DOI: http://dx.doi.org/10.12989/eas.2011.2.1.089

Free vibration analysis of tall buildings with outrigger-belt truss system

Mohsen Malekinejad and Reza Rahgozar*

Department of Civil Engineering, University of Kerman, Kerman, Iran

(Received June 29, 2010, Accepted October 22, 2010)

Abstract. In this paper a simple mathematical model is presented for estimating the natural frequencies and corresponding mode shapes of a tall building with outrigger-belt truss system. For this purposes an equivalent continuum system is analyzed in which a tall building structure is replaced by an idealized cantilever continuum beam representing the structural characteristics. The equivalent system is comprised of a cantilever shear beam in parallel to a cantilever flexural beam that is constrained by a rotational spring at outrigger-belt truss location. The mathematical modeling and the derivation of the equation of motion are given for the cantilevers with identically paralleled and rotational spring. The equation of motion and the associated boundary conditions are analytically obtained by using Hamilton's variational principle. After obtaining non-trivial solution of the eigensystem, the resulting is used to determine the natural frequencies and associated mode shapes of free vibration analysis. A numerical example for a 40 story tall building has been solved with proposed method and finite element method. The results of the proposed mathematical model have good adaptation with those obtained from finite element analysis. Proposed model is practically suitable for quick evaluations during the preliminary design stages.

Keywords: tall buildings; shear beam; outrigger-belt truss system; Hamilton principle; free vibration; natural frequency.

1. Introduction

Structural development of tall buildings has been a continuously evolving process. Undoubtedly, the factor that governs the design of a tall and slender structure most of the times is not the fully stressed state, but the drift of the building for wind loading (Lavan and Levy 2010). There are numerous structural lateral systems used in high-rise building design such as: shear frames, shear trusses, frames with shear core, framed tubes, trussed tubes, super frames etc. (Ali and Moon 2007). However, the outrigger and belt truss system is the one providing significant drift control for the building. The tallest building in the world, the 101-storey-high Taipei building with its composite structure is excellent example of this system (Poon *et al.* 2004). The outrigger trusses which are connected to the core and to the columns outboard of the core in framed tube system, restrain the rotation of the core and convert part of its the bending moment into a vertical couple of forces at the columns. The shortening or the elongation of the columns and the bending deformation of the outrigger trusses permit some rotation of the core at the outriggers (Kian and Siahaan 2001). The

* Corresponding author, Associate Professor, E-mail: rahgozar@mail.uk.ac.ir

outrigger and belt truss system is usually utilized with other resisting systems such as framed tube, shear core, shear wall, braced frame etc. Combined system of framed tube, shear core and outrigger-belt truss is one of the most effective system that used in tall building structures under lateral loads (Halis and Emer 2007).

Most of the tall building structural models are based on the continuum medium technique as well as to framed-tube structures as in the papers by (Coull and Bose 1975, 1976, 1977, Coull and Ahmed 1978, Connor and Pouangare 1991, Takabatake *et al.* 1993, Kwan 1994). They developed an orthotropic membrane model of transforming the frame tube panels into equivalent orthotropic membranes. Many researchers have studied behavior of outrigger-belt truss systems such as Smith and Salim (1983), Moudarres (1984), Rutenberg and Tal (1987), Taranath (1998), Stafford Smith and Coull (1991), Kian and Siahaan (2001) and Gerasimidis *et al.* (2009). In their work, a simplified analytical method for outrigger structure is presented. Swaddiwudhipong *et al.* (2001) employed the continuum model to study the behavior of core-frame interaction in tall buildings. Also Hoenderkamp and Bakker (2003) and Hoenderkamp (2004) investigated the effects of shear wall with outrigger trusses on wall and column foundations using a simplified model. An analytical model was developed to study the lateral drift of wall-frame structures with outriggers. Later, Lee *et al.* (2008) presented an analytical model for high-rise wall-frame structures with outriggers.

A mathematical model for static analysis of combined system of framed tube, shear core and outrigger-belt truss is developed with the objective of determining the axial stress and displacement distribution functions along the height of the building by Rahgozar and Sharifi (2009). In another study Rahgozar *et al.* (2010) calculated optimum location of outrigger-belt truss system along the height of the combined system of framed tube, shear core and outrigger-belt truss system. A great number of exact and approximate methods have been developed to investigate the behavior, deflection, modification and vibration of different types of tall building subjected to lateral loads in the past few decades (Tso and Biswas 1972, Swaddiwudhipong *et al.* 2002, Lee and Loo 2001, Tarjan and Kollar 2004, Eisenberger 1994, Wang 1996a, 1996b, Bozdogan 2006, 2009, Georgoussis 2006, Dym and Wiliams 2007, Kaviani *et al.* 2008, Lee *et al.* 2008, Wu *et al.* 2008, Kuang and Ng 2004, 2009). The effects of varying natural frequency and structural damping values on the along wind response for two reinforced concrete buildings that are among the tallest residential buildings in the world have been studied with respect to serviceability-level vibrations. For typical tall building designs using a gust response factor approach, reasonable predictions of wind-induced loads have probably been achieved through the somewhat fortuitous offsetting of the effects of overestimating both natural periods and structural damping values (Campbell *et al.* 2007).

A modal analysis procedure has been presented for the seismic response of belted high-rise building structures within the framework of the response spectrum technique (Rutenberg 1979).

A few methods are available to deal with vibration of outrigger system or a combined system of framed tube, shear core and outrigger-belt truss elements in tall building. Some researchers have studied the vibration of beams with a spring at a point along the beam such as Maurizi *et al.* (1976), Rutenberg (1978), Takahashi (1980), Lau (1984), Matsuda (1992). In this paper a simple mathematical model for calculation of natural frequencies and associated mode shapes of combined framed tube, shear core, and outrigger-belt truss system is presented. Framed tube system is modeled as cantilever box beam with four orthotropic membranes (Kwan 1994) (Fig. 1). The equivalent system is comprised of a cantilever shear beam in parallel to a cantilever flexural beam. The effect of outrigger-belt truss system and shear core on framed tube structure is modeled as a concentrated moment applied at the outrigger-belt truss location, this moment acts in a direction

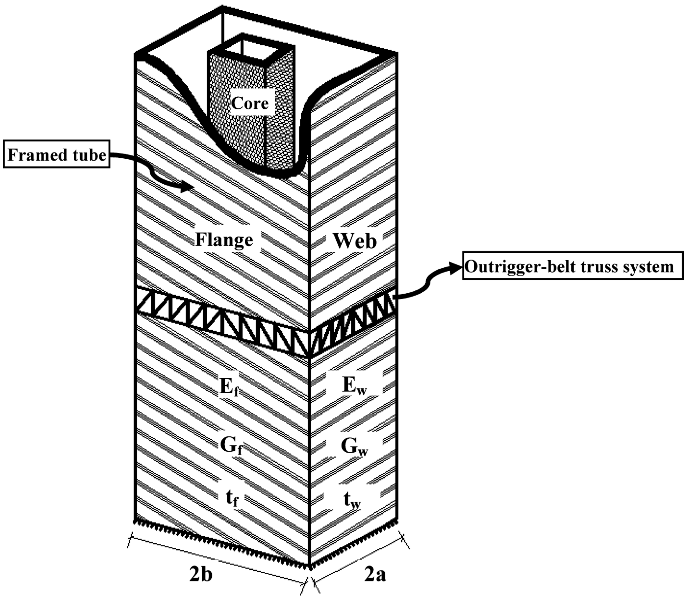


Fig. 1 Orthotropic membrane tube in combined system

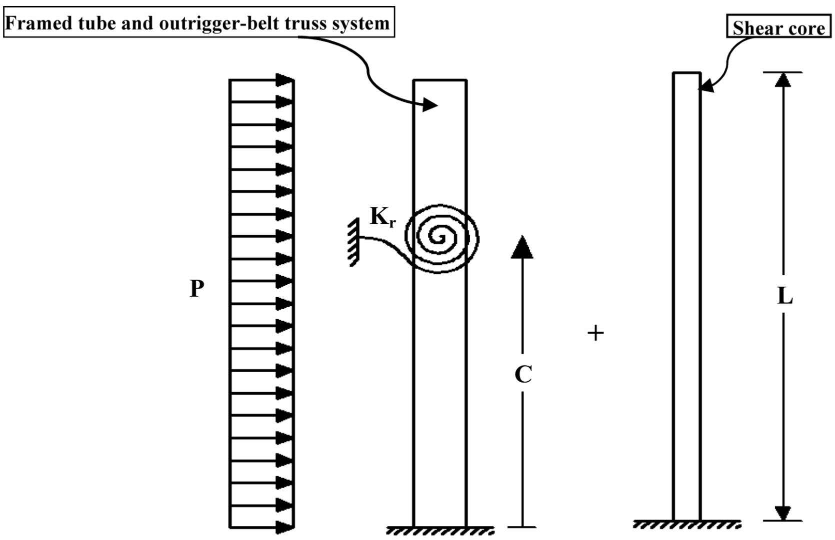


Fig. 2 Behavior of framed tube, shear core and outrigger-belt truss system

opposite to rotation created by lateral loads (Fig. 2). Here by using the energy method and Hamilton's principle, the governing equations are derived. Applying boundary conditions and separation of variables to time and space yields the eigenvalue problem for finding the natural frequencies and corresponding mode shapes of tall buildings. In order to illustrate the efficiency and accuracy of the proposed method a numerical example has been carried out for a 40 storey tall building by the proposed method and SAP2000. The differences between two sets of results are acceptable.

2. Governing equation for combined system of framed tube, shear core and outrigger-belt truss

In order to obtain natural frequencies and corresponding mode shapes of a combined system of framed tube, shear core, and outrigger-belt truss, analytical model of framed tube proposed by Kwan (1994) is used. Here, the framed tube is modeled as equivalent orthotropic plates; and the outrigger-belt truss is modeled as a rotating spring at outrigger-belt truss location (Rahgozar and Sharifi 2009, Rahgozar *et al.* 2010). The assumptions made in this model are as: (*i*) the structure is fixed at the base and properties of the core, beams, columns and spacing of beams and columns are constant along the building's height; (*ii*) the structure undergoes small strain deformations, and the material is homogeneous, isotropic and linear elastic; (*iii*) the floor slab displacements resemble rigid-body movements in horizontal plane; (*iv*) the out-of-plane deformations are negligible compared to the in-plane deformations; (*v*) the interaction between outrigger-belt truss and shear core are modeled as a rotating spring at outrigger-belt truss location; (*vi*) the effects of stress concentrations occurring at outrigger-belt truss locations are neglected; (*vii*) the belt trusses are attached rigidly to core and pinned to the framed tube structure; (*viii*) the structure is assumed symmetric in plan and height, and cannot twist.

The motion of this structure can be express by two variables of position and time. In this section, Lagrange's equation of motion for the vibration of continuous systems and the associated eigenvalue problem are presented. One of the most effective methods to derive the governing equations for a mechanical system is the energy method. Having accounted for system's potential and kinetic energy, Hamilton principle can be applied to obtain the governing equations (Piersol and Paez 2010).

Since there are at least two independent variables, motion of the distributed system is governed by a set of partial differential equations as opposed to discrete systems which are governed by ordinary differential equations. By considering a distributed system defined over the closed domains $0 \le x \le L$, where x is the spatial position of any material point of the system, and let $w(x, t)$ be the dependent variable, which represents displacement of points from a given reference position (see Fig. 2).

It has been pointed out by Meirovitch (1980) that Hamilton's principle over time span $t \in [t_1, t_2]$ can be written as

$$\delta \int_{t_1}^{t_2} (L + W_c) dt + \int_{t_1}^{t_2} \delta W_{nc} dt = \delta \int_{t_1}^{t_2} (T - V + W_c) dt + \int_{t_1}^{t_2} \delta W_{nc} dt = 0 \tag{1}$$

where δ is the variation, L is the Lagrangian, t is time, T is the kinetic energy, V is the potential energy, W_c is the work done by conservative forces, and W_{nc} is the work done by non-conservative forces. For the presented model in Fig. 2, the above items could be presented as follows
the kinetic energy

$$T(t) = \frac{1}{2} \int_0^L m(x) [\dot{w}(x, t)]^2 dx \tag{2}$$

the potential energy

$$V(t) = \frac{1}{2} \int_0^L EI(x) [w''(x, t)]^2 dx + \frac{1}{2} \int_0^L S(x) [w'(x, t)]^2 dx + K_r [w'(C, t)]^2 \tag{3}$$

the nonconservative virtual work

$$\delta\hat{w}_{nc}(x,t) = P(x,t)\,\delta w(x,t) \tag{4}$$

the conservative virtual work is zero.

where L, $m(x)$, $EI(x)$ and $P(x, t)$ are the length, mass per unit length, bending stiffness, and distributed external force of the beam respectively. $w(x, t)$ is the displacement, $S(x)$ is the shear stiffness $GA(x)$, in which G is the shear modulus of elasticity and $A(x)$ is the cross-sectional area, K_r is the equivalent stiffness of the rotational spring which include the effects of belt truss on frame tube acting at $x = C$.

Utilizing the δ operator properties and integration by parts, provide the (*a*) differential equation of motion known as the Lagrange's equation, (*b*) boundary displacements (kinematic boundary conditions), (*c*) boundary forces (natural boundary conditions), and (*d*) eigenvalue solution form (Piersol and Paez 2010).

The Hamilton principle can be written in the form

$$\int_{t_1}^{t_2}\left\{\int_0^L(\delta\hat{L}+\delta\hat{w}_{nc})dx+\delta L_o\right\}dt = 0 \quad \delta w = 0 \;\text{ at }\; t = t_1, t_2 \quad 0 \le x \le L \tag{5}$$

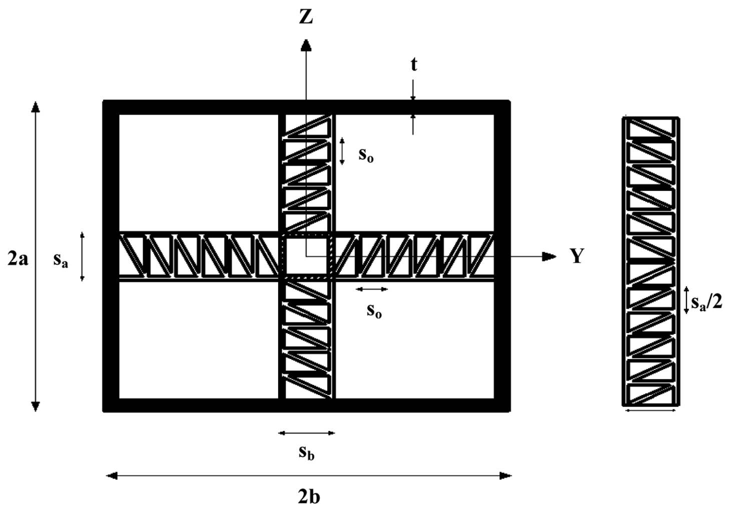


Fig. 3 Schematic plan of combined system in location of outrigger-belt truss

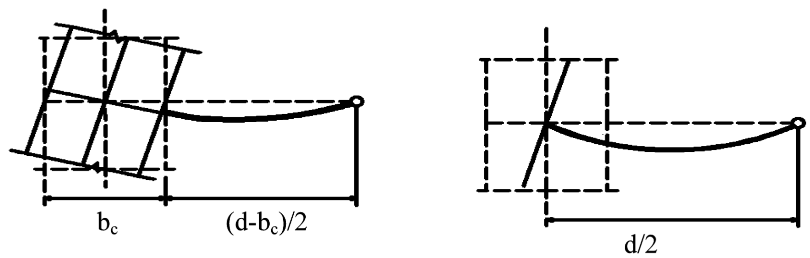


Fig. 4. Wide-column effect of core and outrigger-belt truss

where $\hat{L}$ is the Lagrangian density as follows

$$\hat{L} = \frac{1}{2}m\dot{w}^2 - \frac{1}{2}EIw''^2 - \frac{1}{2}Sw'^2 \tag{6}$$

L_o is a discrete component of the Lagrangian as follows (Taranath 1988, Rahgozar *et al.* 2010)

$$L_o = -K_r w'^2(C, t) \tag{7}$$

where primes and dots denote total derivatives with respect to x and t, respectively.
And $\delta\hat{w}_{nc}$ is the nonconservative virtual work density as follows

$$\delta\hat{w}_{nc} = P\,\delta w \tag{8}$$

By using the Lagrangian density variation, $\delta\hat{L}$ can be obtained as following equation

$$\delta\hat{L} = \frac{\partial L}{\partial w}\delta w + \frac{\partial L}{\partial w'}\delta w' + \frac{\partial L}{\partial w''}\delta w'' + \frac{\partial L}{\partial \dot{w}}\delta \dot{w} \tag{9}$$

and the discrete component of the Lagrangian variation is

$$\delta L_o = \frac{\partial L_o}{\partial w'}\partial w' \tag{10}$$

Substituting Eqs. (8) and (9) into Eq. (5), the following equation is obtained

$$\int_{t_1}^{t_2}\left\{\int_0^L\left(\frac{\partial L}{\partial w}\partial w + \frac{\partial L}{\partial w'}\partial w' + \frac{\partial L}{\partial w''}\partial w'' + \frac{\partial L}{\partial \dot{w}}\partial \dot{w} + P\,\delta w\right)dx + \delta L_o\right\}dt = 0 \tag{11}$$

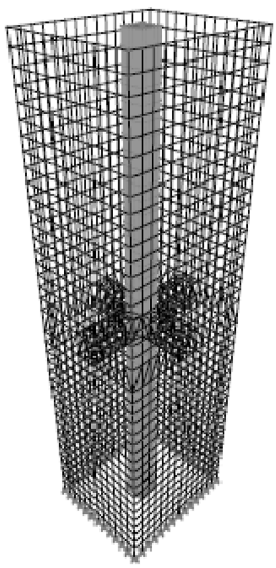


Fig. 5. Model of numerical example

The next step is to transform the integrand in Eq. (11) into one containing only δw i.e. one that is free of $\delta w'$, $\delta w''$ and $\delta \dot{w}$. This can be accomplished by integration by parts, both respect to space and time, as follows

$$\int_{t_1}^{t_2}\left\{\int_0^L\left[\frac{\partial \hat{L}}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w'}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{L}}{\partial w''}\right)-\frac{\partial}{\partial t}\left(\frac{\partial \hat{L}}{\partial \dot{w}}\right)+P\right]\delta w dx+\left(\frac{\partial \hat{L}}{\partial w'}+\frac{\partial \hat{L}_o}{\partial w}\right)\delta w\bigg|_{x=L}-\frac{\partial \hat{L}}{\partial w'}\delta w\bigg|_{x=0}\right\}dt = 0 \quad (12)$$

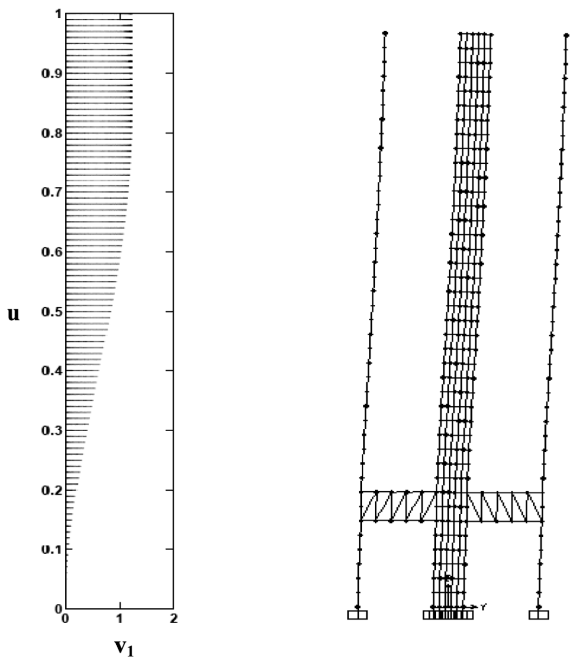
Eq. (12) leads to general Lagrangian equation and boundary equations for the virtual displacement δw as follows
the eigenvalue equation

$$\frac{\partial L}{\partial w}-\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial w'}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial w''}\right)-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{w}}\right)+P = 0, \quad 0<x<L \quad (13)$$

and the following boundary conditions are be obtained

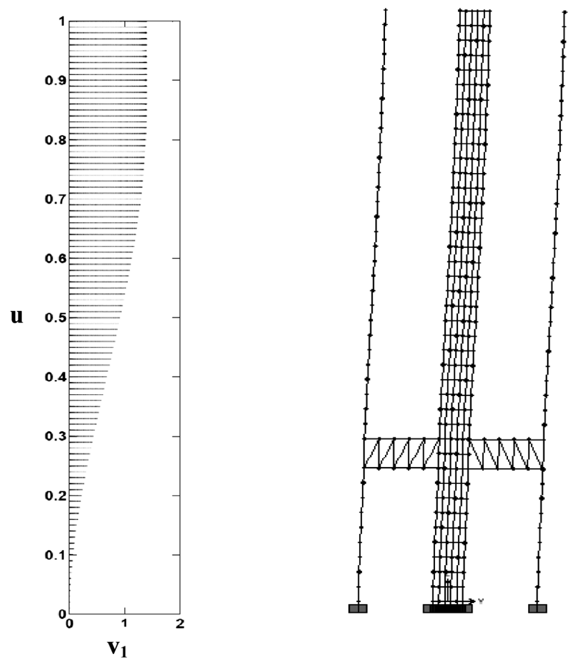
$$\frac{\partial \hat{L}}{\partial w'}\delta w = 0 \ \text{ at } \ x = 0 \quad (14a)$$

$$\left(\frac{\partial \hat{L}}{\partial w'}+\frac{\partial \hat{L}_o}{\partial w'}\right)\delta w = 0 \ \text{ at } \ x = L \quad (14b)$$



Approximate Method **SAP2000**

Fig. 6. First natural mode shape of free vibration of the 40 storey building (B.L. = L/6)



Approximate Method **SAP2000**

Fig. 7. First natural mode shape of free vibration of the 40 storey building (B.L. = L/4)

Substituting Eq. (6) into Eq. (13) yields

$$\frac{\partial}{\partial x}(Sw') - \frac{\partial^2}{\partial x^2}(EIw'') - m\ddot{w} + P = 0, \quad 0 < x < L \tag{15}$$

Substituting Eqs. (6) and (7) into Eqs. (14a) and (14b) gives

$$\left[-Sw' + \frac{\partial}{\partial x}(EIw'')\right]\Big|_{x=L} = 0 \tag{16a}$$

and

$$[-EIw'' - 2K_r w'(C,t)]\big|_{x=L} = 0 \tag{16b}$$

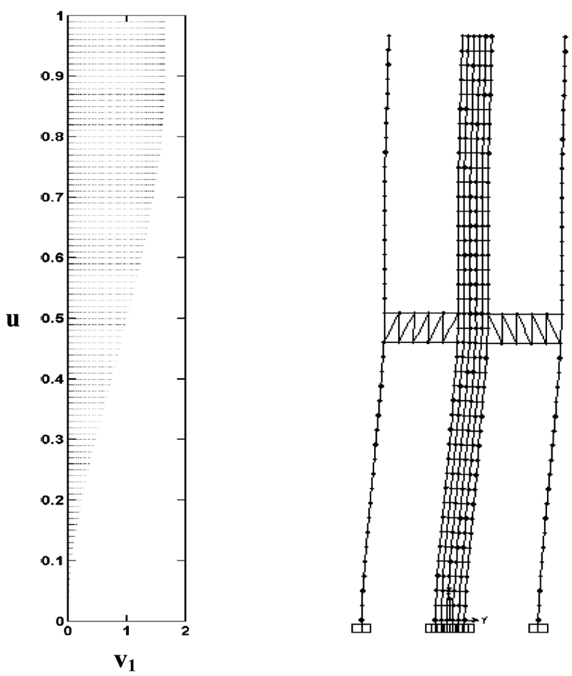
and, considering that the base support is fixed $x = 0$ the boundary conditions concluded are

$$w(0, t) = 0 \tag{17a}$$

$$w'(0, t) = 0 \tag{17b}$$

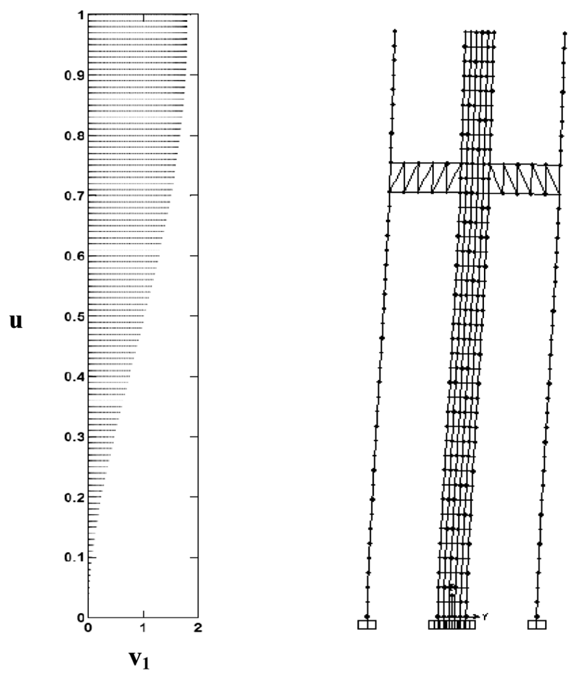
The above Eqs. (17a) and (17b) represent the geometric and dynamic boundary conditions, respectively.

In order to study the free vibration the external force supposed to be zero ($P = 0$).



Approximate Method **SAP2000**

Fig. 8. First natural mode shape of free vibration of the 40 storey building (B.L. = L/2)



Approximate Method **SAP2000**

Fig. 9. First natural mode shape of free vibration of the 40 storey building (B.L. = 3L/4)

Next, it is considered that the free vibration motion at any point of the structural height is harmonic and the deflected shapes are independent in time. Mathematically, the assumption implies that the motion is separable in space and time. Hence, the solution of above equations can be expressed in following form

$$w(x, t) = W(x)F(t) \tag{18}$$

where $W(x)$ depends on the spatial position alone and $F(t)$ is harmonic with frequency ω and depends on time alone.

Substituting Eq. (18) into Eq. (15) the following form is obtained

$$\left[\frac{d}{dx}(SW') - \frac{d^2}{dx^2}(EIW'')\right]F = mW(x)\ddot{F}, \quad 0 < x < L \tag{19}$$

In a similar manner, Eqs. (17a) and (17b) yields

$$W_{(0)}F = 0 \tag{20a}$$

and

$$W'_{(0)}F = 0 \tag{20b}$$

also Eqs. (16a) and (16b) are written as

$$\left[-SW' + \frac{\partial}{\partial x}(EIW'')\right]\bigg|_{x=L} F = 0 \tag{21a}$$

and

$$[-EIW'' - 2K_r W'(C, t)]|_{x=L} F = 0 \tag{21b}$$

By dividing Eq. (19) by $(-mWF)$, the following equation is obtained

$$\frac{\frac{-d}{dx}(SW') + \frac{d^2}{dx^2}(EIW'')}{mW} = -\frac{\ddot{F}}{F} \tag{22}$$

Since the left side of Eq. (22) depends on x alone and the right side on t alone, and the fact that x and t are independent, then Eq. (22) can be satisfied if and only if the two sides of the equation are equal to a constant. Equating the right side of Eq. (22) to λ, it can be expressed as

$$\ddot{F} + \lambda\ F = 0 \tag{23}$$

According to the theory of differential equations, solution of Eq. (23) is of the following form

$$F(t) = Ae^{st} \tag{24}$$

where A and s are constants. Substituting Eq. (24) into Eq. (23) and dividing through by Ae^{st}, the characteristic equation is obtained

$$s^2 + \lambda = 0 \tag{25}$$

which has the two roots

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = \pm\sqrt{-\lambda} \tag{26}$$

Eq. (23) has two solutions, one corresponding to the root s_1 and the other one to the root s_2. Hence, the general solution of Eq. (23) is

$$F(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} = A_1 e^{\sqrt{-\lambda}\ t} + A_2 e^{-\sqrt{-\lambda}\ t} \tag{27}$$

The next step is to determine the sign of λ. If λ is negative, then the exponents s_1 and s_2 are real, where s_1 is positive and s_2 is negative. In this case the motion is unbounded, which is inconsistent with motion in the neighborhood of stable equilibrium, this possibility must be ruled out. If λ is positive, then by introducing the notation $\lambda = \omega^2$ one obtains

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = \pm i\omega \tag{28}$$

and Eq. (27) becomes

$$F(t) = A_1 e^{-i\omega t} + A_2 e^{-i\omega t} \tag{29}$$

Eq. (29) represents harmonic oscillation at frequency ω.

Furthermore the left side of Eq. (22) must be equal to λ as well; hence the following relationship is obtained

$$-\frac{d}{dx}(SW') + \frac{d^2}{dx^2}(EIW'') = \lambda m W, \quad 0 < x < L \tag{30}$$

Moreover, Eqs. (20a) and (20b) yield

$$W_{(x=0)} = 0 \tag{31a}$$

$$W'_{(x=0)} = 0 \tag{31b}$$

whereas, Eqs. (21a) and (21b) changed into the following forms

$$\left[SW' - \frac{\partial}{\partial x}(EIW'')\right]\Bigg|_{x=L} = 0 \tag{32a}$$

$$[-EIW'' - 2K_r W'(C,t)]|_{x=L} = 0 \tag{32b}$$

The problem of determining parameter λ for which the differential Eq. (30) admits a nontrivial solution W, where the solution is subject to boundary conditions (31-32), is known as a characteristic-value problem or an eigenvalue problem or simply eigen problem. The eigen problem (30-32) proceeds by first solving the differential Eq. (30). Because the equation is fourth order in x, the solution cannot be unique (Meirovitch 1980). Applying boundary conditions (31-32) in Eq. (30), yields four homogeneous equations. By solving the homogenous equations the unknown constant can be obtained in terms of the fourth, but the fourth cannot be determined uniquely. Instead, one obtains an equation for λ, generally transcendental, which is known as the characteristic equation, or frequency equation. There is a denumerably infinite set of values of λ satisfying the characteristic equation, namely, λ_1, λ_2,... . They are called characteristic values or eigenvalues. Associated with these values there is a denumerably infinite set of functions W_1, W_2,... known as characteristic functions or eigenfunctions. Because one of the four constants of integration cannot be determined uniquely, the functions W_1, W_2,... can only be determined within a multiplying constant. The implication is that shape of a given eigenfunction is unique but the amplitude is not. The square roots of the eigenvalues, $\omega_r = \sqrt{\lambda_r}\,(r = 1,2,...)$, are the natural frequencies of the system, and, consistent with this, the functions $W_r(r = 1,2,...)$ are also referred to as natural modes.

To simplify Eq. (30), non-dimensional parameters are introduced as follows

$$u = \frac{x}{L} \text{ for } 0 \le x \le L \text{ then } 0 \le u \le 1 \tag{33a}$$

thus

$$\frac{d^{(n)}}{dx^{(n)}}W(x) = W(x)^{(n)} = \frac{1}{L^{(n)}}W(u)^{(n)} \tag{33b}$$

To simplify these equations, non-dimensional parameters are introduced as follows

$$\alpha^2 = \frac{S}{EI}L^2 \tag{34}$$

$$\beta^2 = \frac{m}{EI}L^4 \tag{35}$$

$$\bar{K}_r = \frac{2K_r}{EI}L \tag{36}$$

when values of m, EI and S are constants with the length of structure, and by substituting Eqs. (33)-(36) and $\lambda = \omega^2$, Eq. (30) becomes

$$\frac{d^4W}{du^4} - \alpha^2\frac{d^2W}{du^2} - \beta^2\omega^2 W = 0, \quad 0 < u < 1 \tag{37}$$

and boundary conditions become

$$W_{(u=0)} = 0 \tag{38a}$$

$$W'_{(u=0)} = 0 \tag{38b}$$

and

$$[W''' - \alpha^2 W']\big|_{u=1} = 0 \tag{39a}$$

$$\left[W'' + \bar{K}_r W'\left(\frac{C}{L}\right)\right]\Big|_{u=1} = 0 \tag{39b}$$

Letting

$$W(u) = Be^{pu} \tag{40}$$

Its solution is

$$p^2 = \frac{\alpha^2}{2} \pm \sqrt{\beta^2\omega^2 + \frac{\alpha^4}{4}} \tag{41}$$

then

$$p_{1,2} = \pm\lambda_1;\ p_{3,4} = \pm i\lambda_2 \tag{42}$$

where

$$\lambda_1 = \sqrt{\sqrt{\beta^2\omega^2 + \frac{\alpha^4}{4}} + \frac{\alpha^2}{2}} \tag{43}$$

$$\lambda_2 = \sqrt{\sqrt{\beta^2\omega^2 + \frac{\alpha^4}{4}} - \frac{\alpha^2}{2}} \tag{44}$$

The solution of Eq. (38), $W(u)$ and its related derivatives can be written as follows

$$\begin{bmatrix} W_{(u)} \\ W'_{(u)} \\ W''_{(u)} + \bar{K}_r W'_{(u)} \\ W'''_{(u)} - \alpha^2 W'_{(u)} \end{bmatrix} = P(u,\omega)\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \tag{45}$$

where

$$P(u,\omega) = \begin{pmatrix} \cosh\lambda_1 u & \sinh\lambda_1 u & \cos\lambda_2 u & \sin\lambda_2 u \\ \lambda_1\sinh\lambda_1 u & \lambda_1\cosh\lambda_1 u & -\lambda_2\sin\lambda_2 u & \lambda_2\cos\lambda_2 u \\ (\lambda_1^2\cosh\lambda_1 u + \bar{K}_r\lambda_1\sinh\lambda_1 u) & (\lambda_1^2\sinh\lambda_1 u + \bar{K}_r\lambda_1\cosh\lambda_1 u) & (-\lambda_2^2\cos\lambda_2 u - \bar{K}_r\lambda_2\sin\lambda_2 u) & (-\lambda_2^2\sin\lambda_2 u + \bar{K}_r\lambda_2\cos\lambda_2 u) \\ \lambda_1\lambda_2^2\sinh\lambda_1 u & \lambda_1\lambda_2^2\cosh\lambda_1 u & \lambda_2\lambda_1^2\sin\lambda_2 u & -\lambda_2\lambda_1^2\cos\lambda_2 u \end{pmatrix} \tag{46}$$

By substituting all of the boundary conditions into Eq. (45) yields

$$\begin{bmatrix} W(0) \\ W'(0) \\ W''(1) + \bar{K}_r W'\left(\frac{C}{L}\right) \\ W'''(1) - \alpha^2 W'(1) \end{bmatrix} = P(\omega)\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{47}$$

where

$$P(\omega) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ (\lambda_1^2\cosh\lambda_1 + \bar{K}_r\lambda_1\sinh\frac{\lambda_1 C}{L}) & (\lambda_1^2\sinh\lambda_1 + \bar{K}_r\lambda_1\cosh\frac{\lambda_1 C}{L}) & (-\lambda_2^2\cos\lambda_2 - \bar{K}_r\lambda_2\sin\frac{\lambda_2 C}{L}) & (-\lambda_2^2\sin\lambda_2 + \bar{K}_r\lambda_2\cos\frac{\lambda_2 C}{L}) \\ \lambda_1\lambda_2^2\sinh\lambda_1 u & \lambda_1\lambda_2^2\cosh\lambda_1 & \lambda_2\lambda_1^2\sin\lambda_2 & -\lambda_2\lambda_1^2\cos\lambda_2 \end{pmatrix} \tag{48}$$

thus

$$C = -A; D = -\frac{\lambda_1}{\lambda_2}B \tag{49}$$

where

$$B = -\frac{\lambda_1^2\cosh\lambda_1 + \bar{K}_r\lambda_1\sinh\frac{\lambda_1 C}{L} + \lambda_2^2\cos\lambda_2 + \bar{K}_r\lambda_2\sinh\frac{\lambda_2 C}{L}}{\lambda_1^2\sinh\lambda_1 + \bar{K}_r\lambda_1\cosh\frac{\lambda_1 C}{L} + \beta\omega\cos\lambda_2 - \bar{K}_r\lambda_1\cos\frac{\lambda_2 C}{L}}A \tag{50}$$

Mathematically, the nontrivial solution to Eq. (48) can only be obtained when the determinant of the coefficients is equal to zero, i.e.

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \lambda_1 & 0 & \lambda_2 \\ (\lambda_1^2\cosh\lambda_1 + \bar{K}_r\lambda_1\sinh\frac{\lambda_1 C}{L}) & (\lambda_1^2\sinh\lambda_1 + \bar{K}_r\lambda_1\cosh\frac{\lambda_1 C}{L}) & (-\lambda_2^2\cos\lambda_2 - \bar{K}_r\lambda_2\sin\frac{\lambda_2 C}{L}) & (-\lambda_2^2\sin\lambda_2 + \bar{K}_r\lambda_2\cos\frac{\lambda_2 C}{L}) \\ \lambda_1\lambda_2^2\sinh\lambda_1 u & \lambda_1\lambda_2^2\cosh\lambda_1 & \lambda_2\lambda_1^2\sin\lambda_2 & -\lambda_2\lambda_1^2\cos\lambda_2 \end{vmatrix} = 0 \tag{51}$$

The equation for frequency can be obtained by Mathematica 7.0.0 software as follows

$$\begin{aligned} &\frac{1}{L}\lambda_1\lambda_2\Big\{L\cosh\lambda_1\Big[(\lambda_1^4+\lambda_2^4)\cos\lambda_2+\bar{K}_r\lambda_2^3\sin\Big(\frac{\lambda_2 C}{L}\Big)+\bar{K}_r\lambda_1\lambda_2^2\sinh\Big(\frac{\lambda_1 C}{L}\Big)\Big] \\ &+\lambda_1\Big[\lambda_2+(C\bar{K}_r\lambda_2)\cos\lambda_2+L(\lambda_1^2-\lambda_2^2)\sin\lambda_2)\sinh\lambda_1+L\bar{K}_r\lambda_2\cosh\Big(\frac{\lambda_1 C}{L}\Big)(\lambda_1\sin\lambda_2-\lambda_2\sinh\lambda_1) \\ &+\lambda_1\Big(\lambda_2(2L\lambda_2-C\bar{K}_r\cos\lambda_2\sin\lambda_2)+L\bar{K}_r\cos\lambda_2\sin\Big(\frac{\lambda_2 C}{L}\Big)+L\bar{K}_r\lambda_1\cos\lambda_2\sinh\Big(\frac{\lambda_1 C}{L}\Big)\Big)\Big]\Big\} = 0 \end{aligned} \tag{52}$$

and the corresponding natural mode shape is

$$W^{(i)}(u) = a^{(i)}v^{(i)} \tag{53}$$

where $a^{(i)}$ are unknown constants, and $v^{(i)}$ the non-normalized mod shape functions, given by

$$\begin{aligned} v^{(i)} &= \cosh\lambda_1^{(i)}u - \cos\lambda_2^{(i)}u \\ &-\Big(\frac{\lambda_1^{(i)^2}\cosh\lambda_1^{(i)} + \bar{K}_r\lambda_1^{(i)}\sinh\frac{\lambda_1^{(i)}C}{L} + \lambda_2^{(i)^2}\cos\lambda_2^{(i)} + \bar{K}_r\lambda_2^{(i)}\sinh\frac{\lambda_2^{(i)}C}{L}}{\lambda_1^{(i)^2}\sinh\lambda_1^{(i)} + \bar{K}_r\lambda_1^{(i)}\cosh\frac{\lambda_1^{(i)}C}{L} + \beta\omega\sin\lambda_2^{(i)} - \bar{K}_r\lambda_1^{(i)}\cos\frac{\lambda_2^{(i)}C}{L}}\Big) \\ &\times\Big(\sinh\lambda_1^{(i)}u - \frac{\lambda_1^{(i)}}{\lambda_2^{(i)}}\sin\lambda_2^{(i)}u\Big) \end{aligned} \tag{54}$$

where $i(1,2,3...,n)$ is the vibrating mode's number.

In order to determine the stiffness of outrigger-belt truss (K_r), the works that are done by (Stafford Smith and Coull 1991) and Lee *et al.* (2008) can be utilized. It has been pointed out by Lee *et al.* (2008) that magnitude of the reductions depend on flexural rigidities of the core, the outrigger-belt truss, and the columns acting axially about the core's centroid. Reductions also depend on the location of outrigger-belt truss along the height of the core (Stafford Smith and Coull 1991).

In this paper, outrigger-bet truss is modeled as rotational spring (Fig. 2), with equivalent stiffness (K_r), given by

$$K_r = 1/\theta \tag{55}$$

where θ denotes the total rotation in outrigger-belt truss due to the restraining moment, and can be obtained by splitting up the action, i.e.

$$\theta = \theta_a + \theta_b + \theta_s \tag{56}$$

First, the restraining forces in exterior columns will cause rotation of the outrigger-belt truss resulting from axial lengthening and shortening of columns. The outrigger-belt truss rotation θ_a due to the resulting restraining moment can then be defined as the column's change in length divided by the length of outrigger, d

$$\theta_a = (2\ C)/(d^2 E_f A_c') \tag{57}$$

where d is the distance between center to center of exterior columns and $E_f A_c'$ is the axial rigidity of these columns. Flexural deformation of outrigger-belt truss caused by the action of column force will cause additional drift between the adjacent floors. The resulting rotation θ_b is given by

$$\theta_b = (d)/(12EHI_{oe}) \tag{58}$$

where EI_{oe} is the effective flexural stiffness of the outrigger-belt truss, modeled as though its length extended from the column to core's centroid. EI_{oe} can be obtained from outrigger-belt truss's actual flexural rigidity EI_o by converting the flexural rigidity of a wide-column beam, (Fig. 4(a)), to that of an equivalent full-span beam, (Fig. 4(b)) as follows: (Stafford Smith and Salim 1983). By using parallel axes theory, EI_o can be calculated as

$$EI_{oe} = EI_0[1 + (b_c/2)/((d\text{-}b_c)/2))]^3 \tag{59}$$

The rotation caused by the shear force in the outrigger, θ_s, results from strain in diagonals, and can be expressed as

$$\theta_s = 1/(2hGA_o) \tag{60}$$

where $2h$ is the height of the outrigger, and GA_o is racking shear stiffness of the outrigger and belt truss. This racking shear stiffness can be calculated for specific outrigger-belt truss types. The racking shear rigidity is a property for which the method of determination is most particular to the type of bent. It is depend on the deformation of the web members as the structure racks under shearing action. For various type of bents, the values of GA_o have been given by (Stafford Smith and Coull 1991).

The value of K_r which corresponds to stiffness of the spring at $x = C$, can be derived as follows

$$K_r = [2\,C/(d^2 E_f A_c')+d/(12EI_{oe})+(1/2hGA_o)]^{-1} \tag{61}$$

where $A_c' = \sum_{i=1}^{n} A_i$ is the area of exterior columns which are perpendicular to direction of vibration and can be derived from following equation (Rahgozar and Sharifi 2009)

$$\sum_{i=1}^{n} A_i = \left(1+\frac{2b}{s}\right)d_c^2 \tag{62}$$

$2b$ = length of flange
C = location of outrigger-belt truss from the base of structure
d_c = width of column
s = space of spans

3. Examples and comparisons with computer analysis

Numerical example is given to demonstrate the ease in application and accuracy of the proposed approximate method. A high-rise 40 storey reinforced concrete building which consists of framed tube, shear core and outrigger-belt truss, as shown in Figs. 1 and 5, is analyzed. All beams, columns, belt truss and outrigger members are of size 0.8 m × 0.8 m. The height of each storey is 3.0 m and the center-to-center spacing of columns is 2.5 m. Young's and shear modules of the material are 20×10^8(Kg/m^2) and 8×10(Kg/m^2) respectively. The location of outrigger-belt truss is selected based on optimum location of outrigger-belt truss in combined system of framed tube, shear core and outrigger-belt truss (Rahgozar and Sharifi 2009). Other specifications that are used in numerical example are as follows

$2b = 35$(m), $2a = 30$(m), $s = 2.5$(m), $\rho = 2400$(Kg/m^2)
$s_a = 5$(m), $s_b = 5$(m), $s_o = 2.5$(m)

$t_{slab} = 0.3$(m), $L = 120$(m), $h = 3.0$(m)

Dimensions of the core are 5.0 m × 5.0 m, thickness of shear core panels is 0.250 m and Poisson ratio is assumed to be 0.25.

m is mass per unit height the building and is derived from output of SAP2000

$m_{total} = 48145006$(Kg)

The equivalent elastic parameters for the analogous orthotropic membrane tube, as evaluated by Kwan (1994), are as follows

$E = E_m = 20 \times 10^8$(Kg/m^2), $G_m = 8 \times 10^8$(Kg/m^2)

$$t = \frac{A_c}{s} = 0.256(m)$$

$$G = \frac{\frac{h}{st}}{\frac{\Delta_b}{Q} + \frac{\Delta_s}{Q}} = 1.726 \times 10^8 (\mathrm{kg/m^2})$$

The flexural stiffness is obtained as follows

$$EI = (EI)_{core} + (EI)_{Framed\ tude} = 1.0151 \times 10^{13} (\mathrm{Kg\ m^2})$$

and the shear stiffness is calculated as follows

$$S = (GA)_{core} + (GA)_{Framed\ tude} = 4.651 \times 10^9 (\mathrm{Kg\ m^2})$$

By using Eqs. (62) and (63), the value of K_r is calculated as follows
$K_r = [(2\times C)/(30^2\times 20\times 10^8\times 9.6) + (30)/(12\times 1759.0724\times 10^8) + (1/6\times 916467325.4)]^{-1}$(Kg m) then by substituting in Eq. (37), the value of $\overline{K}_r$ is calculated as follows

$$\overline{K}_r = 2.364\times 10^{-11}\times K_r$$

The structure has been analyzed by the proposed method is compared to SAP2000 for the following cases i) framed tube, ii) framed tube and shear core and iii) framed tube, shear core and outrigger-belt truss the results of natural frequencies listed in Table 1 and 2. The first natural mode shape of free vibration of combined system is calculated using Eq. (55), then the results is compared between proposed method and results from SAP2000 in Figs. 6-9, where B.L. is the

Table 1 Comparison of the results between SAP2000 and the proposed approximate method form analysis of a 40 storey building

Type of Building	α	β	$\omega(rad/sec)$ (Proposed method)	$\omega(rad/sec)$ (SAP2000)	% of error in ω
Framed tube	1.94	2.78	1.92	1.98	3
Framed tube with shear core	2.56	2.82	2.23	2.03	9

Table 2 Comparison of natural frequencies for 40 storey building with combined system ($\alpha = 2.5680$, $\beta = 2.8852$)

Position of outrigger-belt truss from the base of building	$\overline{K}_r$	ω (rad/sec) (Proposed method)	ω (rad/sec) (SAP2000)	% of error in ω
L/6	0.1191	2.191	2.185	0.27
L/4	0.1185	2.190	2.170	0.9
L/2	0.1165	2.220	2.377	6.6
3L/4	0.1090	2.200	2.032	8

location of outrigger-belt truss from the base of the structure.

The main sources of errors between the proposed approximate method and SAP2000 are as: (*i*) All closely spaced perimeter columns tied at each floor level by deep spandrel beams are modeled as a tubular structure; (*ii*) Modeling the frame panels as equivalent orthotropic membranes (framed tube), so it can be analyzed as a continuous structure; (*iii*) The equivalent elastic properties derived for the framed tube, shear core, and outrigger-belt truss; (*iv*) The equivalent stiffness of the rotational spring used to model the effect of outrigger-belt truss on frame tube; (*v*) The approximation to derive *EI* and *GA* parameters; (*vi*) The effect of shear lag has been neglected in approximate method.

4. Conclusions

This paper presents a simple mathematical method for free vibration analysis of a combined system which consists of framed tube; shear core and outrigger-belt truss elements in tall building structures. The analysis is based on continuous system approach, in which a tall building structure maybe modeled as idealized cantilevered beams representing the structural characteristics along with concentrated moments applied at outrigger-belt truss locations. Adopting an analytical approach based on energy methods and Hamilton principle, a general solution to eigenvalue problem of this combined system is derived.

A generalized approximate technique for conducting dynamic analysis of such a system is proposed; which yields the natural frequencies and corresponding mode shapes. Here, natural frequencies and mode shapes of a 40 storey building were computed by the proposed approximate method and SAP2000 was utilized as benchmark. The results demonstrate that the fundamental frequency is underestimated for different cases as shown in Table 2 and the error is within the acceptable range (0.27 - 8%). It has been seen from this study that the proposed method provides a simple, efficient and reasonably accurate means for free vibration analysis at the preliminary stages of designing a combined system of framed tube, shear core, belt trusses and outriggers for a tall building structure.

References

Ali, M.M. and Moon, K.S. (2007), "Structural developments in tall buildings: current trends and future prospects", *Architect. Sci. Rev.*, **50**(3), 205-223.

Bozdogan, K.B. (2006), "A method for free vibration analysis of stiffened multi-bay coupled shear walls", *Asian J. Civil Eng. (Build. Housing)*, **7**(6), 639-649.

Bozdogan, K.B. (2009), "An approximate method for static and dynamic analysis of symmetric wall-frame buildings", *Struct. Des. Tall Spec.*, **18**, 279-290.

Campbell, S., Kwok, K.C.S, Hitchcock, P.A, Tse, K.T. and Leung, H.Y. (2007), "Field measurements of natural periods of vibration and structural damping of wind-excited tall residential buildings", *Wind Struct.*, **10**(5), 401-420.

Connor, J.J. and Pouangare, C.C. (1991), "Simple model for design of framed tube structures", *J. Struct. Eng. - ASCE*, **117**(12), 3623-3644.

Coull, A. and Bose, B. (1975), "Simplified analysis of framed-tube structures", *J. Struct. Division - ASCE*, **101**(11), 2223-2240.

Coull, A. and Bose, B. (1976), "Torsion of frame-tube structures", *J. Struct. Division - ASCE*, **102**(12), 2366-2370.

Coull, A. and Bose, B. (1977), "Discussion of simplified analysis of frame-tube structures", *J. Struct. Division - ASCE*, **103**(1), 297-299.

Coull, A. and Ahmed, K. (1978), "Deflection of framed-tube structures", *J. Struct. Division - ASCE*, **104**(5), 857-862.

Dym, C.L. and Williams, H.E. (2007), "Estimating fundamental frequencies of tall buildings", *J. Struct. Eng. - ASCE*, **133**(10), 1-5.

Eisenberger, M. (1994), "Vibration frequencies for beams on variable one and two parameter elastic foundations", *J. Sound Vib.*, **175**(5), 577-584.

Gerasimidis, S., Efthymiou, E. and Baniotopoulos C.C. (2009), "Optimum outrigger locations of high-rise steel buildings for wind loading", *EACWE 5*, Florence, Italy.

Geourgoussis, K.G. (2006), "A simple model for assessing and modal response quantities in symmetrical buildings", *Struct. Des. Tall Spec.*, **15**, 139-151.

Halis Gunel, M. and Emer Ilgin, H. (2007), "A proposal for the classification of structural systems of tall buildings", *J. Build. Environ.*, **42**, 2667-2675.

Hoenderkamp, J.C.D. and Bakker, M.C.M. (2003), "Analysis of high-rise braced frames with outriggers", *Struct. Des. Tall Spec.*, **12**, 335-350.

Hoenderkamp, J.C.D. (2004), "Shear wall with outrigger trusses on wall and column foundations", *Struct. Des. Tall Spec.*, **12**, 73-87.

Kaviani, P., Rahgozar, R. and Saffari, H. (2008), "Approximate analysis of tall buildings using sandwich beam models with variable cross-section", *Struct. Des. Tall Spec.*, **17**, 401-418.

Kian, P.S. and Siahaan, T.S. (2001), "The use of outrigger and belt truss system for high-rise concrete buildings", *Dimensi Teknik Sipil*, **3**(1), 36-41.

Kuang, J.S. and Ng, S.C. (2004), "Coupled vibration of tall buildings structures", *Struct. Des. Tall Spec.*, **13**, 291-303.

Kuang, J.S. and Ng, S.C. (2009), "Lateral shear St. Venant torsion coupled vibration of asymmetric-plan frame structures", *Struct. Des. Tall Spec.*, **18**(6), 647-656.

Kwan, A.K.H. (1994), "Simple method for approximate analysis of framed tube structures", *J. Struct. Eng. - ASCE*, **120**(4), 1221-1239.

Lau, J.H. (1984), "Vibration frequencies and mode shapes for a constrained cantilever", *J. Appl. Mech. - ASME*, **51**, 182-187.

Lavan, O. and Levy, R. (2010), "Performance based optimal seismic retrofitting of yielding plane frames using added viscous damping", *J. Earthq. Struct.*, **1**(3), 307-326.

Lee, K. and Loo, Y. (2001), "Simple analysis of framed-tube structures with multiple internal tubes", *J. Struct. Eng. - ASCE*, **127**, 450-460.

Lee, J., Bang, M. and Kim, J.Y. (2008), "An analytical model for high-rise wall-frame structures with outriggers", *Struct. Des. Tall Spec.*, **17**, 839-851.

Maurizi, M.J., Rossi, R.E. and Reyes, J.A. (1976), "Vibration frequencies for a uniform beam with one end spring-hinged and subjected to a translational restraint at the other end", *J. Sound Vib.*, **48**(4), 565-568.

Mathematica, 7.0.0, Wolfram Research Inc.

Matsuda, H., Morita, C. and Sakiyama, T. (1992), "A method for vibration analysis of a tapered Timoshenko beam with constraint at any points and carrying a heavy tip body", *J. Sound Vib.*, **158**(2), 331-339.

Meirovitch, L. (1980), *Computational methods in structural dynamics*, The Netherland Rockville, Maryland, U. S. A.

Moudarres, F.R. (1984), "Outrigger-braced coupled shear walls", *J. Struct. Eng. - ASCE*, **10**(12), 2876-2890.

Piersol, A.G. and Paez, T.L. (2010), *Harri's shock and vibration handbook*, McGraw-Hill, New York.

Poon, D.C.K., Shieh, S. and Joseph, L.M. (2004), "Structural design of Taipei 101, the world's tallest building", *Proceedings of the CTBUH 2004*, Seoul Conference, Seoul, Korea, 271-278.

Rahgozar, R. and Sharifi, Y. (2009), "An approximate analysis of combined system of framed tube, shear core and belt truss in high-rise buildings", *Struct. Des. Tall Spec.*, **18**(6), 607-624.

Rahgozar, R., Ahmadi, A. and Sharifi, Y. (2010), "A simple mathematical model for approximate analysis of tall buildings", *J. Appl. Math. Model.*, **34**, 2437-2451.

Rutenberg, A. (1978), "Vibration frequencies for a uniform cantilever with a rotational constraint at a point", *J.*

Appl. Mech. - ASME, **45**, 422-423.
Rutenberg, A. and Tal, D. (1987), "Lateral load response of belted tall building structures", *J. Eng. Struct.*, **9**, 53-67.
Rutenberg, A. (1979), "Earthquake analysis of belted high-rise building structures", *J. Eng. Struct.*, **1**, 191-196.
SAP2000 Advanced 12.0.0, *Computers and structures*, Berkeley, California, USA.
Stafford Smith, B. and Coull, A. (1991), *Tall building structures: analysis and design*, Wiley, New York.
Stafford Smith, B. and Salim, I. (1983), "Formulae for optimum drift resistance of outrigger braced tall building structures", *Comput. Struct.*, **17**(1), 45-50.
Swaddiwudhipong, S., Zhou, Q. and Lee, SL. (2001), "Effect of axial deformation on vibration of tall buildings", *Struct. Des. Tall Spec.*, **10**, 79-91.
Swaddiwudhipong, S., Soelarno, Sidji, S. and Lee, S.L. (2002), "The effects of axial deformation and axial force on vibration characteristics of tall buildings", *Struct. Des. Tall Spec.*, **11**, 309-328.
Takabatake, H., Mukai, H. and Hirano, T. (1993), "Doubly symmetric tube structures. I: static analysis", *J. Struct. Eng. - ASCE*, **119**(7), 1981-2001.
Takahashi, K. (1980), "Eigenvalue problem of a beam with a mass and spring at the end subjected to an axial force", *J. Sound Vib.*, **71**(3), 453-457.
Taranath, B.S. (1988), *Structural analysis and design of tall buildings*, McGraw-Hill, New York.
Tarjan, G. and Kollar, L.P. (2004), "Approximate analysis of building structures with identical stories subjected to earthquakes", *Int. J. Solids Struct.*, **41**, 1411-1433.
Tso, W.K. and Biswas, J.K. (1972), "An approximate seismic analysis of coupled shear walls", *Build. Sci.*, **7**(4), 249-256.
Wang, Q. (1996), "Sturm-Liouville equation for free vibration of a tube-in-tube tall building", *J. Sound Vib.*, **191**(3), 349-355.
Wang, Q. (1996), "Modified ODE-solver for vibration of tube-in-tube structures", *Comput. Method. Appl. M.*, **129**, 151-156.
Wu, J.R., Li, Q.S. and Tuan Alex, Y. (2008), "Wind-induced lateral-torsional coupled responses of tall buildings", *Wind Struct.*, **11**(2), 153-178.

CC