier and comparable (Soldatos *et al.* 1985). Gupta *et al*. (2006) had focused on the vibration and buckling response of polar orthotropic circular plates with linearly varying thickness. Auersch *et al.* (2008) presented a study about infinite beams on half-space compared with finite and infinite beams on a Winkler support. The consideration of soil-structure interaction of an Euler-Bernoulli beam had been studied by Kacar *et al.* (2011). The governing differential equations of the beam are solved by using Differential Transform Method (DTM).

Winkler foundation used in the another study by Motaghian *et al.* (2011) in which they had considered a mathematical approach to achieve an accurate solution of Euler-Bernoulli beam with mixed boundary conditions. Ghannadias *et al.* (2015) tried to prepare an accurate and direct modeling technique for modeling uniform Timoshenko beam with arbitrary boundary conditions.

Zahedinejad (2016) studied the soil-structure interaction of FG beams on elastic foundations and thermal environment effects. Many researchers have been worked on the Winkler elastic foundation modeling in the past few decades (Lohar *et al.* 2016, Tsiatas 2010, Akgöz *et al.* 2015, Niknam *et al.* 2015, Mirzabeigy *et al.* 2014, Ying *et al.* 2008, Xing *et al.* 2013, Shariyat *et al.* 2011, He *et al.* 2016, Rabia *et al.* 2016, Hadji *et al.* 2015, Bayat *et al.* 2018, 2017a, b, c).

To solve nonlinear vibration problems analytically, the only way is to use or prepare some analytical or semi analytical approaches for these problems. In last few decades, many researches have proposed some new approaches such as: Parameter Expansion Method (Xu

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2007), Differential Transform Method (Arikoglu *et al.* 2006), Variational iteration Method (Liu 2011), Homotopy Perturbation Method (Shou 2009). In this study, Max-Min Approach (MMA) is applied to solve the nonlinear vibration equation of Euler-Bernoulli beam resting on a Winkler elastic foundation which was proposed by He (2008). It has been tried to solve the problem by suing only one iteration. The results are compared with other analytical and numerical solutions.

## 2. Description of the problem

Consider a straight beam on an elastic foundation with length *L*, a cross-section *A*, a mass per unit length *M*, moment of inertia *I*, and modulus of elasticity *E* that subjected to an axial force of magnitude $\bar{P}$ as shown in Fig. 1. In Fig. 1, a scheme of simply supported buckled Euler-Bernoulli beam fixed at one end resting on Winkler foundation is presented. The most advantages of using linear Winkler springs is to make the calculations easier.

The basic assumptions of the beam theory are considered such as:

1) It has been considered the isotropic and elastic state.

2) The beam deformation is dominated by bending and the distribution and rotation are negligible.

3) The beam is along as slender with a constant section along the axis.

The equation of motion for an axially loaded Euler-Bernoulli beam by considering the mid-plane stretching effect is

$$EI\frac{\partial^4 W'}{\partial X'^4}+M\frac{\partial^2 W'}{\partial t'^2}+\bar{P}\frac{\partial^2 W'}{\partial X'^2}+K'(X)W'-\frac{EA}{2L}\frac{\partial^2 W'}{\partial X'^2}\int_0^L\left(\frac{\partial W'}{\partial X'}\right)^2 dX'=U(X',t') \quad (1)$$

where $K'$ is a foundation modulus and *U* is a distributed load in the transverse direction.

Assume the non-conservative forces were equal to zero. Therefore, Eq. (1) can be written as follows

$$EI\frac{\partial^4 W'}{\partial X'^4}+M\frac{\partial^2 W'}{\partial t'^2}+\bar{P}\frac{\partial^2 W'}{\partial X'^2}+K'(X)W'-\frac{EA}{2L}\frac{\partial^2 W'}{\partial X'^2}\int_0^L\left(\frac{\partial W'}{\partial X'}\right)^2 dX'=0. \quad (2)$$

Here we introduce the following non-dimensional variables

$$X=\frac{X'}{L},\quad W=\frac{W'}{R},\quad t=t'\sqrt{\frac{EI}{ML^4}},\quad P=\frac{\bar{P}L^2}{EI},\quad K=\frac{K'L^4}{EI} \quad (3)$$

where $R=\sqrt{(I/A)}$ is the radius of gyration of the cross-section. We assume the elastic coefficient of Winkler foundation is constant $K'(X)=K_0$ Then Eq. (1) can be written as follows

$$\frac{\partial^4 W}{\partial W^4}+\frac{\partial^2 W}{\partial t^2}+P\frac{\partial^2 W}{\partial W^2}+K_0 W-\frac{1}{2}\frac{\partial^2 W}{\partial X^2}\int_0^L\left(\frac{\partial W}{\partial X}\right)^2 dX=0 \quad (4)$$

If we assume $W(X,t)=w(t)\phi(X)$ in which $\phi(X)$ is the first Eigen mode of the beam and using the Galerkin method, then we will have the following governing nonlinear vibration equation of motion for an axially loaded Euler-Bernoulli beam (Javanmard *et al.* 2013)

$$\frac{d^2 w(t)}{dt^2}+(\alpha_1+P\alpha_2+K_0)w(t)+\alpha_3 w^3(t)=0 \quad (5)$$

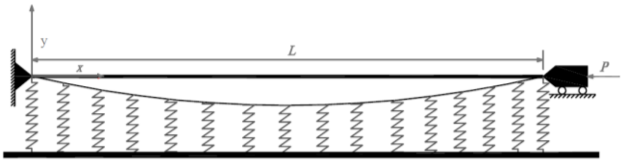



Fig. 1 Schematic representation of an axially loaded Euler-Bernoulli beam resting on Winkler foundation

The initial conditions for center of the beam are

$$w(0)=\Delta,\quad dw(0)/dt=0 \quad (6)$$

The value of the $\alpha_1$, $\alpha_2$ and $\alpha_3$ can be obtained as followd

$$\alpha_1=\left(\int_0^1\left(\frac{\partial^4\phi(X)}{\partial X^4}\right)\phi(X)\,dX\right)\Big/\int_0^1\phi^2(X)\,dX \quad (7a)$$

$$\alpha_2=\left(\int_0^1\left(\frac{\partial^2\phi(X)}{\partial X^2}\right)\phi(X)\,dX\right)\Big/\int_0^1\phi^2(X)\,dX \quad (7b)$$

$$\alpha_3=\left(\left(-\frac{1}{2}\right)\int_0^1\left(\frac{\partial^2\phi(X)}{\partial X^2}\int_0^1\left(\frac{\partial^2\phi(X)}{\partial X^2}\right)^2 dX\right)\phi(X)\,dX\right)\Big/\int_0^1\phi^2(X)\,dX \quad (7c)$$

## 3. Basic idea of max-min approach (MMA)

Consider a generalized nonlinear oscillator in the form

$$\ddot{w}+w\,f(w)=0,\quad w(0)=\Delta,\quad \dot{w}(0)=0, \quad (8)$$

where $f(w)$ is a non-negative function of *w*. According to the idea of the max–min method, we choose a trial-function in the form

$$w(t)=\Delta\cos(\omega t), \quad (9)$$

where $\omega$ the unknown frequency to be further is determined.

Observe that the square of frequency, $\omega^2$, is never less than that in the solution

$$w_1(t)=\Delta\cos(\sqrt{f_{\min}}\,t), \quad (10)$$

of the following linear oscillator

$$\ddot{w}+w\,f_{\min}=0,\quad w(0)=\Delta,\quad \dot{w}(0)=0, \quad (11)$$

where $f_{min}$ is the minimum value of the function $f(w)$.

In addition, $\omega^2$ never exceeds the square of frequency of the solution

$$w_1(t)=\Delta\cos(\sqrt{f_{\max}}\,t), \quad (12)$$

of the following oscillator

$$\ddot{w}+w\,f_{\max}=0,\quad w(0)=\Delta,\quad \dot{w}(0)=0, \quad (13)$$

where $f_{max}$ is the maximum value of the function $f(w)$.

Hence, it follows that

$$\frac{f_{\min}}{1}<\omega^2<\frac{f_{\max}}{1}. \quad (14)$$

According to He Chentian interpolation (He 2008), we obtain

$$\omega^2 = \frac{m f_{\min} + n f_{\max}}{m+n}, \tag{15}$$

or

$$\omega^2 = \frac{f_{\min} + k f_{\max}}{1+k}, \tag{16}$$

where $m$ and $n$ are weighting factors, $k=n/m$. So the solution of Eq. (8) can be expressed as

$$w(t) = \Delta \cos \sqrt{\frac{f_{\min} + k f_{\max}}{1+k}}\, t, \tag{17}$$

The value of $k$ can be approximately determined by various approximate methods (Ozis and Yildirim 2007, 2009). Among others, hereby we use the residual method. Substituting (17) into (8) results in the following residual

$$R(t;k) = -\omega^2 \Delta \cos(\omega t) + \left(\Delta \cos(\omega t)\right) \cdot f\left(\Delta \cos(\omega t)\right) \tag{18}$$

where $\omega = \sqrt{\dfrac{f_{\min} + k f_{\max}}{1+k}}$.

If, by chance, Eq. (17) is the exact solution, then $R(t;k)$ is vanishing completely. Since our approach is only an approximation to the exact solution, we set

$$\int_0^T R(t;k) \cos \sqrt{\frac{f_{\min} + k f_{\max}}{1+k}}\, t\, dt = 0, \tag{19}$$

where $T=2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{f_{\max} - f_{\min}}{1 - \sqrt{\dfrac{\Delta}{\pi} \displaystyle\int_0^{\pi} \cos^2 t . f\left(\Delta \cos t\right) dt}}. \tag{20}$$

Substituting the above equation into Eq. (17), we obtain the approximate solution of Eq. (8).

## 4. Application of MMA

We can re-write Eq. (5) in the following form

$$\ddot{W} + \left(\alpha_1 + P\alpha_2 + K_0 + \alpha_3 W^2\right) W = 0 \tag{21}$$

We choose a trial-function in the form

$$W(t) = \Delta \cos(\omega t) \tag{22}$$

where $\omega$ the frequency to be is determined. By using the trial-function, the maximum and minimum values of $\omega^2$ will be

$$\omega_{\min} = \frac{\alpha_1 + P\alpha_2 + K_0}{1}, \quad \omega_{\max} = \frac{\alpha_1 + P\alpha_2 + K_0 + \alpha_3 \Delta^2}{1}. \tag{23}$$

So we can write

$$\frac{\alpha_1 + P\alpha_2 + K_0}{1} < \omega^2 < \frac{\alpha_1 + P\alpha_2 + K_0 + \alpha_3 \Delta^2}{1} \tag{24}$$

According to the Chengtian's inequality (He 2008), we have

$$\omega^2 = \frac{m.\left(\alpha_1 + P\alpha_2 + K_0\right) + n.\left(\alpha_1 + P\alpha_2 + K_0 + \alpha_3 \Delta^2\right)}{m+n} = \alpha_1 + P\alpha_2 + K_0 + k\alpha_3 \Delta^2 \tag{25}$$

where $m$ and $n$ are weighting factors, $k=n/m+n$. Therefore the frequency can be approximated as

$$\omega = \sqrt{\left(\alpha_1 + p\alpha_2 + K_0\right) + k\alpha_3 \Delta^2} \tag{26}$$

Its approximate solution reads

$$w(t) = \Delta \cos \sqrt{\left(\alpha_1 + p\alpha_2 + K_0\right) + k\alpha_3 \Delta^2}\, t \tag{27}$$

In view of the approximate solution, Eq. (26), we re-write Eq.(21) in the form

$$\ddot{w} + (\alpha_1 + P\alpha_2 + K_0 + k\alpha_3 \Delta) w = (k\alpha_3 \Delta) w - \alpha_3 w^3 \tag{28}$$

If by any chance Eq. (26) is the exact solution, then the right side of Eq. (28) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^{T/4} \left((k\alpha_3 \Delta) w - \alpha_3 w^3\right) \cos \omega t\, dt = 0 \tag{29}$$

where $T=2\pi/\omega$. Solving the above equation, we can easily obtain

$$k = \frac{3}{4} \tag{30}$$

Finally, the frequency is obtained as

$$\omega_{Nonlinear} = \frac{1}{2}\sqrt{4\left(\alpha_1 + p\alpha_2 + K_0\right) + 3\alpha_3 \Delta^2} \tag{31}$$

$$\omega_{Linear} = \sqrt{\alpha_1 + P\alpha_2 + K_0} \tag{32}$$

Hence, the approximate solution can be readily obtained

$$w(t) = \Delta \cos\left(\frac{1}{2}\sqrt{4\left(\alpha_1 + p\alpha_2 + K_0\right) + 3\alpha_3 \Delta^2}\; t\right) \tag{33}$$

The ratio of Non-linear frequency ($\omega_{NL}$) to Linear frequency $\omega_L$ is

$$\frac{\omega_{NL}}{\omega_L} = \frac{1}{2} \frac{\sqrt{4\left(\alpha_1 + p\alpha_2 + K_0\right) + 3\alpha_3 \Delta^2}}{\sqrt{\alpha_1 + p\alpha_2 + K_0}} \tag{34}$$

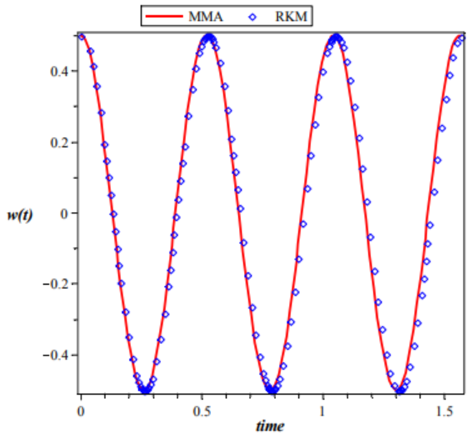
## 5. Results and discussion

The Max-Min Approach (MMA) is used to obtain an analytical solution for simply supported beam at constant elastic modulus. To obtain numerical solution we must specify the parameter $\beta$. This parameter depends on value of $\alpha_1$, $\alpha_2$, $\alpha_3$ and $p$, then we have

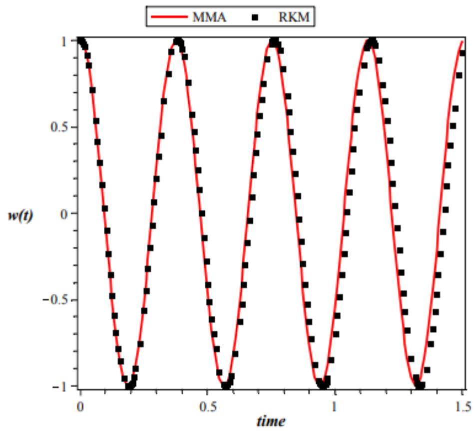
$$\beta = \frac{\alpha_3}{\left(p\alpha_2 + \beta_1 + K_0\right)} \tag{35}$$

Table 1 Comparison of nonlinear to linear frequency ratio ($\omega_{NL}/\omega_L$) for simply-supported beams

| $\alpha$ | $\beta$ | Present Study (MMA) | EBM (Javanmard *et al.* 2013) | Pade approximate P{4,2}(Azrar *et al.* 1999) | Pade approximate P{6,4}(Azrar *et al.* 1999) |
|---|---|---|---|---|---|
| 0.2 | 3 | 1.04403 | 1.04403 | 1.04388 | 1.04388 |
| 0.6 | 3 | 1.34536 | 1.34536 | 1.33973 | 1.33970 |
| 1 | 3 | 1.80277 | 1.80277 | 1.78468 | 1.78442 |
| 1.5 | 3 | 2.46221 | 2.46221 | 2.42618 | 2.42541 |
| 2 | 3 | 3.16227 | 3.16227 | 3.10845 | 3.10712 |
| 2.5 | 3 | 3.88104 | 3.88104 | 3.80991 | 3.80802 |
| 3 | 3 | 4.60977 | 4.60977 | 4.52172 | 4.51927 |




(a)




(b)

Fig. 2 Comparison of MMA solution of $w(t)$ based on time with the Runge-kutta solution for simply supported beam (a) $a$=0.5, $p$=20, $K_0$=200 and (b) $a$=1, $p$=10, $K_0$=100

So Eq. (34) become

$$\frac{\omega_{NL}}{\omega_L} = \sqrt{1+\frac{3}{4}\beta\Delta^2} \tag{36}$$

For simply supported beam the trial function $\phi(X)=sin(\pi X)$ is assumed. The first mode of the vibration give the most behavior of the systems during any excitations.

Table 1 represents the comparison of present study with the results obtained by (Azrar *et al.* 1999) for different values of amplitude and $\beta$.

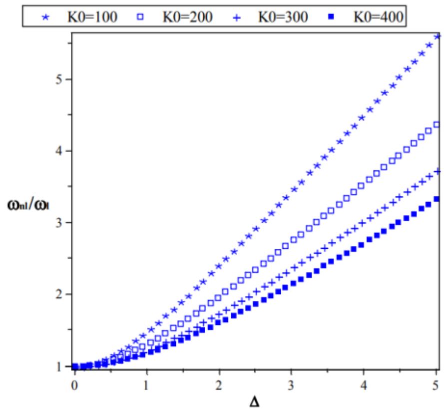



Fig. 3 Influence of $K_0$ on nonlinear to linear frequency base on $\Delta$ for $p$=5

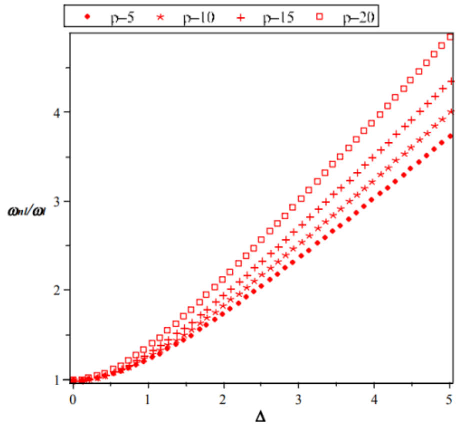



Fig. 4 Influence of axial load on nonlinear to linear frequency base on $\Delta$ for $K_0$=300

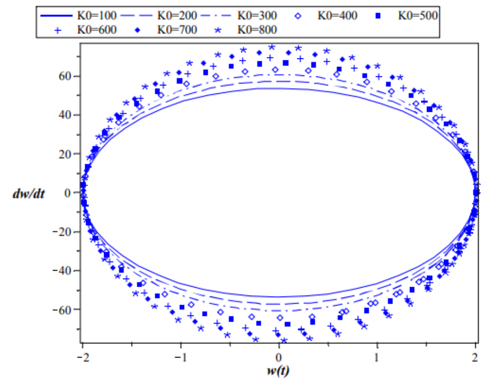
The numerical procedure of the problem has been presented in Appendix A. The presented analytical approach are compared with the results of EBM and Azrar *et al.* (1999).

The displacement comparison are shown in Fig. 2 for simply supported beam (a) $a$=0.5, $p$=20, $K_0$=200 (b) $a$=1, $p$=10, $K_0$=100. Fig. 3 is Influence of $K_0$ on nonlinear to linear frequency base on $\Delta$ for $p$=5. The influence of axial load on nonlinear to linear frequency base on $\Delta$ for $K_0$=300 are shown in figure 4.

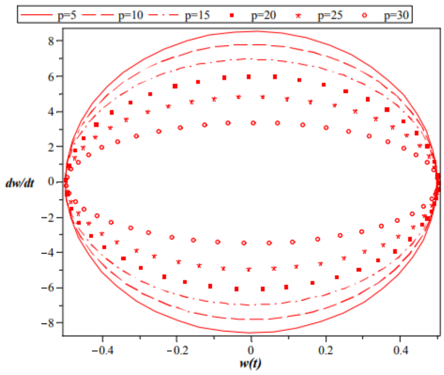
Fig. 5 is the Effect of $K_0$ and $p$ parameters on phase-plan diagram for the cases (A): $a$=2, $p$=20 and (B): $a$=0.5, $K_0$=200.

Fig. 6 is shown the sensitivity analysis of frequency MMA solution for various parameters.

By increasing $\Delta$ the ratio of nonlinear to linear frequency are increased and decreasing the stiffness it leads to lower values of frequency. In general, large vibration amplitude will yield a higher frequency ratio. In large amplitude ratios, the effect of the non-linearity due to mid-plane stretching is dominant and neglecting it introduces error in the results. It has been shown that the only one iteration of the proposed method is preparing a high accurate solution for whole domain.

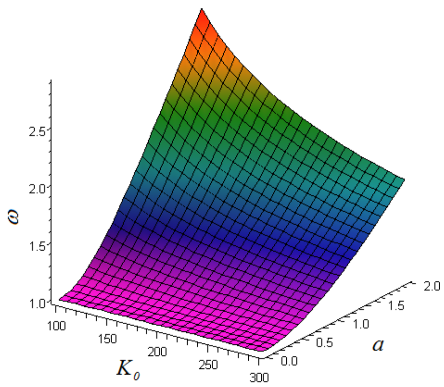



(a)

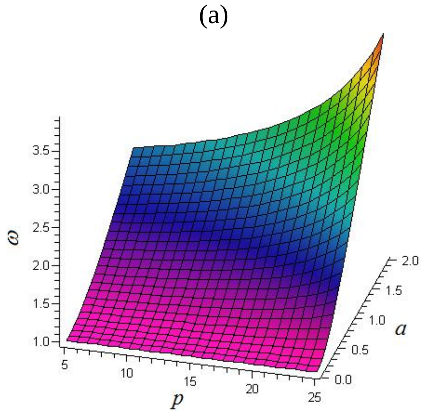



(b)

Fig. 5 Effect of $K_0$ and $p$ parameters on phase-plan diagram for the cases (a) $a$=2, $p$=20 and (b) $a$=0.5, $K_0$=200




(a)




(b)

Fig. 6 Sensitivity analysis of frequency MMA solution for various parameters

## 6. Conclusions

Nonlinear dynamic response of an Euler-Bernoulli beam resting on a Winkler elastic foundation and subjected to the axial loads has been solved analytically by using a new novel method called Max-Min Approach (MMA) in time domain. Winkler approach is used widely to the beams and pipelines resting on an elastic soil. In the present work, we assume the elastic coefficient of the springs is constant. As shown in this paper, the results of Max-Min Approach (MMA) have an excellent agreement with the numerical solutions. Its excellent accuracy in the whole range of oscillation amplitude values is one of the most significant features of this method. The successful application of the Max-Min Approach (MMA) for the large-amplitude beam vibration problem is considered in this study. The Max-Min Approach (MMA) could be easily extended to any nonlinear vibration problems, which provide an easy and direct procedure for determining approximations to the periodic solutions.

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## Nomenclature

$A$ cross-sectional area

$L$ beam length

$W'$ normal displacement

$E$ Young's modulus

$X$ axial coordinate

$\overline{P}$ axial load

$M$ mass per unit length

$\phi(X)$ trial function

$t$ time

$K'$ elastic coefficient of Winkler foundation

$EA$ axial rigidity of the beam cross section

$EI$ bending rigidity of the beam cross section

$w(t)$ time-dependent deflection parameter

$A$ dimensionless maximum amplitude of oscillation

$\beta$ parameter of boundary condition of beam

$\omega_{NL}$ nonlinear frequency

$\omega_L$ linear frequency

## Appendix A: Basic idea of Runge-Kutta's algorithm

For such a boundary value problem given by boundary condition, some numerical methods have been developed. Here we apply the fourth-order RK algorithm to solve governing equations subject to the given boundary conditions. RK iterative formulae for the second-order differential equations are

$$\begin{aligned} \dot{w}_{(i+1)} &= \dot{w}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + h_4\right), \\ w_{(i+1)} &= w_i + \Delta t \left[\dot{w}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + h_3\right)\right], \end{aligned} \tag{A.1}$$

where $\Delta t$ is the increment of the time and $h_1$,$h_2$,$h_3$ and $h_4$ are determined from the following formulas

$$\begin{aligned} h_1 &= f\left(t_i, w_i, \dot{w}_i\right), \\ h_2 &= f\left(t_i + \frac{\Delta t}{2},\ w_i + \frac{\Delta t}{2},\ \dot{w}_i + \frac{\Delta t}{2} h_1\right), \\ h_3 &= f\left(t_i + \frac{\Delta t}{2},\ w_i + \frac{\Delta t}{2}\dot{w}_i,\ \frac{1}{4}\Delta t^2 h_1,\ \dot{w}_i + \frac{\Delta t}{2} h_2\right), \end{aligned} \tag{A.2}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative is determined from the initial conditions. Then, with a small time increment [$\Delta t$], the displacement function and its first-order derivative at the new position can be obtained using (A.2). This process continues to the end of time.