Earthquakes and Structures, *Vol. 17, No. 1 (2019) 31-37*
DOI: https://doi.org/10.12989/eas.2019.17.1.031

Numerical analysis for free vibration of functionally graded beams using an original HSDBT

Abdelkader Sahouane[1,2], Lazreg Hadji[*3,4] and Mohamed Bourada[2]

[1]*Department of Civil Engineering, Ibn Khaldoun University, BP 78 Zaaroura, Tiaret, 14000, Algeria*
[2]*Material and Hydrology Laboratory, Faculty of Technology, Civil Engineering Department, University of Sidi Bel Abbes, Sidi Bel Abbes, Algeria*
[3]*Department of Mechanical Engineering, Ibn Khaldoun University, BP 78 Zaaroura, Tiaret, 14000, Algeria*
[4]*Laboratory of Geomatics and Sustainable Development, Ibn Khaldoun University of Tiaret, Algeria*

(Received March 23, 2019, Revised April 6, 2019, Accepted April 10, 2019)

Abstract. This work presents a free vibration analysis of functionally graded beams by employing an original high order shear deformation theory (HSDBT). This theory use only three unknowns, but it satisfies the stress free boundary conditions on the top and bottom surfaces of the beam without requiring any shear correction factors. The mechanical properties of the beam are assumed to vary continuously in the thickness direction by a simple power-law distribution in terms of the volume fractions of the constituents. In order to investigate the free vibration response, the equations of motion for the dynamic analysis are determined via the Hamilton's principle. The Navier solution technique is adopted to derive analytical solutions for simply supported beams. The accuracy and effectiveness of proposed model are verified by comparison with previous research.

Keywords: free vibration; functionally graded materials; Hamilton's principle; Navier solution

1. Introduction

A new class of materials known as ‘‘functionally graded materials’’ (FGMs) has attracted much attention as advanced structural materials in many structural members used in situations where large temperature gradients are encountered. FGMs are designed so that material properties vary smoothly and continuously through the thickness from the surface of a ceramic exposed to high temperature to that of a metal on the other surface. The composition of the material changes gradually throughout the thickness direction. The FGMs are widely used in mechanical, aerospace, nuclear, and civil engineering. Consequently, studies devoted to understand the dynamic behavior of FGM beams and plates have being paid more and more attentions in recent years. Thai and Vo (2012) obtained Navier-type analytical solution for bending and vibration of functionally graded beams based on various higher order shear deformation beam theories. Tounsi *et al.* (2013) use a refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates. Li and Batra (2013) derived analytical relations between the critical buckling load of a functionally graded Timoshenko beam and Euler-Bernoulli for various boundary conditions. Nguyen *et al*. (2013) applied the first-order shear deformation theory for the static and free vibration analysis of functionally graded beams and obtained an analytical solution according to Navier solution procedure. Ansari *et al.* (2013) studied the size-dependent bending, buckling and free vibration of functionally graded Timoshenko microbeams based on the most general strain gradient theory. Pradhan *et al.* (2013) analyze the free vibration of Euler and Timoshenko functionally graded beams by Rayleigh-Ritz method. Xu *et al.* (2014) presented the twodimensional elasticity solutions of functionally graded beams with varying thickness. Bourada *et al.* (2015) used a new simple shear and normal deformations theory for functionally graded beams. Bennai *et al.* (2015) used a new higher-order shear and normal deformation theory for functionally graded sandwich beams. Akbaş (2015) investigated the wave propagation of a functionally graded beam in thermal environments. Bounouara *et al.* (2016) used a nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation. Recently, Sayyad and Ghugal (2017) reviewed of all those numerical and analytical methods available in the literature for the analysis of composite beams and plates. Ghumare and Sayyad (2017) have developed a new fifth-order shear and normal deformation theory for the bending and free vibration analysis of functionally graded beams. Kahya and Turan (2017) have developed finite-element formulation for the buckling and vibration analysis of functionally graded beams based on the first-order shear deformation theory. Akbaş (2017a) studied the thermal effects on the vibration of functionally graded deep beams with porosity. Akbaş (2017b) developed the free vibration of edge cracked functionally graded microscale beams based on the modified couple stress theory. Akbaş (2018a) analyze the forced vibration of cracked functionally graded microbeams. Akbaş (2018b) analyze the forced vibration of

∗Corresponding author, Ph.D.
E-mail: had_laz@yahoo.fr

Copyright © 2019 Techno-Press, Ltd.
http://www.techno-press.com/journals/eas&subpage=7
ISSN: 2092-7614 (Print), 2092-7622 (Online)

functionally graded porous deep beams", Composite Structures, 186, 293-302. El-Haina *et al.* (2017) used a simple analytical approach for thermal buckling of thick functionally graded sandwich plates. Menasria *et al.* (2017) used a new and simple HSDT for thermal stability analysis of FG sandwich plates. Recently, Tounsi and his co-workers (Fourn *et al.* 2018, Chikh *et al.* 2017, Abdelaziz *et al.* 2017, Attia *et al.* 2018, Bellifa *et al.* 2017, Mokhtar *et al.* 2018, Bouhadra *et al.* 2018, Khiloun *et al.* 2019) developed new shear deformation plates theories involving only four unknown functions. Mahmoud *et al.* (2017) used a new shear deformation plate theory with stretching effect for buckling analysis of functionally graded sandwich plates. Zouatnia *et al.* (2017) developed an analytical solution for bending and vibration responses of functionally graded beams with porosities. Younsi *et al.* (2018) used a novel quasi-3D and 2D shear deformation theories for bending and free vibration analysis of FGM plates. Ould Larbi *et al.* (2018) investigated an analytical solution for free vibration of functionally graded beam using a simple first-order shear deformation theory. Zouatnia *et al.* (2019) studied the effect of the micromechanical models on the bending of FGM beam using a new hyperbolic shear deformation theory. Hadji *et al.* (2019) developed an analytical solution for bending and free vibration responses of functionally graded beams with porosities: Effect of the micromechanical models.

In the present paper, the free vibration analysis of FG beams is investigated. The proposed theory has only three unknowns and three governing equations, but it satisfies the stress free boundary conditions on the top and bottom surfaces of the beam without requiring any shear correction factors. The mechanical properties of the plates are supposed to vary in the thickness direction according to a power law distribution in terms of the volume fractions of the constituents. The interesting beam equations of motion for the free vibration analysis are determined through the Hamilton's principle. These equations are then solved using Navier's procedure. The accuracy of the results of this theory is verified by comparing with other HSDBTs available in the literature.

2. Kinematics

Consider a functionally graded beam with length L and rectangular cross section $b\times h$, with b being the width and h being the height. The x, y and z coordinates are taken along the length, width, and height of the beam, respectively, as shown in Fig. 1.

The formulation is limited to linear elastic material behavior. The displacement field of present original shear deformation beam theorie is chosen based on following assumptions:

- The axial and transverse displacements are partitioned into bending and shear components;
- The bending component of axial displacement is similar to that given by the CBT;
- The shear component of axial displacement gives rise to the higher-order variation of shear strain and hence to shear stress through the depth of the beam in such a way that shear stress vanishes on the top and bottom surfaces.

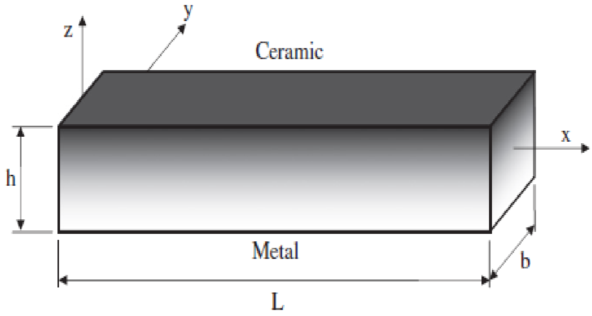

Fig. 1 Geometry and coordinate of a FG beam

Based on these assumptions, the displacement fields of the present original shear deformation beam theory is given in a general form as

$$u(x,z,t)=u_0(x,t)-z\frac{\partial w_b}{\partial x}-f(z)\frac{\partial w_s}{\partial x} \tag{1a}$$

$$v(x,z,t)=0 \tag{1b}$$

$$w(x,z,t)=w_b(x,t)+w_s(x,t) \tag{1c}$$

Where u_0 is the axial displacement of a point on the midplane of the beam; w_b and w_s are the bending and shear components of transverse displacement of a point on the midplane of the beam; and $f(z)$ is a shape function determining the distribution of the transverse shear strain and shear stress through the depth of the beam. The shape functions $f(z)$ is chosen to satisfy the stress-free boundary conditions on the top and bottom surfaces of the beam, thus a shear correction factor is not required. In this work, the present original HSDBT is obtained by setting

$$f(z)=\frac{1}{2}h\tanh\left(\frac{2z}{h}\right)-\frac{4}{3}\frac{z^3}{h^2\cosh(1)^2} \tag{2}$$

The strains associated with the displacements in Eq. (1) are

$$\varepsilon_x=\varepsilon_x^0+z\,k_x^b+f(z)\,k_x^s \tag{3a}$$

$$\gamma_{xz}=g(z)\,\gamma_{xz}^s \tag{3b}$$

where

$$\varepsilon_x^0=\frac{\partial u_0}{\partial x},\ k_x^b=-\frac{\partial^2 w_b}{\partial x^2},\ k_x^s=-\frac{\partial^2 w_s}{\partial x^2},\ \gamma_{xz}^s=\frac{\partial w_s}{\partial x} \tag{3c}$$

$$g(z)=1-f'(z)\ \text{ and }\ f'(z)=\frac{df(z)}{dz} \tag{3d}$$

The state of stress in the beam is given by the generalized Hooke's law as follows

$$\sigma_x=Q_{11}(z)\,\varepsilon_x\ \text{ and }\ \tau_{xz}=Q_{55}(z)\,\gamma_{xz} \tag{4a}$$

where

$$Q_{11}(z)=E(z)\ \text{and}\ Q_{55}(z)=\frac{E(z)}{2(1+\nu)} \tag{4b}$$

3. Material variation laws

Table 1 Material properties used in the FG beam

Properties	Metal	Ceramic		
	Aluminum (Al)	Alumina (Al_2O_3)	Alumina (ZrO_2)	Silicon nitride (Si_3N_4)
E (GPa)	70	380	200	322.2
ρ (kg/m^3)	2702	3800	5700	2370

The material properties of FG beam such as the Young's modulus E and the mass density ρ are considered to vary continuously within the thickness of the beam according to the power law variation as follows (Benahmed *et al*. 2017, Bouafia *et al*. 2017)

$$E(z)=(E_c-E_m)\left(\frac{z}{h}+\frac{1}{2}\right)^{p1}+E_m \tag{5a}$$

$$\rho(z)=(\rho_c-\rho_m)\left(\frac{z}{h}+\frac{1}{2}\right)^{p2}+\rho_m \tag{5b}$$

Where (E_c, ρ_c) and (E_m, ρ_m) are the corresponding properties of the ceramic and metal, respectively, and p_1, p_2 are constants. The Poisson's ratio v is considered to be constant and equal to 0.3 throughout the analyses (Zidi *et al*. 2017, Menasria *et al*. 2017). The value of p (p_1 or p_2) equal to zero represents a fully ceramic beam and infinite p, a fully metallic beam. The distribution of the composition of ceramics and metal is linear for p=1. Typical values for metal and ceramics used in the FG beam are listed in Table 1.

3. Equations of motion

Hamilton's principle is used herein to derive the equations of motion. The principle can be stated in analytical form as (Reddy 2002)

$$\delta\int_{t_1}^{t_2}(U-T)dt=0 \tag{6}$$

where t is the time; t_1 and t_2 are the initial and end time, respectively; δU is the virtual variation of the strain energy; and δT is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned}\delta U&=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}(\sigma_x\delta\varepsilon_x+\tau_{xz}\delta\gamma_{xz})dzdx\\&=\int_0^L\left(N\frac{d\delta u_0}{dx}-M_b\frac{d^2\delta w_b}{dx^2}-M_s\frac{d^2\delta w_s}{dx^2}+Q\frac{d\delta w_s}{dx}\right)dx\end{aligned} \tag{7}$$

where N_x, M_b, M_s and Q_{xz} are the stress resultants defined as

$$(N,M_b,M_s)=\int_{-\frac{h}{2}}^{\frac{h}{2}}(1,z,f)\,\sigma_x dz \quad\text{and}\quad Q_{xz}=\int_{-\frac{h}{2}}^{\frac{h}{2}}g\,\tau_{xz}dz \tag{8}$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned}\delta T&=\int_0^L\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)[\dot{u}\delta\dot{u}+\dot{w}\delta\dot{w}]dzdx\\&=\int_0^L\left\{I_1[\dot{u}_0\delta\dot{u}_0+(\dot{w}_b+\dot{w}_s)(\delta\dot{w}_b+\delta\dot{w}_s)]-I_2\left(\dot{u}_0\frac{d\delta\dot{w}_b}{dx}+\frac{d\dot{w}_b}{dx}\delta\dot{u}_0\right)\right.\\&\quad+I_4\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_b}{dx}\right)-I_3\left(\dot{u}_0\frac{d\delta\dot{w}_s}{dx}+\frac{d\dot{w}_s}{dx}\delta\dot{u}_0\right)+I_6\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_s}{dx}\right)\\&\quad\left.+I_5\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_s}{dx}+\frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_b}{dx}\right)\right\}dx\end{aligned} \tag{9}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z)$ is the mass density; and (I_1, I_2, I_3, I_4, I_5, I_6) are the mass inertias defined as

$$(I_1,I_2,I_3,I_4,I_5,I_6)=\int_{-\frac{h}{2}}^{\frac{h}{2}}(1,z,f,z^2,zf,f^2)\rho(z)dz \tag{10}$$

Substituting the expressions for δU, and δT from Eqs. (6), and (8) into Eq. (5) and integrating by parts versus both space and time variables, and collecting the coefficients of δu_0, δw_b, and δw_s, the following equations of motion of the functionally graded beam are obtained

$$\delta u_0:\ \frac{dN_x}{dx}=I_1\ddot{u}_0-I_2\frac{d\ddot{w}_b}{dx}-I_3\frac{d\ddot{w}_s}{dx} \tag{11a}$$

$$\delta w_b:\ \frac{d^2M_b}{dx^2}=I_1(\ddot{w}_b+\ddot{w}_s)+I_2\frac{d\ddot{u}_0}{dx}-I_4\frac{d^2\ddot{w}_b}{dx^2}-I_5\frac{d^2\ddot{w}_s}{dx^2} \tag{11b}$$

$$\delta w_s:\ \frac{d^2M_s}{dx^2}+\frac{dQ_{xz}}{dx}=I_1(\ddot{w}_b+\ddot{w}_s)+I_3\frac{d\ddot{u}_0}{dx}-I_5\frac{d^2\ddot{w}_b}{dx^2}-I_6\frac{d^2\ddot{w}_s}{dx^2} \tag{11c}$$

By substituting the stress resultants in Eq. (8) into Eq. (11), the equations of motion can be expressed in terms of displacements (u_0, w_b, w_s) as

$$\begin{aligned}&A_{11}\frac{\partial^2u_0}{\partial x^2}-B_{11}\frac{\partial^3w_b}{\partial x^3}-B_{11}^s\frac{\partial^3w_s}{\partial x^3}\\&=I_1\ddot{u}_0-I_2\frac{d\ddot{w}_b}{dx}-I_3\frac{d\ddot{w}_s}{dx}\end{aligned} \tag{12a}$$

$$\begin{aligned}&B_{11}\frac{\partial^3u_0}{\partial x^3}-D_{11}\frac{\partial^4w_b}{\partial x^4}-D_{11}^s\frac{\partial^4w_s}{\partial x^4}\\&=I_1(\ddot{w}_b+\ddot{w}_s)+I_2\frac{d\ddot{u}_0}{dx}-I_4\frac{d^2\ddot{w}_b}{dx^2}-I_5\frac{d^2\ddot{w}_s}{dx^2}\end{aligned} \tag{12b}$$

$$\begin{aligned}&B_{11}^s\frac{\partial^3u_0}{\partial x^3}-D_{11}^s\frac{\partial^4w_b}{\partial x^4}-H_{11}^s\frac{\partial^4w_s}{\partial x^4}+A_{55}^s\frac{\partial^2w_s}{\partial x^2}\\&=I_1(\ddot{w}_b+\ddot{w}_s)+I_3\frac{d\ddot{u}_0}{dx}-I_5\frac{d^2\ddot{w}_b}{dx^2}-I_6\frac{d^2\ddot{w}_s}{dx^2}\end{aligned} \tag{12c}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$\begin{aligned}&(A_{11},B_{11},D_{11},B_{11}^s,D_{11}^s,H_{11}^s)\\&=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{11}(1,z,z^2,f(z),z\,f(z),f^2(z))dz\end{aligned} \tag{13a}$$

and

$$A_{55}^{s}=\int_{-\frac{h}{2}}^{\frac{h}{2}}Q_{55}\left[g(z)\right]^{2}dz \tag{13b}$$

4. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix}=\sum_{m=1}^{\infty}\begin{Bmatrix} U_m\cos(\lambda\, x)\,e^{i\,\omega\, t} \\ W_{bm}\sin(\lambda\, x)\,e^{i\,\omega\, t} \\ W_{sm}\sin(\lambda\, x)\,e^{i\,\omega\, t} \end{Bmatrix} \tag{14}$$

where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with m th eigenmode, $i=\sqrt{-1}$ and $\lambda=m\pi/L$.

Substituting the expansions of u_0, w_b, w_s from Eqs. (14) into the equations of motion Eq. (12), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}-\omega^{2}\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}\right)\begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{Bmatrix}=\begin{Bmatrix} 0 \\ Q_m \\ Q_m \end{Bmatrix} \tag{15}$$

where

$$a_{11}=A_{11}\lambda^{2},\ a_{12}=-B_{11}\lambda^{3},\ \ a_{13}=-B_{11}^{s}\lambda^{3},\ a_{22}=D_{11}\lambda^{4},$$
$$a_{23}=D_{11}^{s}\lambda^{4},\ \ a_{33}=H_{11}^{s}\lambda^{4}+A_{55}^{s}\lambda^{2} \tag{16a}$$

$$m_{11}=I_1,\ \ m_{12}=-I_2\lambda,\ \ m_{13}=-I_3\lambda,$$
$$m_{22}=I_1+I_4\lambda^{2},\ \ m_{23}=I_1+I_5\lambda^{2},\ \ m_{33}=I_1+I_6\lambda^{2} \tag{16b}$$

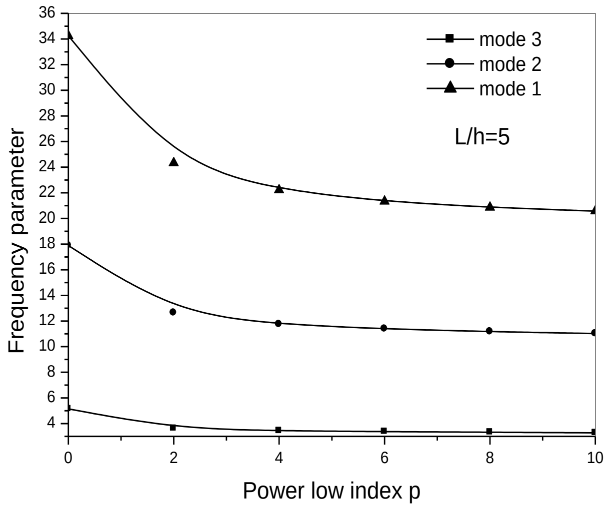

Fig. 2 Variation of first three nondimensional frequencies $\bar{\omega}$ with respect to power law index p (L/h=5)

Table 2 Nondimensional fundamental frequency $\bar{\omega}$ of FG beams

L/h	Theory	k = 0	0.5	1	2	5	10
5	Present	5.1534	4.4112	3.9909	3.6265	3.3997	3.2815
	TBT	5.1527	4.4107	3.9904	3.6264	3.4012	3.2816
	SBT	5.1531	4.4110	3.9907	3.6263	3.3998	3.2811
	HBT	5.1527	4.4107	3.9904	3.6265	3.4014	3.2817
	EBT	5.1542	4.4118	3.9914	3.6267	3.3991	3.2814
	CPT	5.3953	4.5931	4.1484	3.7793	3.5949	3.4921
20	Present	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	TBT	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
	SBT	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	HBT	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
	EBT	5.4604	4.6512	4.2051	3.8361	3.6483	3.5390
	CPT	5.4777	4.6641	4.2163	3.8472	3.6628	3.5547

5. Results and discussion

In this section, various numerical examples are presented and discussed to verify the accuracy of the present theory in predicting the bending and free vibration of simply supported FG beams.

In this section, various numerical examples are presented and discussed for checking the accuracy of the present HSDBT in predicting the dynamic behaviors of simply supported FG beams. For the verification purpose, the results obtained by the proposed HSDBT are compared with the existing data in the literature and discussed for checking the accuracy of the present HSDBT in predicting the dynamic behaviors of simply supported FG beams. For the verification purpose, the results obtained by the proposed HSDT are compared with the existing data in the literature.

Table 2 shows the nondimensional fundamental frequencies $\bar{\omega}$ of the simply supported Al/Al_2O_3 beams for different values span-to-depth ratio L/h with p_1=p_2=p=0, 0,5, 1, 2, 5, 10. The calculated frequencies are compared with those given by Tai *et al.* (2012) using various beam theories. An excellent agreement between the present solutions and results of Tai *et al.* (2012) is found. The first three nondimensional frequencies $\bar{\omega}$ of FG beams predicted by the present theory are presented in Table 3 for different values of power law index p and span-to depth ratio L/h. The obtained results are compared with those of Tai *et al.* (2012) using various beam theories. It can be seen that the present theory and all shear deformation beam theories of Tai *et al.* (2012) give the same frequencies, whereas the CBT overestimates them for all cases considered. The difference between the frequencies of CBT and shear deformation beam theories is significant for higher modes and for small span-to-depth ratios L/h.

The effect power law index p on the first three frequencies of FG beams is shown in Fig. 2. It is observed that an increase in the value of the power law index p leads to a reduction of frequency. The highest frequency values are obtained for full ceramic beams (p=0) while the lowest frequency values are obtained for full metal beams ($p\rightarrow\infty$). This is due to the fact that an increase in the value of the

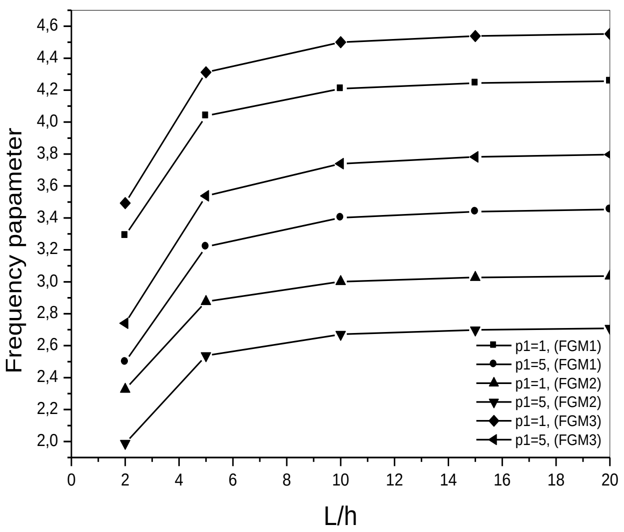

Fig. 3 Variation of frequency parameter with *L/h* ratio and p_1 index (p_2=1)

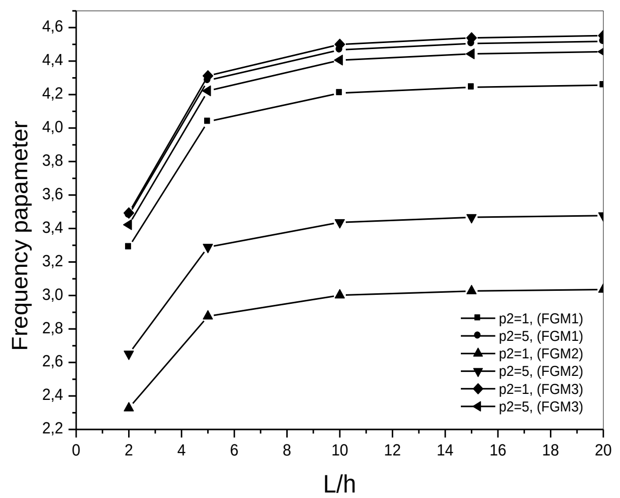

Fig. 4 Variation of frequency parameter with *L/h* ratio and p_2 index (p_1=1)

power law index results in a decrease in the value of elasticity modulus.

To make the effects of ratio and power indices more apparent, Figs. 3 and 4 are shown for Aluminum/lumina (FGM1), Aluminum/Zirconia (FGM2) and Aluminum/ Silicon nitride (FGM3) beams, to show the variation of the non-dimensional fundamental frequency with *L/h* ratio and p_i (i=1,2) power indices, respectively.

According to these results the non-dimensional fundamental frequency increases with increasing *L/h* ratio when $L/h \prec 10$. The non-dimensional frequency is found to be independent of the length-thickness ratio *L/h* for $L/h \succ 10$. It is shown from Fig. 3 that the effect of p_1 is to make the beam stiffer when this gradient index is reduced. However, decreasing the second power index p_2, makes the beam soften as is presented in Fig. 4. In addition, it is observed that the non-dimensional fundamental frequency is approximately insensitive to p_2 for Aluminum/Silicon nitride (FGM3) beam.

Table 3 First three nondimensional frequencies $\bar{\varpi}$ of FG beams

L/h	Mode	Theory	*k* = 0	0.5	1	2	5	10
5	1	Present	5.1534	4.4112	3.9909	3.6265	3.3997	3.2815
		TBT	5.1527	4.4107	3.9904	3.6264	3.4012	3.2816
		SBT	5.1531	4.4110	3.9907	3.6263	3.3998	3.2811
		HBT	5.1527	4.4107	3.9904	3.6265	3.4014	3.2817
		EBT	5.1542	4.4118	3.9914	3.6267	3.3991	3.2814
		CPT	5.3953	4.5931	4.1484	3.7793	3.5949	3.4921
	2	Present	17.8906	15.4659	14.0163	12.6427	11.5322	11.0257
		TBT	17.8812	15.4588	14.0100	12.6405	11.5431	11.0240
		SBT	17.8868	15.4631	14.0138	12.6411	11.5324	11.0216
		HBT	17.8810	15.4587	14.0098	12.6407	11.5444	11.0246
		EBT	17.8996	15.4728	14.0224	12.6466	11.5281	11.0264
		CPT	20.6187	17.5415	15.7982	14.3260	13.5876	13.2376
	3	Present	34.2481	29.8675	27.1246	24.3307	21.6958	20.5715
		TBT	34.2097	29.8382	27.0979	24.3152	21.7158	20.5561
		SBT	34.2344	29.8569	27.1152	24.3237	21.6943	20.5581
		HBT	34.2085	29.8373	27.0971	24.3151	21.7187	20.5569
		EBT	34.2819	29.8929	27.1480	24.3482	21.6924	20.5815
		CPT	43.3483	36.8308	33.0278	29.7458	28.0850	27.4752
20	1	Present	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
		TBT	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
		SBT	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
		HBT	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
		EBT	5.4604	4.6512	4.2051	3.8361	3.6483	3.5390
		CPT	5.4777	4.6641	4.2163	3.8472	3.6628	3.5547
	2	Present	21.5740	18.3968	16.6349	15.1619	14.3726	13.9260
		TBT	21.5732	18.3962	16.6344	15.1619	14.3746	13.9263
		SBT	21.5736	18.3965	16.6347	15.1617	14.3728	13.9255
		HBT	21.5732	18.3962	16.6344	15.1619	14.3748	13.9264
		EBT	21.5748	18.3974	16.6355	15.1621	14.3718	13.9258
		CPT	21.8438	18.5987	16.8100	15.3334	14.5959	14.1676
	3	Present	47.5967	40.6555	36.7704	33.4688	31.5694	30.5360
		TBT	47.5930	40.6526	36.7679	33.4689	31.5780	30.5369
		SBT	47.5950	40.6542	36.7692	33.4681	31.5699	30.5337
		HBT	47.5930	40.6526	36.7679	33.4691	31.5789	30.5373
		EBT	47.6008	40.6586	36.7730	33.4701	31.5655	30.5349
		CPT	48.8999	41.6328	37.6173	34.2954	32.6357	31.6883

6. Conclusions

This work presents a free vibration analysis for FG beams by employing an original HSDBT with only 3 unknown variables. The equations of motion are obtained through the Hamilton's principle. These equations are solved via Navier's procedure. The results were compared with the solutions of several theories. It is concluded that the results of the proposed original HSDBT has an excellent agreement with the other theories used for comparison for free vibration problems.

References

Abdelaziz, H.H., Ait Amar Meziane, M., Bousahla, A.A., Tounsi, A., Mahmoud, S.R. and Alwabli, A.S. (2017), "An efficient hyperbolic shear deformation theory for bending, buckling and

free vibration of FGM sandwich plates with various boundary conditions", *Steel Compos. Struct.*, **25**(6) 693-704. https://doi.org/10.12989/scs.2017.25.6.693.
Akbaş, S.D. (2015), "Wave propagation of a functionally graded beam in thermal environments", *Steel Compos. Struct.*, **19**(6), 1421-1447. http://dx.doi.org/10.12989/scs.2015.19.6.1421.
Akbaş, S.D. (2017a), "Thermal effects on the vibration of functionally graded deep beams with porosity", *Int. J. Appl. Mech.*, **9**(05), 1750076. https://doi.org/10.1142/S1758825117500764.
Akbaş, S.D. (2017b), "Free vibration of edge cracked functionally graded microscale beams based on the modified couple stress theory", *Int. J. Struct. Stab. Dyn.*, **17**(03), 1750033. https://doi.org/10.1142/S021945541750033X.
Akbaş, S.D. (2018a), "Forced vibration analysis of cracked functionally graded microbeams", *Adv. Nano Res.*, **6**(1), 39-55. https://doi.org/10.12989/anr.2018.6.1.039.
Akbaş, S.D. (2018b), "Forced vibration analysis of functionally graded porous deep beams", *Compos. Struct.*, **186**, 293-302. https://doi.org/10.1016/j.compstruct.2017.12.013.
Ansaria, R., Gholami, R., Shojaeia, M.F., Mohammadia, V. and Sahmania, S. (2013), "Size-dependent bending, buckling and free vibration of functionally graded Timoshenko microbeams based on the most general strain gradient theory", *Compos. Struct.*, **100**, 385-397. https://doi.org/10.1016/j.compstruct.2012.12.048.
Attia, A., Bousahla, A.A., Tounsi, A., Mahmoud, S.R. and Afaf, A.S. (2018), "A refined four variable plate theory for thermoelastic analysis of FGM plates resting on variable elastic foundations", *Struct. Eng. Mech.*, **65**(4), 453-464. https://doi.org/10.12989/sem.2018.65.4.453.
Bellifa, H., Bakora, A., Tounsi, A., Bousahla, A.A. and Mahmoud, S.R. (2017), "An efficient and simple four variable refined plate theory for buckling analysis of functionally graded plates", *Steel Compos. Struct.*, **25**(3), 257-270. https://doi.org/10.12989/scs.2017.25.3.257.
Bennai, R., Ait Atmane, H. and Tounsi, A. (2015), "A new higher-order shear and normal deformation theory for functionally graded sandwich beams", *Steel Compos. Struct.*, **19**(3), 521-546. https://doi.org/10.12989/scs.2015.19.3.521.
Bouhadra, A., Tounsi, A., Bousahla, A.A., Benyoucef, S. and Mahmoud, S.R. (2018), "Improved HSDT accounting for effect of thickness stretching in advanced composite plates", *Struct. Eng. Mech.*, **66**(1), 61-73. https://doi.org/10.12989/sem.2018.66.1.061.
Bounouara, F., Benrahou, K.H., Belkorissat, I. and Tounsi, A. (2016), "A nonlocal zeroth-order shear deformation theory for free vibration of functionally graded nanoscale plates resting on elastic foundation", *Steel Compos. Struct.*, **20**(2), 227-249. https://doi.org/10.12989/scs.2016.20.2.227.
Bourada, M., Amara, K., Bousahla, A.A. Tounsi, A. and Mahmoud, S.R. (2018), "A novel refined plate theory for stability analysis of hybrid and symmetric S-FGM plates", *Struct. Eng. Mech.*, **68**(6), 661-675. https://doi.org/10.12989/sem.2018.68.6.661.
Bourada, M., Kaci, A., Houari, M.S.A. and Tounsi, A. (2015), "A new simple shear and normal deformations theory for functionally graded beams", *Steel Compos. Struct.*, **18**(2), 409-423. https://doi.org/10.12989/scs.2015.18.2.409.
Chikh, A., Tounsi, A., Hebali, H. and Mahmoud, S.R. (2017), "Thermal buckling analysis of cross-ply laminated plates using a simplified HSDT", *Smart Struct. Syst.*, **19**(3), 289-297. https://doi.org/10.12989/sss.2017.19.3.289.
El-Haina, F., Bakora, A., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2017), "A simple analytical approach for thermal buckling of thick functionally graded sandwich plates", *Struct. Eng. Mech.*, **63**(5), 585-595. https://doi.org/10.12989/sem.2017.63.5.585.
Fourn, H., Ait Atmane, H., Bourada, M., Bousahla, A.A., Tounsi, A. and Mahmoud, S.R. (2018), "A novel four variable refined plate theory for wave propagation in functionally graded material plates", *Steel Compos. Struct.*, **27**(1), 109-122. https://doi.org/10.12989/scs.2018.27.1.109.
Ghumare, S.M. and Sayyad, A.S. (2017), "A new fifth-order shear and normal deformation theory for static bending and elastic buckling of P-FGM beams", *Lat. Am. J. Solid. Struct.*, **14**, 1-19. http://dx.doi.org/10.1590/1679-78253972.
Hadji, L., Zouatnia, N. and Bernard, F. (2019), "An analytical solution for bending and free vibration responses of functionally graded beams with porosities: Effect of the micromechanical models", *Struct. Eng. Mech.*, **69**(2), 231-241. https://doi.org/10.12989/sem.2019.69.2.231.
Kahya, V. and Turan, M. (2017), "Finite element model for vibration and buckling of functionally graded beams based on the first order shear deformation theory", *Compos. Part B Eng.*, **109**, 108-115. https://doi.org/10.1016/j.compositesb.2016.10.039.
Li, S.R. and Batra, R.C. (2013), "Relations between buckling loads of functionally graded Timoshenko and homogeneous Euler-Bernoulli beams", *Compos. Struct.*, **95**, 5-9. https://doi.org/10.1016/j.compstruct.2012.07.027.
Mahmoud, S.R. and Tounsi, A. (2017), "A new shear deformation plate theory with stretching effect for buckling analysis of functionally graded sandwich plates", *Steel Compos. Struct.*, **24**(5), 569-578. https://doi.org/10.12989/scs.2017.24.5.569.
Menasria, A., Bouhadra, A., Tounsi, A., Bousahla, A.A. and Mahmoud, S.R. (2017), "A new and simple HSDT for thermal stability analysis of FG sandwich plates", *Steel Compos. Struct.*, **25**(2), 157-175. https://doi.org/10.12989/scs.2017.25.2.157.
Mokhtar, Y., Heireche, H., Bousahla, A.A., Houari, M.S.A., Tounsi, A. and Mahmoud, S.R. (2018), "A novel shear deformation theory for buckling analysis of single layer graphene sheet based on nonlocal elasticity theory", *Smart Struct. Syst.*, **21**(4), 397-405. https://doi.org/10.12989/sss.2018.21.4.397.
Nguyen, T.K., Vo, T.P. and Thai, H.T. (2013), "Static and free vibration of axially loaded functionally graded beams based on the first-order shear deformation theory", *Compos. Part B Eng.*, **55**, 147-157. https://doi.org/10.1016/j.compositesb.2013.06.011.
Ould Larbi, L., Hadji, L., Ait Amar Meziane, M. and Adda Bedia, E.A. (2018), "An analytical solution for free vibration of functionally graded beam using a simple first-order shear deformation theory", *Wind Struct.*, **27**(4), 247-254. https://doi.org/10.12989/was.2018.27.4.247.
Pradhan, K.K. and Chakraverty, S. (2013), "Free vibration of Euler and Timoshenko functionally graded beams by Rayleigh-Ritz method", *Compos. Part B: Eng.*, **51**, 175-184. https://doi.org/10.1016/j.compositesb.2013.02.027.
Sayyad, A.S. and Ghugal, Y.M. (2017), "Bending, buckling and free vibration of laminated composite and sandwich beams : A critical review of literature", *Compos. Struct.*, **171**, 486-504. https://doi.org/10.1016/j.compstruct.2017.03.053.
Thai, H.T. and Vo, T.P. (2012), "Bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories", *Int. J. Mech. Sci.*, **62**, 57-66. https://doi.org/10.1016/j.ijmecsci.2012.05.014.
Tounsi, A., Houari, M.S.A., Benyoucef, S. and Adda Bedia, E.A. (2013), "A refined trigonometric shear deformation theory for thermoelastic bending of functionally graded sandwich plates", *Aerosp. Sci. Technol.*, **24**(1), 209-220. https://doi.org/10.1016/j.ast.2011.11.009.
Xu, Y., Yu, T. and Zhou, D. (2014), "Two-dimensional elasticity solution for bending of functionally graded beams with variable thickness", *Meccanica*, **49**, 2479-2489.

https://doi.org/10.1007/s11012-014-9958-1.
Younsi, A., Tounsi, A., Zaoui, F.Z., Bousahla, A.A. and Mahmoud, S.R. (2018), "Novel quasi-3D and 2D shear deformation theories for bending and free vibration analysis of FGM plates", *Geomech. Eng.*, **14**(6), 519-532. https://doi.org/10.12989/gae.2018.14.6.519.
Zouatnia, N. and Hadji, L. (2019), "Effect of the micromechanical models on the bending of FGM beam using a new hyperbolic shear deformation theory", *Earthq. Struct.*, **16**(2), 177-183. https://doi.org/10.12989/eas.2019.16.2.177.
Zouatnia, N., Hadji, L. and Kassoul, A. (2017), "Effect of the micromechanical models on the bending of FGM beam using a new hyperbolic shear deformation theory", *Wind Struct.*, **25**(4), 329-342. https://doi.org/10.12989/eas.2019.16.2.177.

CC