Earthquakes and Structures, *Vol. 9, No. 6 (2015) 1181-1192*
DOI: http://dx.doi.org/10.12989/eas.2015.9.6.1181

Investigation of vibration and stability of cracked columns under axial load

Masoud Ghaderi[1], Hosein Ghaffarzadeh[*2] and Vahid A. Maleki[3]

[1]Department of Civil Engineering, Ghermi Branch, Islamic Azad University, Ghermi, Iran
[2]Department of Structural Engineering, Faculty of Civil Engineering, University of Tabriz, Tabriz, Iran
[3]Department of Mechanical Engineering, University of Tabriz, Tabriz, Iran

(Received April 22, 2014, Revised June 8, 2015, Accepted July 7, 2015)

Abstract. In this paper, an analytical method is proposed to study the effect of crack and axial load on vibration behavior and stability of the cracked columns. Using the local flexibility model, the crack has been simulated by a torsional spring with connecting two segments of column in crack location. By solving governing eigenvalue equation, the effects of crack parameters and axial load on the natural frequencies and buckling load as well as buckling load are investigated. The results show that the presents of crack cause to reduction in natural frequencies and buckling load whereas this reduction is affected by the location and depth of the crack. Furthermore, the tensile and compressive axial load increase and decrease the natural frequencies, respectively. In addition, as the compression load approaches to certain value, the fundamental natural frequency reaches zero and instability occurs. The accuracy of the model is validated through the experimental data reported in the literature.

Keywords: cracked column; stability; critical buckling load; natural frequency; vibration analysis

1. Introduction

Structures undergo different changes such as formation and expansion of cracks, exhaustion, corrosion, and other probable damages. The effects of these parameters on the structure load capacity and safety should be considered in its design. Existence of cracks in structures may impact their mechanical and dynamic behavior, and considerably reduce the load capacity and strength of them. Column under axial load are structures that the existence of cracks impacts their behavior. Any defect in these structures causes changes in their behavior and stability, and if not detected may lead to destruction and disastrous damages.

Crack in columns may be formed because of impurity, impact, cyclic loads, vibration, aerodynamic loads and etc. it is obvious that cracks ruins the continuity of the column, weakens it, and decreases its load capacity. Many studies have investigated the stability, buckling critical load (Anifantis and Dimaragonas 1981, Nikpour 1990, Gurel and Kisa 2005, Jiki and Karim 2011, Krauberger *et al*. 2011, Jiki 2012, Vadillo *et al*. 2012) and dynamic response of the cracked

*Corresponding author, Associate Professor, E-mail: ghaffar@tabrizu.ac.ir

http://www.techno-press.org/journals/eas&subpage=7 ISSN: 2092-7614 (Print), 2092-7622 (Online)

columns (Kim and Cho 2006, Jiki 2007, Rahai and Kazemi 2008, Butcher 2010, Deliang *et al.* 2011, Gürkan *et al.* 2012, Guirong *et al.* 2013), Kisa (2011) studied the vibration behavior and stability of cracked beams under axial loads by using the finite element method. He modeled the crack using the torsion spring, and performed the stability analysis to calculate the buckling critical load. Caddemi *et al.* (2012) investigated the vibration behavior of cracked Euler-Bernoulli beam under axial load. In their study crack was assumed to be open edge and its effect was applied using Dirac delta function in equation of motion. Jena *et al.* (2012) studied the effect of crack on vibration behavior of cracked beams. Yazdchi *et al.* (2008) calculated the buckling load of cracked columns with different sections. Their results show that increasing the crack depth reduces the buckling critical load of the columns. Ranjbaran *et al.* (2008) investigated the buckling and free vibration of the beam with varying and cracked sections. Using the calculus of variations, they modeled the problem as an optimization problem, and studied the effect of the crack on vibration behavior of the cracked beams. Toygar *et al.* (2012) investigated the impact of crack on buckling critical load of composite beams using the experimental tests and the finite element method. Binici (2005) studied the lateral vibrations of cracked Euler-Bernoulli under axial load. Results showed that compressive loads up to 30% of the buckling loads can change the first natural frequency up to 15%. This effect is lower for other frequencies. Okamura (1969) performed studies on narrow columns with one crack to define the load capacity and fracture load of the column.

In many studies accomplished in this field, the effect of crack parameters on vibration behavior and stability of cracked beams and columns under axial loads (Yazdchi and Gowhari 2008, Kisa 2011, Caddemi and Caliò 2012, Jena *et al.* 2012) and in absence of axial loads (Kim and Cho 2006, Butcher 2010, Deliang *et al.* 2011, Gürkan *et al.* 2012, Guirong *et al.* 2013) was investigated. Most of the studies were based on numerical methods such as transfer matrix (Gurel and Kisa 2005) and the finite element method (Jiki and Karim 2011, Jiki 2012) which cause many errors in calculations. By using these methods, detecting the cracks and investigation of different parameters effects cannot be easily accomplished.

In the present study, a new analytical method is presented to investigate the vibration behavior and buckling critical load of cracked columns under axial loads. The crack is modeled using torsion spring that connects the two intact part of the column at the crack location. Applying the fracture mechanics theory, equivalent stiffness of the torsion spring is derived as a function of crack depth. The governing differential equation of the column lateral vibration under the axial load is derived from Hamilton principle. After applying the boundary and compatibility conditions at the crack location, the corresponding eigenvalue value problem is obtained. Then, effect of crack parameters on vibration behavior and critical load of the cracked column for different boundary conditions is investigated. Comparing the results of the presented model and experimental results from the literature review, shows that the new model despite the simplicity, predicts the vibration behavior and stability of the cracked columns under axial loads for a vast range of crack parameters and axial load, accurately.

2. Local flexibility in column caused by crack

Fig. 1 shows the section of a column having an open edge crack with constant length. The traditional method to apply the crack effect on column behavior is the local flexibility model in which crack is modeled using mass-less torsion spring, and the spring equivalent stiffness is derived from the fracture mechanics theory. The additional strain energy caused by crack is expressed by local flexibility coefficient, which is a function of stress intensity coefficient. Local flexibility coefficient for a crack with the width b and depth a_c is derived using Castigliano theorem as

$$c_{ij} = \frac{\partial u_{ij}}{\partial P_i} = \frac{\partial^2}{\partial P_i \partial P_j} \int_0^b \int_0^{a_c} J(\alpha) d\alpha d\zeta \tag{1}$$

P_i is the external load component along the corresponding displacement u_i, and $J(\alpha)$ is the function of strain energy density which is

$$J(\alpha) = \frac{1-v^2}{E} K_I^2(\alpha) \tag{2}$$

where E and v is Young's modulus and Poisson's coefficient, respectively. $K_I(\alpha)$ is stress intensity factor for first mode of fracture in correspondence with bending moment M, which for rectangular section is (Tada *et al.* 2004)

$$K_I(a_c) = \frac{Mh}{2I_o} \sqrt{\pi a_c} F\left(\frac{a_c}{h}\right) \tag{3}$$

where h, I_o are height and inertia moment of column section. For rectangular section, the function $\mathrm{F}(a_c/h)$ is expressed as

$$\mathrm{F}(\frac{a_c}{h}) = \frac{0.923 + 0.199(1-\sin(\frac{\pi}{2}\frac{a_c}{h}))^4}{\cos(\frac{\pi}{2}\frac{a_c}{h})} \sqrt{\frac{\tan(\frac{\pi}{2}\frac{a_c}{h})}{\frac{\pi}{2}\frac{a_c}{h}}} \tag{4}$$

The equivalent stiffness of the rotational spring K_t using the fundamentals of fracture mechanics presented the corresponding compliance equation in the following form

$$K_t = \frac{1}{C} \tag{5}$$

where C is local flexibility coefficient for the first mode of loading in correspondence with bending moment, and equal to

$$C = \frac{b(1-v^2)}{E} \frac{\partial^2}{\partial M^2} \int_0^{a_c} K_I^2(\alpha) d\alpha = \frac{bh^2}{EI_o} \pi (1-v^2) \int_0^{a_c} \alpha F^2(\alpha) d\alpha \tag{6}$$

3. Free vibration of cracked column under axial load

Cracked Euler-Bernoulli column under constant axial load and applied mathematical model are shown in Fig. 1. The crack is located at x_c, and its effect is modeled by using a torsion spring which connects the two intact parts at the crack location.

Using Hamilton principle and assumptions of Euler-Bernoulli beam theory, the governing equation of lateral vibration behavior related to each of the intact parts is derived as

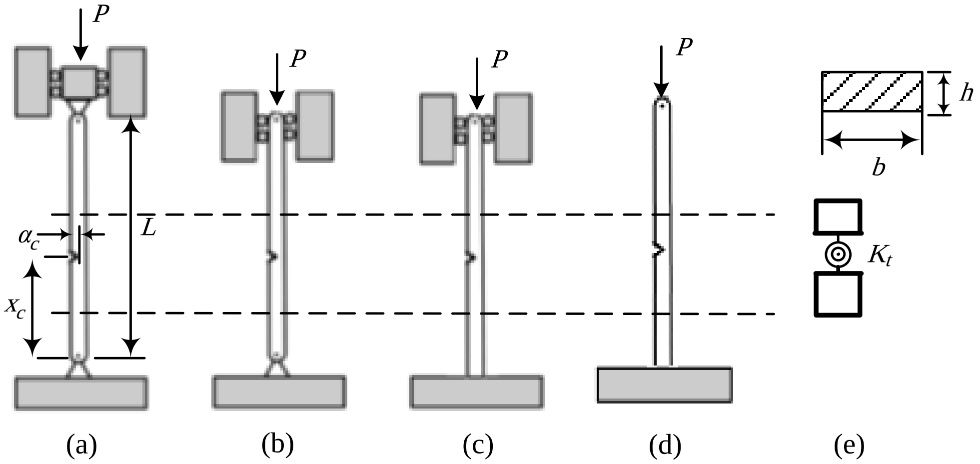

Fig. 1 Schematic view of the cracked column (a) Simply supported, (b) Simply-Clamped, (c) Clamped-Clamped, (d) clamped-Free and (e) mathematical model of the crack

$$EI\frac{\partial^4 y(x,t)}{\partial x^4}+P\frac{\partial^2 y(x,t)}{\partial x^2}+\rho A\frac{\partial^2 y(x,t)}{\partial t^2}=0 \tag{7}$$

where EI, P, ρA are bending rigidity of the column, compressive axial load and mass per unit volume of the column. For using the torsion spring model, first it is necessary to find the analytical solution of the above equation. Harmonic solution of Eq. (7) is shown as $y(x,t)=Y(x)e^{i\omega t}$. Substituting this solution in Eq. (7) leads to

$$EI\frac{d^4Y(x)}{dx^4}+P\frac{d^2Y(x)}{dx^2}-\rho A\omega^2 Y(x)=0 \tag{8}$$

where ω is natural frequency of intact column under axial load. The above equation is linear differential equation of second order with constant coefficients. To solve the Eq. (8), the solution can be assumed as

$$Y(x)=A_1\sinh(\frac{\eta x}{L})+A_2\cosh(\frac{\eta x}{L})+A_3\sin(\frac{\zeta x}{L})+A_4\cos(\frac{\zeta x}{L}) \tag{9}$$

where coefficients $A_i(i=1\rightarrow 4)$ are unknown constants obtained from boundary and ompatibility conditions at the crack location. ζ, η are dimensionless parameters defined as follows

$$\zeta=\sqrt{-\phi+\sqrt{\phi^2+\psi^2}} \tag{10}$$

$$\eta=\sqrt{\phi+\sqrt{\phi^2+\psi^2}} \tag{11}$$

where dimensionless parameters defined as follows

$$\phi=\frac{PL^2}{2EI},\ \psi=L^2\sqrt{\frac{\rho A}{EI}}\omega \tag{12}$$

3.1 Solving the governing differential equation of cracked column vibration

To derive the frequency equation of the cracked column lateral vibration under axial load, the mathematical model of the cracked column shown in Fig. 1 is applied. Using Eq. (9), the solution of Eq. (8) for the two intact parts of the column at the crack sides can be derived as

$$Y_L(x) = B_1 \sin(\frac{\zeta x}{L}) + B_2 \cos(\frac{\zeta x}{L}) + B_3 \sinh(\frac{\eta x}{L}) + B_4 \cosh(\frac{\eta x}{L}) \tag{13}$$

$$Y_R(x) = B_5 \sin(\frac{\zeta x}{L}) + B_6 \cos(\frac{\zeta x}{L}) + B_7 \sinh(\frac{\eta x}{L}) + B_8 \cosh(\frac{\eta x}{L}) \tag{14}$$

The system vibration modes (Eqs. (13) and (14)) include 8 unknowns B_i(i=1→8) which are obtained from boundary and compatibility conditions at the crack location. The compatibility conditions at x=x_c are found from continuity condition of deflection, moment and shear load, and slope difference at the crack sides that are respectively as follows

$$\begin{gathered} Y_L(x_c)=Y_R(x_c), \quad Y_L''(x_c)=Y_R''(x_c), \\ Y_L'''(x_c)+\frac{2\phi}{L}Y_L'(x_c)=Y_R'''(x_c)+\frac{2\phi}{L}Y_R'(x_c), \\ K_tY_L'(x_c)-K_tY_R'(x_c)=\mathrm{EI}Y_R'(x_c) \end{gathered} \tag{15}$$

The presented analytical model can be applied for different boundary conditions. Therefore, in the present study the standard boundary conditions (simple, cantilever, end-supported cantilever and fixed) are investigated.

3.2 Buckling of cracked column under axial load

The governing equation of cracked column deflection under axial load is expressed as

$$EI\frac{d^4Y(x)}{dx^4} + P\frac{d^2Y(x)}{dx^2} = 0 \tag{16}$$

Solution of the Eq. (16) can be written as follows

$$Y(x) = C_1 + C_2x + C_3\sinh(\frac{\eta x}{L}) + C_4\cosh(\frac{\eta x}{L}) \tag{17}$$

Considering the above equation, solution for the two intact parts of the column at the crack sides are as

$$Y_L(x) = D_1 + D_2x + D_3\sinh(\frac{\eta x}{L}) + D_4\cosh(\frac{\eta x}{L}) \tag{18}$$

$$Y_R(x) = D_5 + D_6x + D_7\sinh(\frac{\eta x}{L}) + D_8\cosh(\frac{\eta x}{L}) \tag{19}$$

Eqs. (18) and (19) include 8 unknowns D_i, i=1, 2,…, 8 which are obtained from boundary and compatibility conditions at the crack location.

The eigenvalue problem in correspondence with lateral vibration and buckling of the cracked column is obtained from substituting the equations found for the two intact parts of the cracked column in continuity conditions of Eq. (15), Applying the boundary conditions, 8 algebraic equation for the unknown coefficients $B=\{B_i\}$ or $D=\{D_i\}(i=1\rightarrow 8)$ as follows

$$[\Delta]\{B\}=0 \tag{20}$$

$$[\Delta]\{D\}=0 \tag{21}$$

In the Eq. (21), elements of the coefficient matrix [Δ] depend on geometric and mechanical characteristics, boundary conditions, crack parameters and axial load. To have non-trivial solution, the determinant of coefficient matrix should be equal to zero. So, the eigenvalue problem for the cracked column under axial load is obtained as

$$\det[\Delta(K_t, x_c, P, \omega)]=0 \tag{22}$$

Solving the last equation natural frequencies of the cracked column is obtained. Also, considering the statically case (Eq. (21)), the buckling critical load of the cracked column is found.

4. Analytical results

For validating the presented model, experimental tests results of (Vakil *et al.* 2012) are used. In the mentioned reference the tests were accomplished on cracked cantilever beam with length L=820 mm, width b=20 mm, height h=10 mm, elasticity module E=70 GPa, mass per unit volume ρ=2700 Kg/m^3, and in absence of axial load. Table 1 demonstrates natural frequencies of cracked cantilever beam in absence of axial load. In this table, a comparison between the results from experimental tests of (Vakil *et al.* 2012) and the results of the current study model is performed for different parameters of crack. Results show that maximum error of the model in calculating first, second and third natural frequencies are 0.95, 0.9 and 0.74, respectively. It can be seen that the presented model predicts the vibration behavior of the cracked column with proper accuracy. The dimensionless crack parameters are $\alpha_c=a_c/h$ and $\beta=x_c/L$, where α_c and β are dimensionless crack depth and location, respectively.

Diagram of the first and second frequency ratio changes with relative location of the crack is demonstrated in Figs. 2 and 3. Results show that in the second vibration mode, the lowest decrease of the second frequency is for the crack located at relative location β=0.2. The mentioned location is the point of inflection of the second vibration mode function. At this location the second derivative of deflection function is zero. It means that bending moment at that point during the column vibration at the second mode is zero. Since the major factor of decrease in natural frequencies due to crack is bending moment therefore, at the second vibration mode effect of the crack located at β=0.2 on ratio of the second natural frequency decrease is negligible. Fig. 4 shows diagram of frequency ratio at the first vibration mode with crack relative depth for different crack relative locations. As seen the crack causes natural frequency reduction, and this reduction has a direct relation with the crack depth. Increasing the crack depth and consequently increasing local flexibility of the column at the section where the crack exists, more decrease is seen in natural frequencies. Also, results show that for crack closer to the fixed end, effect of the crack on frequency reduction is more.

Table 1 Natural frequencies of clamped-free cracked column and Comparisons of the results with experimental data of (Vakil *et al.* 2012)

Crack parameters, mm		**Natural Frequencies, Hz**								
Crack location	Crack depth	f_1			f_2			f_3		
		Exp. [a]	Presented method	Error%	Exp. [a]	Presented method	Error%	Exp. [a]	Presented method	Error%
30	1	11.49	11.58	0.73	72.11	72.60	0.69	201.99	203.37	0.68
30	2	11.47	11.50	0.23	72.04	72.23	0.27	201.92	202.61	0.34
600	1	11.51	11.60	0.83	72.11	72.69	0.81	201.98	203.47	0.74
600	2	11.49	11.60	0.95	71.93	72.57	0.90	201.73	203.01	0.63

a (Vakil *et al.* 2012)

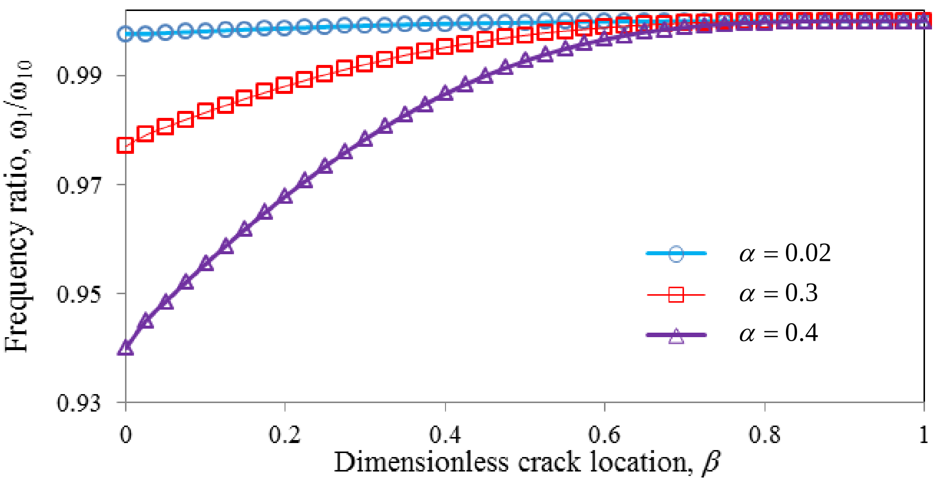

Fig. 2 First frequency ratio of the clamped-free cracked column for different crack depth

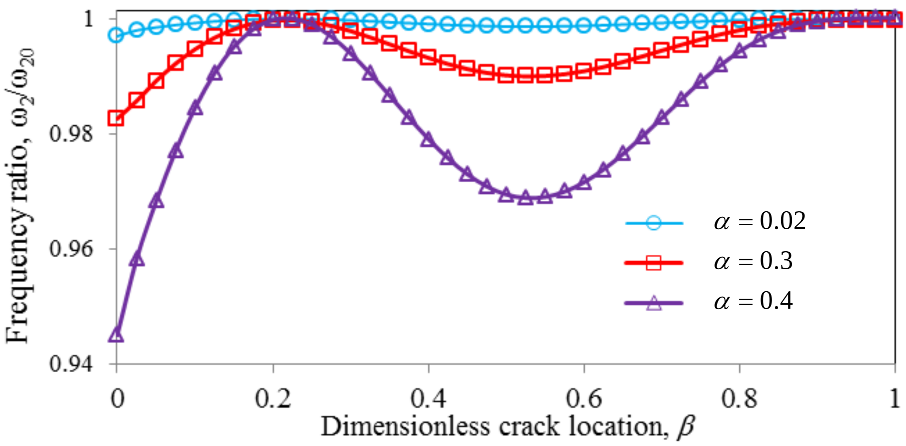

Fig. 3 Second frequency ratio of the clamped-free cracked column for different crack depth

It is obvious from the Eq. (22) that natural frequencies of cracked column also depend on axial load. Diagram of first frequency ratio of the cracked column under axial load with the axial load magnitude for both compressive and tensile loadings is shown in Fig. 5. Results show that tensile load increases natural frequencies. Also, compressive load reduces natural frequencies. At buckling load the first natural frequency becomes zero, and the system becomes unstable.

From other advantages of the presented model is the possibility of calculating the buckling load of the cracked columns. For intact column, buckling load is calculated from Euler formula as follows (Surya and Dale 2004)

$$P_e = \frac{\pi^2 EI}{(KL)^2} \tag{23}$$

where K is column effective length factor. For cantilever, simple, end-supported cantilever and fixed columns, K is 2, 1, 0.7 and 0.5, respectively (Surya and Dale 2004), Using Eq. (21),

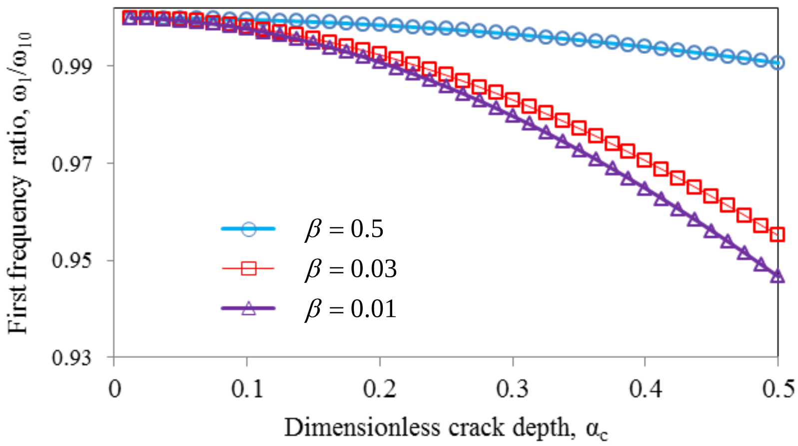

Fig. 4 First frequency ratios for different crack location

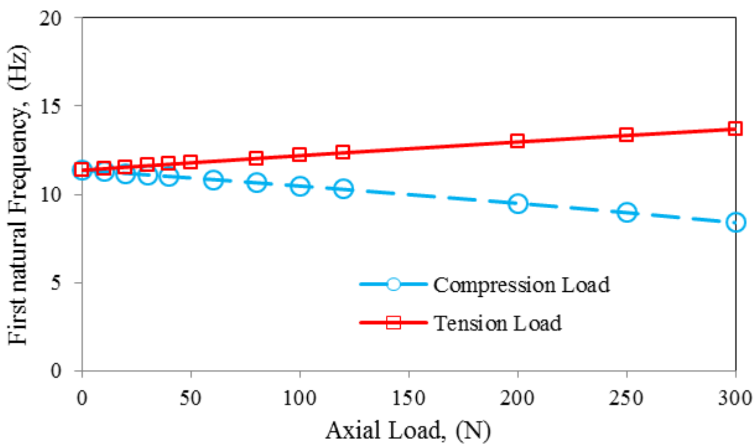

Fig. 5 First frequency ratio of clamped-free cracked column for different axial loads

buckling load of cracked columns can be calculated. Dimensionless buckling load changes of cantilever cracked column with crack relative location for different relative depth of the crack is demonstrated in Fig. 6. P_{cr} and P_e is buckling load of cracked and intact columns, respectively. Results show that crack decreases buckling load and load capacity of the column. For instance, buckling load for studied intact cantilever from Euler formula is P_e=3.67EI, and for the cracked column with relative depth 0.6 and at relative location 0.001 is P_{cr}=3.35EI. Therefore, the crack decreases the buckling load of the cracked column 9.67% in comparison with intact column. It is seen that at a given crack location, increasing the crack depth reduces the column buckling load.

Effect of crack location on column buckling load is different based on boundary conditions. Figs. (7)-(8) show buckling load changes of the cracked columns having end-supported cantilever

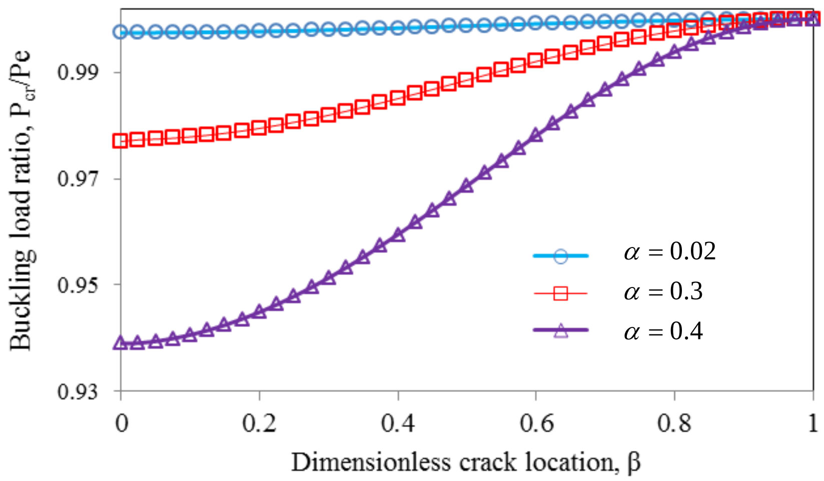

Fig. 6 Buckling load of the clamped-free cracked column vs. the crack location for various crack depth

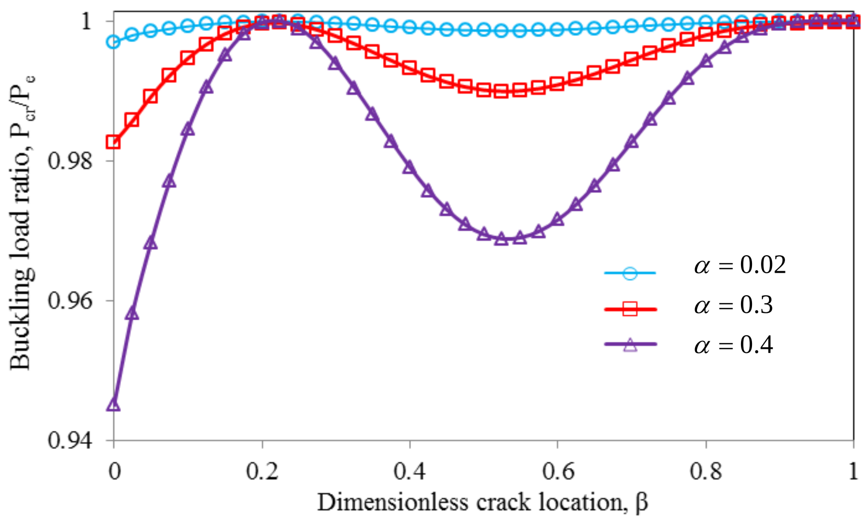

Fig. 7 Buckling load of the clamped-simply supported cracked column vs. the crack location

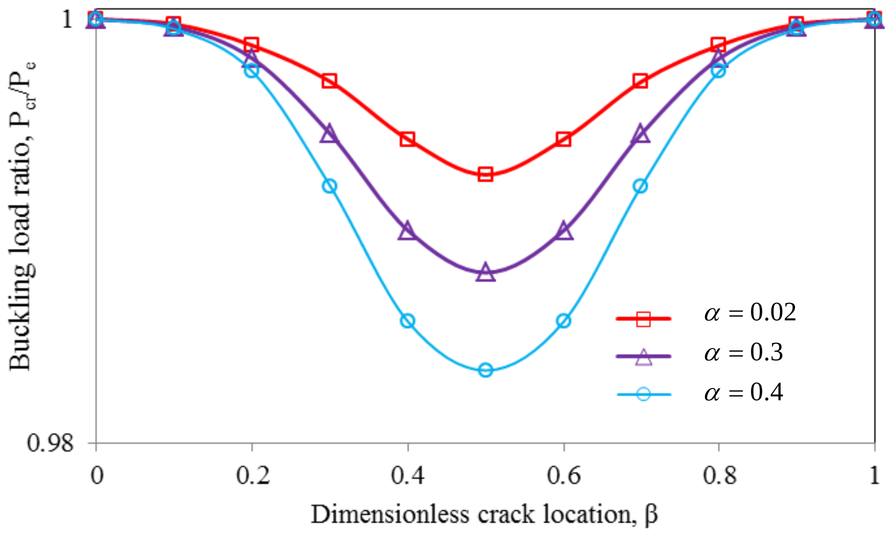

Fig. 8 Buckling load of the simply support cracked column vs. the crack location for various crack depth

and fixed boundary conditions for different crack relative depth. It is seen that crack decreases the buckling load of the column. According to fracture theory, it is obvious that strain energy stored in elastic materials under bending moment is a function of applied bending moment. Therefore, for crack with a given depth, crack at location in correspondence with the maximum bending moment, has the maximum effect on reduction of buckling load and load capacity. In addition, crack located at points of inflection where bending moment is zero, has no effect on buckling load of the cracked column. For cantilever column as the crack, location is closer to the fixed end, the crack effect on buckling load increases.

5. Conclusions

In the present study, a new analytical method is presented to investigate stability and vibration behavior of cracked columns under axial load. The presented model for the crack uses the torsion spring with equivalent stiffness calculate from fracture mechanics theory. After deriving the governing differential equation of lateral vibration behavior of the column under axial load, by applying boundary and compatibility conditions the corresponding eigenvalue problem is derived to investigate the effect of axial load and crack parameters on vibration behavior and buckling load of cracked columns. Results show that tensile axial load increases the natural frequencies. On the other hand, compressive axial load decreases the natural frequencies, and at the buckling load the first natural frequency becomes zero, and the system becomes unstable. Also, results show that crack reduces local stiffness of the column, and at a given crack location, increasing the crack depth raises the effect on vibration behavior and buckling load of the column. For a given crack depth, the crack location changes the natural frequencies reduction. Since the major factor of reduction of natural frequencies and buckling load caused by crack is bending moment, at locations where bending moment becomes zero the effect of crack on natural frequencies and buckling load decreases. Comparison between results from the presented analytical solution and experimental tests shows the very good accordance of the two methods for a vast range of crack parameters.

References

Anifantis, N. and Dimaragonas, A. (1981), "Stability of columns with a single crack subject to follower and vertical loads", *Int. J. Solid. Struct.*, **19**(4), 281-291.

Binici, B. (2005), "Vibration of beams with multiple open cracks subjected to axial force", *J. Sound Vib*., **287**(1), 277-295.

Butcher, E.A. (2010), "Natural frequencies and critical loads of beams and columns with damaged boundaries using Chebyshev polynomials", *Int. J. Eng. Sci*., **48**(10), 862-873.

Caddemi, S. and Caliò, I. (2012), "The influence of the axial force on the vibration of the Euler-Bernoulli beam with an arbitrary number of cracks", *Arch. Appl. Mech.*, **82**(6), 827-839.

Chen, D.L., Wang, W.T. and Liu, F. (2011), "Vibration analysis method of cracked beam based on the principle of energy", *Appl. Mech. Mater*., **94**, 1633-1637.

Gurel, M.A. and Kisa, M. (2005), "Buckling of slender prismatic columns with a single edge crack under concentric vertical loads", *Turkish J. Eng. Envir. Sci.*, **29**(3), 185-193.

Jena, P.K., Thatoi, D.N., Nanda, J. and Parhi, D.R.K. (2012), "Effect of damage parameters on vibration signatures of a cantilever beam", *Procedia Eng*., **38**, 3318-3330.

Jiki, P.N. (2007), "Buckling analysis of pre-cracked beam-columns by Liapunov's second method", *Eur. J. Mech. A Solid.*, **26**(3), 503-518.

Jiki, P.N. (2011), "A Finite element java program for stability analysis of pre-cracked beam-columns", *Eur. J. Appl. Sci*., **3**(4), 162-168.

Jiki, P.N. and Karim, U. (2011), "A Numerical model for stability of fre-cracked beam-columns", *1st International Technology, Education and Environment Conference*.

Kim, S.M. and Cho, Y.H. (2006), "Vibration and dynamic buckling of shear beam-columns on elastic foundation under moving harmonic loads", *Int. J. Solid. Struct*., **43**(3), 393-412.

Kisa, M. (2011), "Vibration and stability of multi-cracked beams under compressive axial loading", *Int. J. Phys. Sci*., **6**(11), 2681-2696.

Krauberger, N., Bratina, S., Saje, M., Schnabl, S. and Planinc, I. (2012), "Inelastic buckling load of a locally weakened reinforced concrete column", *Eng. Struct*., **34**, 278-288.

Nikpour, K. (1990), "Buckling of cracked composite columns", *Int. J. Solid. Struct.*, **26**(12), 71-86.

Okamura, H., Liu, H.W., Chu, C.S. and Liebowitz, H. (1969), "A cracked column under compression", Eng. Fracture Mech., **1**(3), 547-564.

Patnaik, S.N. and Hopkins, D.A. (2004), *Strength of Materials*, New York, Elsevier.

Rahai, A.R. and Kazemi, S. (2008), "Buckling analysis of non-prismatic columns based on modified vibration modes", *Comm. Nonlin. Sci. Numer. Simulation*, **13**(8), 1721-1735.

Ranjbaran, A., Hashemi, S. and Ghaffarian, A.R. (2008), "A new approach for buckling and vibration analysis of cracked column", *Int. J. Eng. Trans. A: Basic*., **21**(3), 225-230.

Şakar, G., Öztürk, H. and Sabuncu, M. (2012), "Dynamic stability of multi-span frames subjected to periodic loading", *J. Constr. Steel Res*., **70**, 65-70.

Tada, H., Paris, P. and Irwin, G.R. (2004), *The Stress Analysis of Cracks Handbook*, New York, ASME Press.

Toygar, M.E., Kıral, Z., Sayman, O., Arman, Y. and Özen, M. (2012), "Effect of interface crack on lateral buckling behavior and free vibration response of a sandwich composite beam", *J. Compos. Mater.*, **1**, 1-9.

Vadillo, G., Loya, J.A. and Fernandez-Saez, J. (2012), "First order solutions for the buckling loads of weakened Timoshenko columns", *Comput. Math. Appl.*, **64**, 2395-2407.

Vakil Baghmisheh, M.T., Peimani M., Homayoun Sadeghi, M., Ettefaghb, M.M. and Fakheri Tabrizi, A.

(2012), “A hybrid particle swarm-Nelder-Mead optimization method for crack detection in cantilever beams”, *Appl. Soft Comput.*, **12**, 2217-2226.
Yan, G., De Stefano, A., Matta, E. and Feng, R. (2013), “A novel approach to detecting breathing-fatigue cracks based on dynamic characteristics”, *J. Sound Vib*., **332**(2), 407-422.
Yazdchi, K. and Gowhari Anaraki, A.R. (2008), “Carrying capacity of edge-cracked columns under concentric vertical loads”, *Acta Mechanica*, **198**, 1-19.

CC