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Analytical solution of stress-strain relationship of modified Cam clay in undrained shear

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Abstract. The modified Cam clay (MCC) model is used to study the response of virgin compressed clay in undrained compression. The MCC deviatoric stress-strain relationship is obtained in closed form. Elastic and plastic deviatoric strains are taken into account in the analysis. For the determination of the elastic strain components, both a variable shear modulus and constant shear modulus are considered. Constitutive relationships are applied to the well-known London and Weald clays sheared in undrained compression.

Keywords: modified Cam clay model; constitutive relations; elastic and plastic deviatoric strains; virgin compressed clay; undrained compression; comparisons.

1. Introduction

The original and modified Cam clay models allow the determination of effective stress paths in undrained triaxial compression tests on normally consolidated clay (Roscoe and Schofield 1963, Burland 1965, Roscoe and Burland 1968, Schofield and Wroth 1968). Pore water pressures can be deduced from the difference between applied total and resulting effective mean stresses. Schofield and Wroth (1968) also obtained the deviatoric stress-plastic strain relationship in the case of normally consolidated clay which obeys the original Cam clay model.

The constitutive relationship of modified Cam clay (MCC) in undrained shear is generally presented only in incremental form (Wood 2007). It is precisely this form that is implemented in several commercial numerical codes available on the market.

This paper presents the closed-form solution of the MCC model in undrained shear. Elastic and plastic deviatoric strains are taken into account in the analysis.

The results which are applied for illustration purposes to the well-known reconstituted London and Weald clays, are also compared to a theoretical solution obtained by Perić and Ayari (2002).

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2. Theoretical considerations

The focus of the paper being on the undrained behaviour of virgin compressed clay, only the response of the MCC model in such condition is addressed in the present paper.

2.1 Modified Cam clay model

2.1.1 Stress and strain invariants

Let the total major, intermediate, and minor principal stresses acting on a soil element be denoted by σ_1, σ_2, σ_3; the corresponding effective principal stresses by σ_1', σ_2', σ_3'; and the natural principal strain increments by $d\varepsilon_1$, $d\varepsilon_2$, $d\varepsilon_3$. The stress and strain increment invariants, which have been commonly used for the response of the MCC model under generalized stress conditions, are the following (See for example, Wood 2007):

$$p = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{1a}$$

$$p' = \frac{1}{3}(\sigma_1' + \sigma_2' + \sigma_3') \tag{1b}$$

$$q = q' = \frac{1}{\sqrt{2}}[(\sigma_1' - \sigma_2')^2 + (\sigma_2' - \sigma_3')^2 + (\sigma_3' - \sigma_1')^2]^{1/2} \tag{1c}$$

$$d\varepsilon_v = d\varepsilon_1 + d\varepsilon_2 + d\varepsilon_3 \tag{1d}$$

$$d\varepsilon_q = \frac{\sqrt{2}}{3}[(d\varepsilon_1 - d\varepsilon_2)^2 + (d\varepsilon_2 - d\varepsilon_3)^2 + (d\varepsilon_3 - d\varepsilon_1)^2]^{1/2} \tag{1e}$$

where p, p' = total and effective mean principal stresses; $q = q'$ = deviator stress; $d\varepsilon_v$ = natural volumetric strain increment; and $d\varepsilon_q$ = natural deviatoric strain increment.

It should be also noted that the strain increments, $d\varepsilon_v$ and $d\varepsilon_q$, comprise elastic and plastic components, that is,

$$d\varepsilon_v = d\varepsilon_v^e + d\varepsilon_v^p \tag{2a}$$

$$d\varepsilon_q = d\varepsilon_q^e + d\varepsilon_q^p \tag{2b}$$

where the superscripts "*e*" and "*p*" refer to elastic and plastic components.

The pore pressures is found from the difference between the total mean stress, p, and the effective mean stress, p':

$$u = p - p' \tag{3}$$

Hence, the excess pore pressure, Δu, may be expressed as the difference between the change in the mean total stress Δp, and the change in the mean effective stress, $\Delta p'$. That is:

$$\Delta u = \Delta p - \Delta p' = (p - p_0) - (p' - p_0') = u - u_0 \tag{4}$$

where p_0, p_0' = initial, total and effective mean stresses, and u_0 = initial (ambient) pore pressure.

2.1.2 Yield curve and effective stress path

The response behaviour of virgin compressed clay that obeys the modified Cam clay model is shown in Fig. 1. The curves which describe the relationships between the specific volume v, where $v = 1 + e$, with e = void ratio, and the mean effective stress p' for the normal or isotropic consolidation line, the swelling line, and the critical state line are shown in Fig. 1a. It should be noted that these relationships are considered as straight lines in a v versus ln p' diagram. The slopes of these straight lines are referred to respectively as λ for the normal consolidation and critical state curves, and κ for the swelling or unloading-reloading line. Fig. 1a also shows a typical undrained stress path (ESP), $\overline{AB}$, followed in a compression test. The undrained stress path starts at point A on the isotropic consolidation line and ends at point B on the critical state line. Fig. 1b presents the critical state line, the effective stress path followed in undrained test, and the yield curve of modified Cam clay in q versus p' diagram.

Burland (1965) showed that, since normally consolidated clay deform irrecoverably under isotropic stress, the expression for the increment of work dissipated per unit bulk volume of an isotropic medium during deformation leads to the following flow rule:

$$\frac{d\varepsilon_v^p}{d\varepsilon_q^p} = \frac{M^2 - \left(\frac{q}{p'}\right)^2}{2\left(\frac{q}{p'}\right)} \tag{4a}$$

or

$$\frac{d\varepsilon_v^p}{d\varepsilon_q^p} = \frac{M^2 - \eta^2}{2\eta^2} \tag{4b}$$

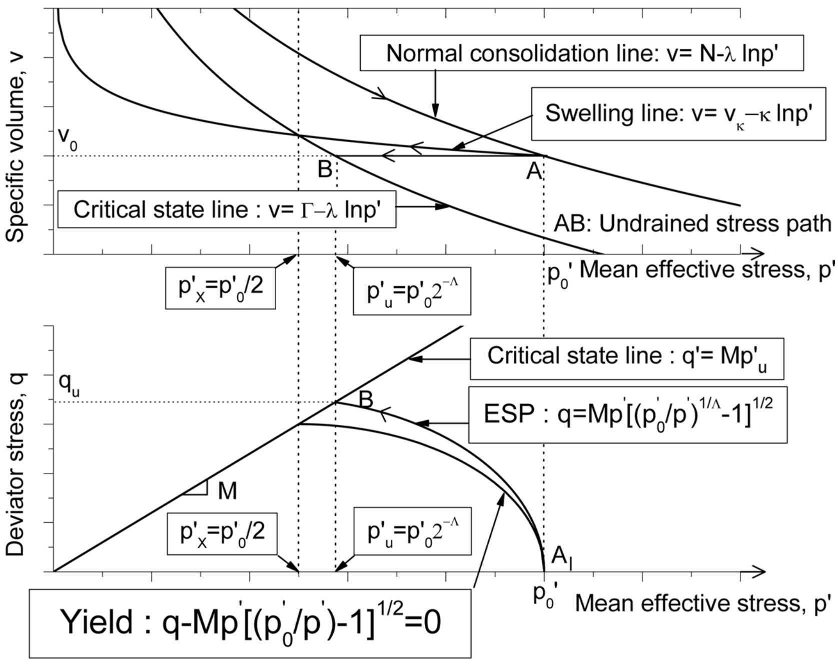

Fig. 1 Modified Cam clay model

where $d\varepsilon_v^p$ is the plastic volumetric strain increment, $d\varepsilon_q^p$ is the plastic deviatoric strain increment, M is the slope of the critical state line (Fig. 1b), and $\eta = q/p'$. The equation of the yield curve or yield locus is obtained from Eq. (4) by applying the normality condition and integrating:

$$q - Mp'\left[\left(\frac{p_0'}{p'}\right) - 1\right]^{1/2} = 0 \tag{5a}$$

or

$$\eta - M\left[\left(\frac{p_0'}{p'}\right) - 1\right]^{1/2} = 0 \tag{5b}$$

where p_0' is the initial isotropic consolidation pressure, as shown in Fig. 1b. Eq. (5b) represents an ellipse in a $p'\!-\!q$ space with its centre at $p_0'/2$ on the hydrostatic axis. When the soil is yielding, the change in size of the yield locus is linked with changes in effective stresses p' et $q = \eta p'$ (Wood 2007).

Since the overall natural volumetric strain increment, $d\varepsilon_v$, is zero in an undrained test, the sum of the elastic, $d\varepsilon_v^e$, and the plastic, $d\varepsilon_v^p$, natural volumetric strain increments must be zero, *i.e.*, $d\varepsilon_v^e + d\varepsilon_v^p = 0$, implying that $d\varepsilon_v^e = -d\varepsilon_v^p$ from Eq. (2a). As the elastic and plastic volumetric strain increments are given, respectively, by (See for example, Wood 2007):

$$d\varepsilon_v^e = \frac{\kappa dp'}{vp'} = \frac{dp'}{K'} \tag{6a}$$

and

$$d\varepsilon_v^p = \frac{\lambda - \kappa}{vp'(M^2 + \eta^2)}[(M^2 - \eta^2)dp' + 2\eta dq] \tag{6b}$$

substitution of these expressions into Eq. (6) and integrating gives:

$$q = Mp'\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2} \tag{7a}$$

or

$$\eta = M\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2} \tag{7b}$$

where $\Lambda = (\lambda - \kappa)\lambda$. Eq. (7) specifies the shape of the undrained stress path in the p'-q plane. As shown in Fig. 1, the undrained stress path begins at an isotropic effective stress state, p_0', on the normal consolidation line and ends at a mean effective stress $p_u' = p_0' 2^{-\Lambda}$ on the critical state line. The ultimate deviator stress $q_u = p_u' M = p_0' 2^{-\Lambda} M$.

2.1.3. Plastic strains

Substitution of $d\varepsilon_v^p = -d\varepsilon_v^e = -\kappa dp'/vp'$ from Eq. (6a) into the flow rule given by Eq. (4a) yields the following expression for the plastic deviatoric strain increment:

$$d\varepsilon_q^p = -\frac{\kappa dp'}{vp'}\frac{2(q/p')}{M^2 - (q/p')^2} \tag{8}$$

Combining Eq. (8) with Eq. (7a) gives

$$d\varepsilon_q^p = -\frac{2\kappa}{vM} \frac{\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2}}{\left[2 - \left(\frac{p_0'}{p'}\right)^{1/\Lambda}\right]} \frac{dp'}{p'} \tag{9}$$

Although integration of this equation is not straightforward, it is however facilitated by introducing a variable $y = (p_0'/p')^{1/\Lambda}$. This allows the determination of the deviatoric plastic strain ε_q^p (See Appendix), that is:

$$\varepsilon_q^p = \frac{2\kappa\Lambda}{vM} \left\{ \tanh^{-1} \left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2} - \tan^{-1} \left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2} \right\} \tag{10a}$$

which may also be written, on the basis of Eq. (5b), as

$$\varepsilon_q^p = \frac{2\kappa\Lambda}{vM} \left\{ \tanh^{-1}\left(\frac{q}{Mp'}\right) - \tan^{-1}\left(\frac{q}{Mp'}\right) \right\} \tag{10b}$$

2.1.4. Elastic strains

The expression in Eq. (6a) for the volumetric strain implies that a constant slope κ of the unloading-reloading line in a semi-logarithmic compression plane results in a bulk modulus K' that increases with the mean stress p'.

Changes in deviator stress q within the yield locus, for an isotropic elastic soil, cause no changes in volume but do produce elastic deviatoric strains $d\varepsilon_q^e$ which are calculated from (Atkinson 1993, Wood 2007).

$$d\varepsilon_q^e = \frac{dq}{3G} \tag{11}$$

where G = shear modulus. The shear modulus G is equal to (Atkinson and Bransby 1978):

$$G = \frac{3(1-2\mu')K'}{2(1+\mu')} \tag{12a}$$

or

$$G = \frac{3vp'(1-2\mu')}{2\kappa(1+\mu')} \tag{12b}$$

since $K' = vp'/\kappa$ and μ' is Poisson's ratio.

With a bulk modulus dependent on mean stress, there are limitations on the choice of a variable or constant shear modulus. Zytynski *et al.* (1978) showed that the use of a pressure-dependent shear modulus results in a non-conservative elastic model.

Alternatively, a constant value of shear modulus might be assumed, in which case the variation of bulk modulus with mean stress implies a variation of Poisson's ratio. The model is then conservative, but may lead to negative values of Poisson's ratio for some stress hystories, which is physically unreasonable.

In order to arrive at an analytical solution, the elastic deviatoric strain component ε_q^e will be obtained for two cases: a) one with a variable shear modulus, and b) the other with a constant shear modulus.

a) Variable shear modulus

Substitution of Eq. (12b) into Eq. (11) yields

$$d\varepsilon_q^e = \frac{2\kappa(1+\mu')dq}{9v(1-2\mu')p'} \tag{13}$$

An expression must be first found for dq in order to integrate this equation. This is done by differentiating Eq. (7a) with respect to p', yielding

$$dq = M\left\{\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda}-1\right]^{1/2} - \frac{1}{2\Lambda}\left(\frac{p_0'}{p'}\right)^{1/\Lambda}\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda}-1\right]^{-1/2}\right\}dp' \tag{14}$$

Introducing this equation into Eq. (13) leads to

$$d\varepsilon_q^e = \frac{2\kappa(1+\mu')M}{9v(1-2\mu')}\left\{\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda}-1\right]^{1/2} - \frac{1}{2\Lambda}\left(\frac{p_0'}{p'}\right)^{1/\Lambda}\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda}-1\right]^{-1/2}\right\}\frac{dp'}{p'} \tag{15}$$

Again, integration of this equation is facilitated by introducing the variable $y = (p_0'/p')^{1/\Lambda}$, leading to the elastic deviatoric strain ε_q^e (See Appendix), that is:

$$\varepsilon_q^e = \frac{2\kappa(1+\mu')M}{9v(1-2\mu')}\left\{\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda}-1\right]^{1/2}(1-2\Lambda) + 2\Lambda\tan^{-1}\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda}-1\right]^{1/2}\right\} \tag{16a}$$

or, from Eq. (7a),

$$\varepsilon_q^e = \frac{2\kappa(1+\mu')M}{9v(1-2\mu')}\left\{\left(\frac{q}{Mp'}\right)(1-2\Lambda) + 2\Lambda\tan^{-1}\left(\frac{q}{Mp'}\right)\right\} \tag{16b}$$

b) Constant shear modulus

In order to obtain realistic values of the shear modulus G, especially for overconsolidated clays, Randolph *et al.* (1979) suggested to select G as some fixed proportion of the maximum value of the elastic bulk modulus, $K'_{\max}$, that was ever reached during the history of the soil. The proportionality relation adopted by these authors is

$$G = 0.5K'_{\max} \tag{17a}$$

Because $K'_{\max} = v\,p_0'/\kappa$ in the case of virgin compressed clay, where p_0' is the initial isotropic consolidation pressure, the latter equation becomes

$$G = 0.5\frac{v\,p_0'}{\kappa} \tag{17b}$$

Substitution of this expression in Eq. (11) and integrating yields:

$$\varepsilon_q^e = \frac{2q\kappa}{3v\,p_0'} \tag{18}$$

2.2 Total deviatoric strain

The total deviatoric strain is given by

$$\varepsilon_q = \varepsilon_q^e + \varepsilon_q^p \tag{19}$$

which is found by adding Eqs. (16) or (18) to Eq. (10). Eq. (19) is valid under generalized stress conditions. The relationship between the deviator stress q and the deviatoric strain ε_q which is illustrated in Figs. 2 and 3 for a variable and a constant shear modulus, respectively, provides the complete description of an undrained test on a virgin compressed clay that obeys the MCC model. In Figs. 2 and 3 are also shown separately the elastic component, ε_q^e, given by either Eq. (16) or Eq. (18), and the plastic component ε_q^p, given by Eq. (10).

The relationships in these two figures were drawn for illustration purposes, by using the soil parameters of reconstituted virgin compressed London clay (Schofield and Bransby 1978): $N = 2.858$ is the value of the specific volume v at $p' = 1$ kPa on the normal consolidation line, $\Gamma = 2.759$ is the value of the specific volume v on the critical state line, $\lambda = 0.161$, $\kappa = 0.062$, $\Lambda = (\lambda - \kappa)/\lambda = 0.615$, $\mu' = 0.3$, $\phi' = 22.75^\circ$, and $M = 0.888$. It will be assumed that the initial isotropic consolidation pressure $p_0' = 206.3$ kPa, resulting in a specific volume $v = N - \lambda \ln p' = 2.0$.

In Fig. 2 in which the shear modulus has been considered to vary with the mean stress p', the elastic deviatoric stress-strain relationship is slightly non-linear. The value of the shear modulus at the origin is equal to 3071 kPa on the basis of Eq. (12b).

In Fig. 3 the elastic deviatoric stress-strain relationship is linear because the shear modulus G was assumed constant, equal to $0.5K'_{\max}$. As $K'_{\max} = vp_0'/\kappa = 2\times206.3/0.062 = 6655$ kPa, then $G = 0.5K'_{\max} = 3327$ kPa. Because the ultimate value of the mean stress, p_u', equals $p_0' 2^{-\Lambda} = 206.3\times2^{-0.615} = 134.7$ kPa and the ultimate value of the deviator stress, q_u, equals Mp_u', then $q_u = 134.7\times0.888 = 119.6$ kPa.

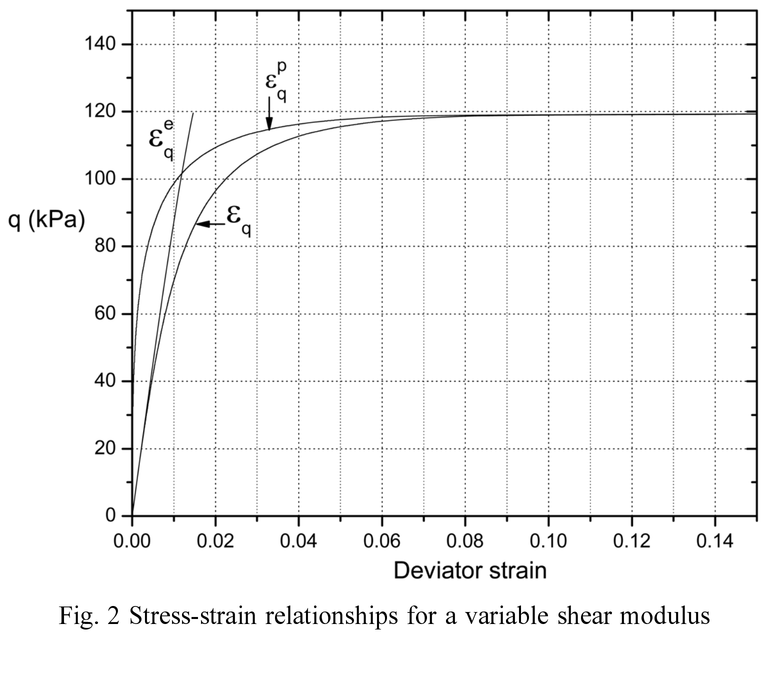

Fig. 2 Stress-strain relationships for a variable shear modulus

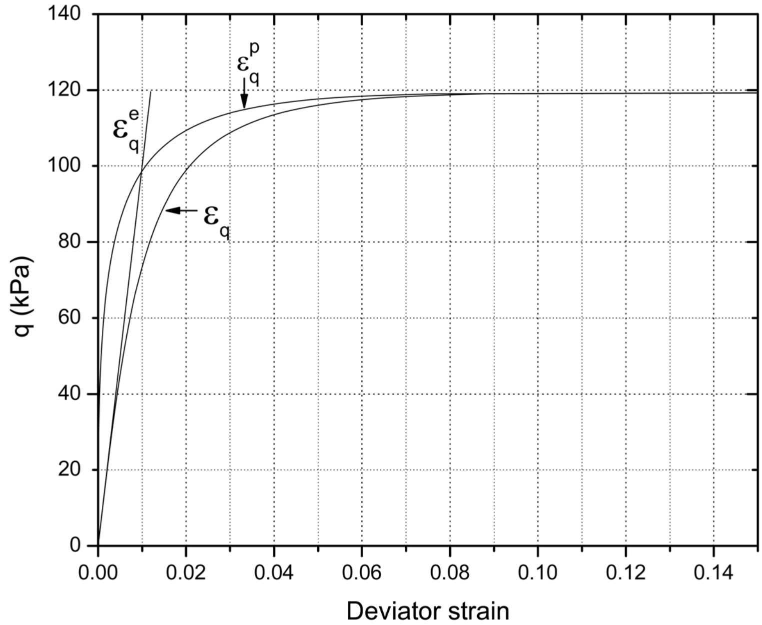

Fig. 3 Stress-strain relationships for a constant shear modulus

3. Comparison

The constitutive relationship derived in the present paper was compared with an analytical solution obtained by Perić and Ayari (2002). The two solutions were then applied to predict the response of normally consolidated remolded Weald clay observed in a conventional undrained triaxial compression test. The results of this test which were originally reported by Bishop and Henkel (1957) may be found in Wood (2007). The various stress-strain curves are grouped in Fig. 4.

The specimen of Weald clay was isotropically consolidated at $p_0' = 207$ kPa. The specific volume, v_0, at the end of the consolidation phase was 1.640. The specimen was sheared undrained by increasing the axial stress. The deviator stress at failure, q_u, was approximately equal to 119 kPa and failure occured at a deviatoric strain, ε_q, of 17.5%.

The critical state parameters used by Perić and Ayari (2000) to predict the stress-strain curve were the following: $\lambda = 0.088$, $\kappa = 0.031$, and $\mu' = 0.41$. The stress-strain curve based upon the analytical solution of Perić and Ayari (2002) was obtained by means of Eq. (32) of these authors. The result which is shown in Fig. 4 indicates that the predicted curve is stiffer than the experimental stress-strain relationship. The reason for such response is due to the choice of the relatively high value of $\mu' = 0.41$ retained for Poisson's ratio. Indeed, a smaller value of μ' would have resulted in a stiffer response at small strain.

The solution obtained in this paper was similarly applied to Weald clay. Two cases were considered and are also reported in Fig. 4: The first, with $\mu' = 0.41$, for direct comparison with the solution of Perić and Ayari (2002); the second, with $\mu' = 0.35$, in order to see whether better agreement would be obtained with the experimental stress-strain curves. Comparison between the theoretical stress-strain curves corresponding to $\mu' = 0.41$ indicates that the present solution is essentially the same as that derived by Prrić and Ayari (2002), even though slightly different approaches were used to

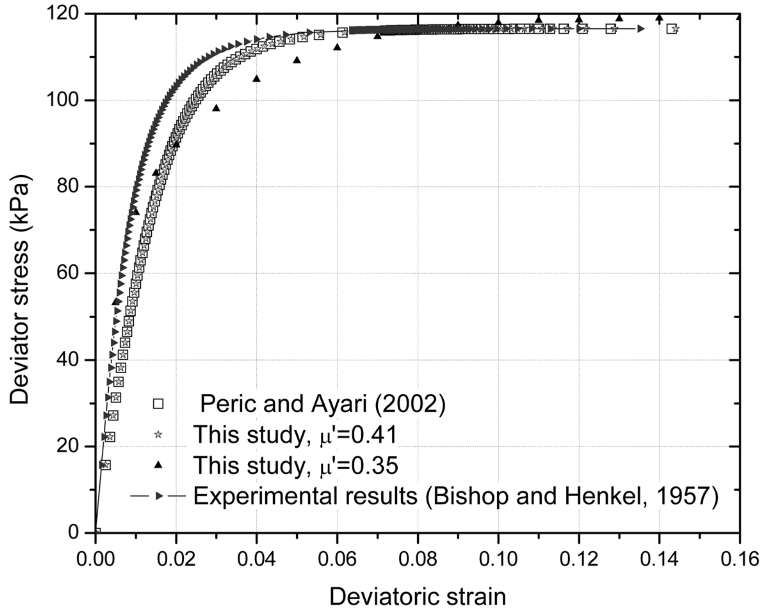

Fig. 4 Comparison of predicted stress-strain curves with experimental test results on reconstituted Weald clay

obtain the constitutive relationships. In addition, the solution with $\mu' = 0.35$ provides a better agreement with the initial segment of the experimental stress-strain curve.

4. Conclusions

The following main conclusions are drawn on the basis of the content of this paper:

a) An equation is obtained for the description of the deviatoric stress-strain response of the modified Cam clay model in undrained triaxial compression.

b) The closed-form solution was applied for illustration purposes, to virgin compressed London clay. Two cases were considered. While the first case involved a variable shear modulus, the second was drawn by using a constant shear modulus.

c) The solution was also applied to predict the stress-strain curve of virgin compressed Weald clay sheared in undrained triaxial compression. Comparison was then carried out with a theoritical solution obtained by Perić and Ayari (2002), and applied to the same specimen.

d) It appears that the closed-form stress-strain relashionship of modified Cam clay adequately simulates the response of normally consolidated clays.

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Appendix

Integration of Eq. (9)

Substitution of the variable $y = (p_0'/p')^{1/\Lambda}$ into Eq. (9) and noting that

$$dy = -\frac{1}{\Lambda} y \frac{dp'}{p'} \tag{A1}$$

yields

$$d\varepsilon_q^p = \frac{2\kappa\Lambda}{vM} \frac{(y-1)^{1/2}}{(2-y)y} dy \tag{A2}$$

Integration of this equation which is carried out between the limits $y = 1$ or $p' = p_0'$ and $y = (p_0'/p')^{1/\Lambda}$ leads to

$$\varepsilon_q^p = \frac{2\kappa\Lambda}{vM} \int_{y=1}^{y=\left(\frac{p_0'}{p'}\right)^{1/\Lambda}} \frac{(y-1)^{1/2}}{y(2-y)} dy \tag{A3}$$

The integral in Eq. (A3) is of the form (Gradshteyn and Ryzhik 1980):

$$I = \int \frac{\sqrt{z}}{(x+p)(x+q)} \tag{A4}$$

where $z = (a + bx)$ whose solution is

$$I = \frac{1}{q-p} \int \frac{\sqrt{z}}{x+p} dx + \frac{1}{p-q} \int \frac{\sqrt{z}}{x+q} dx \tag{A5}$$

Integration of this equation and substitution into Eq. (A3) yields:

$$\varepsilon_q^p = \frac{2\kappa\Lambda}{vM} \left\{ \tanh^{-1} \left[\left(\frac{p_0'}{p'} \right)^{1/\Lambda} - 1 \right]^{1/2} - \tan^{-1} \left[\left(\frac{p_0'}{p'} \right)^{1/\Lambda} - 1 \right]^{1/2} \right\} \tag{A6a}$$

or, from Eq. (7a),

$$\varepsilon_q^p = \frac{2\kappa\Lambda}{vM} \left\{ \tanh^{-1}\left(\frac{q}{Mp'} \right) - \tan^{-1}\left(\frac{q}{Mp'} \right) \right\} \tag{A6b}$$

Integration of Eq. (14) (Elastic strains with variable shear modulus)

Substitution of the variable $y = (p_0'/p')^{1/\Lambda}$ and using Eq. (A1), Eq. (14) integrates to:

$$\varepsilon_q^e = \frac{-2\kappa(1+\mu')\Lambda M}{9v(1-2\mu')} \int \frac{(y-1)^{1/2}}{y} dy + \frac{\kappa(1+\mu)M}{9v(1-2\mu)} \int \frac{dy}{(y-1)^{1/2}} \tag{A7}$$

As the integrals in this equation are of the form

$$I_1 = \int \frac{\sqrt{z}}{x} dx \tag{A8}$$

and

$$I_2 = \int \frac{dx}{\sqrt{z}} \tag{A9}$$

respectively, where $z = (a + bx)$, their integration is straightforward and leads to:

$$\varepsilon_q^e = \frac{2\kappa(1+\mu')M}{9v(1-2\mu')}\left\{(1-2\Lambda)\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2} + 2\Lambda \tan^{-1}\left[\left(\frac{p_0'}{p'}\right)^{1/\Lambda} - 1\right]^{1/2}\right\} \tag{A10a}$$

or, from Eq. (7a),

$$\varepsilon_q^e = \frac{2\kappa(1+\mu')M}{9v(1-2\mu')}\left\{(1-2\Lambda)\left(\frac{q}{Mp'}\right) + 2\Lambda \tan^{-1}\left(\frac{q}{Mp'}\right)\right\} \tag{A10b}$$