Earthquakes and Structures, *Vol. 9, No. 1 (2015) 221-232*
DOI: http://dx.doi.org/10.12989/eas.2015.9.1.221

Analytical study of nonlinear vibration of oscillators with damping

Mahmoud Bayat*[1], Mahdi Bayat[1] and Iman Pakar[2]

[1]*Department of Civil Engineering, Mashhad Branch, Islamic Azad University, Mashhad, Iran*
[2]*Young Researchers and Elite Club, Mashhad Branch, Islamic Azad University, Mashhad, Iran*

(Received July 1, 2014, Revised September 19, 2014, Accepted November 6, 2014)

Abstract. In this study, Homotopy Perturbation Method (HPM) is used to solve the nonlinear oscillators with damping. We have considered two strong nonlinear equations to show the application of the method. The Runge-Kutta's algorithm is used to obtain the numerical solution for the problems. The method works very well for the whole range of initial amplitudes and does not demand small perturbation and also sufficiently accurate to both linear and nonlinear physics and engineering problems. Finally to show the accuracy of the HPM, the results have been shown graphically and compared with the numerical solution.

Keywords: Homotopy Perturbation Method (HPM); nonlinear vibrations; damping

1. Introduction

Dynamical differential equations can be parted into linear and nonlinear differential equations. For linear dynamical differential equations it is possible to prepare exact solution but for nonlinear ones, it is very hard to solve them analytically. In the past few decades many new analytical and numerical approaches have been used for solving nonlinear differential equations such as: Homotopy perturbation method (Bayat *et al.* 2013a, 2014a),Hamiltonian approach (He 2010, Xu 2010, Bayat *et al*. 2014b, c, d, e, f, 2013b, Bayat and Pakar 2013c), Energy balance method (Jamshidi *et al*. 2010, Bayat *et al*. 2014g, Mehdipour *et al*. 2010),Variational iteration method (Dehghan 2008), Amplitude frequency formulation (He 2008), Max-Min approach (Shen *et al*. 2009, Zeng *et al*. 2009), Variational approach (He 2007, Bayat and Pakar 2012a, Bayat *et al*. 2012b, Bayat and Pakar 2013a, Bayat *et al*. 2013b, Shahidi *et al*. 2011, Pakar and Bayat 2013), and the other analytical and numerical (Bayat and Abdollahzade 2011, Pakar *et al*. 2014a, b, 2011, Xu 2009, Alicia *et al*. 2010, Bor-Lih *et al*. 2009, Wu 2011, Odibat *et al*. 2008, Zhifeng *et al*. 2013, Rajasekaran 2013, Akgoz 2013, Akgoz and Civalek 2011, Atmane *et al*. 2011, Cunedioglu and Beylergil 2014).

Among of these methods, Homotopy perturbation method is considered to solve the nonlinear equations with damping. In this study, first we describe the basic concept of the Homotopy perturbation method and Runge-Kutta algorithm. In the second section apply this method

∗Corresponding author, Researcher, E-mail: mbayat14@yahoo.com

http://www.techno-press.org/journals/eas&subpage=7 ISSN: 2092-7614 (Print), 2092-7622 (Online)

for two examples. And in the last section, some comparisons between analytical and numerical solutions are presented to show the accuracy of the HPM.

2. Basic idea of the HPM

To illustrate the basic ideas of this method, we consider the following equation

$$A(x)-f(r)=0 \quad r\in\Omega \tag{2.1}$$

With the boundary condition of

$$B\left(x,\frac{\partial x}{\partial t}\right)=0 \quad r\in\Gamma \tag{2.2}$$

Where A is a general differential operator, B a boundary operator, $f(r)$ a known analytical function and Γ is the boundary of the domain Ω. A Can be divided into two parts of L and N, where L is linear and N is nonlinear. Eq. (2.1) can therefore be rewritten as follows

$$L(x)+N(x)-f(r)=0 \quad r\in\Omega \tag{2.3}$$

Homotopy perturbation structure is shown as follows

$$H(\nu,p)=(1-p)\left[L(\nu)-L(x_0)\right]+p\left[A(\nu)-f(r)\right]=0 \tag{2.4}$$

Where,

$$\nu(r,p):\ \Omega\times[0,1]\to R \tag{2.5}$$

In Eq. (2.4), $p\in[0,1]$ is an embedding parameter and x_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (2.1) can be written as a power series in p, as following

$$\nu=\nu_0+p\nu_1+p^2\nu_2+\cdots=\sum_{i=0}^{n}\nu_i p^i \tag{2.6}$$

And the best approximation for the solution is

$$x=\lim_{p\to 1}\nu=\nu_0+\nu_1+\nu_2+\cdots \tag{2.7}$$

3. Basic idea of Runge-Kutta

The fourth RK (Runge-Kutta) method has been used to verify the homotopy perturbation solution. This iterative algorithm is written in the form of the following formulae for the second-order differential equation

$$\dot{x}_{i+1} = \dot{x}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + h_4\right)$$
$$x_{i+1} = x_i + \Delta t\left(\dot{x}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + h_3\right)\right) \tag{3.1}$$

Where, Δt is the increment of the time and h_1, h_2, h_3, and h_4 are determined from the following formulae

$$\begin{aligned}
h_1 &= f\left(\dot{x}, x_i, \dot{x}_i\right),\\
h_2 &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2}\dot{x}_i, \dot{x}_i + \frac{\Delta t}{2}h_1\right),\\
h_3 &= f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{\Delta t}{2}\dot{x}_i, \frac{1}{4}\Delta t^2 h_1, \dot{x}_i + \frac{\Delta t}{2}h_2\right),\\
h_4 &= f\left(t_i + \Delta t, x_i + \Delta t\dot{x}_i, \frac{1}{2}\Delta t^2 h_2, \dot{x}_i + \Delta t\, h_3\right).
\end{aligned} \tag{3.2}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Eq. (3.1). This process continues to the end of the time limit.

4. Application

In order to assess the advantages and the accuracy of the Homotopy Perturbation Method (HPM), we will consider the following examples:

4.1 Example 1

The general equation of an oscillator with a nonlinear spring, a linear spring and a damper under a harmonic load is as follow

$$m\ddot{x} + c\dot{x} + k_1 x + k_2 x^3 = F_0\cos\left(\omega t\right) \tag{4.1}$$

Subject to the following initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0 \tag{4.2}$$

Where m is the mass, c is a viscous damping coefficient, k_1 is a linear stiffness coefficient, and k_2 is a nonlinear stiffness coefficient. The harmonic excitation force is characterized by the force amplitude, F_0, with excitation frequency of ω. A is the initial amplitude of displacement.

ω can be found easily by having the parameters, A, c, m, k_1 and k_2:

As the HPM was applied to the nonlinear Eq. (4.1), we have

$$\left(1-p\right)\left(m\ddot{x} + c\dot{x} + k_1 x\right) + p\left(m\ddot{x} + c\dot{x} + k_1 x + k_2 x^3 - F_0\cos(\omega t)\right) = 0 \tag{4.3}$$

After expanding the equation and collecting it based on the coefficients of p-terms, we have

$$\begin{cases} p^0 : m\ddot{x}_0 + c\dot{x}_0 + k_1 x_0 = 0 \\ p^1 : m\ddot{x}_1 + c\dot{x}_1 + k_1 x_1 + k_2 x_0^{\,3} - F_0\cos(\omega t) = 0 \\ P^2 : m\ddot{x}_2 + c\dot{x}_2 + k_1 x_2 + 3k_2 x_0^{\,2} x_1 = 0 \\ P^3 : m\ddot{x}_3 + c\dot{x}_3 + k_1 x_3 + 3k_2 x_0^{\,2} x_2 + 3k_2 x_0 x_1^{\,2} = 0 \end{cases} \tag{4.4}$$

One can now try to obtain the solution of different iterations (4.4), in the form of

$$\begin{aligned} x_0(t) &= \frac{1}{2}\frac{1}{c^2-4k_1m}\left(c\sqrt{c^2-4k_1m}+c^2-4k_1m\right)e^{\frac{1}{2}\frac{(-c+\sqrt{c^2-4k_1m})t}{m}} \\ &+\frac{1}{2}\frac{1}{c^2-4k_1m}\left(c^2-c\sqrt{c^2-4k_1m}-4k_1m\right)e^{\frac{-(c+\sqrt{c^2-4k_1m})t}{2m}} \end{aligned} \tag{4.5}$$

$$\begin{aligned} x_1(t) &= \frac{1}{2}\frac{1}{(c^2-4k_1m)^{\frac{5}{2}}}\Bigg(\Bigg(\Bigg(\int_0^t \Bigg(-2e^{-\frac{5}{2}\frac{(c+\sqrt{c^2-4k_1m})\_zl}{m}}\Bigg(-\frac{1}{2}\Big(\big(c^3-3mck_1\big)\sqrt{c^2-4k_1m} \\ &+c^4-5c^2k_1m+4k_1^2m^2\Big)\times k_2 e^{\frac{1}{2}\frac{\_zl\,(3c+7\sqrt{c^2-4k_1m})}{m}}+\frac{3}{2}mk_2k_1\frac{(c\sqrt{c^2-4k_1m})}{m} \\ &+c^2-4k_1m)e^{\frac{1}{2}\frac{\_zl\,(3c+5\sqrt{c^2-4k_1m})}{m}}+F_0\cos(\omega t\ \_zl)(c^2-4k_1m)^2 e^{\frac{\_zl\,(3c+2\sqrt{c^2-4k_1m})}{m}} \\ &-\frac{1}{2}(((-c^3+3mck_1)^2 e^{\frac{\_zl\,(3c+2\sqrt{c^2-4k_1m})}{m}}-\frac{1}{2}(((-c^3+3mck_1)\sqrt{c^2-4k_1m}+c4 \\ &-5c^2k_1m+4k_1^2m^2)e^{\frac{1}{2}\frac{\_zl\,(3c+\sqrt{c^2-4k_1m})}{m}}-3mk_1e^{\frac{3}{2}\frac{(c+\sqrt{c^2-4k_1m})\_zl}{m}}\times \\ &(c^2-c\sqrt{c^2-4k_1m}-4k_1m)k_2))d\ \_zl)e^{\frac{t\sqrt{c^2-4k_1m}}{m}} \\ &-\Bigg(\int_0^t\Bigg(-2e^{\frac{-1}{2}\frac{\_zl\,(5c+3\sqrt{c^2-4k_1m})}{m}}\Bigg(-\frac{1}{2}\Big(\big(c^3-3mck_1\big)\sqrt{c^2-4k_1m} \\ &+c^4-5c^2k_1m+4k_1^2m^2\Big)k_2e^{\frac{1}{2}\frac{\_zl\,(3c+7\sqrt{c^2-4k_1m})}{m}} \\ &+\frac{3}{2}mk_2k_1(c\sqrt{c^2-4k_1m}+c^2-4k_1m)e^{\frac{1}{2}\frac{\_zl\,(3c+5\sqrt{c^2-4k_1m})}{m}} \\ &+F_0\cos(\omega t\ \_zl)(c^2-4k_1m)^2e^{\frac{\_zl\,(3c+2\sqrt{c^2-4k_1m})}{m}} \\ &-\frac{1}{2}(((-c^3+3mck_1)\sqrt{c^2-4k_1m}+c^4-5c^2k_1m+4k_1^2m^2)\times e^{\frac{1}{2}\frac{\_zl\,(3c+\sqrt{c^2-4k_1m})}{m}} \\ &-3mk_1e^{\frac{3}{2}\frac{(3c+\sqrt{c^2-4k_1m})\_zl}{m}}(c^2-c\sqrt{c^2-4k_1m}-4k_1m))k_2))d\ \_zl))e^{\frac{-1}{2}\frac{(c+\sqrt{c^2-4k_1m})t}{m}}\Bigg) \end{aligned} \tag{4.6}$$

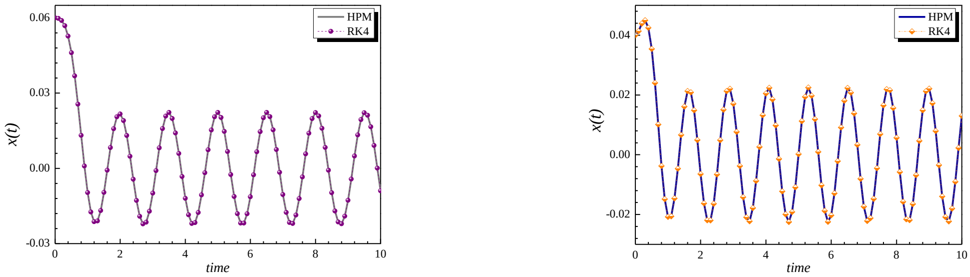


Fig. 1 Displacement x based on time t for (a) f=0.5, A=0.06, ω=4.163379415, (b) f=0.7, A=0.04, ω=5.147879675

And from Eqs. (4.5) and (4.6), $x(t)$ can be obtained

$$
\begin{aligned}
x(t) &= \frac{1}{2}\frac{1}{c^2-4k_1m}((c\sqrt{c^2-4k_1m}+c^2-4k_1m)e^{\frac{1}{2}\frac{(-c+\sqrt{c^2-4k_1m})t}{m}} \\
&+\frac{1}{2}\frac{1}{c^2-4k_1m}(c^2-c\sqrt{c^2-4k_1m}-4k_1m)e^{\frac{-(c+\sqrt{c^2-4k_1m})t}{2m}} \\
&+\frac{1}{2}\frac{1}{(c^2-4k_1m)^{\frac{5}{2}}}(((\int_0^t(-2e^{-\frac{5}{2}\frac{(c+\sqrt{c^2-4k_1m})\_zl}{m}}(-\frac{1}{2}((c^3-3mck_1)\sqrt{c^2-4k_1m} \\
&+c^4-5c^2k_1m+4k_1^2m^2)\times k_2e^{\frac{1}{2}\frac{\_zl(3c+7\sqrt{c^2-4k_1m})}{m}}+\frac{3}{2}mk_2k_1\frac{(c\sqrt{c^2-4k_1m})}{m} \\
&+c^2-4k_1m)e^{\frac{1}{2}\frac{\_zl(3c+5\sqrt{c^2-4k_1m})}{m}}+F_0\cos(\omega t\ \_zl)(c^2-4k_1m)^2e^{\frac{\_zl(3c+2\sqrt{c^2-4k_1m})}{m}} \\
&-\frac{1}{2}(((-c^3+3mck_1)^2e^{\frac{\_zl(3c+2\sqrt{c^2-4k_1m})}{m}}-\frac{1}{2}(((-c^3+3mck_1)\sqrt{c^2-4k_1m}+c4 \\
&-5c^2k_1m+4k_1^2m^2)e^{\frac{1}{2}\frac{\_zl(3c+\sqrt{c^2-4k_1m})}{m}}-3mk_1e^{\frac{3}{2}\frac{(c+\sqrt{c^2-4k_1m})\_zl}{m}}\times \\
&(c^2-c\sqrt{c^2-4k_1m}-4k_1m)k_2))d\ \_zl)e^{\frac{t\sqrt{c^2-4k_1m}}{m}} \\
&-(\int_0^t(-2e^{\frac{-1}{2}\frac{\_zl(5c+3\sqrt{c^2-4k_1m})}{m}}(-\frac{1}{2}((c^3-3mck_1)\sqrt{c^2-4k_1m} \\
&+c^4-5c^2k_1m+4k_1^2m^2)k_2e^{\frac{1}{2}\frac{\_zl(3c+7\sqrt{c^2-4k_1m})}{m}} \\
&+\frac{3}{2}mk_2k_1(c\sqrt{c^2-4k_1m}+c^2-4k_1m)e^{\frac{1}{2}\frac{\_zl(3c+5\sqrt{c^2-4k_1m})}{m}} \\
&+F_0\cos(\omega t\ \_zl)(c^2-4k_1m)^2e^{\frac{\_zl(3c+2\sqrt{c^2-4k_1m})}{m}} \\
&-\frac{1}{2}(((-c^3+3mck_1)\sqrt{c^2-4k_1m}+c^4-5c^2k_1m+4k_1^2m^2)\times e^{\frac{1}{2}\frac{\_zl(3c+\sqrt{c^2-4k_1m})}{m}} \\
&-3mk_1e^{\frac{3}{2}\frac{(3c+\sqrt{c^2-4k_1m})\_zl}{m}}(c^2-c\sqrt{c^2-4k_1m}-4k_1m))k_2))d\ \_zl))e^{\frac{-1}{2}\frac{(c+\sqrt{c^2-4k_1m})t}{m}})
\end{aligned}
\tag{4.7}
$$

The obtained iteration is used to generate the equation for the next iteration, and therefore the second and third iterations are obtained. Since the two other ones and therefore the general solution are too long to be written in this article, we have shown them in graphs.

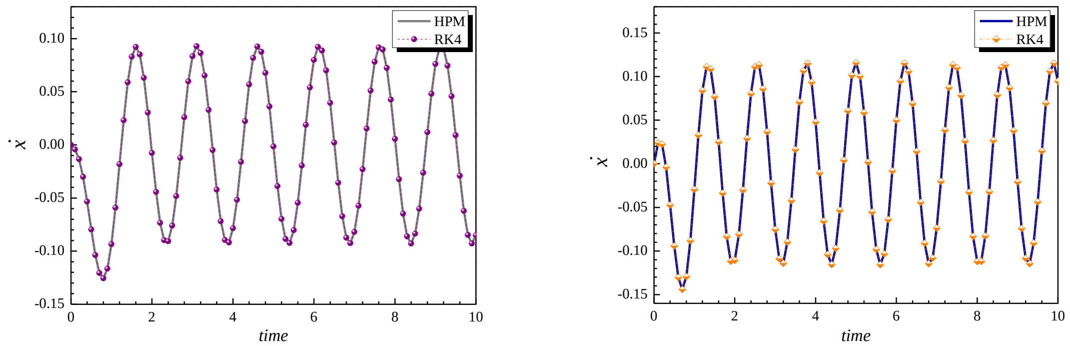


Fig. 2 Velocity $\dot{x}$ based on time t for (a) f=0.5, A=0.06, ω=4.163379415, (b) f=0.7, A=0.04, ω=5.147879675

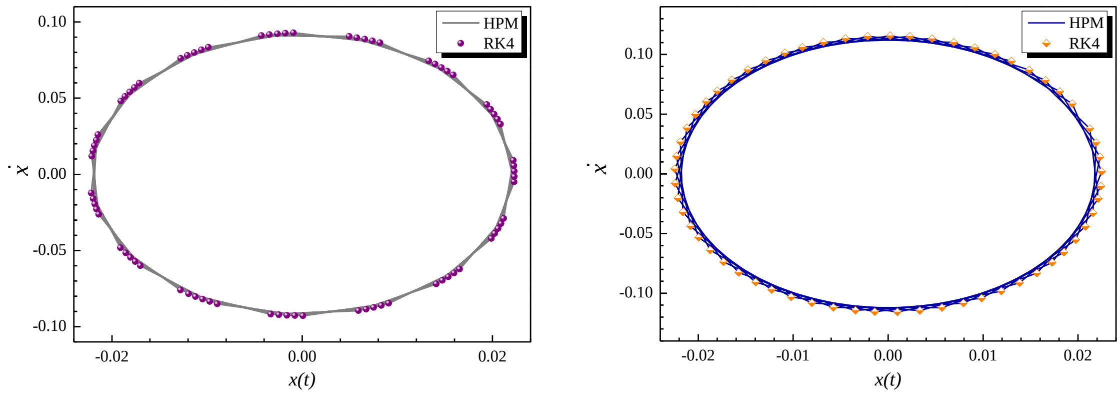


Fig. 3 Velocity $\dot{x}$ based on displacement x for (a) f=0.5, A=0.06, ω=4.163379415, (b) f=0.7, A=0.04, ω=5.147879675

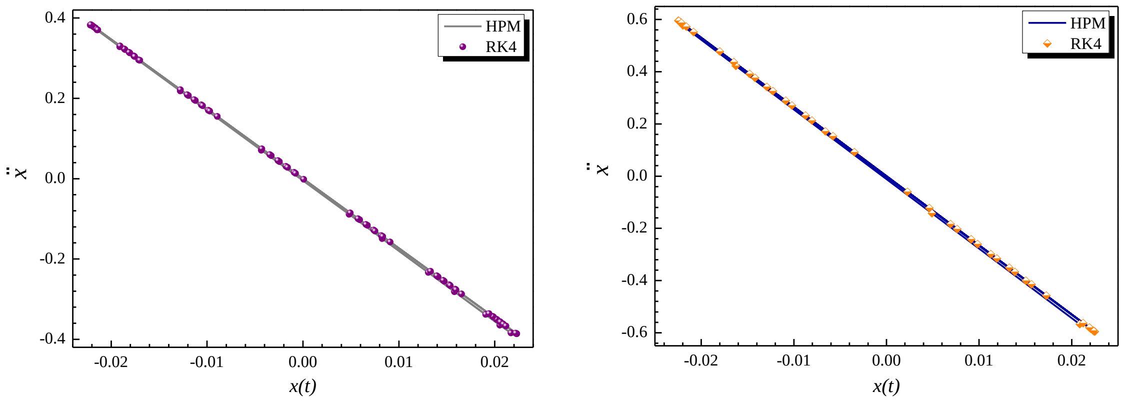


Fig. 4 Acceleration $\ddot{x}$ based on displacement x for (a) f=0.5, A=0.06, ω=4.163379415 (b) f=0.7, A=0.04, ω=5.147879675

Table 1 Comparison of displacement and velocity of HPM with Runge-Kutta for example 1

	Case 1				Case 2			
	Displacement (x)		Velocity $(\dot{x})$		Displacement (x)		Velocity $(\dot{x})$	
Time	HPM	RK4	HPM	RK4	HPM	RK4	HPM	RK4
0	0.06	0.06	0	0	0.04	0.04	0	0
0.5	0.04609	0.04646	-0.07957	-0.08021	0.03542	0.03567	-0.09412	-0.09478
1	-0.00966	-0.00973	-0.09347	-0.09422	-0.02081	-0.02095	-0.02950	-0.02971
1.5	-0.00962	-0.00969	0.08297	0.08364	0.01610	0.01621	0.07691	0.07745
2	0.02169	0.02186	-0.00767	-0.00773	-0.00646	-0.00650	-0.11066	-0.11143
2.5	-0.01280	-0.01290	-0.07599	-0.07660	-0.00660	-0.00665	0.11114	0.11192
3	-0.00986	-0.00994	0.08343	0.08409	0.01706	0.01718	-0.07535	-0.07587
3.5	0.02227	0.02245	-0.00499	-0.00503	-0.02226	-0.02241	0.01635	0.01646
4	-0.01193	-0.01203	-0.07843	-0.07906	0.02048	0.02063	0.04783	0.04817
4.5	-0.01059	-0.01068	0.08168	0.08234	-0.01227	-0.01236	-0.09702	-0.09770
5	0.02230	0.02247	-0.00146	-0.00147	0.00021	0.00022	0.11577	0.11658
5.5	-0.01121	-0.01130	-0.08026	-0.08090	0.01191	0.01199	-0.09821	-0.09889
6	-0.01134	-0.01143	0.07995	0.08059	-0.02030	-0.02044	0.04984	0.05019
6.5	0.02229	0.02247	0.00208	0.00209	0.02232	0.02248	0.01416	0.01426
7	-0.01047	-0.01055	-0.08198	-0.08263	-0.01734	-0.01746	-0.07372	-0.07424
7.5	-0.01206	-0.01216	0.07809	0.07872	0.00692	0.00697	0.11015	0.11093
8	0.02226	0.02244	0.00561	0.00566	0.00567	0.00571	-0.11204	-0.11282
8.5	-0.00971	-0.00978	-0.08358	-0.08425	-0.01648	-0.01660	0.07878	0.07933
9	-0.01276	-0.01287	0.07612	0.07673	0.02212	0.02228	-0.02081	-0.02095
9.5	0.02219	0.02237	0.00914	0.00921	-0.02083	-0.02097	-0.04369	-0.04399
10	-0.00894	-0.00901	-0.08506	-0.08574	0.01300	0.01309	0.09448	0.09514

Case 1: f=0.5, A=0.06, ω=4.163379415
Case 2: f=0.7, A=0.04, ω=5.147879675

4.2 Example 2

In this example we have considered the same oscillators with nonlinear damped behavior

$$m\ddot{x} + \left(\beta_1 + \beta_2 x^2\right)\dot{x} + k_1 x + k_2 x^3 = F_0 \cos(\omega t) \tag{4.8}$$

Subject to the following initial conditions

$$x(0) = A, \quad \dot{x}(0) = 0 \tag{4.9}$$

As the HPM was applied to the nonlinear Eq. (4.8), we have

$$(1-p)\left(m\ddot{x} + \beta_1\dot{x} + k_1 x\right) + p\left(m\ddot{x} + \left(\beta_1 + \beta_2 x^2\right)\dot{x} + k_1 x + k_2 x^3 - F_0 \cos(\omega t)\right) = 0 \tag{4.10}$$

After expanding the equation and collecting it based on the coefficients of p-terms, we have

$$\begin{cases} p^0 : m\ddot{x}_0 + \beta_1\dot{x}_0 + k_1x_0 = 0 \\ p^1 : m\ddot{x}_1 + (\beta_1 + \beta_2 x_0{}^2)\dot{x}_1 + k_1x_1 + k_2x_0{}^3 - F_0\cos(\omega t) = 0 \\ P^2 : m\ddot{x}_2 + \beta_1\dot{x}_2 + \beta_2 x_0{}^2\dot{x}_1 + 2\beta_2 x_0 x_1\dot{x}_0 + k_1x_2 + 3k_2x_0{}^2x_1 = 0 \\ P^3 : .... \end{cases} \tag{4.11}$$

One can now try to obtain the solution of different iterations (4.11), in the form of

$$x_0(t) = \frac{1}{2}\frac{\left(-\beta_1^2 + 4k_1m - \beta_1\sqrt{\beta_1^2 - 4k_1m}\right)e^{\frac{1}{2}\frac{(-\beta_1+\sqrt{\beta_1^2-4k_1m})t}{m}}}{-\beta_1^2 + 4\omega^2 m} + \frac{1}{2}\frac{\left(-\beta_1^2 + 4k_1m + \beta_1\sqrt{\beta_1^2 - 4k_1m}\right)e^{-\frac{1}{2}\frac{(-\beta_1+\sqrt{\beta_1^2-4k_1m})t}{m}}}{-\beta_1^2 + 4k_1m} \tag{4.12}$$

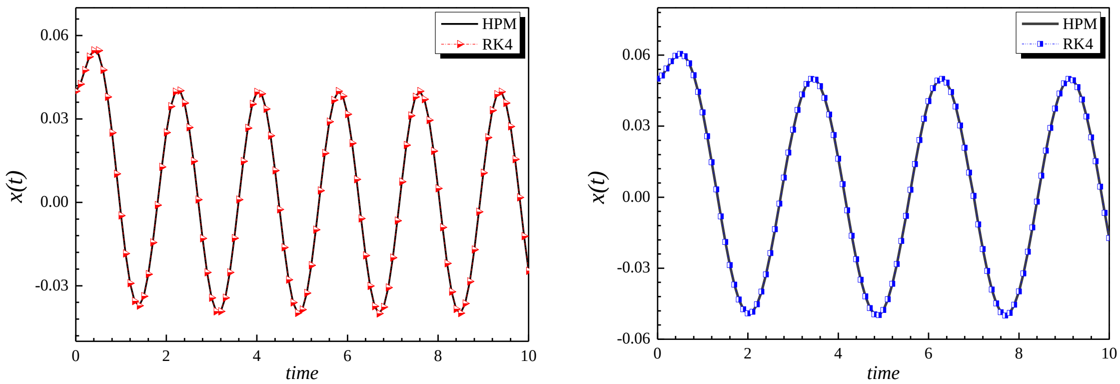


Fig 5. Displacement x based on time t for (a) f=1, A=0.04, ω=3.536163732, (b) f=0.8, A=0.05, ω=2.208420786

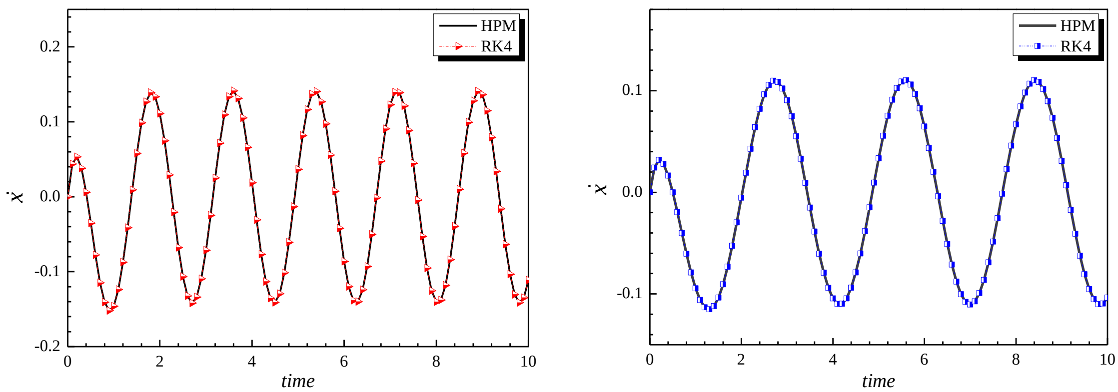


Fig 6. Velocity $\dot{x}$ based on time t for (a) f=1, A=0.04, ω=3.536163732, (b) f=0.8, A=0.05, ω=2.208420786

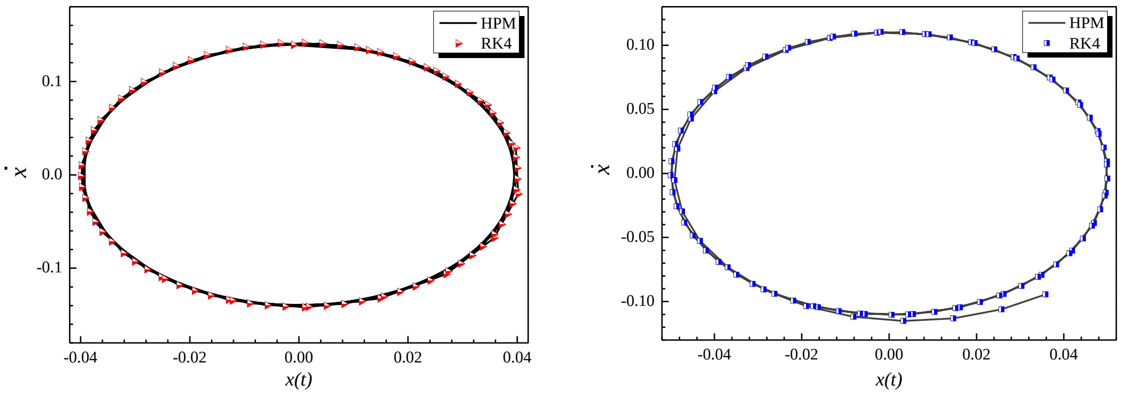
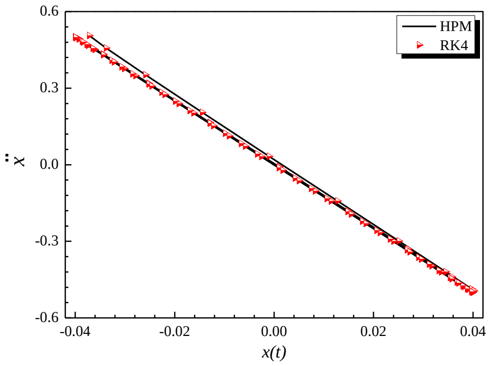


Fig 7. Velocity $\dot{x}$ based on displacement x for (a) f=1, A=0.04, ω=3.536163732, (b) f=0.8, A=0.05, ω=2.208420786

Table 2 Comparison of displacement and velocity of HPM with Runge-Kutta for example 2

	Case 1				Case 2			
	Displacement (x)		Velocity ($\dot{x}$)		Displacement (x)		Velocity ($\dot{x}$)	
Time	HPM	RK4	HPM	RK4	HPM	RK4	HPM	RK4
0	0.04	0.04	0	0	0.05		0	0
0.5	0.05325	0.05372	-0.03502	-0.03533	0.06050	0.06110	-0.00011	-0.00011
1	-0.00479	-0.00483	-0.14615	-0.14746	0.03579	0.03615	-0.09447	-0.09542
1.5	-0.03378	-0.03408	0.05789	0.05841	-0.01896	-0.01915	-0.10364	-0.10468
2	0.02511	0.02533	0.11112	0.11212	-0.04913	-0.04962	-0.00528	-0.00533
2.5	0.02693	0.02717	-0.10656	-0.10752	-0.02368	-0.02392	0.09622	0.09718
3	-0.03438	-0.03469	-0.07152	-0.07217	0.02850	0.02878	0.09071	0.09162
3.5	-0.01290	-0.01302	0.13367	0.13487	0.04961	0.05010	-0.01510	-0.01525
4	0.03968	0.04003	0.01872	0.01889	0.01625	0.01642	-0.10450	-0.10554
4.5	-0.00255	-0.00257	-0.14118	-0.14245	-0.03493	-0.03528	-0.07901	-0.07980
5	-0.03864	-0.03898	0.03655	0.03688	-0.04766	-0.04813	0.03338	0.03371
5.5	0.01771	0.01787	0.12682	0.12796	-0.00794	-0.00801	0.10902	0.11011
6	0.03170	0.03198	-0.08628	-0.08705	0.04052	0.04093	0.06470	0.06535
6.5	-0.03014	-0.03041	-0.09300	-0.09384	0.04439	0.04484	-0.05082	-0.05132
7	-0.01988	-0.02006	0.12273	0.12384	0.00058	0.00059	-0.11042	-0.11152
7.5	0.03793	0.03827	0.04489	0.04529	-0.04492	-0.04536	-0.04853	-0.04901
8	0.00501	0.00506	-0.14033	-0.14159	-0.03983	-0.04023	0.06676	0.06743
8.5	-0.03990	-0.04026	0.01012	0.01021	0.00908	0.00917	0.10859	0.10967
9	0.01063	0.01072	0.13636	0.13759	0.04800	0.04848	0.03093	0.03124
9.5	0.03573	0.03605	-0.06358	-0.06415	0.03410	0.03444	-0.08076	-0.08157
10	-0.02463	-0.02486	-0.11144	-0.11244	-0.01732	-0.01749	-0.10359	-0.10463

Case1: f=1, A=0.04, ω=3.536163732
Case2: f=0.8, A=0.05, ω=2.208420786



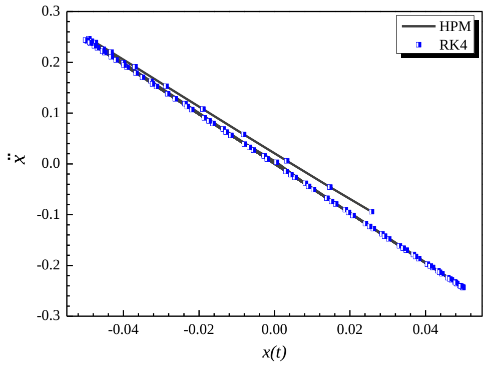


Fig 8. Acceleration $\ddot{x}$ based on displacement x for (a) f=1, A=0.04, ω=3.536163732, (b) f=0.8, A=0.05, ω=2.208420786

5. Results and discussions

To illustrate and verify the accuracy of this new approximate analytical approach, for the problem, a comparison of the time history oscillatory displacement responses with the numerical solution using Runge-Kutta method is presented in Figs. 1 to 4 for example 1 and Figs. 5 to 8 for example 2. Figs. 1 and 5 represent a comparison of analytical solution of $x(t)$ based on time with the numerical solution and Figs. 2 and 6 show comparison of analytical solution of of $\dot{x}$ based on time. The phase plan curves of $\dot{x}$ based on displacement $x(t)$ with the numerical solution is also presented in Figs. 3 and 7. Comparison of $\ddot{x}$ based on displacement x is shown in Figs. 4 and 8. Tables 1 and 2 are shown the point value comparison of displacement and velocity of the problems for different cases. The results compare with the numerical solution for the both cases. It is evident that HPM shows an excellent agreement with the numerical solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the HPM can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

6. Conclusions

In this study we applied the He’s homotopy perturbation method for nonlinear oscillators with damping. Two strong examples have been studied to show the accuracy and convergence of the method. It has been proved that the HPM is very efficient, comfortable and sufficiently exact in engineering problems. Homotopy perturbation method can be simply extended to nonlinear equations for the analysis of nonlinear systems. The obtained results from the approximate analytical solutions are in excellent agreement with the corresponding numerical solutions.

References

Akgoz, B. and Civalek, O. (2011), “Nonlinear vibration analysis of laminated plates resting on nonlinear two-parameters elastic foundations", *Steel Compos. Struct.*, **11**(5), 403-421.

Atmane, H.A., Tounsi, A., Ziane, N. and Mechab, I. (2011), "Mathematical solution for free vibration of sigmoid functionally graded beams with varying cross-section", *Steel Compos. Struct.*, **11**(6), 489-504.

Cordero, A., Hueso, J.L., Martínez, E. and Torregros, J.R. (2010), "Iterative methods for use with nonlinear discrete algebraic models", *Math. Comput. Model.*, **52**(7-8), 1251-1257.

Bayat, M., Pakar, I. and Emadi, A. (2013a), "Vibration of electrostatically actuated microbeam by means of homotopy perturbation method", *Struct. Eng. Mech.*, **48**(6), 823-831.

Bayat, M., Bayat, M. and Pakar, I. (2014a), "The analytic solution for parametrically excited oscillators of complex variable in nonlinear dynamic systems under harmonic loading", *Steel Compos. Struct.*, **17**(1), 123-131.

Bayat, M., Pakar, I. and Bayat, M. (2014b), "An accurate novel method for solving nonlinear mechanical systems", *Struct. Eng. Mech.*, **51**(3), 519-530.

Bayat, M., Bayat, M. and Pakar, I. (2014c), "Forced nonlinear vibration by means of two approximate analytical solutions", *Struct. Eng. Mech.*, **50**(6), 853-862.

Bayat, M., Pakar, I. and Cveticanin, L. (2014d), "Nonlinear free vibration of systems with inertia and static type cubic nonlinearities : An analytical approach", *Mech. Mach. Theo*., **77**, 50-58.

Bayat, M., Pakar, I. and Cveticanin, L. (2014e), "Nonlinear vibration of stringer shell by means of extended Hamiltonian approach", *Arch. Appl. Mech.*, **84**(1), 43-50.

Bayat, M., Bayat, M. and Pakar, I. (2014f), "Nonlinear vibration of an electrostatically actuated microbeam", *Latin Am. J. Solid. Struct.*, **11**(3), 534-544.

Bayat, M., Pakar, I. and Bayat, M. (2013b), "On the large amplitude free vibrations of axially loaded Euler-Bernoulli beams", *Steel Compos. Struct.*, **14**(1), 73-83.

Bayat, M. and Pakar, I. (2013c), "Nonlinear dynamics of two degree of freedom systems with linear and nonlinear stiffnesses", *Earthq. Eng. Eng. Vib.*, **12**(3), 411-420.

Bayat, M., Bayat, M. and Pakar, I. (2014g), "Accurate analytical solutions for nonlinear oscillators with discontinuous", *Struct. Eng. Mech.*, **51**(2), 349-360.

Bayat, M. and Abdollahzadeh, G. (2011), "On the effect of the near field records on the steel braced frames equipped with energy dissipating devices", *Latin Am. J. Solid. Struct.*, **8**(4), 429-443.

Bayat, M. and Pakar, I. (2012a), "Accurate analytical solution for nonlinear free vibration of beams", *Struct. Eng. Mech.*, **43**(3), 337-347.

Bayat, M., Pakar, I. and Domaiirry, G. (2012b), "Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review", *Latin Am. J. Solid. Struct*., **9**(2), 145-234.

Bayat, M. and Pakar, I. (2013a), "On the approximate analytical solution to non-linear oscillation systems", *Shock Vib.*, **20**(1), 43-52.

Bayat, M., Pakar, I. and Bayat, M. (2013b), "Analytical solution for nonlinear vibration of an eccentrically reinforced cylindrical shell", *Steel Compos. Struct.*, **14**(5), 511-521.

Bor-Lih, K. and Cheng-Ying, L. (2009), "Application of the differential transformation method to the solution of a damped system with high nonlinearity", *Nonlinear Anal*., **70**(4), 1732-1737.

Cunedioglu, Y. and Beylergil, B. (2014), "Free vibration analysis of laminated composite beam under room and high temperatures", *Struct. Eng. Mech*., **51**(1), 111-130.

Dehghan, M. and Tatari, M. (2008), "Identifying an unknown function in a parabolic equation with over specified data via He's variational iteration method", *Chaos, Solitons Fractals*, **36**(1), 157-166.

Jamshidi, N. and Ganji, D.D. (2010), "Application of energy balance method and variational iteration method to an oscillation of a mass attached to a stretched elastic wire", *Curr. Appl. Phys*., **10**, 484-486.

He, J.H. (2007), "Variational approach for nonlinear oscillators", *Chaos, Solitons Fractals*, **34**(5), 1430-1439.

He, J.H. (2008), "An improved amplitude-frequency formulation for nonlinear oscillators", *Int. J. Nonlinear Sci. Numer. Simul.*, **9**(2), 211-212.

He, J.H. (2010), "Hamiltonian approach to nonlinear oscillators", *Phys. Lett. A*, **374**(23), 2312-2314.

Mehdipour, I., Ganji, D.D. and Mozaffari, M. (2010), "Application of the energy balance method to

nonlinear vibrating equations", *Curr. Appl. Phys.*, **10**(1), 104-112.
Odibat, Z., Momani, S. and Suat Erturk, V. (2008), "Generalized differential transform method: application to differential equations of fractional order", *Appl. Math. Comput*., **197**(2), 467-477.
Pakar, I., Bayat, M. and Bayat, M. (2014a), "Nonlinear vibration of thin circular sector cylinder: An analytical approach", *Steel Compos. Struct.*, **17**(1), 133-143.
Pakar, I., Bayat, M. and Bayat, M. (2014b), "Accurate periodic solution for nonlinear vibration of thick circular sector slab", *Steel Compos. Struct.*, **16**(5), 521-531.
Pakar, I., Bayat, M. and Bayat, M. (2011), "Analytical evaluation of the nonlinear vibration of a solid circular sector object", *Int. J. Phys. Sci.*, **6**(30), 6861-6866.
Pakar, I. and Bayat, M. (2013), "Vibration analysis of high nonlinear oscillators using accurate approximate methods", *Struct. Eng. Mech.*, **46**(1), 137-151.
Rajasekaran, S. (2013), "Free vibration of tapered arches made of axially functionally graded materials", *Struct. Eng. Mech.*, **45**(4), 569-594.
Shahidi, M., Bayat, M., Pakar, I. and Abdollahzadeh, G.R. (2011), "Solution of free non-linear vibration of beams", *Int. J. Phys. Sci.*, **6**(7), 1628-1634.
Shen, Y.Y. and Mo, L.F. (2009), "The max-min approach to a relativistic equation", *Comput. Math. Appl*., **58**(11), 2131-2133.
Wu, G. (2011), "Adomian decomposition method for non-smooth initial value problems", *Math. Comput. Model.*, **54**(9-10), 2104-2108.
Xu, Nan and Zhang, A. (2009), "Variational approachnext term to analyzing catalytic reactions in short monoliths", *Comput. Math. Appl.*, **58**(11-12), 2460-2463.
Xu, L. (2010), "Application of Hamiltonian approach to an oscillation of a mass attached to a stretched elastic wire", *Math. Comput. Appl*., **15**(5), 901-906.
Zeng, D.Q. and Lee, Y.Y. (2009), "Analysis of strongly nonlinear oscillator using the max-min approach", *Int. J. Nonlinear Sci. Numer. Simul*., **10**(10), 1361-1368.
Zhifeng, L., Yunyao, Y., Feng, W., Yongsheng, Z. and Ligang, C. (2013), "Study on modified differential transform method for free vibration analysis of uniform Euler-Bernoulli beam", *Struct. Eng. Mech.*, **48**(5), 697-709.

CC