**Geomechanics and Engineering**, *Vol. 12, No. 1 (2017) 9-34*
DOI: https://doi.org/10.12989/gae.2017.12.1.009


# A novel quasi-3D hyperbolic shear deformation theory for functionally graded thick rectangular plates on elastic foundation

Abdelkarim Benahmed [1], Mohammed Sid Ahmed Houari [1,2],
Samir Benyoucef [1], Khalil Belakhdar [1] and Abdelouahed Tounsi [*1,3,4]

[1] *Material and Hydrology Laboratory, University of Sidi Bel Abbes, Faculty of Technology, Civil Engineering Department, Algeria*
[2] *Université Mustapha Stambouli de Mascara, Department of Civil Engineering, Mascara, Algeria*
[3] *Laboratoire de Modélisation et Simulation Multi-échelle, Département de Physique, Faculté des Sciences Exactes, Département de Physique, Université de Sidi Bel Abbés, Algeria*
[4] *Algerian National Thematic Agency of Research in Science and Technology (ATRST), Algeria*

(*Received August 14, 2015, Revised August 14, 2016, Accepted September 08, 2016*)

**Abstract.** In this work, an efficient and simple quasi-3D hyperbolic shear deformation theory is developed for bending and vibration analyses of functionally graded (FG) plates resting on two-parameter elastic foundation. The significant feature of this theory is that, in addition to including the thickness stretching effect, it deals with only 5 unknowns as the first order shear deformation theory (FSDT). The foundation is described by the Pasternak (two-parameter) model. The material properties of the plate are assumed to vary continuously in the thickness direction by a simple power law distribution in terms of the volume fractions of the constituents. Equations of motion for thick FG plates are obtained within the Hamilton's principle. Analytical solutions for the bending and free vibration analysis are obtained for simply supported plates. The numerical results are given in detail and compared with the existing works such as 3-dimensional solutions and those predicted by other plate theories. It can be concluded that the present theory is not only accurate but also simple in predicting the bending and free vibration responses of functionally graded plates resting on elastic foundation.

**Keywords:** bending; free vibration; functionally graded plate; elastic foundation; quasi-3D hyperbolic shear deformation theory

## 1. Introduction

Functionally graded materials (FGMs) are a class of composites that have continuous variation of material properties from one surface to another and thus eliminate the stress concentration found in laminated composites. A typical FGM is made from a mixture of two material phases, for example, a ceramic and a metal. The reason for the increasing use of FGMs in a variety of aerospace, automotive, civil, and mechanical engineering structures is that their material properties can be tailored to different applications and working environments (Reddy 2000, Qian and Batra 2005, Bachir Bouiadjra *et al*. 2013, Attia *et al*. 2015, Hamidi *et al*. 2015, Darılmaz 2015, Ebrahimi and Dashti 2015, Bouguenina *et al*. 2015, Akbaş 2015, Arefi 2015, Pradhan and Chakraverty 2015,

*Corresponding author, Professor, E-mail: tou_abdel@yahoo.com


http://www.techno-press.org/?journal=gae&subpage=7 ISSN: 2005-307X (Print), 2092-6219 (Online)

Kar and Panda 2015, Bouderba *et al*. 2016, Beldjelili *et al*. 2016, Ebrahimi and Habibi 2016, Hadji *et al*. 2016, Moradi-Dastjerdi 2016, Laoufi *et al*. 2016, Bousahla *et al*. 2016, Ebrahimi and Salari 2016, Trinh *et al*. 2016).

Plates resting on elastic foundation can be found in various structural engineering fields. A two-parameter model of Pasternak (1954) which considers the shear deformation between the springs had been proposed to describe the interaction between the plate and foundation. The Winkler model (1867) is a special case of Pasternak model by setting the shear modulus to zero.

Mechanical response of functionally graded plates (FGs) resting on a Winkler–Pasternak elastic foundation has been investigated in some research papers. Huang *et al*. (2008) used a 3D theory of elasticity to study FG thick plates on elastic foundation. Based on 3D elasticity theory, Malekzadeh (2009) studied the free vibration of FG plates resting on elastic foundation. Amini *et al*. (2009) studied 3D free vibration response of FG plates supported by elastic foundation. Using the state space approach, Lü *et al*. (2009) proposed exact solutions for free vibrations of FG thick plates resting on two-parameter elastic foundation. Ait Atmane *et al*. (2010) analysed the free vibration of FG plates supported by elastic foundation by using a new shear hyperbolic deformation theory and Navier procedure. Using the parabolic shear deformation theory, Baferani *et al*. (2011) established an accurate method for vibration of FG thick plates supported by elastic foundation. Fallah *et al*. (2013) studied the vibration of FG plates resting on elastic foundation using the extended Kantorovich method together with infinite power series solution. Sheikholeslami and Saidi (2013) employed the higher-order shear and normal deformation plate theory together with an analytical formulation to investigate the free vibration behaviour of simply supported FG plates resting on elastic foundation. Sobhy (2013) investigated the free vibration and buckling responses of exponentially graded material sandwich plate supported by Winkler–Pasternak elastic foundation. An analytical approach based on the first-order shear deformation plate theory is presented by Yaghoobi and Yaghoobi (2013) to study the thermo-mechanical buckling of symmetric sandwich plates with FG face sheets resting on two-parameter elastic foundation. Hadji *et al*. (2011) investigated the free vibration analysis of functionally graded material (FGM) sandwich rectangular plates. The theory presented is variationally consistent and strongly similar to the classical plate theory in many aspects. It does not require the shear correction factor, and gives rise to the transverse shear stress variation so that the transverse shear stresses vary parabolically across the thickness to satisfy free surface conditions for the shear stress. Benachour *et al*. (2011) developed a model for free vibration analysis of plates made of functionally graded materials with an arbitrary gradient. Closed form solutions are obtained by using Navier technique, and then fundamental frequencies are found by solving the results of eigenvalue problems. El Meiche *et al*. (2011) developed a refined hyperbolic shear deformable plate theory for buckling and vibration of FGM sandwich plates. Tounsi *et al*. (2013) studied the thermoelastic bending response of FG sandwich plates using a refined trigonometric shear deformation theory. Zidi *et al*. (2014) studied the bending response of functionally graded material (FGM) plate resting on elastic foundation and subjected to hygro-thermo-mechanical loading. Ait Amar Meziane *et al*. (2014) developed an efficient and simple refined shear deformation theory for the bucking and vibration analyses of EGM sandwich plates supported by elastic foundations with considering various types of boundary conditions. Khalfi *et al*. (2014) examined the thermal buckling behavior of solar FG plates resting on elastic foundation. Hebali *et al*. (2014) developed a new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of FG plates. Mahi *et al*. (2015) proposed a new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates.

Saidi *et al*. (2016) developed a simple hyperbolic shear deformation theory for vibration analysis of thick FG rectangular plates resting on elastic foundations. Wave propagation of porous FG plates using various higher-order shear deformation theories (HSDTs) is studied by Ait Yahia *et al*. (2015). They concluded that higher order theories can accurately predict the wave characteristics of FG structures and only a little difference exists between their results. Also, many papers are published concerning with analysis of FGM structures based on HSDTs (Bourada *et al*. 2012, Belabed *et al*. 2014, Bousahla *et al*. 2014, Bennoun *et al*. 2016, Bellifa *et al*. 2016, Al-Basyouni *et al*. 2015, Ait Atmane *et al*. 2015, Bourada *et al*. 2015, Merazi *et al*. 2015). Additional shear and normal deformation theories are presented in the literature (Ould Larbi *et al*. 2013, Hadji *et al*. 2014, Nedri *et al*. 2014, Draiche *et al*. 2014, Bennai *et al*. 2015, Akavci 2015, Meradjah *et al*. 2015, Bakora and Tounsi 2015, Belkorissat *et al*. 2015, Bouchafa *et al*. 2015, Sallai *et al*. 2015, Tagrara *et al*. 2015, Larbi Chaht *et al*. 2015, Zemri *et al*. 2015, Tebboune *et al*. 2015, Hadji and Adda Bedia 2015a, b, Nguyen *et al*. 2015, Mouaici *et al*. 2016, Boukhari *et al*. 2016, Chikh *et al*. 2016, Barati *et al*. 2016). Bounouara *et al*. (2016) presented a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Bourada *et al*. (2016) analyzed the buckling behavior of isotropic and orthotropic plates using a novel four variable refined plate theory. Recently, Tounsi *et al*. (2016) presented a new 3-unknowns non-polynomial plate theory for buckling and vibration of FG sandwich plate.

According to the works of literature survey, the only work on the natural frequency of FG plates supported on elastic foundation based on the higher-order shear and normal deformation plate theory seems to be presented by Sheikholeslami and Saidi (2013). This latter theory has six unknowns and it is still more complicated than the first shear deformation theory (FSDT). Thus, developing a simple quasi-3D theory is necessary.

The purpose of this work is to develop a simple quasi-3D theory with only five unknowns for free vibration of FG plates resting on a Winkler–Pasternak elastic foundation. The displacement field is chosen based on a hyperbolic variation of in-plane and transverse displacements through the thickness. The partition of the transverse displacement into the bending, shear and stretching parts leads to a reduction of the number of unknowns, and subsequently, makes the new theory simple to use. Using Hamilton's principle, the equations of motion are obtained. Numerical results are illustrated to check the accuracy of the present formulation.

## 2. Theoretical formulations

### *2.1 Kinematics*

The displacement field of the present formulation is obtained based on the following assumptions: (1) The transverse displacement is composed of three components namely: bending, shear and stretching parts; (2) the in-plane displacement is partitioned into extension, bending and shear components; (3) the bending parts of the in-plane displacements are similar to those given by classical plates theory (CPT); and (4) the shear parts of the in-plane displacements give rise to the hyperbolic variations of shear strains and hence to shear stresses through the thickness of the plate in such a way that the shear stresses vanish on the top and bottom surfaces of the plate. Based on these assumptions, the following displacement field relations can be obtained

$$u(x,y,z,t) = u_0(x,y,t) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{1}$$

$$\begin{aligned} v(x,y,z,t) &= v_0(x,y,t) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\ w(x,y,z,t) &= w_b(x,y,t) + w_s(x,y,t) + g(z)\,\varphi(x,y,t) \end{aligned} \tag{1}$$

where $u_0$ and $v_0$ denote the displacements along the $x$ and $y$ coordinate directions of a point on the mid-plane of the plate; $w_b$ and $w_s$ are the bending and shear components of the transverse displacement, respectively; and the additional displacement $\varphi$ accounts for the effect of normal stress (stretching effect). The shape functions $f(z)$ and $g(z)$ are given as follows

$$f(z) = z\left[1 + \frac{3\pi}{2}\sec h^2\left(\frac{1}{2}\right)\right] - \frac{3\pi}{2}h\tanh\left(\frac{z}{h}\right) \tag{2}$$

and

$$g(z) = 1 - f'(z) \tag{3}$$

The non-zero strains associated with the displacement field in Eq. (1) are

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z)\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z)\begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix}, \quad \varepsilon_z = g'(z)\,\varepsilon_z^0 \tag{4}$$

where

$$\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} \\ \dfrac{\partial v_0}{\partial x} \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_b}{\partial x^2} \\ -\dfrac{\partial^2 w_b}{\partial y^2} \\ -2\dfrac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial^2 w_s}{\partial x^2} \\ -\dfrac{\partial^2 w_s}{\partial y^2} \\ -2\dfrac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^0 \\ \gamma_{xz}^0 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial w_s}{\partial y} + \dfrac{\partial \varphi}{\partial y} \\ \dfrac{\partial w_s}{\partial x} + \dfrac{\partial \varphi}{\partial x} \end{Bmatrix}, \tag{5}$$

(5)

$$\varepsilon_z^0 = \varphi$$

and

$$g'(z) = \frac{dg(z)}{dz} \tag{6}$$

### *2.2 Basic equations*

Functionally graded ceramic–metal plates resting on a Winkler–Pasternak elastic foundation are considered in this study as shown in Fig. 1. The Young's modulus $E$ and mass density $\rho$ of such plates are assumed to vary according to a power law distribution in terms of the volume fractions of the constituents as

$$E(z) = E_m + (E_c - E_m)\left(\frac{2z + h}{2h}\right)^k \tag{7a}$$

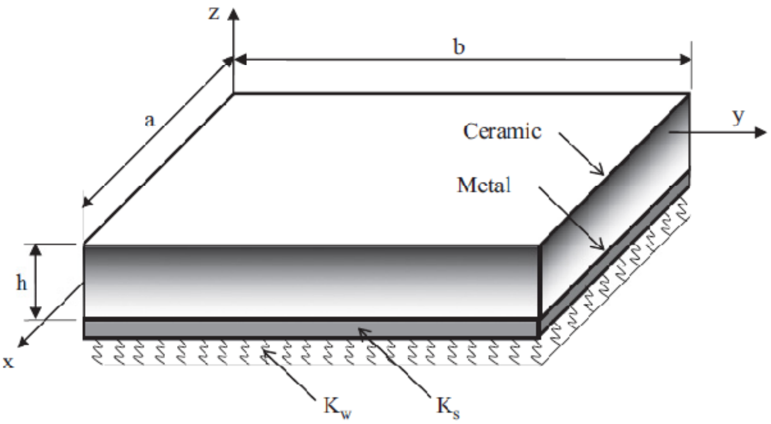


Fig. 1 Geometry and coordinates of the considered FG plate which is resting on elastic foundation

$$\rho(z) = \rho_m + (\rho_c - \rho_m)\left(\frac{2z+h}{2h}\right)^k \tag{7b}$$

where the subscripts $m$ and $c$ indicate the metallic and ceramic constituents, respectively; and $k$ is the power law index. The value of $k$ equal to zero represents a fully ceramic plate, whereas infinite $k$ indicates a fully metallic plate. Since the effects of the variation of Poisson's ratio $\nu$ on the response of FG plates are very small (Yang *et al*. 2005, Kitipornchai *et al*. 2006), it is assumed to be constant for convenience. The linear constitutive relations of a FG plate can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \tag{8}$$

where ($\sigma_x$, $\sigma_y$, $\sigma_z$, $\tau_{yz}$, $\tau_{xz}$, $\tau_{xy}$) and ($\varepsilon_x$, $\varepsilon_y$, $\varepsilon_z$, $\gamma_{yz}$, $\gamma_{xz}$, $\gamma_{xy}$) are the stress and strain components, respectively.

The computation of the elastic constants $C_{ij}$ depends on which assumption of $\varepsilon_z$ we consider. If $\varepsilon_z = 0$, then $C_{ij}$ are the plane stress reduced elastic constants, defined as

$$C_{11} = C_{22} = \frac{E(z)}{1-\nu^2}, \qquad C_{12} = \nu\, C_{11} \tag{9a}$$

$$C_{44} = C_{55} = C_{66} = G(z) = \frac{E(z)}{2(1+\nu)}, \tag{9b}$$

If $\varepsilon_z \neq 0$ (thickness stretching), then $C_{ij}$ are the three-dimensional elastic constants, given by

$$C_{11} = C_{22} = C_{33} = \frac{(1-\nu)}{\nu}\lambda(z)\,, \quad C_{12} = C_{13} = C_{23} = \lambda(z) \tag{10a}$$

$$C_{44} = C_{55} = C_{66} = G(z) = \mu(z) = \frac{E(z)}{2(1+\nu)}, \tag{10b}$$

where $\lambda(z) = \dfrac{\nu\, E(z)}{(1-2\nu)(1+\nu)}$ and $\mu(z) = G(z) = \dfrac{E(z)}{2(1+\nu)}$ are Lamé's coefficients.

Considering the displacement components of the present simple quasi-3D theory in Eq. (1) and using the Hamilton's principle, the equations of motion of FG plates resting on elastic foundation can be obtained as

$$\begin{aligned}
&\delta\, u_0: \ \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = I_0 \ddot{u}_0 - I_1 \frac{\partial \ddot{w}_b}{\partial x} - J_1 \frac{\partial \ddot{w}_s}{\partial x} \\
&\delta\, v_0: \ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = I_0 \ddot{v}_0 - I_1 \frac{\partial \ddot{w}_b}{\partial y} - J_1 \frac{\partial \ddot{w}_s}{\partial y} \\
&\delta\, w_b: \ \frac{\partial^2 M_x^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_y^b}{\partial y^2} + K_w w - K_s \nabla^2 w = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) \\
&\qquad - I_2 \nabla^2 \ddot{w}_b - J_2 \nabla^2 \ddot{w}_s + J_1^s \ddot{\varphi} \\
&\delta\, w_s: \ \frac{\partial^2 M_x^s}{\partial x^2} + 2\frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \frac{\partial^2 M_y^s}{\partial y^2} + \frac{\partial S_{xz}^s}{\partial x} + \frac{\partial S_{yz}^s}{\partial y} + K_w w - K_s \nabla^2 w = I_0(\ddot{w}_b + \ddot{w}_s) \\
&\qquad + J_1\left(\frac{\partial \ddot{u}_0}{\partial x} + \frac{\partial \ddot{v}_0}{\partial y}\right) - J_2 \nabla^2 \ddot{w}_b - K_2 \nabla^2 \ddot{w}_s + J_1^s \ddot{\varphi} \\
&\delta\varphi: \ \frac{\partial S_{xz}^s}{\partial x} + \frac{\partial S_{yz}^s}{\partial y} - N_z = J_1^s(\ddot{w}_b + \ddot{w}_s) + K_2^s \ddot{\varphi}
\end{aligned} \tag{11}$$

In the above equations dot above each parameter denotes partial differentiating with respect to time. The parameters $K_w$ and $K_s$ are the Winkler and Pasternak parameters for elastic foundation. Also the stress resultants ($N$, $M^b$, $M^s$, $S^s$ and $N_z$) and the mass inertias ($I_0$, $I_1$, $J_1$, $I_2$, $J_2$, $K_2$) are as follows

$$\begin{aligned}
&(N_i, M_i^b, S_i^b) = \int_{-h/2}^{h/2} (1, z, f)\sigma_i dz\,, \quad (i = x, y, xy), \\
&S_i^s = \int_{-h/2}^{h/2} \tau_i g(z) dz.\,, \quad (i = xz, yz) \quad \text{and} \quad N_z = \int_{-h/2}^{h/2} \sigma_z g'(z) dz,
\end{aligned} \tag{12}$$

$$(I_0, I_1, J_1, J_1^s, I_2, J_2, K_2, K_2^s) = \int_{-h/2}^{h/2} (1, z, f, g, z^2, z\, f, f^2, g^2)\rho(z) dz \tag{13}$$

Substituting Eqs. (12) and (13) into (11) and using stress–strain relations, the governing equations of motion are obtained as

$$A_{11}d_{11}u_0 + A_{66}d_{22}u_0 + (A_{12} + A_{66})d_{12}v_0 - B_{11}d_{111}w_b - (B_{12} + 2B_{66})d_{122}w_b \\ - (B_{12}^s + 2B_{66}^s)d_{122}w_s - B_{11}^s d_{111}w_s + Ld_1\varphi = I_0\ddot{u}_0 - I_1 d_1\ddot{w}_b - J_1 d_1\ddot{w}_s, \tag{14a}$$

$$A_{22}d_{22}v_0 + A_{66}d_{11}v_0 + (A_{12} + A_{66})d_{12}u_0 - B_{22}d_{222}w_b - (B_{12} + 2B_{66})d_{112}w_b \\ - (B_{12}^s + 2B_{66}^s)d_{112}w_s - B_{22}^s d_{222}w_s + Ld_2\varphi = I_0\ddot{v}_0 - I_1 d_2\ddot{w}_b - J_1 d_2\ddot{w}_s, \tag{14b}$$

$$B_{11}d_{111}u_0 + (B_{12} + 2B_{66})d_{122}u_0 + (B_{12} + 2B_{66})d_{112}v_0 + B_{22}d_{222}v_0 \\ - D_{11}d_{1111}w_b - 2(D_{12} + 2D_{66})d_{1122}w_b - D_{22}d_{2222}w_b - D_{11}^s d_{1111}w_s \\ - 2(D_{12}^s + 2D_{66}^s)d_{1122}w_s - D_{22}^s d_{2222}w_s + L^a(d_{11}\varphi + d_{22}\varphi) \\ + K_w(w_b + w_s) - K_s\nabla^2(w_b + w_s) = I_0(\ddot{w}_b + \ddot{w}_s) + I_1(d_1\ddot{u}_0 + d_2\ddot{v}_0) \\ - I_2(d_{11}\ddot{w}_b + d_{22}\ddot{w}_b) - J_2(d_{11}\ddot{w}_s + d_{22}\ddot{w}_s) + J_1^s\ddot{\varphi} \tag{14c}$$

$$B_{11}^s d_{111}u_0 + (B_{12}^s + 2B_{66}^s)d_{122}u_0 + (B_{12}^s + 2B_{66}^s)d_{112}v_0 + B_{22}^s d_{222}v_0 - D_{11}^s d_{1111}w_b \\ - 2(D_{12}^s + 2D_{66}^s)d_{1122}w_b - D_{22}^s d_{2222}w_b - H_{11}^s d_{1111}w_s \\ - 2(H_{12}^s + 2H_{66}^s)d_{1122}w_s - H_{22}^s d_{2222}w_s + A_{44}^s d_{11}w_s + A_{55}^s d_{22}w_s + R(d_{11}\varphi + d_{22}\varphi) \\ + A_{44}^s d_{11}\varphi + A_{55}^s d_{22}\varphi + K_w(w_b + w_s) - K_s\nabla^2(w_b + w_s) = I_0(\ddot{w}_b + \ddot{w}_s) \\ + J_1(d_1\ddot{u}_0 + d_2\ddot{v}_0) - J_2(d_{11}\ddot{w}_b + d_{22}\ddot{w}_b) - K_2(d_{11}\ddot{w}_s + d_{22}\ddot{w}_s) + J_1^s\ddot{\varphi} \tag{14d}$$

$$L(d_1u_0 + d_2v_0) - L^a(d_{11}w_b + d_{22}w_b) + (R - A_{44}^s)d_{11}w_s + (R - A_{55}^s)d_{22}w_s \\ + R^a\varphi - A_{44}^s d_{11}\varphi - A_{55}^s d_{22}\varphi = J_1^s(\ddot{w}_b + \ddot{w}_s) + K_2^s\ddot{\varphi} \tag{14e}$$

where $d_{ij}$, $d_{ijl}$ and $d_{ijlm}$ are the following differential operators

$$d_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}, \quad d_{ijl} = \frac{\partial^3}{\partial x_i \partial x_j \partial x_l}, \quad d_{ijlm} = \frac{\partial^4}{\partial x_i \partial x_j \partial x_l \partial x_m}, \quad d_i = \frac{\partial}{\partial x_i}, \qquad (i, j, l, m = 1,2). \tag{15}$$

The stiffness coefficients used in Eq. (14) are defined as

$$\begin{Bmatrix} A_{11} & B_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & B_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{66} & B_{66} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix} = \int_{-h/2}^{h/2} \lambda(z)\left(1, z, z^2, f(z), z\,f(z), f^2(z)\right) \begin{Bmatrix} \dfrac{1-\nu}{\nu} \\ 1 \\ \dfrac{1-2\nu}{2\nu} \end{Bmatrix} dz \tag{16a}$$

and

$$\left(A_{22}, B_{22}, D_{22}, B_{22}^s, D_{22}^s, H_{22}^s\right) = \left(A_{11}, B_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s\right) \tag{16b}$$

$$A_{44}^s = A_{55}^s = \int_{-h/2}^{h/2} \mu(z)[g(z)]^2 dz, \tag{16c}$$

## 3. Closed-form solution for simply supported plates

Based on Navier procedure, the following expansions of generalized displacements are chosen to automatically satisfy the simply supported boundary conditions

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_b \\ w_s \\ \varphi \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} U_{mn} e^{i\omega t} \cos(\lambda\, x)\sin(\mu\, y) \\ V_{mn} e^{i\omega t} \sin(\lambda\, x)\cos(\mu\, y) \\ W_{bmn} e^{i\omega t} \sin(\lambda\, x)\sin(\mu\, y) \\ W_{smn} e^{i\omega t} \sin(\lambda\, x)\sin(\mu\, y) \\ \Phi_{mn} e^{i\omega t} \sin(\lambda\, x)\sin(\mu\, y) \end{Bmatrix} \tag{17}$$

where $U_{mn}$, $V_{mn}$, $W_{bmn}$, $W_{smn}$ and $\Phi_{mn}$ unknown parameters must be determined, $\omega$ is the eigenfrequency associated with $(m, n)^{\text{th}}$ eigenmode, and $\lambda = m\,\pi\,/\,a$ and $\mu = m\,\pi\,/\,b$.

Substituting Eq. (17) into Eq. (14), the analytical solutions can be obtained by

$$\left( \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{13} & a_{23} & a_{33} & a_{34} & a_{35} \\ a_{14} & a_{24} & a_{34} & a_{44} & a_{45} \\ a_{15} & a_{25} & a_{35} & a_{45} & a_{55} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & 0 & m_{13} & m_{14} & 0 \\ 0 & m_{22} & m_{23} & m_{24} & 0 \\ m_{13} & m_{23} & m_{33} & m_{34} & m_{35} \\ m_{14} & m_{24} & m_{34} & m_{44} & m_{45} \\ 0 & 0 & m_{35} & m_{45} & m_{55} \end{bmatrix} \right) \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \\ \Phi_{mn} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{18}$$

in which

$$\begin{aligned}
a_{11} &= -\left(A_{11}\lambda^2 + A_{66}\mu^2\right) \\
a_{12} &= -\lambda\,\mu\left(A_{12} + A_{66}\right) \\
a_{13} &= \lambda\,[B_{11}\lambda^2 + (B_{12} + 2B_{66})\,\mu^2] \\
a_{14} &= \lambda\,[B^s_{11}\lambda^2 + (B^s_{12} + 2B^s_{66})\,\mu^2] \\
a_{15} &= L\lambda \\
a_{22} &= -\left(A_{66}\lambda^2 + A_{22}\mu^2\right) \\
a_{23} &= \mu\,[(B_{12} + 2B_{66})\,\lambda^2 + B_{22}\mu^2] \\
a_{24} &= \mu\,[(B^s_{12} + 2B^s_{66})\,\lambda^2 + B^s_{22}\mu^2] \\
a_{25} &= L\mu \\
a_{33} &= -\left(D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 + K_w + J_1\lambda^2 + J_2\mu^2\right) \\
a_{34} &= -\left(D^s_{11}\lambda^4 + 2(D^s_{12} + 2D^s_{66})\lambda^2\,\mu^2 + D^s_{22}\,\mu^4 + K_w + J_1\lambda^2 + J_2\mu^2\right) \\
a_{35} &= -L^a\left(\lambda^2 + \mu^2\right)
\end{aligned} \tag{19}$$

$$
\begin{aligned}
&a_{44} = -\left(H_{11}^s\lambda^4 + 2(H_{11}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 + A_{55}^s\lambda^2 + A_{44}^s\mu^2 + K_w + J_1\lambda^2 + J_2\mu^2\right)\\
&a_{45} = -\left(A_{44}^s\lambda^2 + A_{55}^s\mu^2 + R\left(\lambda^2 + \mu^2\right)\right)\\
&a_{55} = -\left(A_{44}^s\lambda^2 + A_{55}^s\mu^2 + R^a\right)\\
&m_{11} = m_{22} = -I_0\,,\;\; m_{13} = \lambda I_1\,,\;\; m_{14} = \lambda J_1\,,\;\; m_{23} = \mu I_1\,,\;\; m_{24} = \mu J_1\,,\\
&m_{33} = -\left(I_0 + I_2\left(\lambda^2 + \mu^2\right)\right),\;\; m_{34} = -\left(I_0 + J_2\left(\lambda^2 + \mu^2\right)\right),\\
&m_{44} = -\left(I_0 + K_2\left(\lambda^2 + \mu^2\right)\right),\;\; m_{35} = m_{45} = -J_1^s\,,\;\; m_{55} = -K_2^s
\end{aligned}
\tag{19}
$$

## 4. Numerical results

In this section, various numerical examples are presented and discussed to verify the accuracy of the present theory in predicting the bending and free vibration responses of simply supported isotropic and FG plates resting on Winkler-Pasternak elastic foundations. For bending analysis, a plate subjected to a uniformly distributed load (UDL) and sinusoidal load is considered.

Numerical results are presented in terms of non-dimensional stresses and deflection. The various nondimensional parameters used are

$$
\hat{w} = \frac{100E}{q_0 h S^4} w\left(\frac{a}{2},\frac{b}{2},\bar{z}\right),\quad \hat{\sigma}_x = \frac{1}{q_0 S^2}\sigma_x\left(\frac{a}{2},\frac{b}{2},\bar{z}\right),\quad \hat{\sigma}_y = \frac{1}{q_0 S^2}\sigma_y\left(\frac{a}{2},\frac{b}{2},\bar{z}\right),
$$

$$
\hat{\tau}_{xy} = \frac{1}{q_0 S^2}\tau_{xy}\left(0,0,\bar{z}\right),\quad \bar{\tau}_{yz} = \frac{1}{q_0 S}\tau_{yz}\left(\frac{a}{2},0,\bar{z}\right),\quad \bar{\tau}_{xz} = \frac{1}{q_0 S}\tau_{xz}\left(0,\frac{b}{2},\bar{z}\right).
$$

$$
S = a/h\,,\quad \bar{z} = z/h
$$

$$
\overline{u_x} = \frac{100D}{q_0 a^4}u_x\left(0,\frac{b}{2},-\frac{h}{2}\right),\quad \overline{u_z} = \frac{100D}{q_0 a^4}u_z\left(\frac{a}{2},\frac{b}{2},0\right),
$$

$$
\overline{\sigma_x} = -\frac{h^2}{q_0 a^2}\sigma_x\left(\frac{a}{2},\frac{b}{2},-\frac{h}{2}\right),\quad \overline{\sigma_z} = \frac{1}{q_0}\sigma_z\left(\frac{a}{2},\frac{b}{2},z\right),\quad \overline{\tau_{xy}} = \frac{h^2}{q_0 a^2}\tau_{xy}\left(0,0,-\frac{h}{2}\right)
$$

$$
\overline{\tau_{xz}} = \frac{1}{q_0}\tau_{xz}\left(0,\frac{b}{2},0\right),\quad k_w = \frac{K_w a^4}{h^3},\quad k_s = \frac{K_s a^2}{h^3\nu} = \frac{K_2 b^2}{h^3\nu},\quad D = Eh^3/12\left(1-\nu^2\right).
$$

$$
\tau_{xy}^{\ *} = \frac{1}{10q_0}\tau_{xy}\left(0,0,-\frac{h}{3}\right),\quad \tau_{xz}^{\ *} = \frac{1}{10q_0}\tau_{xz}\left(0,\frac{b}{2},0\right)
$$

$$
\tilde{\omega} = \frac{\omega a^2}{\pi^2}\sqrt{\rho_C h/D_C}\,,\quad D_C = E_C h^3/12\left(1-\nu^2\right)
$$

$$
\overline{\omega} = \frac{\omega\, a^2}{\mathrm{h}}\sqrt{\rho_m/E_m}\,,\quad \hat{\omega} = \omega h\sqrt{\rho_m/E_m}
$$

### *4.1 Bending analysis*

As a first example, the deflections and dimensionless stresses of a square isotropic plate ($a$ / $h$ =

Table 1 Material Properties for the FG Plate

| | Metal | Ceramic | |
|---|---|---|---|
| Properties | AL | $Al_2O_3$ | $ZrO_2$ |
| $E$ (GPa) | 66.2 | 380 | 117.0 |
| $\nu$ | 1/3 | 1/3 | 1/3 |
| $\rho$ (kg/m$^3$) | | | |

Table 2 Effect of normal strain $\varepsilon_z$ on the dimensionless stresses and transversal displacement for isotropic square plate ($a$ / $h$ = 10) subjected to a UDL

| Theory | $\hat{w}(a/2,b/2,0)$ | $\hat{\sigma}_x(h/2)$ | $\hat{\sigma}_y(h/2)$ | $\hat{\tau}_{xy}(h/2)$ | $\bar{\tau}_{xz}(0,b/2,0)$ | $\bar{\tau}_{yz}(a/2,0,0)$ |
|---|---|---|---|---|---|---|
| Present $\varepsilon_z \neq 0$ | 4.633 | 0.302 | 0.302 | 0.197 | 0.481 | 0.502 |
| Shimpi *et al*. (2003) $\varepsilon_z \neq 0$ | 4.625 | 0.307 | 0.307 | 0.195 | 0.505 | 0.505 |
| Exact 3D (Srinivas *et al*. 1970a) | 4.639 | 0.290 | 0.290 | / | 0.488 | / |

Table 3 Comparison of nondimensional deflection $D10^3\, w(0.5a, 0.5b, z = 0)/qa^4$ of simply supported isotropic thin square plate under uniformly distributed load ($a/h$ = 100)

| $k_w$ | $k_s$ | $D10^3\, w(0.5a, 0.5b, z = 0)/qa^4$ | | | |
|---|---|---|---|---|---|
| | | Present $\varepsilon_z \neq 0$ | Benyoucef *et al*. (2010) | 3D (Huang *et al*. 2008) | Lam *et al*. (2000) |
| 1 | 1 | 3.8490 | 3.8550 | 3.8546 | 3.853 |
| | $3^4$ | 0.7628 | 0.7630 | 0.7630 | 0.763 |
| | $5^4$ | 0.1153 | 0.1153 | 0.1153 | 0.115 |
| $3^4$ | 1 | 3.2067 | 3.2108 | 3.2105 | 3.210 |
| | $3^4$ | 0.7316 | 0.7317 | 0.7317 | 0.732 |
| | $5^4$ | 0.1145 | 0.1145 | 0.1145 | 0.115 |
| $5^4$ | 1 | 1.4759 | 1.4765 | 1.4765 | 1.476 |
| | $3^4$ | 0.5703 | 0.5704 | 0.5704 | 0.570 |
| | $5^4$ | 0.1095 | 0.1095 | 0.1095 | 0.109 |

10) subjected to a UDL are compared in Table 2 with those given by the quasi-3D solutions of Shimpi *et al*. (2003), and the exact solution carried out by Srinivas *et al*. (1970a). It can be seen from this table that the results are in close agreement.

The second example to validate the present method for plates resting on an elastic foundation, the results for dimensionless deflections of a thick isotropic plate are compared with results published previously. Table 3 presents the center deflections of a uniformly loaded homogeneous square plate simply supported on a Winkler-Pasternak foundation. The results are compared with those of Benyoucef *et al*. (2010), 3D solution (Huang *et al*. 2008) and Lam *et al*. (2000). It can be seen that the results agree closely.

In the third example, we present in Table 4 the deflections of a uniformly loaded homogeneous square plate simply supported on a Winkler foundation. The results are compared with those obtained by Benyoucef *et al*. (2010), Lam *et al*. (2000) employing Green’s functions and

Table 4 Comparison of the deflection $D10^3 w(0.5a, 0.5b, z = 0)/qa^4$ of a uniformly loaded simply supported homogeneous square plate on a Winkler foundation ($a/h = 100$)

| $k_w$ | $D10^3 w(0.5a, 0.5b, z = 0)/qa^4$ | | | |
|---|---|---|---|---|
| | Present $\varepsilon_z \neq 0$ | Benyoucef *et al*. (2010) | Lam *et al*. (2000) | Kobayashi and Sonoda (1989) |
| 1 | 4.0472 | 4.053 | 4.053 | 4.052 |
| $3^4$ | 3.344 | 3.348 | 3.349 | 3.347 |
| $5^4$ | 1.505 | 1.506 | 1.507 | 1.506 |

Table 5 Comparison of the deflection w of a homogeneous square plate under uniformly distributed load resting on Winkler's elastic foundation ($a/h = 20$)

| $k_w$ | Present $\varepsilon_z \neq 0$ | Zenkour and Sobhy (2012) | Buczkowsi and Torbacki (2001) |
|---|---|---|---|
| 0 | 4.1026 | 4.1149 | 4.1197 |
| 1 | 4.0917 | 4.1039 | 4.1088 |
| $3^4$ | 3.3747 | 3.3813 | 3.3855 |
| $5^4$ | 1.5107 | 1.5094 | 1.5114 |
| $10^4$ | 0.1110 | 0.1108 | 0.1096 |
| $15^4$ | 0.0196 | 0.0196 | 0.0191 |

Table 6 Comparison of the displacements and stresses of simply supported $Al/Al_2O_3$ rectangular plate under uniformly distributed load ($a = 10h$, $b = 3a$)

| $k$ | $k_w$ | $k_S$ | Theory | $\overline{u}_x$ | $\overline{u}_z$ | $\overline{\sigma}_x$ | $\overline{\sigma}_{xy}$ | $\overline{\tau}_{xz}$ |
|---|---|---|---|---|---|---|---|---|
| 0.5 | 0 | 0 | Thai and Choi (2011) | 0.3491 | 1.9345 | 0.2337 | 0.0941 | — |
| | | | Zenkour and Sobhy (2013) | 0.34919 | 1.93441 | 0.23372 | 0.09415 | 7.68354 |
| | | | Present $\varepsilon_z \neq 0$ | 0.33498 | 1.90215 | 0.23941 | 0.09007 | 7.56253 |
| | 100 | 0 | Thai and Choi (2011) | 0.3358 | 1.8590 | 0.2242 | 0.0916 | — |
| | | | Zenkour and Sobhy (2013) | 0.33586 | 1.85907 | 0.22424 | 0.09167 | 7.42978 |
| | | | Present $\varepsilon_z \neq 0$ | 0.32246 | 1.82955 | 0.22989 | 0.08774 | 7.31675 |
| | 100 | 100 | Thai and Choi (2011) | 0.3012 | 1.6640 | 0.1999 | 0.0850 | — |
| | | | Zenkour and Sobhy (2013) | 0.30131 | 1.66399 | 0.19989 | 0.08503 | 6.76069 |
| | | | Present $\varepsilon_z \neq 0$ | 0.28991 | 1.64138 | 0.20536 | 0.08151 | 6.66745 |
| 2 | 0 | 0 | Thai and Choi (2011) | 0.6564 | 3.2266 | 0.4395 | 0.1766 | — |
| | | | Zenkour and Sobhy (2013) | 0.65655 | 3.22672 | 0.43961 | 0.17666 | 6.91072 |
| | | | Present $\varepsilon_z \neq 0$ | 0.60340 | 3.07560 | 0.44695 | 0.16202 | 6.79513 |
| | 100 | 0 | Thai and Choi (2011) | 0.6156 | 3.0218 | 0.4105 | 0.1690 | — |
| | | | Zenkour and Sobhy (2013) | 0.61576 | 3.02190 | 0.41060 | 0.16906 | 6.53895 |
| | | | Present $\varepsilon_z \neq 0$ | 0.56771 | 2.88981 | 0.41881 | 0.15538 | 6.44548 |
| | 100 | 100 | Thai and Choi (2011) | 0.5186 | 2.5364 | 0.3423 | 0.1501 | — |
| | | | Zenkour and Sobhy (2013) | 0.51872 | 2.53642 | 0.34233 | 0.15020 | 5.63882 |
| | | | Present $\varepsilon_z \neq 0$ | 0.48189 | 2.44460 | 0.35187 | 0.13875 | 5.59033 |

Table 6 Continued

| $k$ | $k_w$ | $k_S$ | Theory | $\overline{u_x}$ | $\overline{u_z}$ | $\overline{\sigma_x}$ | $\overline{\sigma_{xy}}$ | $\overline{\tau_{xz}}$ |
|---|---|---|---|---|---|---|---|---|
| 5 | 0 | 0 | Thai and Choi (2011) | 0.7802 | 3.8506 | 0.5223 | 0.2103 | — |
| | | | Zenkour and Sobhy (2013) | 0.78046 | 3.85174 | 0.52237 | 0.21044 | 6.14557 |
| | | | Present $\varepsilon_z \neq 0$ | 0.72061 | 3.69376 | 0.53104 | 0.19389 | 6.03129 |
| | 100 | 0 | Thai and Choi (2011) | 0.7230 | 3.5620 | 0.4816 | 0.1996 | — |
| | | | Zenkour and Sobhy (2013) | 0.72323 | 3.56296 | 0.48167 | 0.19975 | 5.75485 |
| | | | Present $\varepsilon_z \neq 0$ | 0.66999 | 3.42857 | 0.49132 | 0.18445 | 5.66241 |
| | 100 | 100 | Thai and Choi (2011) | 0.5922 | 2.9046 | 0.3897 | 0.1740 | — |
| | | | Zenkour and Sobhy (2013) | 0.59231 | 2.90518 | 0.38971 | 0.17410 | 4.84302 |
| | | | Present $\varepsilon_z \neq 0$ | 0.55294 | 2.81786 | 0.40060 | 0.16159 | 4.79288 |
| 0.5 | 0 | 0 | Thai and Choi (2011) | 0.3491 | 1.9345 | 0.2337 | 0.0941 | — |
| | | | Zenkour and Sobhy (2013) | 0.34919 | 1.93441 | 0.23372 | 0.09415 | 7.68354 |
| | | | Present $\varepsilon_z \neq 0$ | 0.33498 | 1.90215 | 0.23941 | 0.09007 | 7.56253 |
| | 100 | 0 | Thai and Choi (2011) | 0.3358 | 1.8590 | 0.2242 | 0.0916 | — |
| | | | Zenkour and Sobhy (2013) | 0.33586 | 1.85907 | 0.22424 | 0.09167 | 7.42978 |
| | | | Present $\varepsilon_z \neq 0$ | 0.32246 | 1.82955 | 0.22989 | 0.08774 | 7.31675 |
| | 100 | 100 | Thai and Choi (2011) | 0.3012 | 1.6640 | 0.1999 | 0.0850 | — |
| | | | Zenkour and Sobhy (2013) | 0.30131 | 1.66399 | 0.19989 | 0.08503 | 6.76069 |
| | | | Present $\varepsilon_z \neq 0$ | 0.28991 | 1.64138 | 0.20536 | 0.08151 | 6.66745 |
| 2 | 0 | 0 | Thai and Choi (2011) | 0.6564 | 3.2266 | 0.4395 | 0.1766 | — |
| | | | Zenkour and Sobhy (2013) | 0.65655 | 3.22672 | 0.43961 | 0.17666 | 6.91072 |
| | | | Present $\varepsilon_z \neq 0$ | 0.60340 | 3.07560 | 0.44695 | 0.16202 | 6.79513 |
| | 100 | 0 | Thai and Choi (2011) | 0.6156 | 3.0218 | 0.4105 | 0.1690 | — |
| | | | Zenkour and Sobhy (2013) | 0.61576 | 3.02190 | 0.41060 | 0.16906 | 6.53895 |
| | | | Present $\varepsilon_z \neq 0$ | 0.56771 | 2.88981 | 0.41881 | 0.15538 | 6.44548 |
| | 100 | 100 | Thai and Choi (2011) | 0.5186 | 2.5364 | 0.3423 | 0.1501 | — |
| | | | Zenkour and Sobhy (2013) | 0.51872 | 2.53642 | 0.34233 | 0.15020 | 5.63882 |
| | | | Present $\varepsilon_z \neq 0$ | 0.48189 | 2.44460 | 0.35187 | 0.13875 | 5.59033 |
| 5 | 0 | 0 | Thai and Choi (2011) | 0.7802 | 3.8506 | 0.5223 | 0.2103 | — |
| | | | Zenkour and Sobhy (2013) | 0.78046 | 3.85174 | 0.52237 | 0.21044 | 6.14557 |
| | | | Present $\varepsilon_z \neq 0$ | 0.72061 | 3.69376 | 0.53104 | 0.19389 | 6.03129 |
| | 100 | 0 | Thai and Choi (2011) | 0.7230 | 3.5620 | 0.4816 | 0.1996 | — |
| | | | Zenkour and Sobhy (2013) | 0.72323 | 3.56296 | 0.48167 | 0.19975 | 5.75485 |
| | | | Present $\varepsilon_z \neq 0$ | 0.66999 | 3.42857 | 0.49132 | 0.18445 | 5.66241 |
| | 100 | 100 | Thai and Choi (2011) | 0.5922 | 2.9046 | 0.3897 | 0.1740 | — |
| | | | Zenkour and Sobhy (2013) | 0.59231 | 2.90518 | 0.38971 | 0.17410 | 4.84302 |
| | | | Present $\varepsilon_z \neq 0$ | 0.55294 | 2.81786 | 0.40060 | 0.16159 | 4.79288 |

Kobayashi and Sonoda (1989) using the Levy series method. It can be concluded that a good agreement between the results is seen.

Table 5 present similar results as those given in Table 4 but for $a/h$ = 20. The obtained results are compared with those predicted by Zenkour and Sobhy (2012) and Buczkowski and Torbacki (2001) based on finite element method. An excellent agreement is obtained between the different results for all values of the Winkler parameter.

Numerical results are tabulated in Tables 6-7 and plotted in Figs. 2 to 5 for FG plates using the present theory. The material properties of FG plates are listed in Table 1.

Table 6 gives a comparison of displacement and stress of the present method with those of Thai and Choi (2011) and Zenkour and Sobhy (2013) in which the thickness stretching effect is neglected ($\varepsilon_z = 0$). The results are given for FG $Al/Al_2O_3$ rectangular plate subjected to a uniformly distributed load in terms of the power index $k$ and elastic foundation parameters. The results presented in Table 6, show that the results given by Thai and Choi (2011) and Zenkour and Sobhy (2013) overestimates the deflections and stresses, and this is attributable to the thickness stretching effect, which is omitted in the theories developed by these references.

Another example is presented in Table 7 for FGM plate under sinusoidal loading. The results of the method are compared with the refined trigonometric shear deformation theory developped by Bouderba *et al*. (2013), the first order shear deformation theory (FSDT) and classical plate theory (CPT) and this for different values of the power index $k$ and elastic foundation parameters. It can be seen that an excellent agreement is obtained for all values of the power-law index $k$ and foundation parameters $K_w$ and $K_s$. In addition, it can be shown that the deflection and stresses are decreasing with the existence of the elastic foundations.

Table 7 Effect of the volume fraction exponent and elastic foundation parameters on the dimensionless and stresses of an FGM rectangular plate under sinusoidal load. ($a = 10h$, $b = 2a$, $q_0 = 100$)

| $k$ | $k_w$ | $k_S$ | Theory | $\overline{w}$ | $\overline{\sigma}_x$ | $\tau^*_{xy}$ | $-\tau^*_{xz}$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | Present $\varepsilon_z \neq 0$ | 0.67669 | 0.44410 | 0.85538 | -0.38933 |
| | | | Bouderba *et al*. (2013) | 0.68131 | 0.42424 | 0.86240 | -0.39400 |
| | | | FSDPT[(a)] | 0.68135 | 0.42148 | 0.86459 | -0.30558 |
| | | | CPT[(a)] | 0.65704 | 0.42148 | 0.86459 | - |
| | 100 | 0 | Present $\varepsilon_z \neq 0$ | 0.40481 | 0.26567 | 0.51170 | -0.23290 |
| | | | Bouderba *et al*. (2013) | 0.40523 | 0.25233 | 0.51296 | -0.23435 |
| | | | FSDPT[(a)] | 0.40525 | 0.25070 | 0.51426 | -0.18175 |
| | | | CPT[(a)] | 0.39652 | 0.25437 | 0.52183 | - |
| | 0 | 100 | Present $\varepsilon_z \neq 0$ | 0.084133 | 0.055215 | 0.10635 | -0.048406 |
| | | | Bouderba *et al*. (2013) | 0.083654 | 0.052093 | 0.10589 | -0.048377 |
| | | | FSDPT[(a)] | 0.083655 | 0.051750 | 0.10615 | -0.037518 |
| | | | CPT[(a)] | 0.083277 | 0.053420 | 0.10959 | - |
| | 100 | 100 | Present $\varepsilon_z \neq 0$ | 0.077649 | 0.050960 | 0.098154 | -0.044675 |
| | | | Bouderba *et al*. (2013) | 0.077197 | 0.048071 | 0.097724 | -0.044643 |
| | | | FSDPT[(a)] | 0.077198 | 0.047754 | 0.097959 | -0.034622 |
| | | | CPT[(a)] | 0.076875 | 0.049316 | 0.101160 | - |

Table 7 Continued

| $k$ | $k_w$ | $k_S$ | Theory | $\overline{w}$ | $\overline{\sigma_x}$ | $\tau_{xy}^*$ | $-\tau_{xz}^*$ |
|---|---|---|---|---|---|---|---|
| 0.5 | 100 | 100 | Present $\varepsilon_z \neq 0$ | 0.079180 | 0.048733 | 0.080257 | -0.038219 |
| | | | Bouderba *et al*. (2013) | 0.078729 | 0.045788 | 0.081728 | -0.038066 |
| | | | FSDPT[(a)] | 0.078732 | 0.045460 | 0.081870 | -0.029835 |
| | | | CPT[(a)] | 0.078463 | 0.046927 | 0.084506 | - |
| 1 | 100 | 100 | Present $\varepsilon_z \neq 0$ | 0.079761 | 0.047891 | 0.071203 | -0.035252 |
| | | | Bouderba *et al*. (2013) | 0.079321 | 0.044892 | 0.073054 | -0.035023 |
| | | | FSDPT[(a)] | 0.079322 | 0.044575 | 0.073208 | -0.027163 |
| | | | CPT[(a)] | 0.079069 | 0.046036 | 0.075608 | - |
| 2 | 100 | 100 | Present $\varepsilon_z \neq 0$ | 0.080200 | 0.047581 | 0.065302 | -0.032442 |
| | | | Bouderba *et al*. (2013) | 0.079758 | 0.044595 | 0.067185 | -0.032215 |
| | | | FSDPT[(a)] | 0.079753 | 0.044297 | 0.067395 | -0.024345 |
| | | | CPT[(a)] | 0.079503 | 0.045808 | 0.069690 | - |
| 5 | 100 | 100 | Present $\varepsilon_z \neq 0$ | 0.080628 | 0.048596 | 0.062746 | -0.030046 |
| | | | Bouderba *et al*. (2013) | 0.080150 | 0.045736 | 0.064125 | -0.029922 |
| | | | FSDPT[(a)] | 0.080141 | 0.045462 | 0.064399 | -0.022053 |
| | | | CPT[(a)] | 0.079892 | 0.047100 | 0.066720 | - |
| ∞ | 100 | 100 | Present $\varepsilon_z \neq 0$ | 0.081721 | 0.030345 | 0.058449 | -0.026603 |
| | | | Bouderba *et al*. (2013) | 0.081190 | 0.050559 | 0.058148 | -0.026565 |
| | | | FSDPT[(a)] | 0.081191 | 0.050227 | 0.058294 | -0.020603 |
| | | | CPT[(a)] | 0.080989 | 0.051956 | 0.060300 | - |

*Data taken from Bouderba *et al*. (2013)

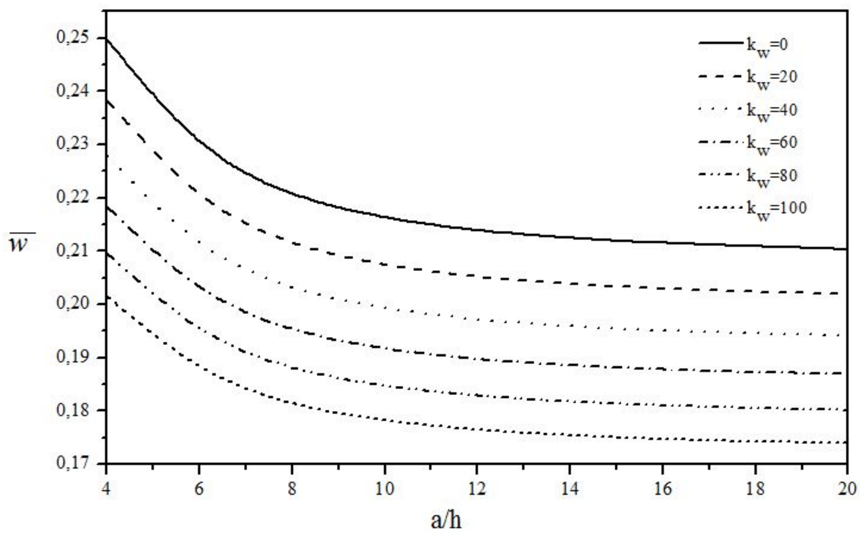


Fig. 2 Effect of Winkler modulus parameter on the dimensionless center deflection of a square FG plate ($k$ = 2, $k_s$ = 10) for different side-to-thickness ratio

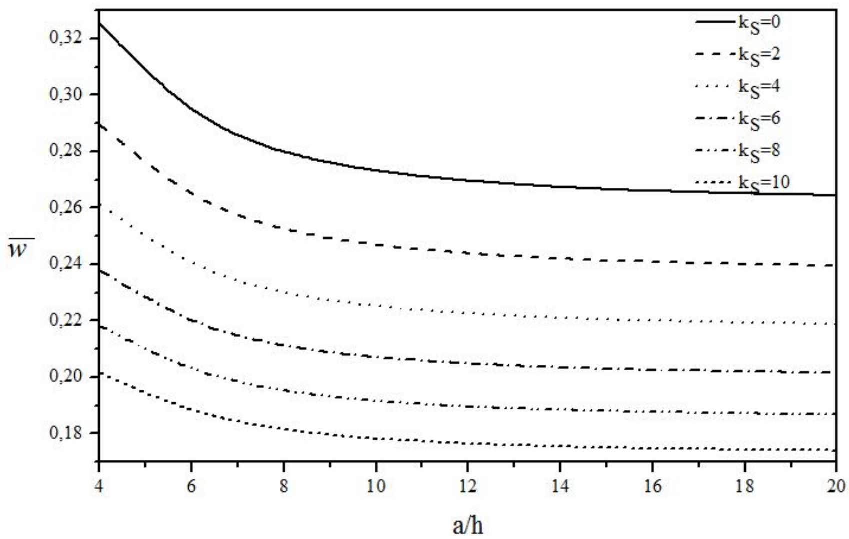


Fig. 3 Effect of Pasternak Shear modulus parameter on the dimensionless center deflection of a square FG plate ($k$ = 2, $k_w$ = 100) for different side-to-thickness ratio

Figs. 2 and 3 exhibit the deflection $\overline{w}$ of the plate centroid versus the side-to-thickness ratio $a/h$ for different values of the foundation stiffness of FG square plate ($k$ = 2). It can be seen that the increase of side-to thickness ratio $a$ / $h$ leads to a decrease of the center deflection of the FG plate. Furthermore, it is seen from Figs. 2 and 3 that as the foundation modulus parameter increase the center deflection of the FG plate decreases. However, for thin plates, the effect of foundation stiffness tends to become less.

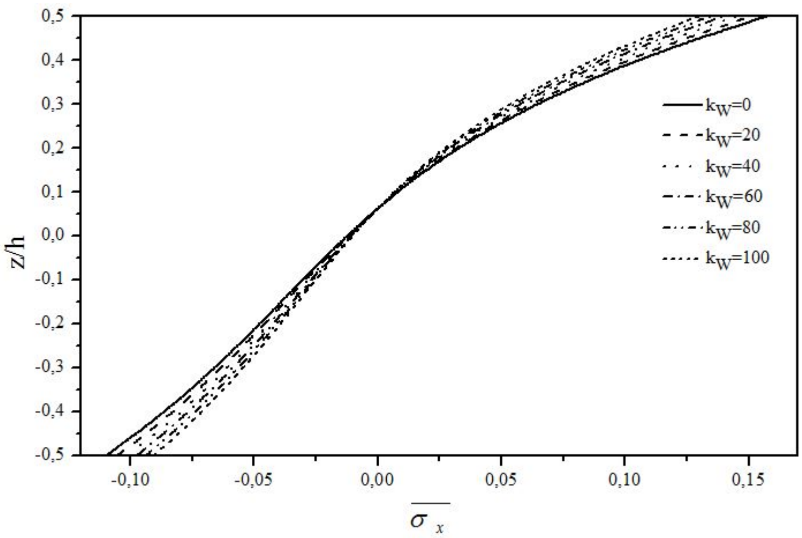


Fig. 4 Variation of dimensionless axial stress $(\overline{\sigma_x})$ through-the-thickness of a square FG plate ($k$ = 2, $k_s$ = 10, $a/h$ = 10) for different values of Winkler modulus parameter

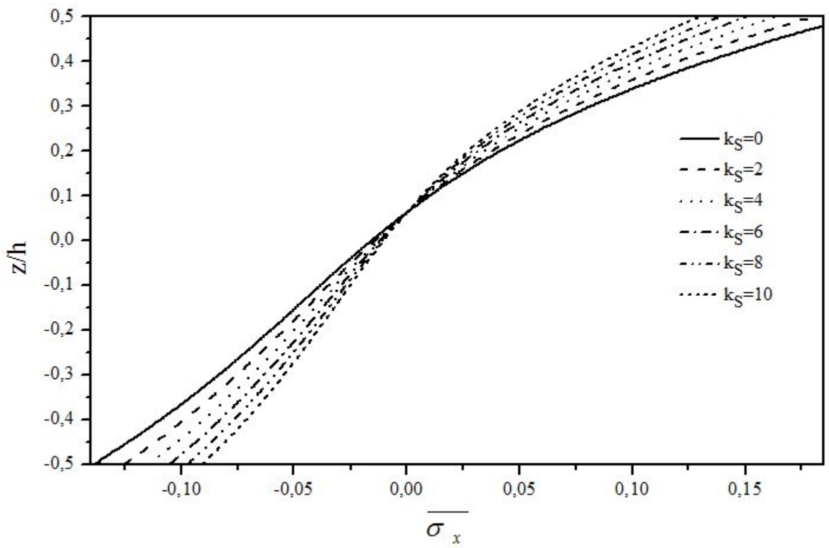


Fig. 5 Variation of dimensionless axial stress $(\overline{\sigma_x})$ through-the-thickness of a square FG plate ($k$ = 2, $k_w$ = 100, $a/h$ = 10) for different values of Pasternak modulus parameter

Variations of the axial stress $\overline{\sigma_x}$ through the thickness of FG plate are shown graphically in Figs. 4 and 5 for different values of the elastic foundation parameter. It can be seen that the maximum compressive stresses occur at a point near the top surface and the maximum tensile stresses occur, of course, at a point near the bottom surface of the FG plate. It is observed that normal stress increases gradually with the values of the foundation stiffness. However, the effect of Pasternak shears modulus parameter is more significant than Winkler modulus parameter.

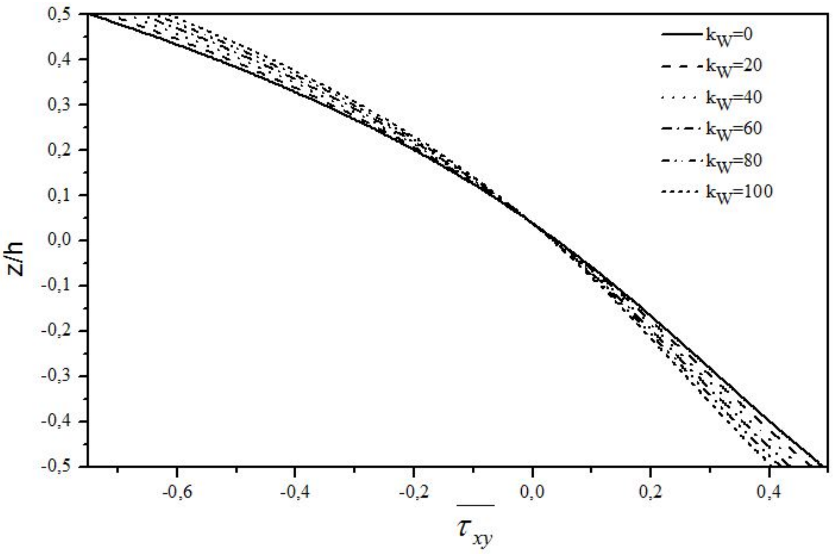


Fig. 6 Variation of dimensionless shear stress $(\overline{\tau_{xy}})$ through-the-thickness of a square FG plate ($k$ = 2, $k_s$ = 10, $a/h$ = 10) for different values of Winkler modulus parameter

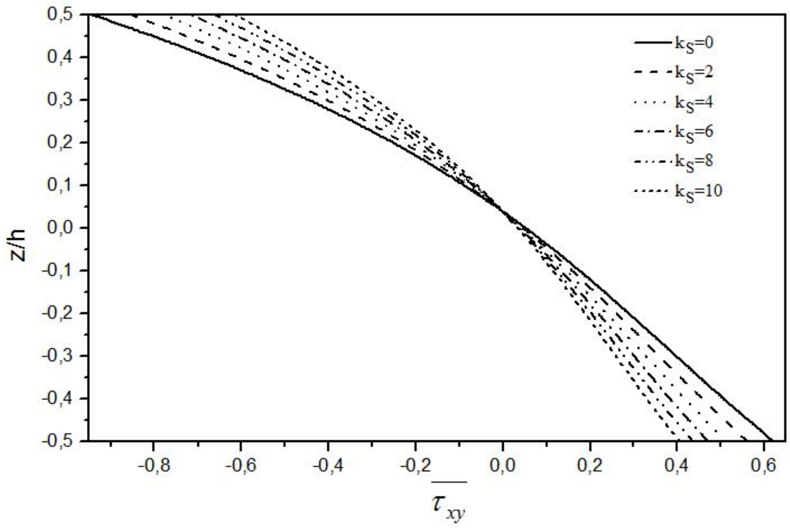


Fig. 7 Variation of dimensionless shear stress $(\overline{\tau_{xy}})$ through-the-thickness of a square FG plate ($k$ = 2, $k_w$ = 100, $a/h$ = 10) for different values of Pasternak modulus parameter

In Figs. 6 and 7, we present the evolution of the dimensionless shear stress $(\overline{\tau_{xy}})$ through-the-thickness of a square FG plate ($k$ = 2) for different values of the elastic foundation parameter.

The results reveal that the maximum compressive stresses occur at a point near the bottom surface and the maximum tensile stresses occur at a point near the top surface of the FG plate.

Again, in the case of an elastic foundation type Winkler, the evolution of the dimensionless shear stress $\overline{\tau_{xy}}$ is little affected by the variation of the parameter of this foundation compared to that of Pasternak.

### *4.2 Vibration analysis*

In order to verify the accuracy of the present theory in predicting the free vibration responses of plates, several numerical examples are presented and discussed in this section.

In Table 8, the results computed by the present higher order shear and normal deformation theory are compared with those obtained using the 3D solution of Srinivas *et al*. (1970b), the higher-order shear deformation theory of Reddy and Phan (1985) and the first shear deformation theory of Whitney and Pagano (1970). Comparisons are given for an isotropic square plate with $a/h$ = 10. It can be seen that the present theory which takes into account both the transverse shear and transverse normal deformation, predicts the natural frequencies with the same degree of accuracy as that of 3D elasticity solutions (Srinivas *et al*. 1970b) and Refs (Reddy and Phan 1985, Whitney and Pagano 1970).

Table 9 displays the comparison between the first three nondimensional frequencies of simply supported square plate resting on elastic foundation computed using the present theory and those presented in Refs (Zhou *et al*. 2004, Matsunaga 2000, Lü *et al*. 2009). It is seen that, the results of the present theory that takes into account the stretching effect are in excellent agreement with those of Refs (Zhou *et al*. 2004, Matsunaga 2000, Lü *et al*. 2009) for the first frequencies. However, for the two other frequencies, solution presented in Refs (Zhou *et al*. 2004, Matsunaga 2000, Lü *et al*. 2009) slightly underestimate frequency compared to the present theory.

Table 8 Natural frequencies $\hat{\omega} = \omega\, h\sqrt{\rho/G}$ of an isotropic plate with $\nu = 0.3$, $a/h = 10$ and $a/h = 1$

| $M$ | $n$ | Present $\varepsilon_z \neq 0$ | 3D (Srinivas *et al.* 1970b) | Reddy and Phan (1985) | Whitney and Pagano (1970) |
|---|---|---|---|---|---|
| 1 | 1 | 0.0932 | 0.0932 | 0.0931 | 0.0930 |
| 1 | 2 | 0.2229 | 0.2226 | 0.2222 | 0.2220 |
| 2 | 2 | 0.3425 | 0.3421 | 0.3411 | 0.3406 |
| 1 | 3 | 0.4178 | 0.4171 | 0.4158 | 0.4149 |
| 2 | 3 | 0.5248 | 0.5239 | 0.5221 | 0.5206 |
| 3 | 3 | 0.6904 | 0.6889 | 0.6862 | 0.6834 |
| 2 | 4 | 0.7528 | 0.7511 | 0.7481 | 0.7447 |
| 1 | 5 | 0.9294 | 0.9268 | 0.9230 | 0.9174 |
| 4 | 4 | 1.0924 | 1.0889 | 1.0847 | 1.0764 |

Table 9 Comparison of the first three nondimensional frequencies $\tilde{\omega}/\pi^2$ $(\tilde{\omega} = \omega a^2\sqrt{\rho_m h/D_m})$ of simply supported isotropic square plate ($k_s = 10$)

| | | $\tilde{\omega}_{11}$ | | | | $\tilde{\omega}_{12}$ | | | | $\tilde{\omega}_{13}$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $a/h$ | $k_w$ | 3D Ref[(a)] | HSDPT Ref[(b)] | Sheikholeslami and Saidi (2013) | Present $\varepsilon_z \neq 0$ | 3D Ref[(a)] | HSDPT Ref[(b)] | Sheikholeslami and Saidi (2013) | Present $\varepsilon_z \neq 0$ | 3D Ref[(a)] | HSDPT Ref[(b)] | Sheikholeslami and Saidi (2013) | Present $\varepsilon_z \neq 0$ |
| 5 | 0 | 2.2334 | 2.2334 | 2.2334 | 2.2383 | 4.4056 | 4.4056 | 4.4056 | 4.4220 | 7.2436 | 7.2436 | 7.2436 | 7.2864 |
| | 10 | 2.2539 | 2.2539 | 2.2539 | 2.2590 | 4.4150 | 4.4150 | 4.4150 | 4.4317 | 7.2487 | 7.2488 | 7.2488 | 7.2919 |
| | $10^2$ | 2.4300 | 2.4300 | 2.4300 | 2.4377 | 4.4986 | 4.4986 | 4.4986 | 4.5182 | 7.2948 | 7.2948 | 7.2948 | 7.3412 |
| | $10^3$ | 3.7111 | 3.7112 | 3.7111 | 3.7726 | 5.2285 | 5.2285 | 5.2285 | 5.2959 | 7.7191 | 7.7191 | 7.7191 | 7.8096 |

Ref[(a)] : (Zhou *et al.* 2004)
Ref[(b)] : (Matsunaga 2000)

Another example to verify the accuracy of the proposed theory compared to three-dimensional theory of elasticity (Lü *et al.* 2009) and the refined plate theory of Thai and Choi (2011) is presented in Table 10. This table present the nondimensional fundamental frequencies of simply supported square plate ($a = b = 10h$, $k = 2.3$). It can be seen that the results of the proposed theory agree well with three-dimensional solutions.

The non-dimensional natural frequency $\hat{\omega}$ of FG square plate versus the shear and Winkler parameters, power law index and thickness–length ratio are listed in Table 11. These results are predicted by the shear and normal deformation theory which takes in account the stretching effect as well as theories of Refs (Lü *et al.* 2009, Whitney and Pagano 1970). There is a slight difference between the results. This is due to the stretching effect which is taken into account by the present theory and neglected by the two others.

The non-dimensional fundamental natural frequency of simply supported square FG plates and

Table 10 Comparison of nondimensional fundamental frequency $\bar{\omega} = \frac{\omega\, a^2}{\mathrm{h}}\sqrt{\rho_m / E_m}$ of simply supported square plate ($a = b = 10$ h)

| $k_w$ | $k_s$ | $a/h = 10$ | | |
|---|---|---|---|---|
| | | 3D (Lü *et al.* 2009) | Thai and Choi (2011) | Present $\varepsilon_z \neq 0$ |
| 0 | 0 | 5.1295 | 5.2385 | 5.1638 |
| | 10 | 5.5560 | 5.6576 | 5.6059 |
| | 25 | 6.1404 | 6.2336 | 6.2103 |
| 10 | 0 | 5.1520 | 5.2605 | 5.1871 |
| | 10 | 5.5767 | 5.678 | 5.6274 |
| | 25 | 6.1591 | 6.2521 | 6.2297 |
| 100 | 0 | 5.3498 | 5.4548 | 5.3923 |
| | 10 | 5.7600 | 5.8584 | 5.8171 |
| | 25 | 6.3255 | 6.4164 | 6.4015 |
| 1000 | 0 | 7.0281 | 7.1116 | 7.1262 |
| | 10 | 7.3450 | 7.4257 | 7.4527 |
| | 25 | 7.7962 | 7.8734 | 7.9172 |

Table 11 The non-dimensional natural frequency $\hat{\omega} = \omega h\sqrt{\rho_m / E_m}$ of FG square plate versus the shear and Winkler parameters, power law index and thickness–length ratio

| $k_w$ | $k_s$ | $h/a$ | Power law index | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | $k = 0$ | | | $k = 1$ | | | $k = 2$ | | |
| | | | Ref(a) | Ref(b) | Present $\varepsilon_z \neq 0$ | Ref Ref(a) | Ref Ref(b) | Present $\varepsilon_z \neq 0$ | Ref Ref(a) | Ref Ref(b) | Present $\varepsilon_z \neq 0$ |
| 0 | 0 | 0.05 | 0.0291 | 0.0291 | 0.0291 | 0.0222 | 0.0227 | 0.0226 | 0.0202 | 0.0209 | 0.0207 |
| | | 0.1 | 0.1135 | 0.1134 | 0.1136 | 0.0870 | 0.0891 | 0.0883 | 0.0789 | 0.0819 | 0.0807 |
| | | 0.15 | 0.2459 | 0.2454 | 0.2461 | 0.1891 | 0.1939 | 0.1918 | 0.1711 | 0.1778 | 0.1748 |
| | | 0.20 | 0.4169 | 0.4154 | 0.4174 | 0.3222 | 0.3299 | 0.3264 | 0.2906 | 0.3016 | 0.2965 |
| 0 | 100 | 0.05 | 0.0405 | 0.0406 | 0.0406 | 0.0377 | 0.0382 | 0.0380 | 0.0373 | 0.0380 | 0.0376 |
| | | 0.1 | 0.1593 | 0.1599 | 0.1594 | 0.1482 | 0.1517 | 0.1497 | 0.1463 | 0.1508 | 0.14829 |
| | | 0.15 | 0.3487 | 0.3515 | 0.3492 | 0.3236 | 0.3365 | 0.3295 | 0.3180 | 0.3351 | 0.3265 |
| | | 0.20 | 0.5988 | 0.6080 | 0.6011 | 0.5509 | 0.5876 | 0.5699 | 0.5370 | 0.5861 | 0.5650 |
| 100 | 0 | 0.05 | 0.0298 | 0.0298 | 0.0298 | 0.0233 | 0.0238 | 0.0236 | 0.0214 | 0.0221 | 0.0218 |
| | | 0.1 | 0.1163 | 0.1162 | 0.1164 | 0.0911 | 0.0933 | 0.0924 | 0.0836 | 0.0867 | 0.0854 |
| | | 0.15 | 0.2521 | 0.2519 | 0.2524 | 0.1983 | 0.2036 | 0.2011 | 0.1817 | 0.1889 | 0.1855 |
| | | 0.20 | 0.4281 | 0.4273 | 0.4286 | 0.3383 | 0.3476 | 0.3431 | 0.3092 | 0.3219 | 0.3158 |
| 100 | 100 | 0.05 | 0.0410 | 0.0411 | 0.0411 | 0.0384 | 0.0388 | 0.0386 | 0.0380 | 0.0386 | 0.0383 |
| | | 0.1 | 0.1613 | 0.1619 | 0.1614 | 0.1506 | 0.1542 | 0.1521 | 0.1489 | 0.1535 | 0.1509 |
| | | 0.15 | 0.3531 | 0.3560 | 0.3537 | 0.3288 | 0.3422 | 0.3349 | 0.3236 | 0.3412 | 0.3323 |
| | | 0.20 | 0.6070 | 0.6162 | 0.6089 | 0.5598 | 0.5978 | 0.5794 | 0.5460 | 0.5970 | 0.5752 |

Ref(a) : (Sheikholeslami and Saidi 2013); Ref(b) : (Baferani *et al.* 2011)

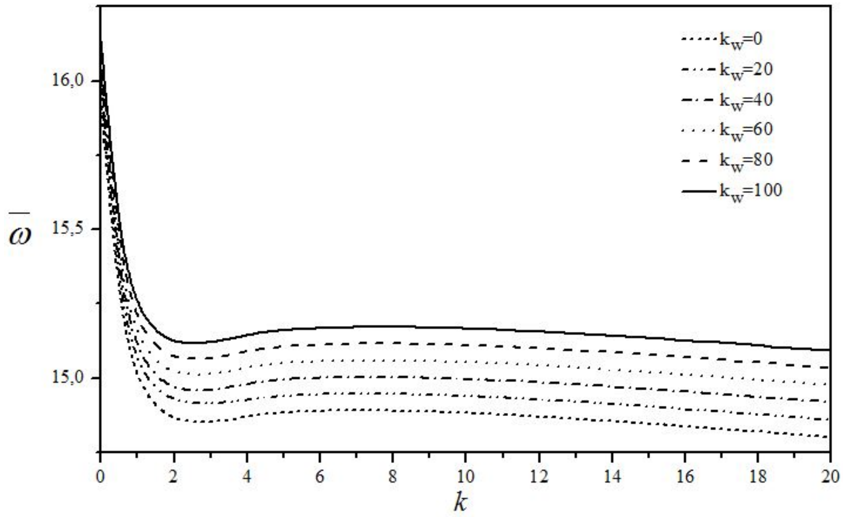


Fig. 8 The variation of non-dimensional fundamental natural frequency $\overline{\omega}$ versus the power law index $k$ for different values of Winkler parameter ($a/b$ = 1, $k_s$ = 10, $a/h$ = 10)

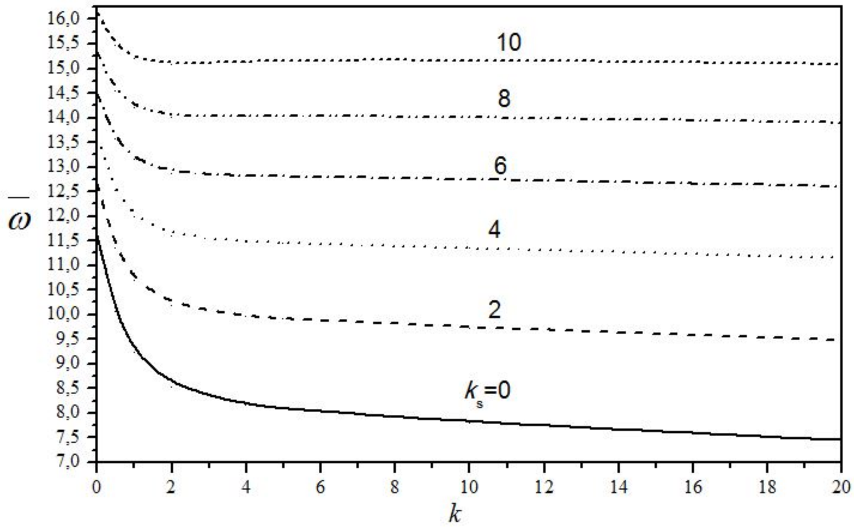


Fig. 9 The variation of non-dimensional fundamental natural frequency $\overline{\omega}$ versus the power law index $k$ for different values of Pasternak parameter ($a/b$ = 1, $k_s$ = 100, $a/h$ = 10)

power law index $k$ for various values of the Winkler foundation parameter are plotted in Fig. 8 based on the present new quasi-3D hyperbolic shear deformation theory. The same non-dimensional parameter is shown in Fig. 9 for different values of Pasternak parameter.

There is a rapid change of the non-dimensional fundamental natural frequency for the values of the power law index $k$ less than 2. Then the curves maintain a more or less constant pace but remain close to each other which show that this parameter does not affect too much the non-dimensional fundamental natural frequency for values of $k$ greater than 2.

In the case of Fig. 9, there is no sudden change but the curves remain remote separate. This indicates that this parameter influence the non-dimensional fundamental natural frequency compared to the Winkler parameter. In addition, when the power law index $k$ takes values greater than 4, the non-dimensional fundamental natural frequency keep constant values except where $k_s$ = 0.

## 5. Conclusions

A new quasi-3D hyperbolic shear deformation theory for bending and free vibration of FG plates resting on Winkler-Pasternak elastic foundations is presented. The theory contains only five unknown displacements and satisfies the zero traction boundary conditions at the plate's surfaces without requiring a shear correction factor. Thus, a considerably lower computational time is reached. The accuracy of the present work is ascertained by comparing it with existing solutions, and excellent agreement was observed in all cases. The inclusion of thickness stretching effect makes a plate stiffer, and hence leads to a reduction of deflection and an increase of frequency. Hence, it can be said that the proposed theory is accurate and simple in solving the bending and free vibration behavior of FG plates resting on elastic foundations.

## Acknowledgments

This research was supported by the Algerian National Thematic Agency of Research in Science and Technology (ATRST) and university of Sidi Bel Abbes (UDL SBA) in Algeria.

## References

Ait Amar Meziane, M., Abdelaziz, H.H. and Tounsi, A. (2014), "An efficient and simple refined theory for buckling and free vibration of exponentially graded sandwich plates under various boundary conditions", *J. Sandw. Struct. Mater.*, **16**(3), 293-318.

Ait Atmane, H., Tounsi, A., Mechab, I. and Adda Bedia, E.A. (2010), "Free vibration analysis of functionally graded plates resting on Winkler-Pasternak elastic foundations using a new shear deformation theory", *Int. J. Mech. Mater. Des.*, **6**(2), 113-121.

Ait Atmane, H., Tounsi, A., Bernard, F. and Mahmoud, S.R. (2015), "A computational shear displacement model for vibrational analysis of functionally graded beams with porosities", *Steel Compos. Struct., Int. J.*, **19**(2), 369-384.

Ait Yahia, S., Ait Atmane, H., Houari, M.S.A. and Tounsi, A. (2015), "Wave propagation in functionally graded plates with porosities using various higher-order shear deformation plate theories", *Struct. Eng. Mech., Int. J.*, **53**(6), 1143-1165.

Akavci, S.S. (2015), "An efficient shear deformation theory for free vibration of functionally graded thick rectangular plates on elastic foundation", *Compos. Struct.*, **108**, 667-676.

Akbaş, Ş.D. (2015), "Wave propagation of a functionally graded beam in thermal environments", *Steel Compos. Struct., Int. J.*, **19**(6), 1421-1447.

Amini, M.H., Soleimani, M. and Rastgoo, A. (2009), "Three-dimensional free vibration analysis of functionally graded material plates resting on an elastic foundation", *Smart Mater. Struct.*, **18**(8), 1-9.

Al-Basyouni, K.S., Tounsi, A. and Mahmoud, S.R. (2015), "Size dependent bending and vibration analysis of functionally graded micro beams based on modified couple stress theory and neutral surface position", *Compos. Struct.*, **125**, 621-630.

Arefi, M. (2015), "Elastic solution of a curved beam made of functionally graded materials with different cross sections", *Steel Compos. Struct., Int. J.*, **18**(3), 659-672.

Attia, A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2015), "Free vibration analysis of functionally graded plates with temperature-dependent properties using various four variable refined plate theories", *Steel Compos. Struct., Int. J.*, **18**(1), 187-212.

Bachir Bouiadjra, R., Adda Bedia, E.A. and Tounsi, A. (2013), "Nonlinear thermal buckling behavior of functionally graded plates using an efficient sinusoidal shear deformation theory", *Struct. Eng. Mech., Int. J.*, **48**(4), 547-567.

Baferani, A.H., Saidi, A.R. and Ehteshami, H. (2011), "Accurate solution for free vibration analysis of functionally graded thick rectangular plates resting on elastic foundation", *Compos. Struct.*, **93**(7), 1842-1853.

Bakora, A. and Tounsi, A. (2015), "Thermo-mechanical post-buckling behavior of thick functionally graded plates resting on elastic foundations", *Struct. Eng. Mech., Int. J.*, **56**(1), 85-106.

Barati, M.R., Zenkour, A.M. and Shahverdi, H. (2016), "Thermo-mechanical buckling analysis of embedded nanosize FG plates in thermal environments via an inverse cotangential theory", *Compos. Struct.*, **141**, 203-212.

Belabed, Z., Houari, M.S.A., Tounsi, A., Mahmoud, S.R. and Anwar Bég, O. (2014), "An efficient and simple higher order shear and normal deformation theory for functionally graded material (FGM) plates", *Compos.: Part B*, **60**, 274-283.

Beldjelili, Y., Tounsi, A. and Mahmoud, S.R. (2016), "Hygro-thermo-mechanical bending of S-FGM plates resting on variable elastic foundations using a four-variable trigonometric plate theory", *Smart Struct. Syst., Int. J.*, **18**(4), 755-786.

Belkorissat, I., Houari, M.S.A., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2015), "On vibration properties of functionally graded nano-plate using a new nonlocal refined four variable model", *Steel Compos. Struct., Int. J.*, **18**(4), 1063-1081.

Bellifa, H., Benrahou, K.H., Hadji, L., Houari, M.S.A. and Tounsi, A. (2016), "Bending and free vibration analysis of functionally graded plates using a simple shear deformation theory and the concept the neutral surface position", *J. Braz. Soc. Mech. Sci. Eng.*, **38**(1), 265-275.

Benachour, A., Daouadji, H.T., Ait Atmane, H., Tounsi, A. and Meftah, S.A. (2011), "A four variable refined plate theory for free vibrations of functionally graded plates with arbitrary gradient", *Compos. Part B*, **42**(6), 1386-1394.

Bennai, R., Ait Atmane, H. and Tounsi, A. (2015), "A new higher-order shear and normal deformation theory for functionally graded sandwich beams", *Steel Compos. Struct., Int. J.*, **19**(3), 521-546.

Bennoun, M., Houari, M.S.A. and Tounsi, A. (2016), "A novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates", *Mech. Adv. Mater. Struct.*, **23**(4), 423-431.

Benyoucef, S., Mechab, I., Tounsi, A., Fekrar, A., Ait Atmane, H. and Adda Bedia, E.A. (2010), "Bending of thick functionally graded plates resting on Winkler-Pasternak elastic foundations", *Mech. Compos. Mater.*, **46**(4), 425-434.

Bouchafa, A., Bachir Bouiadjra, M., Houari, M.S.A. and Tounsi, A. (2015), "Thermal stresses and deflections of functionally graded sandwich plates using a new refined hyperbolic shear deformation theory", *Steel Compos. Struct., Int. J.*, **18**(6), 1493-1515.

Bouderba, B., Houari, M.S.A. and Tounsi, A. (2013), "Thermomechanical bending response of FGM thick plates resting on Winkler-Pasternak elastic foundations", *Steel Compos. Struct., Int. J.*, **14**(1), 85-104.

Bouderba, B., Houari, M.S.A. and Tounsi, A. and Mahmoud, S.R. (2016), "Thermal stability of functionally graded sandwich plates using a simple shear deformation theory", *Struct. Eng. Mech., Int. J.*, **58**(3), 397-422.

Bouguenina, O., Belakhdar, K., Tounsi, A. and Adda Bedia, E.A. (2015), "Numerical analysis of FGM plates with variable thickness subjected to thermal buckling", *Steel Compos. Struct., Int. J.*, **19**(3), 679-695.

Boukhari, A., Ait Atmane, H., Tounsi, A., Adda Bedia, E.A. and Mahmoud, S.R. (2016), "An efficient shear deformation theory for wave propagation of functionally graded material plates", *Struct. Eng. Mech., Int.*

*J*., **57**(5), 837-859.

Bounouara, F., Benrahou, K.H., Belkorissat, I. and Tounsi, A. (2016), "A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation", *Steel Compos. Struct., Int. J.*, **20**(2), 227-249.

Bourada, M., Tounsi, A., Houari, M.S.A. and Adda Bedia, E.A. (2012), "A new four-variable refined plate theory for thermal buckling analysis of functionally graded sandwich plates", *J. Sandw. Struct. Mater.*, **14**(1), 5-33.

Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), "A new simple shear and normal deformations theory for functionally graded beams", *Steel Compos. Struct., Int. J.*, **18**(2), 409-423.

Bourada, F., Amara, K. and Tounsi, A. (2016), "Buckling analysis of isotropic and orthotropic plates using a novel four variable refined plate theory", *Steel Compos. Struct., Int. J.*, **21**(6), 1287-1306.

Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A. (2014), "A novel higher order shear and normal deformation theory based on neutral surface position for bending analysis of advanced composite plates", *Int. J. Computat. Method.*, **11**(6), 1350082.

Bousahla, A.A., Benyoucef, S. Tounsi, A. and Mahmoud, S.R. (2016), "On thermal stability of plates with functionally graded coefficient of thermal expansion", *Struct. Eng. Mech., Int. J.*, **60**(2), 313-335.

Buczkowski, R. and Torbacki, W. (2001), "Finite element modeling of thick plates on two-parameter elastic foundation", *Int. J. Numer. Anal. Method. Geomech.*, **25**(14), 1409-1427.

Chikh, A., Bakora, A., Heireche, H., Houari, M.S.A., Tounsi, A. and Adda Bedia, E.A. (2016), "Thermo-mechanical postbuckling of symmetric S-FGM plates resting on Pasternak elastic foundations using hyperbolic shear deformation theory", *Struct. Eng. Mech., Int. J.*, **57**(4), 617-639.

Darılmaz, K. (2015), "Vibration analysis of functionally graded material (FGM) grid systems", *Steel Compos. Struct., Int. J.*, **18**(2), 395-408.

Draiche, K., Tounsi, A. and Khalfi, Y. (2014), "A trigonometric four variable plate theory for free vibration of rectangular composite plates with patch mass", *Steel Compos. Struct., Int. J.*, **17**(1), 69-81.

Ebrahimi, F. and Dashti, S. (2015), "Free vibration analysis of a rotating non-uniform functionally graded beam", *Steel Compos. Struct., Int. J.*, **19**(5), 1279-1298.

Ebrahimi, F. and Habibi, S. (2016), "Deflection and vibration analysis of higher-order shear deformable compositionally graded porous plate", *Steel Compos. Struct., Int. J.*, **20**(1), 205-225.

Ebrahimi, F. and Salari, E. (2016), "Thermal loading effects on electro-mechanical vibration behavior of piezoelectrically actuated inhomogeneous size-dependent Timoshenko nanobeams", *Adv. Nano Res., Int. J.*, **4**(3), 197-228.

El Meiche, N., Tounsi, A., Ziane, N., Mechab, I. and Adda Bedia, E.A. (2011), "A new hyperbolic shear deformation theory for buckling and vibration of functionally graded sandwich plate", *Int. J. Mech. Sci.*, **53**(4), 237-247.

Fallah, A., Aghdam, M.M. and Kargarnovin, M.H. (2013), "Free vibration analysis of moderately thick functionally graded plates on elastic foundation using the extended Kantorovich method", *Arch. Appl. Mech.*, **83**(2), 177-191.

Hadji, L. and Adda Bedia, E.A. (2015a), "Influence of the porosities on the free vibration of FGM beams", *Wind Struct., Int. J.*, **21**(3), 273-287.

Hadji, L. and Adda Bedia, E.A. (2015b), "Analyse of the behavior of functionally graded beams based on neutral surface position", *Struct. Eng. Mech., Int. J.*, **55**(4), 703-717.

Hadji, L., Ait Atmane, H., Tounsi, A. and Adda Bedia, E.A. (2011), "Free vibration of functionally graded sandwich plates using four-variable refined plate theory", *Appl. Math. Mech. -Engl. Ed.*, **32**(7), 925-942.

Hadji, L., Daouadji, T.H., Tounsi, A. and Bedia, E.A. (2014), "A higher order shear deformation theory for static and free vibration of FGM beam", *Steel Compos. Struct., Int. J.*, **16**(5), 507-519.

Hadji, L., Hassaine Daouadji, T., Ait Amar Meziane, M., Tlidji, Y. and Adda Bedia, E.A. (2016), "Analysis of functionally graded beam using a new first-order shear deformation theory", *Struct. Eng. Mech., Int. J.*, **57**(2), 315-325.

Hamidi, A., Houari, M.S.A., Mahmoud, S.R. and Tounsi, A. (2015), "A sinusoidal plate theory with 5-unknowns and stretching effect for thermomechanical bending of functionally graded sandwich plates",

*Steel Compos*. *Struct*., *Int*. *J*., **18**(1), 235-253.
Hebali, H., Tounsi, A., Houari, M.S.A., Bessaim, A. and Adda Bedia, E.A. (2014), "A new quasi-3D hyperbolic shear deformation theory for the static and free vibration analysis of functionally graded plates", *ASCE J*. *Eng*. *Mech*., **140**(2), 374-383.
Huang, Z.Y., Lu, C.F. and Chen, W.Q. (2008), "Benchmark solutions for functionally graded thick plates resting on Winkler–Pasternak elastic foundations", *Compos*. *Struct*., **85**(2), 95-104.
Kar, V.R. and Panda, S.K. (2015), "Nonlinear flexural vibration of shear deformable functionally graded spherical shell panel", *Steel Compos*. *Struct*., *Int*. *J*., **18**(3), 693-709.
Khalfi, Y., Houari, M.S.A. and Tounsi, A. (2014), "A refined and simple shear deformation theory for thermal buckling of solar functionally graded plates on elastic foundation", *Int*. *J*. *Computat*. *Methods*, **11**(5), 135007.
Kitipornchai, S., Yang, J. and Liew, K.M. (2006), "Random vibration of the functionally graded laminates in thermal environments", *Comput*. *Method*. *Appl*. *Mech*. *Eng*., **195**(9-12), 1075-1095.
Kobayashi, H. and Sonoda, K. (1989), "Rectangular Mindlin plates on elastic foundations", *Int*. *J*. *Mech*. *Sci*., **31**(9), 679-692.
Lam, K.Y., Wang, C.M. and He, X.Q. (2000), "Canonical exact solutions for Levy-plates on two parameter foundation using Green's functions", *Eng*. *Struct*., **22**(4), 364-378.
Laoufi, I., Ameur, A., Zidi, M., Adda Bedia, E.A. and Bousahla, A.A. (2016), "Mechanical and hygro-thermal behaviour of functionally graded plates using a hyperbolic shear deformation theory", *Steel Compos*. *Struct*., *Int*. *J*., **20**(4), 889-912.
Larbi Chaht, F., Kaci, A., Houari, M.S.A., Tounsi, A., Anwar Bég, O. and Mahmoud, S.R. (2015), "Bending and buckling analyses of functionally graded material (FGM) size-dependent nanoscale beams including the thickness stretching effect", *Steel Compos*. *Struct*., *Int*. *J*., **18**(2), 425-442.
Lü, C.F., Lim, C.W. and Chen, W.Q. (2009), "Exact solutions for free vibrations of functionally graded thick plates on elastic foundations", *Mech*. *Adv*. *Mater*. *Struct*., **16**(8), 576-584.
Mahi, A., Adda Bedia, E.A. and Tounsi, A. (2015), "A new hyperbolic shear deformation theory for bending and free vibration analysis of isotropic, functionally graded, sandwich and laminated composite plates", *Appl*. *Math*. *Model*., **39**(9), 2489-2508.
Malekzadeh, P. (2009), "Three-dimensional free vibration analysis of thick functionally graded plates on elastic foundations", *Compos*. *Struct*., **89**(3), 367-373.
Matsunaga, H. (2000), "Vibration and stability of thick plates on elastic foundations", *J*. *Eng*. *Mech*. (*ASCE*), **126**(1), 27-34.
Meradjah, M., Kaci, A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2015), "A new higher order shear and normal deformation theory for functionally graded beams", *Steel Compos*. *Struct*., *Int*. *J*., **18**(3), 793-809.
Merazi, M., Hadji, L., Daouadji, T.H., Tounsi, A. and Adda Bedia, E.A. (2015), "A new hyperbolic shear deformation plate theory for static analysis of FGM plate based on neutral surface position", *Geomech*. *Eng*., *Int*. *J*., **8**(3), 305-321.
Moradi-Dastjerdi, R. (2016), "Wave propagation in functionally graded composite cylinders reinforced by aggregated carbon nanotube", *Struct*. *Eng*. *Mech*., *Int*. *J*., **57**(3), 441-456.
Mouaici, F., Benyoucef, S., Ait Atmane, H. and Tounsi, A. (2016), "Effect of porosity on vibrational characteristics of non-homogeneous plates using hyperbolic shear deformation theory", *Wind Struct*., *Int*. *J*., **22**(4), 429-454.
Nedri, K., El Meiche, N. and Tounsi, A. (2014), "Free vibration analysis of laminated composite plates resting on elastic foundations by using a refined hyperbolic shear deformation theory", *Mech*. *Compos*. *Mater*., **49**(6), 641-650.
Nguyen, K.T., Thai, T.H. and Vo, T.P. (2015), "A refined higher-order shear deformation theory for bending, vibration and buckling analysis of functionally graded sandwich plates", *Steel Compos*. *Struct*., *Int*. *J*., **18**(1), 91-120.
Ould Larbi, L, Kaci, A., Houari, M.S.A. and Tounsi, A. (2013), "An efficient shear deformation beam theory based on neutral surface position for bending and free vibration of functionally graded beams",

*Mech*. *Based Des*. *Struct*. *Mach*., **41**(4), 421-433.
Pasternak, P.L. (1954), "On a new method of analysis of an elastic foundation by means of two foundation constants", *Gosudarstvennoe Izdatelstvo Literaturi po Stroitelstvu I Arkhitekture*, **1**, Moscow: USSR, 1-56. [In Russian]
Pradhan, K.K. and Chakraverty, S. (2015), "Free vibration of functionally graded thin elliptic plates with various edge supports", *Struct*. *Eng*. *Mech*., *Int*. *J*., **53**(2), 337-354.
Qian, L.F. and Batra, R.C. (2005), "Three-dimensional transient heat conduction in a functionally graded thick plate with a higher-order plate theory and a meshless local Petrov-Galerkin Method", *Computat*. *Mech*., **35**(3), 214-226.
Reddy, J.N. (2000), "Analysis of functionally graded plates", *Int*. *J*. *Numer*. *Method*. *Eng*., **47**(1-3), 663-684.
Reddy, J.N. and Phan, N.D. (1985). "Stability and vibration of isotropic, orthotropic and laminated plates according to a higher-order shear deformation theory", *J*. *Sound Vib*., **98**, 157-170.
Saidi, H., Tounsi, A. and Bousahla, A.A. (2016), "A simple hyperbolic shear deformation theory for vibration analysis of thick functionally graded rectangular plates resting on elastic foundations", *Geomech*. *Eng*., *Int*. *J*., **11**(2), 289-307.
Sallai, B., Hadji, L., Hassaine Daouadji, T. and Adda Bedia, E.A. (2015), "Analytical solution for bending analysis of functionally graded beam", *Steel Compos*. *Struct*., *Int*. *J*., **19**(4), 829-841.
Sheikholeslami, S.A. and Saidi, A.R. (2013), "Vibration analysis of functionally graded rectangular plates resting on elastic foundation using higher-order shear and normal deformable plate theory", *Compos*. *Struct*., **106**, 350-361.
Shimpi, R.P., Arya, H. and Naik, N.K. (2003), "A higher order displacement model for the plate analysis", *J*. *Reinf*. *Plast*. *Compos*., **22**(22), 1667-1688.
Sobhy, M. (2013), "Buckling and free vibration of exponentially graded sandwich plates resting on elastic foundations under various boundary conditions", *Compos*. *Struct*., **99**, 76-87.
Srinivas S., Joga, C.V. and Rao, A.K. (1970a), "Bending, vibration and buckling of simply supported thick orthotropic rectangular plate and laminates", *Int*. *J*. *Solid*. *Struct*., **6**(11), 1463-1481.
Srinivas, S., Joga Rao, C.V. and Rao, A.K. (1970b), "An exact analysis for vibration of simply supported homogeneous and laminated thick rectangular plates", *J*. *Sound Vib*., **12**(2), 187-199.
Tagrara, S.H., Benachour, A., Bachir Bouiadjra, M. and Tounsi, A. (2015), "On bending, buckling and vibration responses of functionally graded carbon nanotube-reinforced composite beams", *Steel Compos*. *Struct*., *Int*. *J*., **19**(5), 1259-1277.
Tebboune, W., Benrahou, K.H., Houari, M.S.A. and Tounsi, A. (2015), "Thermal buckling analysis of FG plates resting on elastic foundation based on an efficient and simple trigonometric shear deformation theory", *Steel Compos*. *Struct*., *Int*. *J*., **18**(2), 443-465.
Thai, H.T. and Choi, D.H. (2011), "A refined plate theory for functionally graded plates resting on elastic foundation", *Compos*. *Sci*. *Technol*., **71**(16), 1850-1858.
Tounsi, A., Houari, M.S.A., Benyoucef, S. and Adda Bedia, E.A. (2013), "A refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates", *Aerosp*. *Sci*. *Technol*., **24**(1), 209-220.
Tounsi, A., Houari, M.S.A. and Bessaim, A. (2016), "A new 3-unknowns non-polynomial plate theory for buckling and vibration of functionally graded sandwich plate", *Struct*. *Eng*. *Mech*., *Int*. *J*., **60**(4), 547-565.
Trinh, T.H., Nguyen, D.K., Gan, B.S. and Alexandrov, S. (2016), "Post-buckling responses of elastoplastic FGM beams on nonlinear elastic foundation", *Struct*. *Eng*. *Mech*., *Int*. *J*., **58**(3), 515-532.
Whitney, J.M. and Pagano, N.J. (1970), "Shear deformation in heterogeneous anisotropic plates", *J*. *Appl*. *Mech*., **37**(4), 1031-1036.
Winkler, E. (1867), "Die Lehre von der Elasticitaet und Festigkeit", Prag Dominicus.
Yaghoobi, H. and Yaghoobi, P. (2013), "Buckling analysis of sandwich plates with FGM face sheets resting on elastic foundation with various boundary conditions: An analytical approach", *Meccanica*, **48**(8), 2019-2035.
Yang, J., Liew, K.M. and Kitipornchai, S. (2005), "Stochastic analysis of compositionally graded plates with system randomness under static loading", *Int*. *J*. *Mech*. *Sci*., **47**(10), 1519-1541.

Zemri, A., Houari, M.S.A., Bousahla, A.A. and Tounsi, A. (2015), "A mechanical response of functionally graded nanoscale beam: an assessment of a refined nonlocal shear deformation theory beam theory", *Struct*. *Eng*. *Mech*., *Int*. *J*., **54**(4), 693-710.

Zenkour, A.M. and Sobhy, M. (2012), "Elastic foundation analysis of uniformly loaded functionally graded viscoelastic sandwich plates", *J*. *Mech*., **28**(3), 439452.

Zenkour, A.M. and Sobhy, M. (2013), "Dynamic bending response of thermoelastic functionally graded plates resting on elastic foundations", *Aerosp*. *Sci*. *Technol*., **29**(1), 7-17.

Zhou, D., Cheung, Y.K., Lo, S.H. and Au, F.T.K. (2004), "Three-dimensional vibration analysis of rectangular thick plates on Pasternak foundation", *Int*. *J*. *Numer*. *Meth*. *Eng*., **59**(10), 1313-1334.

Zidi, M., Tounsi, A., Houari, M.S.A., Adda Bedia, E.A. and Anwar Bég, O. (2014), "Bending analysis of FGM plates under hygro-thermo-mechanical loading using a four variable refined plate theory", *Aerosp*. *Sci*. *Technol*., **34**, 24-34.

*CC*