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Analytical modeling of bending and free vibration of thick advanced composite beams resting on Winkler-Pasternak elastic foundation

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Abstract. This work presents an efficient and original hyperbolic shear deformation theory for the bending and dynamic behavior of functionally graded (FG) beams resting on Winkler - Pasternak foundations. The theory accounts for hyperbolic distribution of the transverse shear strains and satisfies the zero traction boundary conditions on the surfaces of the beam without using shear correction factors. Based on the present theory, the equations of motion are derived from Hamilton's principle. Navier type analytical solutions are obtained for the bending and vibration problems. The accuracy of the present solutions is verified by comparing the obtained results with the existing solutions. It can be concluded that the present theory is not only accurate but also simple in predicting the bending and vibration behavior of functionally graded beams.

Keywords: functionally graded material; Winkler-Pasternak elastic foundation; bending; free vibration; Hamilton's principle

1. Introduction

Functionally graded materials (FGMs) are a class of composites that have continuous variation of material properties from one surface to another and thus eliminate the stress concentration found in laminated composites. A typical FGM structure presents a continuous variation of material properties over the thickness direction by mixing two different materials, for example, a ceramic and a metal. The gradual variation of properties avoids the delamination failure that is common in laminated composites. Functionally graded materials have been used in the aerospace, mechanical, nuclear, and civil engineering due to superior heat shielding properties of ceramics and high toughness of metals.

Structures like plates resting on elastic foundation can be found in various structural engineering fields. A two-parameter model of Pasternak (1954) which considers the shear deformation between the springs had been proposed to describe the interaction between the plate and foundation. The Winkler model (1867) is a special case of Pasternak model by setting the shear modulus to zero.

Bending and free vibration of functionally graded structures resting on elastic foundation had been studied by many scholars. Thai and Choi (2012) studied the free vibration of functionally graded plates on elastic foundation using a refined shear deformation theory which contains only four unknowns. Jodaei and Yas (2012) investigated the elastic foundations based on the three-dimensional theory of free vibration of functionally graded annular plates on elasticity using state-space based differential quadrature method. Yas and Tahouneh (2012) investigated the free vibration of functionally graded annular plates on elastic foundations based on the three-dimensional theory of elasticity using the differential quadrature method (DQM). Akbaş (2014) studied the wave propagation analysis of edge cracked beams resting on elastic foundation.

Recently, Akbaş (2015) investigated the free vibration analysis of edge cracked functionally graded beams resting on WinklerPasternak foundation. Ait Atmane *et al.* (2017) studied the effect of thickness stretching and porosity on mechanical response of a functionally graded beams resting on elastic foundations. Sayyad and Ghugal (2018) investigated an inverse hyperbolic theory for FG beams resting on Winkler-Pasternak elastic foundation. Benyoucef *et al.* (2018) studied the effect of the micromechanical models on the free vibration of rectangular FGM plate resting on elastic foundation. Mirza *et al.* (2018) used the various theories for the analysis of Laminated and FGM Beams. Fazzolari (2018) investigated the free vibration and elastic stability behaviour of three-dimensional functionally graded sandwich beams featured by two different types of porosity, with arbitrary boundary conditions and resting on Winkler-Pasternak elastic foundations. Khelifa *et al.* (2018) studied the buckling response with stretching effect of carbon nanotube-reinforced composite beams resting on elastic foundation. Zaoui *et al.* (2019) used a new 2D and quasi-3D shear deformation theories for free vibration of functionally graded plates on elastic foundations. Mahmoud *et al.* (2019) studied the effect of parameters of visco-Pasternak foundation on the bending and vibration properties of a thick FG plate. In addition, in recent years,

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many papers have been published in the studies of FGM structures on elastic foundations: Addou *et al.* (2019) investigated the influences of porosity on dynamic response of FG plates resting on Winkler/Pasternak/Kerr foundation using quasi 3D HSDT. Chaabane *et al.* (2019) developed analytical study of bending and free vibration responses of functionally graded beams resting on elastic foundation. Boulefrakh *et al.* (2019) analyze the effect of parameters of visco-Pasternak foundation on the bending and vibration properties of a thick FG plate. Boukhlif *et al.* (2019) used a simple quasi-3D HSDT for the dynamics analysis of FG thick plate on elastic foundation. Mahmoudi *et al.* (2019) used a refined quasi-3D shear deformation theory for thermo-mechanical behavior of functionally graded sandwich plates on elastic foundations. Semmah *et al.* (2019) analyze the thermal buckling of SWBNNT on Winkler foundation by non local FSDT. Karami *et al.* (2019a) studied the wave propagation of functionally graded anisotropic nanoplates resting on Winkler-Pasternak foundation. Chikr *et al.* (2020) used a novel four-unknown integral model for buckling response of FG sandwich plates resting on elastic foundations under various boundary conditions using Galerkin's approach. Refrafi *et al.* (2020) studied effects of hygro-thermo-mechanical conditions on the buckling of FG sandwich plates resting on elastic foundations. To include certain higher order effects such as thickness stretching effects, various higher order shear deformation theories (HSDT) have been developed (Boutaleb et al. 2019, Addou *et al.* 2019, Khiloun *et al.* 2019, Zarga *et al.* 2019, Boulefrakh *et al.* 2019, Boukhlif *et al.* 2019, Mahmoudi *et al.* 2019, Zaoui *et al.* 2019, Benrahou *et al.* 2019, Kaddari *et al.* 2020).

Since the shear deformation effects are more pronounced in these structures, the first-order shear deformation theory and higher-order shear deformation theories should be used. By using these theories, although many papers have been devoted to study static, vibration and buckling analysis of FG structures such as (Draiche *et al.* 2019, Abualnour *et al.* 2019, Alimirzaei *et al.* 2019, Medani *et al.* 2019, Draoui *et al.* 2019, Berghouti *et al.* 2019, Bourada *et al.* 2019, Batou *et al.* 2019, Tlidji *et al.* 2019, Boussoula *et al.* 2020, Adda Bedia *et al.* 2019, Meksi *et al.* 2019, Hellal *et al.* 2019, Hussain *et al.* 2019, Belbachir *et al.* 2019, Mahmoud and Tounsi 2019, Sahla *et al.* 2019, Safa *et al.* 2019, Karami *et al.* 2019b, 2019c, 2019d, 2019e).

The aim of this work is to propose a new hyperbolic shear deformation beam theory with only three unknowns for bending and dynamics of FG thick beames resting on a Winkler-Pasternak foundation. The kinematics is chosen based on hyperbolic cosine variation of transverse shear stress across the thickness of the beam. The equations of motion for thick FG beams are obtained in the Hamilton principle. Effects of the power-law index, length-to-thickness ratio and foundation parameter on the displacements, stresses, and natural frequencies of FG beams are investigated. Due to the lack of any study on the mechanics of FG beams resting on Winkler-Pasternak foundation, it is hoped that the present study may be employed as a benchmark for future works of such structures.

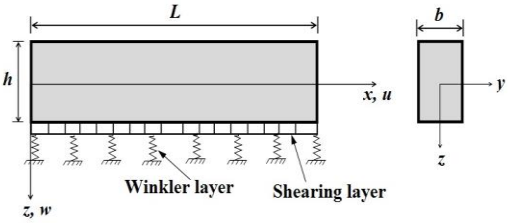

Fig. 1 FGM Beam resting on a two parameters elastic foundation

3. Theoretical formulations

3.1. Basic assumptions

The assumptions of the present theory are as follows:

• The displacements are small in comparison with the height of the beam and, therefore, strains involved are infinitesimal.

• The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinates x, t only.

$$w(x,z,t) = w_b(x,t) + w_s(x,t) \tag{1}$$

• The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x.

• The axial displacement u in x-direction, consists of extension, bending, and shear components.

$$u = u_0 + u_b + u_s \tag{2}$$

The bending component u_b is assumed to be similar to the displacements given by the classical beam theory. Therefore, the expression for u_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x} \tag{3}$$

The shear component u_s gives rise, in conjunction with w_s, to the hyperbolic variation of shear strain γ_{xz} and hence to shear stress τ_{xz} through the thickness of the beam in such a way that shear stress τ_{xz} is zero at the top and bottom faces of the beam. Consequently, the expression for u_s can be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x} \tag{4}$$

where

$$f(z) = z - \frac{z\cosh\left(\frac{1}{2}\pi\right) - \frac{h\sinh\left(\frac{\pi z}{h}\right)}{\pi}}{\cosh\left(\frac{1}{2}\pi\right) - 1} \tag{5}$$

3.2 Kinematics and constitutive equations

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (1) - (5) as

$$u(x,z,t)=u_0(x,t)-z\frac{\partial w_b}{\partial x}-f(z\frac{\partial w_b}{\partial x} \tag{6a}$$

$$w(x,z,t)=w_b(x,t)+w_s(x,t) \tag{6b}$$

The strains associated with the displacements in Eq. (6) are

$$\varepsilon_x=\varepsilon_x^0+z\,k_x^b+f(z)\,k_x^s \tag{7a}$$

$$\gamma_{xz}=g(z)\,\gamma_{xz}^s \qquad 7(b)$$

where

$$\varepsilon_x^0=\frac{\partial u_0}{\partial x},\ k_x^b=-\frac{\partial^2 w_b}{\partial x^2},\ k_x^s=-\frac{\partial^2 w_s}{\partial x^2},\ \gamma_{xz}^s=\frac{\partial w_s}{\partial x} \tag{8a}$$

$$g(z)=1-f'(z)\ \text{and}\ f'(z)=\frac{df(z)}{dz} \tag{8b}$$

By assuming that the material of FG beam obeys Hooke's law, the stresses in the beam become

$$\sigma_x=Q_{11}(z)\,\varepsilon_x \quad \text{and} \quad \tau_{xz}=Q_{55}(z)\,\gamma_{xz} \tag{9}$$

where

$$Q_{11}(z)=E(z)\text{and}\ \ Q_{55}(z)=\frac{E(z)}{2(1+\nu)} \tag{10}$$

3.3. Constitutive equations

In this work, a rectangular beam of uniform thickness h, length L and rectangular cross section b x h, made of FGM and supported by an elastic foundation is considered, as shown in Fig. 1.

The mechanical properties of FGM such as Young's modulus E and mass density ρ can be expressed as

$$\begin{aligned}E(z)&=E_m+(E_c-E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^p\\ \rho(z)&=\rho_m+(\rho_c-\rho_m)\left(\frac{z}{h}+\frac{1}{2}\right)^p\end{aligned} \tag{11}$$

where the subscripts m and c represent the metallic and ceramic constituents, respectively; and p is the volume fraction exponent. The value of p equals to zero represents a fully ceramic beam, whereas infinite p indicates a fully metallic beam. The variation of Poisson's ratio ν is generally small and it is assumed to be a constant for convenience.

3.4. Equations of motion

In order to obtain the equations of motion, the energy method is adopted and the total energy of structure is required. This will include the strain energy of the beam UP? U_B, the strain energy of foundation U_F, the potential energy of the load V, and the kinetic energy of mass system T. The strain energy of the beam can be expressed as

$$\begin{aligned}\delta U&=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x\delta\,\varepsilon_x+\tau_{xz}\delta\,\gamma_{xz})dzdx\\ &=\int_0^L\left(N_x\delta\varepsilon_x^0-M_x^b\delta k_x^b-M_x^s\delta k_x^s+Q_{xz}\delta\gamma_{xz}^s\right)dx\end{aligned} \tag{12}$$

Where N_x, M_x^b, M_x^s and Q_{xz} are the stress resultants defined as

$$(N_x,M_x^b,M_x^s)=\int_{-\frac{h}{2}}^{\frac{h}{2}}(1,z,f(z))\,\sigma_x dz \tag{13a}$$

$$Q_{xz}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\tau_{xz}g(z)dz \tag{13b}$$

The strain energy of the foundation can be expressed as

$$\delta U_{ef}=\int_0^L\begin{bmatrix}K_W(w_b+w_s)\delta(w_b+w_s)\\ -K_P\dfrac{\partial^2(w_b+w_s)}{\partial x^2}\delta(w_b+w_s)\end{bmatrix}dx \tag{14}$$

where K_w and K_p are the transverse and shear stiffness coefficients of the foundation, respectively.

The variation of work done by externally transverse load q can be expressed as

$$\delta V=-\int_0^L q\delta w_0 dx \tag{15}$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned}\delta T&=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)[\dot{u}\delta\,\dot{u}+\dot{w}\delta\,\dot{w}]dz_{ns}dx\\ &=\int_0^L\Bigg\{I_0[\dot{u}_0\delta\dot{u}_0+(\dot{w}_b+\dot{w}_s)(\delta\dot{w}_b+\delta\dot{w}_s)]-I_1\left(\dot{u}_0\frac{d\delta\dot{w}_b}{dx}+\frac{d\dot{w}_b}{dx}\delta\,\dot{u}_0\right)\\ &+I_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\,\dot{w}_b}{dx}\right)-J_1\left(\dot{u}_0\frac{d\delta\dot{w}_s}{dx}+\frac{d\dot{w}_s}{dx}\delta\,\dot{u}_0\right)+K_2\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\,\dot{w}_s}{dx}\right)\\ &+J_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\,\dot{w}_s}{dx}+\frac{d\dot{w}_s}{dx}\frac{d\delta\,\dot{w}_b}{dx}\right)\Bigg\}dx\end{aligned} \tag{16}$$

Where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_0, I_1, J_1, I_2, J_2, K_2) are the mass inertias defined as

$$\left(I_0, I_1, J_1, I_2, J_2, K_2\right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} \left(1, z, f, z^2, zf, f^2\right)\rho(z)dz \quad (17)$$

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in analytical form as

$$\int_{t_1}^{t_2} \left(\delta U + \delta U_{ef} + \delta V - \delta T\right)dt = 0 \quad (18)$$

Substituting the expressions for δU, δU_{ef}, δV and δT from Eqs. (12), (14), (15) and (17) into Eq. (18) and integrating the displacement gradients by parts and setting the coefficients of δu_0, δw_b and δw_s to zero separately, the following equations of motion are obtained

$$\delta u_0 : \frac{dN_x}{dx} = I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \quad (19a)$$

$$\begin{aligned} \delta w_b : &\frac{d^2 M_b}{dx^2} + q + K_P\left(\frac{d^2\left(w_b + w_s\right)}{dx^2}\right) - K_W\left(w_b + w_s\right) \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2\ddot{w}_b}{dx^2} - J_2 \frac{d^2\ddot{w}_s}{dx^2} \end{aligned} \quad (19b)$$

$$\begin{aligned} \delta w_s : &\frac{d^2 M_s}{dx^2} + \frac{dQ_{xz}}{dx} + q + K_P\left(\frac{d^2\left(w_b + w_s\right)}{dx^2}\right) - K_W\left(w_b + w_s\right) \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2\ddot{w}_b}{dx^2} - K_2 \frac{d^2\ddot{w}_s}{dx^2} \end{aligned} \quad (19c)$$

Introducing Eq. (13) into Eq. (19), the equations of motion can be expressed in terms of displacements (u_0, w_b, w_s) and the appropriate equations take the form

$$\begin{aligned} &A_{11} \frac{\partial^2 u_0}{\partial x^2} - B_{11} \frac{\partial^3 w_b}{\partial x^3} - B_{11}^s \frac{\partial^3 w_s}{\partial x^3} \\ &= I_0 \ddot{u}_0 - I_1 \frac{d\ddot{w}_b}{dx} - J_1 \frac{d\ddot{w}_s}{dx} \end{aligned} \quad (20a)$$

$$\begin{aligned} &B_{11} \frac{\partial^3 u_0}{\partial x^3} - D_{11} \frac{\partial^4 w_b}{\partial x^4} - D_{11}^s \frac{\partial^4 w_s}{\partial x^4} + q \\ &+ K_P\left(\frac{d^2\left(w_b + w_s\right)}{dx^2}\right) - K_W\left(w_b + w_s\right) \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + I_1 \frac{d\ddot{u}_0}{dx} - I_2 \frac{d^2\ddot{w}_b}{dx^2} - J_2 \frac{d^2\ddot{w}_s}{dx^2} \end{aligned} \quad (20b)$$

$$\begin{aligned} &B_{11}^s \frac{\partial^3 u_0}{\partial x^3} - D_{11}^s \frac{\partial^4 w_b}{\partial x^4} - H_{11}^s \frac{\partial^4 w_s}{\partial x^4} + A_{55}^s \frac{\partial^2 w_s}{\partial x^2} + q \\ &+ K_P\left(\frac{d^2\left(w_b + w_s\right)}{dx^2}\right) - K_W\left(w_b + w_s\right) \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + J_1 \frac{d\ddot{u}_0}{dx} - J_2 \frac{d^2\ddot{w}_b}{dx^2} - K_2 \frac{d^2\ddot{w}_s}{dx^2} \end{aligned} \quad (20c)$$

Where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$\begin{aligned} &\left(A_{ij}, A_{ij}^s, B_{ij}, D_{ij}, B_{ij}^s, D_{ij}^s, H_{ij}^s\right) \\ &= \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}\left(1, g^2(z), z, z^2, f(z), z\, f(z), f^2(z)\right)dz \end{aligned} \quad (21)$$

4. Analytical solution

Navier-type analytical solutions are obtained for the bending and free vibration analysis of functionally graded beams resting on two parameter elastic foundation. According to the Navier-type solution technique, the unknown displacement variables are expanded in a Fourier series as given below

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x)\, e^{i\omega t} \\ W_{bm} \sin(\lambda x)\, e^{i\omega t} \\ W_{sm} \sin(\lambda x)\, e^{i\omega t} \end{Bmatrix} \quad (22)$$

Where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with mth eigenmode, and $a = m\pi / L$.

The transverse load q is also expanded in Fourier series as

$$q(x) = \sum_{m=1,3,5}^{\infty} Q_m \sin \alpha x \quad (23)$$

Where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L}\int_0^L q(x)\sin(\alpha x)dx \quad (24)$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$m = 1 \text{ and } Q_1 = q_0 \quad (25a)$$

And for the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \ \left(m = 1,3,5\ldots\right) \quad (25b)$$

Substituting Eqs. (22) and (23) into Eq. (20), the analytical solutions can be obtained by the eigenvalue equations below, for any fixed value of m.

For free vibration problem

$$\left([K]-\omega^2[M]\right)\{\Delta\}=\{0\} \tag{26a}$$

For static problems, we obtain the following operator equation

$$\left([K]-\omega^2[M]\right)\{\Delta\}=\{0\} \tag{26b}$$

Where

$$[K]=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}, \tag{27a}$$

$$[M]=\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}, \tag{27b}$$

And

$$\{\Delta\}=\begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{Bmatrix}, \{F\}=\begin{Bmatrix} 0 \\ Q_m \\ Q_m \end{Bmatrix} \tag{27c}$$

With

$$a_{11}=A_{11}\alpha^2,\ a_{12}=-B_{11}\alpha^3,$$

$$a_{13}=-B_{11}^s\alpha^3,\ a_{22}=D_{11}\alpha^4+K_w+K_p\alpha^2, \tag{28a}$$

$$a_{23}=D_{11}^s\alpha^4+K_w+K_p\alpha^2,$$

$$a_{23}=H_{11}^s\alpha^4+A_{55}^s\alpha^2+K_w+K_p\alpha^2$$

$$m_{11}=I_0,\ m_{12}=-I_1\alpha, \tag{28b}$$

$$m_{13}=-J_1\alpha,\ \ m_{22}=I_0+I_2\alpha^2,$$

5. Results and discussions

Numerical results for displacements, stresses and natural frequencies of functionally graded beams resting on two parameter elastic foundation are presented in this section to verify the accuracy of the present formulation. The beam is made of the following material properties

Ceramic : Alumina (Al_2O_3) : $E_c=380$ GPa; $\nu=0.3$; $\rho_c=3960$ kg/m³.

Metal: Aluminium (Al): $E_m=70$ GPa; $\nu=0.3$; $\rho_m=2702$ kg/m³.

For simplicity, the following non-dimensional parameters are used

Axial displacement : $\bar{u}=100\dfrac{E_m h^3}{q_0 L^4}u\left(0,-\dfrac{h}{2}\right)$;

Transverse displacement : $\bar{w}=100\dfrac{E_m h^3}{q_0 L^4}w\left(\dfrac{L}{2},0\right)$;

Axial stress : $\bar{\sigma}_x=\dfrac{h}{q_0 L}\sigma_x\left(\dfrac{L}{2},\dfrac{h}{2}\right)$;

Transverse shear stress : $\bar{\tau}_{xz}=\dfrac{h}{q_0 L}\tau_{xz}(0,0)$;

Fundamental frequency : $\bar{\omega}=\dfrac{\omega L^2}{h}\sqrt{\dfrac{\rho_m}{E_m}}$;

Finally, the non-dimensional elastic foundation parameters are : $\xi_w=\dfrac{K_w l^2}{E_m h}$, $\xi_p=\dfrac{K_p}{E_m h}$.

5.1. Results of bending analysis for sinusoidal load

In this example, the bending response of FG beam under sinusoidal load is investigated. Table 1 contains nondimensional deflection and stresses of FG beams for different values of power law index p and with foundation parameters (ξ_w,ξ_p). Two values of span-to-depth ratio L/h are considered, 5 and 20. The obtained results are compared with various shear deformation beam theories (i.e., the inverse hyperbolic theory (IHBT) of Sayyad and Ghugal (2018), the parabolic beam theory (PSDBT) of Reddy (1984) and first order beam theory (FSDBT) of Timoshenko (1921)). It is seen that the displacements and stresses obtained from the present hyperbolic theory are in excellent agreement with those obtained from Sayyad and Ghugal (2018) and PSDBT. On the contrary, the FSDBT underestimates the displacements and stresses. Furthermore, it is observed from this table that the displacements increase with the increase in power-law index whereas stresses are identical when beam is made of fully ceramic (p=0) or fully metal(p=∞). This is due to the fact that an increase of the power-law index makes FG beams more flexible i.e. reduces their stiffness. It is also observed from Table 1 that the displacement and stresses of FG beam are reduced when it is resting on two parameter elastic foundation i.e. Winkler layer and shearing layer.

Figs. 2 and 3 show effect of power-law index and foundation parameter on axial displacement of FG beam subjected to sinusoidal load using the present hyperbolic beam theory (HBT). Figs. 4 and 5 show non-linear variation of bending stress for p=1, 5 and 10 and linear variation for p=0 and ∞. Through-the-thickness variations of transverse shear stresses are shown in Figs. 6 and 7 for various values of power law index and foundation parameters.

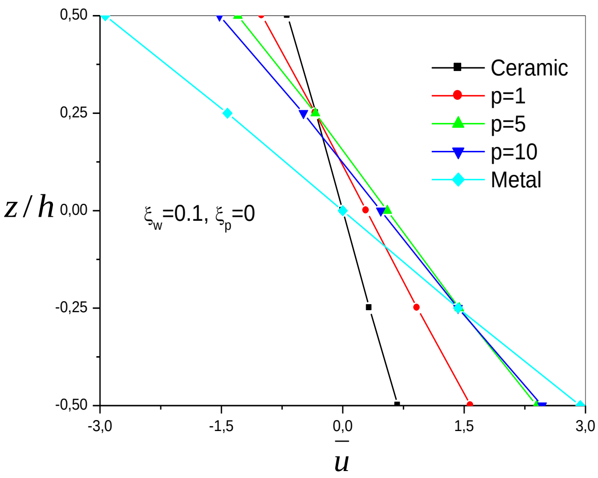

Fig. 2 Non-dimensional axial displacement through the thickness (L/h = 5, non-dimensional elastic foundation parameters are : ξ_w=0.1 and ξ_p=0)

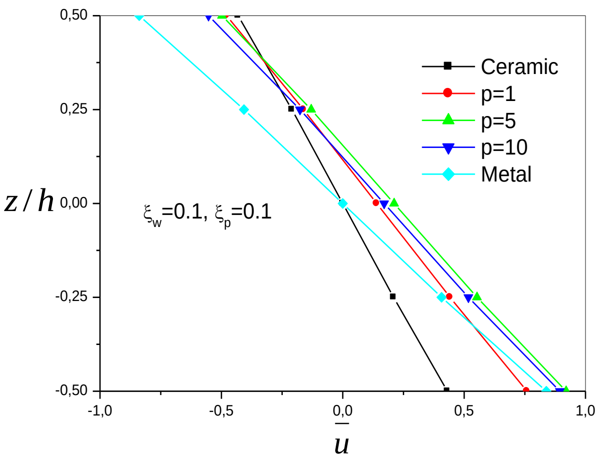

Fig. 3 Non-dimensional axial displacement through the thickness (L/h = 5, non-dimensional elastic foundation parameters are : ξ_w=0.1 and ξ_p=0.1)

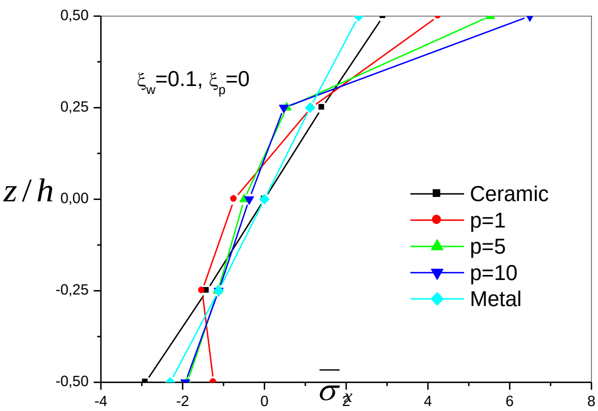

Fig. 4 Non-dimensional axial stress through the thickness (L/h = 5, non-dimensional elastic foundation parameters are : ξ_w=0.1 and ξ_p=0)

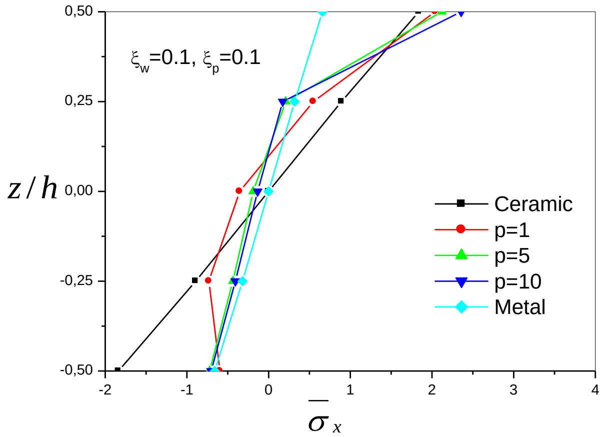

Fig. 5 Non-dimensional axial stress through the thickness (L/h = 5, non-dimensional elastic foundation parameters are : ξ_w=0.1 and ξ_p=0.1)

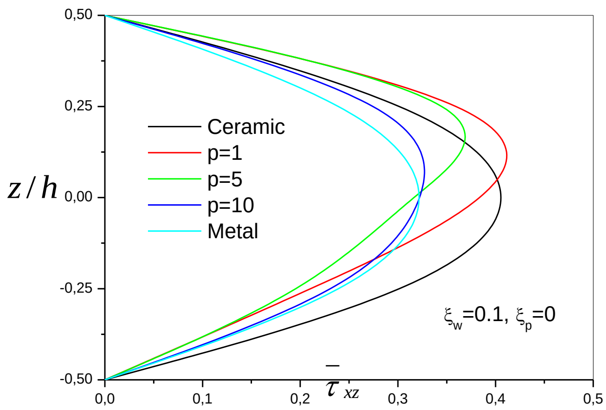

Fig. 6 Non-dimensional transverse shear stress through the thickness (L/h = 5, non-dimensional elastic foundation parameters are : ξ_w=0.1 and ξ_p=0)

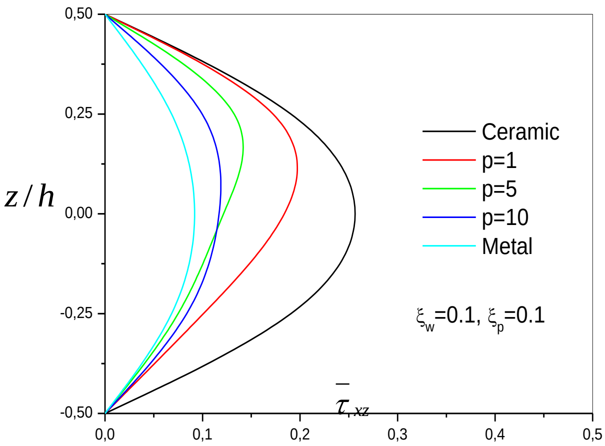

Fig. 7 Non-dimensional transverse shear stress through the thickness (L/h = 5, non-dimensional elastic foundation parameters are : ξ_w=0.1 and ξ_p=0.1)

5.2 Results for free vibration analysis

The non-dimensional natural frequencies of a simply supported FG beam obtained from the present hyperbolic theory (HSDBT) are given in Table 1 for different values of power-law index p. The present results are compared with those presented by Sayyad and Ghugal (2018), Reddy (1984), Simsek (2010), Timoshenko (1921) and Bernoulli-Euler (1744). An excellent agreement between the present solutions with the previously published results are obtained. Table 3 show the first nondimensional frequencies $\bar{\omega}$ of FG beams resting on two parameters elastic foundation obtained from the present hyperbolic beam theory (HSDBT) for different values of power law index p and span-to-depth ratio L/h. The present results are compared with those presented by Sayyad and Ghugal (2018). The examination of table 3 reveal that the frequencies obtained using the present theory are in excellent agreement with the previously published results. It is observed that an increase in the value of the p index leads to a reduction of fundamental frequencies and a decrease in the value of elasticity modulud. Also, it is observed that the natural frequencies are increased when beam is resting on two parameters elastic foundation.

6. Conclusions

A New hyperbolic shear deformation beam theory is proposed for bending and free vibration analysis of functionally graded beams under sinusoidal loads resting on two parameters elastic foundation. The theory gives hyperbolic cosine variation of transverse shear stress across the thickness of the beam. Effects of the power-law index, length-to-thickness ratio and foundation parameter on the displacements, stresses, and natural frequencies of FG beams are investigated. Numerical examples show that the proposed theory gives solutions which are almost identical with those obtained using other shear deformation theories. Consequently, the present model can be used as a reference to check the efficiency of approximate numerical methods. The extension of this study is also envisaged for general boundary conditions and different types of FG beams subjected to different loading (mechanical, thermal, buckling, etc.).

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Appendix

Table 1 Non-dimensional displacements and stresses of functionally graded beam resting on two parameter elastic foundation and subjected to sinusoidal load

p	ξ_w	ξ_p	Theory	L/h=5				L/h=20			
				$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	0	0	Present	0.7244	2.5018	3.0888	0.4641	0.1784	2.2839	12.171	0.4645
			Sayyad *et al.* (2018)	0.7253	2.5019	3.0922	0.4800	0.1784	2.2839	12.171	0.4806
			Reddy 1984	0.7251	2.5020	3.0916	0.4769	0.1784	2.2838	12.171	0.4774
			Timoshenko (1921)	0.7129	2.0523	3.0396	0.2653	0.1782	2.2839	12.158	0.2653
	0.1	0	Present	0.6818	2.3545	2.9069	0.4368	0.0932	1.1935	6.3604	0.2428
			Sayyad *et al.* (2018)	0.6826	2.3547	2.9102	0.4517	0.0932	1.1935	6.3608	0.2511
			Reddy 1984	0.6824	2.3547	2.9096	0.4488	0.0932	1.1935	6.3606	0.2495
			Timoshenko (1921)	0.6716	2.3205	2.8607	0.2499	0.0932	1.1929	6.3539	0.1387
	0.1	0.1	Present	0.4313	1.4893	1.8387	0.2763	0.0163	0.2089	1.1135	0.0425
			Sayyad *et al.* (2018)	0.4317	1.4894	1.8407	0.2857	0.0163	0.2090	1.1136	0.0440
			Reddy 1984	0.4316	1.4894	1.8403	0.2839	0.0163	0.2090	1.1136	0.0437
			Timoshenko (1921)	0.4271	1.4756	1.8093	0.1589	0.0163	0.2089	1.1124	0.0243
1	0	0	Present	1.7782	4.9455	4.7809	0.4641	0.4399	4.5774	18.813	0.4645
			Sayyad *et al.* (2018)	1.7796	4.9441	4.7867	0.5248	0.4400	4.5774	18.814	0.5245
			Reddy 1984	1.7793	4.9458	4.7857	0.5243	0.4400	4.5773	18.813	0.5249
			Timoshenko (1921)	1.7588	4.8807	4.6979	0.5376	0.4397	4.5734	18.792	0.5376
	0.1	0	Present	1.5826	4.4013	4.2548	0.4130	0.1554	1.6169	6.6453	0.1641
			Sayyad *et al.* (2018)	1.5838	4.4015	4.2600	0.4657	0.1554	1.6169	6.6458	0.1851
			Reddy 1984	1.5835	4.4015	4.2591	0.4666	0.1554	1.6169	6.6456	0.1854
			Timoshenko (1921)	1.5675	4.3499	4.1871	0.4791	0.1554	1.6164	6.6418	0.1900
	0.1	0.1	Present	0.7587	2.1099	2.0397	0.1980	0.0211	0.2189	0.9000	0.0222
			Sayyad *et al.* (2018)	0.7592	2.1100	2.0422	0.2232	0.0211	0.2190	0.9001	0.0251
			Reddy 1984	0.7591	2.1100	2.0417	0.2237	0.0211	0.2190	0.9001	0.0251
			Timoshenko (1921)	0.7560	2.0981	2.0195	0.2311	0.0211	0.2190	0.8998	0.0257
5	0	0	Present	2.8621	7.7643	6.5960	0.3713	0.7068	6.9535	25.792	0.3718
			Sayyad *et al.* (2018)	2.8649	7.7739	6.6079	0.5274	0.7069	6.9541	25.795	0.5313
			Reddy 1984	2.8644	7.7723	6.6057	0.5314	0.7069	6.9540	25.794	0.5323
			Timoshenko (1921)	2.8250	7.5056	6.4382	0.9942	0.7062	6.9373	25.752	0.9942
	0.1	0	Present	2.3968	6.5022	5.5238	0.3109	0.1869	1.8389	6.8208	0.0983
			Sayyad *et al.* (2018)	2.3987	6.5089	5.5327	0.4416	0.1869	1.8389	6.8212	0.1397
			Reddy 1984	2.3984	6.5078	5.5310	0.4450	0.1869	1.8389	6.8211	0.1408
			Timoshenko (1921)	2.3786	6.3198	5.4210	0.8371	0.1871	1.8377	6.8221	0.2634
	0.1	0.1	Present	0.9203	2.4967	2.1209	0.1194	0.0226	0.2226	0.8258	0.0119
			Sayyad *et al.* (2018)	0.9205	2.4976	2.1231	0.1694	0.0226	0.2226	0.8258	0.0170
			Reddy 1984	0.9204	2.4975	2.1226	0.1708	0.0226	0.2226	0.8258	0.0170
			Timoshenko (1921)	0.9294	2.4693	2.1181	0.3271	0.0227	0.2226	0.8264	0.0319
10	0	0	Present	2.9961	8.6475	7.8981	0.4082	0.7379	7.6417	30.921	0.4088
			Sayyad *et al.* (2018)	2.9995	8.6539	7.9102	0.4237	0.7380	7.6422	30.923	0.4263
			Reddy 1984	2.9989	8.6530	7.9080	0.4226	0.7379	7.6421	30.999	0.4233
			Timoshenko (1921)	2.9488	8.3259	7.7189	1.2320	0.7372	7.6215	30.875	1.2320
	0.1	0	Present	2.4635	7.1104	6.4942	0.3356	0.1819	1.8837	7.6221	0.1008
			Sayyad *et al.* (2018)	2.4659	7.1147	6.5033	0.3484	0.1819	1.8838	7.6225	0.1051
			Reddy 1984	2.4655	7.1141	6.5016	0.3474	0.1819	1.8838	7.5606	0.1043
			Timoshenko (1921)	2.4408	6.8914	6.3891	1.0197	0.1821	1.8825	7.6262	0.3043
	0.1	0.1	Present	0.8944	2.5814	2.3577	0.1219	0.0216	0.2233	0.9034	0.0119
			Sayyad *et al.* (2018)	0.8950	2.5820	2.3601	0.1264	0.0216	0.2233	0.9035	0.0125
			Reddy 1984	0.8948	2.5819	2.3596	0.1261	0.0215	0.2233	0.8934	0.0124
			Timoshenko (1921)	0.9039	2.5520	2.3660	0.3776	0.0216	0.2233	0.9045	0.0361
∞	0	0	Present	3.9327	13.581	3.0888	0.4641	0.9685	12.398	12.171	0.4645
			Sayyad *et al.* (2018)	3.9371	13.582	3.0922	0.4800	0.9677	12.329	12.171	0.4806
			Reddy 1984	3.9363	13.582	3.0916	0.4769	0.9686	12.398	12.171	0.4774
			Timoshenko (1921)	3.8702	12.552	3.0396	0.3183	0.9676	12.398	12.158	0.3183
	0.1	0	Present	2.9359	10.139	2.3058	0.3465	0.1625	2.0805	2.0423	0.0779
			Sayyad *et al.* (2018)	2.9391	10.139	2.3084	0.3583	0.1631	2.0785	2.0425	0.0806
			Reddy 1984	2.9385	10.139	2.3079	0.3560	0.1625	2.0805	2.0424	0.0801
			Timoshenko (1921)	2.8891	10.140	2.2691	0.2376	0.1624	2.0805	2.0403	0.0534
	0.1	0.1	Present	0.8384	2.8954	0.6585	0.0989	0.0176	0.2258	0.2217	0.0085
			Sayyad *et al.* (2018)	0.8393	2.8955	0.6592	0.1023	0.0177	0.2258	0.2217	0.0088
			Reddy 1984	0.8392	2.8955	0.6591	0.1017	0.0176	0.2258	0.2217	0.0087
			Timoshenko (1921)	0.8250	2.8955	0.6479	0.0679	0.0176	0.2258	0.2214	0.0058

Table 2 Non-dimensional natural frequencies of simply supported functionally graded beam

		p					
L/*h*	Theory	0	1	2	5	10	∞
5	Present	5.1529	3.9905	3.6269	3.4028	3.2826	2.6774
	Sayyad *et al.* (2018)	5.1453	3.9826	3.6184	3.3917	3.2727	2.6734
	Reddy (1984)	5.1527	3.9904	3.6264	3.4012	3.2816	2.6773
	Simsek (2010)	5.1527	3.9904	3.6261	3.4012	3.2816	2.6773
	Timoshenko (1921)	5.1524	3.9902	3.6343	3.4311	3.3134	2.6771
	Bernoulli–Euler (1744)	5.3953	4.1484	3.7793	3.5949	3.4921	2.8033
20	Present	5.4603	4.2051	3.8362	3.6486	3.5390	2.8371
	Sayyad *et al.* (2018)	5.4603	4.2050	3.8361	3.6485	3.5389	2.8371
	Reddy (1984)	5.4603	4.2050	3.8361	3.6485	3.5389	2.8371
	Simsek (2010)	5.4603	4.2050	3.8361	3.6485	3.5389	2.8371
	Timoshenko (1921)	5.4603	4.2050	3.8367	3.6508	3.5415	2.8371
	Bernoulli–Euler (1744)	5.4777	4.2163	3.8472	3.6628	3.5547	2.8461

Table 3 Non-dimensional flexural natural frequencies of functionally graded beams resting on elastic foundation

					p					
L/*h*	Mode	ξ_w	ξ_p	Theory	0	1	2	5	10	∞
5	1	0	0	Present	5.1529	3.9905	3.6269	3.4028	3.2826	2.6774
				Sayyad *et al.* (2018)	5.1453	3.9826	3.6184	3.3917	3.2727	2.6734
		0.1	0	Present	5.3116	4.2299	3.9051	3.7183	3.6199	3.0987
				Sayyad *et al.* (2018)	5.3038	4.2216	3.8961	3.7066	3.6094	3.0942
		0.1	0.1	Present	6.6785	6.1088	5.9937	5.9989	6.0063	5.7979
				Sayyad *et al.* (2018)	6.6689	6.0973	5.9810	5.9830	5.9909	5.7903
	2	0	0	Present	17.8816	14.0102	12.644	11.556	11.031	9.2911
				Sayyad *et al.* (2018)	17.589	13.754	12.388	11.260	10.748	9.1392
		0.1	0	Present	17.926	14.077	12.722	11.649	11.132	9.4160
				Sayyad *et al.* (2018)	17.633	13.820	12.465	11.351	10.848	9.2623
		0.1	0.1	Present	19.603	16.492	15.498	14.845	14.573	13.444
				Sayyad *et al.* (2018)	19.287	16.200	15.200	14.493	14.224	13.240
	3	0	0	Present	34.205	27.095	24.319	21.747	20.569	17.773
				Sayyad *et al.* (2018)	32.324	25.538	22.812	20.117	19.003	16.794
		0.1	0	Present	34.228	27.128	24.359	21.796	20.623	17.837
				Sayyad *et al.* (2018)	32.346	25.570	22.849	20.163	19.053	16.855
		0.1	0.1	Present	36.194	29.939	27.628	25.718	24.929	22.812
				Sayyad *et al.* (2018)	34.223	28.261	25.980	23.881	23.1.7	21.626
20	1	0	0	Present	5.4603	4.2051	3.8362	3.6486	3.5390	2.8371
				Sayyad *et al.* (2018)	5.4603	4.2050	3.8361	3.6484	3.5389	2.8371
		0.1	0	Present	7.5533	7.0752	7.0185	7.0950	7.1281	6.9259
				Sayyad *et al.* (2018)	7.5533	7.0751	7.0184	7.0948	7.1279	6.9259
		0.1	0.1	Present	18.052	19.225	19.753	20.390	20.704	21.023
				Sayyad *et al.* (2018)	18.052	19.224	19.752	20.390	20.703	21.022
	2	0	0	Present	21.573	16.635	15.162	14.377	13.928	11.209
				Sayyad *et al.* (2018)	21.571	16.631	15.158	14.370	13.922	11.208
		0.1	0	Present	22.192	17.575	16.255	15.603	15.232	12.858
				Sayyad *et al.* (2018)	22.189	17.571	16.250	15.596	15.226	12.857
		0.1	0.1	Present	39.517	39.737	40.232	41.168	41.634	41.619
				Sayyad *et al.* (2018)	39.513	39.730	40.223	41.157	41.624	41.615
	3	0	0	Present	47.594	36.769	33.471	31.587	30.542	24.729
				Sayyad *et al.* (2018)	47.569	36.740	33.440	31.543	30.505	24.716
		0.1	0	Present	47.875	37.199	33.975	32.158	31.153	25.513
				Sayyad *et al.* (2018)	47.851	37.171	33.943	32.114	31.116	25.499
		0.1	0.1	Present	68.387	64.919	64.576	65.312	65.696	64.387
				Sayyad *et al.* (2018)	68.353	64.871	64.523	65.245	65.633	64.358