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Fractional wave propagation in radially vibrating non-classical cylinder

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Abstract. This work derives a generalized time fractional differential equation governing wave propagation in a radially vibrating non-classical cylindrical medium. The cylinder is made of a transversely isotropic hyperelastic John's material which obeys frequency-dependent power law attenuation. Employing the definition of the conformable fractional derivative, the solution of the obtained generalized time fractional wave equation is expressed in terms of product of Bessel functions in spatial and temporal variables; and the resulting wave is characterized by the presence of peakons, the appearance of which fade in density as the order of fractional derivative approaches 2. It is obtained that the transversely isotropic structure of the material of the cylinder increases the wave speed and introduces an additional term in the wave equation. Further, it is observed that the law relating the non-zero components of the Cauchy stress tensor in the cylinder under consideration generalizes the hypothesis of plane strain in classical elasticity theory. This study reinforces the view that fractional derivative is suitable for modeling anomalous wave propagation in media.

Keywords: fractional wave; non-classical cylinder; radial vibration

1. Introduction

Cylindrical structures made of transversely isotropic materials find applications in pressure vessels, missiles, space vehicles, oil prospection industry, and other areas of applied geophysics, to name but a few (Akinola 2004, Abd-alla *et al*. 2015). It has been established that any medium having mass and elasticity is capable of vibratory motion. The phenomenon of vibration involves an alternating interchange of potential energy to kinetic energy, and kinetic energy to potential energy. Hence, any vibrating structure must have a component that stores kinetic energy and a component that stores potential energy. The components that store potential and kinetic energies are called elastic and inertial elements respectively (Rao 2007). In some engineering applications, vibrations serve useful purposes such as in vibratory conveyors, washing machine, hoppers, compactors, dentist drill, electric toothbrushes, sieves, electric massaging units, pile drivers, and vibratory testing of material. In many other cases, however, vibration is undesirable. The failure of most mechanical and structural elements and systems can be associated with vibrations. For example, vibration in machine leads to rapid wear of parts such as gears and bearings, loosening of fasteners such as nuts and bolts, poor surface finish during metal cutting, and excessive noise. The blade and disk failures in stream and gas turbines and structural failures in aircraft are usually associated with vibration and the resulting fatigue. In addition, supersonic aircraft create sonic booms that shatter doors and windows; and several spectacular failures of bridges, buildings, and dams are associated with the wind-induced vibration, as well as vibratory ground motion during earthquakes (Rao 2007). The study of vibration analysis of structural members subjected to transient loads such as earthquakes, winds, and traffic loads is important in characterizing their dynamics behavior (Abd-all *et al*. 2015, Honarvar *et al*. 2007, Ponnusamy and Rajagopal, 2010, Daouadji, *et al*. 2016, Benferhat *et al.* 2016, Korabathina and Koppanati 2016). Hadji *et al*. (2017) employed Hamilton's principle to develop higher-order shear deformation beams theories for wave propagation in functionally graded beams, Hadji *et al*. (2016) proposed a new higher-order shear deformation model which accounts for the variation of transverse shear strain for functionally graded beams, Bennoun *et al*. (2016) decomposed transverse displacement into bending, shear, and thickness stretching components; and developed a novel five variable refined plate theory for vibration analysis of functionally graded sandwich plates, while Bourada *et al*. (2016) introduced a new displacement field which incorporates undetermined integral terms and proposed a new four variable refined plate theory for analyzing buckling analysis of isotropic and orthotropic plates. The limitation of the above investigations is that the authors neglect the frequency-dependent attenuation property of the material of the plates and beams; whereas it is well-know that most structural materials obey frequency-dependent power law attenuation (that is, the attenuation coefficient $\alpha(\omega)$ takes the form $\alpha(\omega)=\alpha_0|\omega|^y$, where $0\leq y\leq 2$, α_0, y are the material-specific attenuation parameters, and ω is the angular frequency). For $0<y<2$, the wave propagation in these materials cannot be described by usual integer order wave equations; thus, the use of fractional calculus for modeling anomalous process (Holm and Sinkus 2003, Treeby and Cox 2010). In addition, many integer order differential equations describing physical processes in

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science and engineering have been generalized to fractional order differential equations. The commonly used fractional derivatives in literatures are that of Riemann-Liouville and Caputo (Torres 2013, Yuste 2006, Fu *et al*. 2013, Li 2014, Minardi 1995, Chen *et al*. 2010, Du *et al*. 2010). However, the Caputo and Riemann-Liouville definitions of fractional derivatives are expressed in terms of integrals with singular kernel which limit their applicability. In order to facilitate exact solutions for wider class of fractional differential equations, Khalil *et al*. (2014) gave an elegant and novel definition of fractional derivative involving no integral terms and which seems to be a natural extension of the usual derivative. The present study employs the novel and elegant definition of fractional derivative by Khalil *et al*. (2014) to investigate anomalous wave propagation in a vibrating non-classical cylindrical solid. The research is motivated by the increasing use of non-classical materials in the design of modern structures (Fadodun and Akinola 2017). In this study, we consider anomalous radial vibration of a non-classical cylinder. The cylinder is made of hyperelastic John's material (Akinola 2001, 2004) with the assumption that the material obeys frequency-dependent power law. We derive the generalized time fractional wave equation governing the anomalous radial vibration of the cylinder under consideration. The solution of the obtained wave equation is expressed in terms of product of Bessel functions in spatial and temporal variables. The transversely isotropic structure of the material of the cylinder increases the wave speed and introduces an additional term in the wave equation. The graphical illustration shows that these waves are characterized by the presence of peakons, the appearance of which fade in density as the order of the fractional derivative approaches 2. The rest of the paper is organized as follow: section two details the derivation of generalized time fractional wave equation, section three highlights the solution of the corresponding initial boundary value problem, section four gives the graphical illustration while section five concludes the work.

2. Derivation of time fractional wave equation

Let Ω be a subset of $\Re^3$ occupied by a transversely isotropic solid cylinder of radius *a* obeying frequency dependent power law attenuation. Further, assume the cylindrical body undergoes radial deformation such that the deformation $\vec{\varphi}$ of Ω from a reference configuration Ω_0 onto current configuration Ω takes the form

$$\varphi = \varphi(r,t), \quad \phi = \theta, \qquad Z = z \tag{1}$$

where (r,θ,z) and (φ,ϕ,Z) are the material coordinates in reference and current configurations Ω_0 and Ω respectively, and t is the time.

2.1 Geometry of deformation

The deformation of Ω is given by specifying the position vector $\vec{r}$ of the cylinder particles prior to the deformation in the initial configuration Ω_0 and the position vector $\vec{\varphi}$ (deformation vector) in the current configuration Ω.

Using Eq. (1), the position vectors of every particle of the cylinder in the reference and current configurations Ω_0 and Ω are given respectively as

$$\vec{r} = r\vec{e}_r + z\vec{k}\,, \tag{2}$$

$$\vec{\varphi} = \varphi(r,t)\vec{e}_\varphi + Z\vec{k}\,, \tag{3}$$

where $\vec{e}_r$, $\vec{e}_\varphi$ are the unit vectors along the radial axes of the cylinder in the reference and current configurations respectively, and $\vec{k}$ is the unit vector along the longitudinal axis of the cylinder.

Let the geometry of deformation of Ω from reference configuration Ω_0 onto current configuration Ω be the tensor-gradient (deformation gradient) of position vector $\vec{\varphi}$ in $\Omega(\vec{\varphi})$ taken in the reference configuration $\Omega_0(\vec{r})$. That is, applying the operator of gradient-vector in the reference configuration, $\nabla = \frac{\partial}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\vec{e}_\theta + \frac{\partial}{\partial z}\vec{k}$, on the position vector $\vec{\varphi}$ in the current configuration Ω.

Now, invoking the concept of differentiation in the orthogonal curvilinear coordinates give

$$\vec{R}_r = \frac{\partial\vec{\varphi}}{\partial r}, \quad \vec{R}_\theta = \frac{\partial\vec{\varphi}}{\partial\theta}, \quad \vec{R}_z = \frac{\partial\vec{\varphi}}{\partial z}, \tag{4}$$

$$\vec{J}_r = \frac{\partial\vec{r}}{\partial r}, \quad \vec{J}_\theta = \frac{\partial\vec{r}}{\partial\theta}, \quad \vec{J}_z = \frac{\partial\vec{r}}{\partial r}, \tag{5}$$

$$\vec{J}^r = \frac{\vec{J}_\theta \times \vec{J}_z}{g}, \quad \vec{J}^\theta = \frac{\vec{J}_z \times \vec{J}_\theta}{g}, \quad \vec{J}^z = \frac{\vec{J}_r \times \vec{J}_\theta}{g} \tag{6}$$

where $\vec{R}_r$, $\vec{R}_\theta$, $\vec{R}_z$ are the covariant base vectors in the current configuration Ω, $\vec{J}_r$, $\vec{J}_z$, $\vec{J}_\theta$, are the covariant base vectors in the reference configuration Ω_0, $\vec{J}^r$, $\vec{J}^\theta$, $\vec{J}^z$, are the associated contravariant base vectors in Ω_0, $\times$ and $\cdot$ are the usual vector and scalr products respectively, and $g = \vec{J}_r \cdot (\vec{J}_\theta \times \vec{J}_z)$.

Substituting Eq. (2) and (3) into Eqs. (4) and (5) give

$$\vec{R}_r = \frac{\partial\varphi(r,t)}{\partial r}\vec{e}_r, \quad \vec{R}_\theta = \frac{\varphi(r,t)}{r}\vec{e}_\theta, \quad \vec{R}_z = \vec{k}\,, \tag{7}$$

$$\vec{J}_r = \vec{e}_r\,, \quad \vec{J}_\theta = r\vec{e}_\theta, \quad \vec{J}_z = \vec{k}\,, \tag{8}$$

where $(\vec{e}_r, \vec{e}_\theta, \vec{k})$ are the unit vectors associated with material coordinates (r,θ,z) in Ω_0.

Substituting Eq. (8) into Eq. (6) gives the contravariant base vectors:

$$\vec{J}^r = \vec{e}_r\,, \quad \vec{J}^\theta = \frac{\vec{e}_\theta}{r}\,, \quad \vec{J}^z = \vec{k}\,. \tag{9}$$

Then, the tensor-gradient $\nabla\vec{\varphi}$ is defined as

$$\nabla\vec{\varphi} = \vec{J}^m \otimes \vec{R}_m, \quad m = r, \theta, z \tag{10}$$

$$\nabla\vec{\varphi} = \vec{J}^r \otimes \vec{R}_r + \vec{J}^\theta \otimes \vec{R}_\theta + \vec{J}^z \otimes \vec{R}_z, \tag{11}$$

where $\otimes$ is the usual tensor product.

Substituting Eqs. (7) and (9) into Eq. (11) gives

$$\nabla\vec{\varphi} = \frac{\partial\varphi(r,t)}{\partial r}\vec{e}_r \otimes \vec{e}_r + \frac{\varphi(r,t)}{r}\vec{e}_\theta \otimes \vec{e}_\theta + \vec{k} \otimes \vec{k} \cdot \tag{12}$$

The matrix form of Eq. (12) is

$$\nabla\vec{\varphi} = \begin{pmatrix} \frac{\partial\varphi(r,t)}{\partial r} & 0 & 0 \\ 0 & \frac{\varphi(r,t)}{r} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{13}$$

Polar decomposition theorem: This states that for a real invertible matrix $\nabla\vec{\varphi}$, there exist a unique, symmetric, positive definite stretch tensor $\tilde{U}$ and an orthogonal rotation tensor $\tilde{O}^D$ such that

$$\nabla\vec{\varphi} = \tilde{U} \cdot \tilde{O}^D. \tag{14}$$

Using Eq. (14), the left symmetric tensor $\tilde{U}$ satisfies

$$\tilde{U}^2 = \nabla\vec{\varphi} \cdot \nabla\vec{\varphi}^T, \tag{15}$$

where $\nabla\vec{\varphi}^T$ is the transpose of $\nabla\vec{\varphi}$ and $\cdot$ is the usual scalar product.

Using Eqs. (14) and (15), the orthogonal rotation tensor $\tilde{O}^D$ is obtained from the relation

$$\tilde{O}^D = \tilde{U}^{-1} \cdot \nabla\vec{\varphi}. \tag{16}$$

The form of deformation gradient $\nabla\vec{\varphi}$ in Eq. (12)/(13) suggests that $\nabla\vec{\varphi} = \nabla\vec{\varphi}^T$. Reflecting this condition in Eq. (15) gives

$$\tilde{U}^2 = \nabla\vec{\varphi} \cdot \nabla\vec{\varphi}^T = \nabla\vec{\varphi} \cdot \nabla\vec{\varphi} = \nabla\vec{\varphi}^2, \tag{17}$$

$$\tilde{U} = \nabla\vec{\varphi}. \tag{18}$$

Substituting Eq. (18) into Eq. (16) gives

$$\tilde{O}^D = \tilde{E}, \tag{19}$$

where $\tilde{E}$ is a unit matrix. That is

$$\tilde{O}^D = \tilde{E} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

2.2 Energy function for a non-classical material

The energy function for an isotropic elastic material in small deformation theory (classical elasticity) is given by Cauchy (Fadodun 2014, Amenzade 1979) as

$$W = \mu S_1(\tilde{\varepsilon})^2 + \frac{1}{2}\lambda S_1^2(\tilde{\varepsilon}), \tag{21}$$

where μ, λ are the Lame's constants, $\tilde{\varepsilon}$ is the small strain tensor, and $S_1(\tilde{\varepsilon})$ is the trace of the small strain tensor $\tilde{\varepsilon}$.

On the basis of Eq. (21), John proposed an energy function for an isotropic hyperelastic semi-linear material (non-classical material) in finite deformation (Fadodun and Akinola 2017)

$$W = \mu S_1(\tilde{U} - \tilde{E})^2 + \frac{1}{2}\lambda S_1^2(\tilde{U} - \tilde{E}), \tag{22}$$

where μ, λ are the Lame's constants, $S_1(\tilde{U} - \tilde{E})$ is the trace of second rank tensor $(\tilde{U} - \tilde{E})$, $\tilde{E}$ is the unit second rank tensor, $\tilde{U}$ is the left stretch symmetric second rank tensor such that $\nabla\vec{\varphi} = \tilde{U} \cdot \tilde{O}^D$ and $\tilde{U}^2 = \nabla\vec{\varphi} \cdot \nabla\vec{\varphi}^T$.

Using Eq. (22), Akinola (1999) constructed an energy function for a transversely isotropic hyperelastic semi-linear material (non-classical material) in finite deformation (Akinola 2001, Fadodun 2014) as

$$W = \lambda_2 S_1(\tilde{U} - \tilde{E})^2 + \frac{1}{2}\lambda_1 S_1^2(\tilde{U} - \tilde{E}) + \lambda_0 S_0(\tilde{U} - \tilde{E}) \tag{23}$$

where $S_0 = \vec{c} \cdot \tilde{U}^2 \cdot \vec{c}$ is an additional invariant of deformation due to anisotropy of the medium, $\vec{c}$ is a unit vector characterizing the direction of anisotropy, and λ_0, λ_1, λ_2 are the material effective characteristics (constants). These parameters are defined by

$$\lambda_1 = \langle\lambda\rangle + \frac{\left\langle \lambda/(\lambda+2\mu) \right\rangle}{\left\langle 1/(\lambda+2\mu) \right\rangle} - \left\langle \frac{\lambda^2}{\lambda+2\mu} \right\rangle \tag{24a}$$

$$\lambda_2 = \langle\mu\rangle, \quad \lambda_3 = \frac{1}{\left\langle 1/\mu \right\rangle}, \tag{24b}$$

$$\lambda_0 = \lambda_0(\lambda_2, \lambda_3) = 2(\lambda_3 - \lambda_2), \tag{24c}$$

where for any function $\eta(r,t) \in \Omega \times [0,T)$, $\langle\eta(r,t)\rangle$ denotes its geometric average over Ω with volume $|\Omega|$:

$$\langle\eta(r,t)\rangle = \frac{1}{|\Omega|}\int_\Omega \eta(r,t)d\Omega \cdot$$

It is noted that in the case of degeneracy into isotropic; that is, when the effective moduli take the form

$$\lambda_3 = \lambda_2 = \mu, \ \lambda_1 = \lambda, \text{ and } \lambda_0 = 0, \tag{25}$$

the energy function in Eq. (23) automatically reduces to energy function in Eq. (22).

2.3 Constitutive law and time fractional wave equation

Let $\tilde{P}$ denote the first Piola-Kirchhoff's stress tensor which is an energy conjugate to the deformation gradient (geometry of deformation) $\nabla\vec{\varphi}$, then the Frechet derivative of the energy function in Eq. (23) with respect to the geometry of deformation $\nabla\vec{\varphi}$ gives $\tilde{P}$.

$$\tilde{P} = \frac{\partial W}{\partial \nabla\vec{\varphi}}, \tag{26}$$

$$\tilde{P} = \lambda_2 \frac{\partial}{\partial\nabla\vec{\varphi}} S_1(\tilde{U}-\tilde{E})^2 + \frac{1}{2}\lambda_1 \frac{\partial}{\partial\nabla\vec{\varphi}} S_1^2(\tilde{U}-\tilde{E}) + \lambda_0 \frac{\partial}{\partial\nabla\vec{\varphi}}\left(\vec{c}\cdot\tilde{U}\cdot\vec{c}\right). \tag{27}$$

The Frechet derivatives of terms in Eq. (27) are:

$$\frac{\partial}{\partial\nabla\vec{\varphi}} S_1(\tilde{U}-\tilde{E}) = \frac{\partial S_1(\tilde{U}-\tilde{E})}{\partial\left(\tilde{U}-\tilde{E}\right)}\cdot\frac{\partial}{\partial\nabla\vec{\varphi}}\left(\tilde{U}-\tilde{E}\right),$$

$$\frac{\partial}{\partial\nabla\vec{\varphi}} S_1(\tilde{U}-\tilde{E}) = \tilde{E}\cdot\frac{\partial}{\partial\nabla\vec{\varphi}}\tilde{U} = \tilde{E}\cdot\tilde{O}^D = \tilde{O}^D, \tag{28}$$

$$\frac{\partial}{\partial\nabla\vec{\varphi}} S_1(\tilde{U}-\tilde{E})^2 = \frac{\partial S_1(\tilde{U}-\tilde{E})^2}{\partial\left(\tilde{U}-\tilde{E}\right)}\cdot\frac{\partial}{\partial\nabla\vec{\varphi}}(\tilde{U}-\tilde{E}),$$

$$\frac{\partial S_1(\tilde{U}-\tilde{E})^2}{\partial\nabla\vec{\varphi}} = 2\left(\tilde{U}-\tilde{E}\right)\cdot\tilde{O}^D, \quad = \left(\nabla\vec{\varphi}-\tilde{O}^D\right), \tag{29}$$

$$\frac{\partial S_1^2\left(\tilde{U}-\tilde{E}\right)}{\partial\nabla\vec{\varphi}} = \frac{\partial S_1^2\left(\tilde{U}-\tilde{E}\right)}{\partial S_1\left(\tilde{U}-\tilde{E}\right)}\frac{\partial S_1\left(\tilde{U}-\tilde{E}\right)}{\partial\nabla\vec{\varphi}},$$
$$= 2S_1\left(\tilde{U}-\tilde{E}\right)\tilde{O}^D \tag{30}$$

and

$$\frac{\partial\left(\vec{c}\cdot\tilde{U}^2\cdot\vec{c}\right)}{\partial\nabla\vec{\varphi}} = \vec{c}\vec{c}\cdot\frac{\partial}{\partial\tilde{U}}\tilde{U}^2\cdot\frac{\partial}{\partial\nabla\vec{\varphi}}\tilde{U},$$
$$= 2\vec{c}\vec{c}\cdot\tilde{U}\cdot\tilde{O}^D = 2\vec{c}\vec{c}\cdot\nabla\vec{\varphi}, \tag{31}$$

respectively, where $\vec{c}\vec{c} = \vec{c}\otimes\vec{c}$.

Substituting Eqs. (29)-(31) into Eq. (27) gives the constitutive law for the material of the cylinder.

$$\tilde{P} = 2\lambda_2\nabla\vec{\varphi} + (\lambda_1 S_1(\tilde{U}-\tilde{E}) - 2\lambda_2)\tilde{O}^D + 2\vec{c}\vec{c}\cdot\nabla\vec{\varphi}. \tag{32}$$

In order to derive the time fractional wave equation, we use the fractional generalized motion equation

$$\nabla\cdot\tilde{P} = \frac{1}{\tau^{2-\alpha}}\rho\frac{\partial^\alpha}{\partial t^\alpha}\vec{\varphi}, \quad 1<\alpha\le 2 \tag{33}$$

where ρ is the mass density of the body, α is an arbitrary real number, and t is the time. It must be noted that the quantity τ with dimension of time is introduced to ensure all terms in Eq. (33) have the same dimensions.

In view of the form of deformation function $\vec{\varphi}$ in Eq. (3), the components form of generalized fractional motion equation in Eq. (33) are

$$\frac{\partial P_{rr}}{\partial r} + \frac{1}{r}\left(\frac{\partial P_{r\theta}}{\partial\theta} + P_{rr} - P_{\theta\theta}\right) + \frac{\partial P_{rz}}{\partial z} = \frac{\rho}{\tau^{2-\alpha}}\frac{\partial^\alpha\varphi(r,t)}{\partial t^\alpha} \tag{34}$$

$$\frac{\partial P_{\theta r}}{\partial r} + \frac{1}{r}\left(\frac{\partial P_{\theta\theta}}{\partial\theta} + P_{r\theta} + P_{r\theta}\right) + \frac{\partial P_{\theta z}}{\partial z} = 0, \tag{35}$$

$$\frac{\partial P_{zr}}{\partial r} + \frac{1}{r}\left(\frac{\partial P_{z\theta}}{\partial\theta} + P_{z\theta}\right) + \frac{\partial P_{zz}}{\partial z} = 0, \tag{36}$$

where P_{mn}, $m,n=r,\theta,z$ are the components of the second rank Piola-Kirchhoff's stress tensor $\tilde{P}$.

Substituting Eqs. (12)/(13), (18), and (20) into Eq. (32) give the components P_{mn}

$$P_{rr} = 2\lambda_2\frac{\partial\varphi}{\partial r} + \lambda_1\left(\frac{\partial\varphi}{\partial r} + \frac{\varphi}{r} - 2\right) - 2\lambda_2 + 2\lambda_0\frac{\partial\varphi}{\partial r}$$
$$= \left(2\lambda_0 + \lambda_1 + 2\lambda_2\right)\frac{\partial\varphi}{\partial r} + \lambda_1\frac{\varphi}{r} - 2\left(\lambda_1 + \lambda_2\right), \tag{37}$$

$$P_{\theta\theta} = 2\lambda_2\frac{\varphi}{r} + \lambda_1\left(\frac{\partial\varphi}{\partial r} + \frac{\varphi}{r} - 2\right) - 2\lambda_2,$$
$$= \lambda_1\frac{\partial\varphi}{\partial r} + \left(2\lambda_2 + \lambda_1\right)\frac{\varphi}{r} - 2\left(\lambda_1 + \lambda_2\right), \tag{38}$$

$$P_{zz} = \lambda_1\left(\frac{\partial\varphi}{\partial r} + \frac{\varphi}{r} - 2\right), \tag{39}$$

$$P_{r\theta} = P_{\theta r} = P_{rz} = P_{zr} = P_{\theta z} = P_{z\theta} = 0, \tag{40}$$

where the function $\varphi=\varphi(r,t)$.

Substituting Eqs. (37)-(40) into Eqs. (34)-(36) give the generalized fractional wave equation

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\varphi)\right) = \frac{1}{v^2}\frac{1}{\tau^{2-\alpha}}\frac{\partial^\alpha}{\partial t^\alpha}\varphi - \kappa\frac{\varphi}{r^2}, \tag{41}$$

where $v = \sqrt{\dfrac{2\lambda_2 + \lambda_1 + 2\lambda_0}{\rho}}$ is the speed of the wave, $\kappa = \dfrac{2\lambda_0}{2\lambda_2 + \lambda_1 + 2\lambda_0} < 1$ is a dimensionless number, and $1<\alpha\le 2$.

Remark 1: Eq. (41) is the generalized time fractional wave equation governing the dynamic behavior of the non-classical cylinder obeying frequency-dependent power law attenuation.

Remark 2: In the case of degeneracy to isotropy, $\lambda_0=0$, $\lambda_1=\lambda$, $\lambda_2=\mu$, the generalized fractional wave equation in Eq. (41) reduces to

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\varphi)\right) = \frac{1}{v_0^2}\frac{1}{\tau^{2-\alpha}}\frac{\partial^\alpha}{\partial t^\alpha}\varphi, \tag{42}$$

where the corresponding wave speed v_0 also reduces to $v_0 = \sqrt{\dfrac{2\mu + \lambda}{\rho}}$. This implies that the transversal isotropic structure of the material of the cylinder increases the speed of the wave and introduces an additional term (the second term on the right-hand side of Eq. (41)) in the wave equation.

It is observed that, setting α=2, the generalized time fractional wave equation in Eq. (42) further reduces to the classical wave equation in a cylindrical body

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\varphi)\right)=\frac{1}{v_0^2}\frac{\partial^2}{\partial t^2}\varphi\cdot \tag{43}$$

2.4 Cauchy stress in the non-classical cylinder

The Cauchy stress tensor $\tilde{\sigma}$ in terms of first Piola-Kirchhoff's stress tensor $\tilde{P}$ (Fadodun 2014) is

$$\tilde{\sigma}=\frac{d\Omega_0}{d\Omega}\nabla\vec{\varphi}\cdot\tilde{P}, \tag{44}$$

where $d\Omega_0$ is the elemental volume element in the reference configuration Ω_0 and $d\Omega$ is corresponding elemental volume element in the current configuration Ω.

Using Eqs (7) and (8), the elemental volumes $d\Omega_0$, $d\Omega$ are:

$$d\Omega_0=\vec{J}_r\cdot(\vec{J}_\theta\times\vec{J}_z)=r, \tag{45}$$

and

$$d\Omega=\vec{R}_r\cdot(\vec{R}_\theta\times\vec{R}_z)=\varphi(r,t)\frac{\partial\varphi(r,t)}{\partial r}, \tag{46}$$

respectively.

Using Eqs. (12), (13), (32), (45), and (46) in Eq. (44) gives the symmetric Cauchy stress tensor $\tilde{\sigma}$

$$\tilde{\sigma}=\begin{pmatrix}\sigma_{rr} & \sigma_{r\theta} & \sigma_{rz}\\ \sigma_{r\theta} & \sigma_{\theta\theta} & \sigma_{\theta z}\\ \sigma_{rz} & \sigma_{\theta z} & \sigma_{zz}\end{pmatrix}, \tag{47}$$

where the components σ_{mn}, $m,n=r,\theta,z$ are:

$$\sigma_{rr}=\frac{r}{\varphi(r,t)}P_{rr},\quad \sigma_{\theta\theta}=\left(\frac{\partial\varphi(r,t)}{\partial r}\right)^{-1}P_{\theta\theta}, \tag{48}$$

$$\sigma_{zz}=\frac{r}{\varphi(r,t)}\left(\frac{\partial\varphi(r,t)}{\partial r}\right)^{-1}P_{zz}, \tag{49}$$

$$\sigma_{r\theta}=\sigma_{rz}=\sigma_{\theta z}=0, \tag{50}$$

where P_{rr}, $P_{\theta\theta}$, and P_{zz} are the non-zero components (in Eqs. (37)-(39)) of the first Piola-Kirchhoff's stress tensor $\tilde{P}$.

In view of Eqs. (37)-(39) and Eqs. (48)-(49), the non-zero components σ_{rr}, $\sigma_{\theta\theta}$, and σ_{zz} of Cauchy stress tensor $\tilde{\sigma}$ are related by

$$\sigma_{zz}=\frac{\lambda_1}{2(\lambda_1+\lambda_2)}\left(\left(\frac{\partial\varphi}{\partial r}\right)^{-1}\sigma_{rr}+\frac{r}{\varphi}\left(\sigma_{\theta\theta}-2\lambda_0\right)\right), \tag{51}$$

where $\varphi=\varphi(r,t)$.

In the case of an isotropic non-classical cylinder ($\lambda_0=0$, $\lambda_1=\lambda$, and $\lambda_2=\mu$), the above equation takes the form

$$\sigma_{zz}=\frac{\lambda}{2(\lambda+\mu)}\left(\left(\frac{\partial\varphi}{\partial r}\right)^{-1}\sigma_{rr}+\frac{r}{\varphi}\sigma_{\theta\theta}\right)\cdot \tag{52}$$

Meanwhile, in the case of an isotropic cylinder in classical elasticity, the non-zero components of the Cauchy stress tensor are related by (Amenzade 1979)

$$\sigma_{zz}=\frac{\lambda}{2(\lambda+\mu)}\left(\sigma_{rr}+\sigma_{\theta\theta}\right)\cdot \tag{53}$$

Remark3: If the deformation function $\varphi=\varphi(r,t)=r$ and the material constant $\lambda_0=0$, then, Eq. (51) reduces to Eq. (53). This implies that the law relating the non-zero components of Cauchy stress tensor in non-classical cylinder under consideration (Eq. (51)) generalizes the hypothesis of plane strain in the infinitesimal deformation theory (classical elasticity).

3. Conformable fractional derivative approach

In this section, we highlight a new and an elegant definition of fractional derivative (conformable fractional derivative) proposed by Khalil *et al*. (2014). This definition coincides with the known fractional derivatives such as Riemann-Liouville and Caputo definitions on polynomials. Furthermore, unlike the previous definitions in literature, it enables researchers to study basic analysis theorems like the Rolle's theorem and the mean value theorem.

Definition: Let $\alpha\in(n,n+1]$, and f be an n-differentiable function at t, where $t>0$. Then, the conformable fractional derivative of function f of order α is defined as (Khalil *et al.* 2014)

$$\frac{d^\alpha}{dt^\alpha}f(t)=\lim_{\varepsilon\to 0}\frac{f^{(\lceil\alpha\rceil-1)}(t+\varepsilon t^{(\lceil\alpha\rceil-\alpha)})-f^{(\lceil\alpha\rceil-1)}}{\varepsilon}, \tag{54}$$

where $[\alpha]$ is the smallest integer greater than or equal to α. As a consequence of the above definition in Eq. (54), one notes that

$$\frac{d^\alpha}{dt^\alpha}f(t)=t^{(\lceil\alpha\rceil-\alpha)}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}f(t), \tag{55}$$

where $\alpha\in(n,n+1)$, and f is $(n+1)$-differentiable at $t>0$.

Also, if f is α-differentiable in $(0,\alpha)$, $\alpha>0$, and $\lim_{t\to 0^+}\frac{d^\alpha}{dt^\alpha}f(t)$ exists, then (Khalil *et al.* 2014)

$$\frac{d^\alpha}{dt^\alpha}f(0)=\lim_{t\to 0^+}\frac{d^{\lceil\alpha\rceil}}{dt^{\lceil\alpha\rceil}}f(t)\cdot \tag{56}$$

3.1 Initial boundary value problem

In view of the generalized wave equation in Eq. (41), the conformable fractional derivative definition in Eq. (54), and the subsequent consequence relation in Eq. (55), we proceed to solve the initial boundary value problem modeling the problem under consideration

$$\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\varphi\right)=\frac{1}{v^2}\frac{1}{\tau^{2-\alpha}}\frac{\partial^\alpha}{\partial t^\alpha}\varphi-\kappa\frac{\varphi}{r^2}, \tag{57}$$

$$\varphi(r,t)=0 \text{ and } \lim_{r\to 0^+}\varphi(r,t)=\mathit{finite},\qquad r=a \tag{58}$$

$$\varphi(r,t\to 0^+)=0,\quad \frac{\partial\varphi(r,0)}{\partial t}=\lim_{t\to 0^+}\frac{\partial\varphi(r,t)}{\partial t}. \tag{59}$$

3.2 Conformable fractional wave solution

Let the field variable $\varphi(r,t)$ assume the form

$$\varphi(r,t)=\psi(r)\xi(t), \tag{60}$$

where $\psi(r)$ is a function of radius r only and $\xi(t)$ is similarly a function of time t only.

Substituting Eq. (60) into Eq. (57) leads to two ordinary differential equations

$$r^2\frac{d^2}{dr^2}\psi(r)+r\frac{d}{dr}\psi(r)+((\chi_n r)^2-m^2)\psi(r)=0, \tag{61}$$

$$\frac{d^2}{dt^2}\xi(t)+\chi_n^2 v^2\left(\frac{t}{\tau}\right)^{\alpha-2}\xi(t)=0, \tag{62}$$

where m and χ_n are positive real numbers, and that $m=\sqrt{1-\kappa}=\sqrt{\frac{2\lambda_2+\lambda_1}{2\lambda_2+\lambda_1+2\lambda_2}}<1$.

The above equation in Eq. (61) is the known Bessel equation of order m; its solution $\psi(r)$ takes the form

$$\psi(r)=d_1J_m(\chi_n r)+d_2Y_m(\chi_n r), \tag{63}$$

where $J_m(\chi_n r)$ and $Y_m(\chi_n r)$ are Bessel functions of order m of the first and second kind respectively, and d_1, d_2 are arbitrary constants.

Using the conditions $\lim_{r\to 0^+}\varphi(r,t)=\mathit{finite}$ and $\varphi(a,t)=0$, (where a is the radius of the cylinder) yield $d_2=0$, $d_1\neq 0$, and the n^{th} positive zeros χ_n of $J_m(a\chi_n)$.

Therefore, without loss of generality, the solution $\psi(r)$ in Eq. (63) reduces to

$$\psi(r)=J_m(\chi_n r), \tag{64}$$

where χ_n are the n^{th} positive zeros of $J_m(a\chi_n)$.

In the special case (τ=1), the solution of Eq. (62) together with the conditions Eq. (59) is

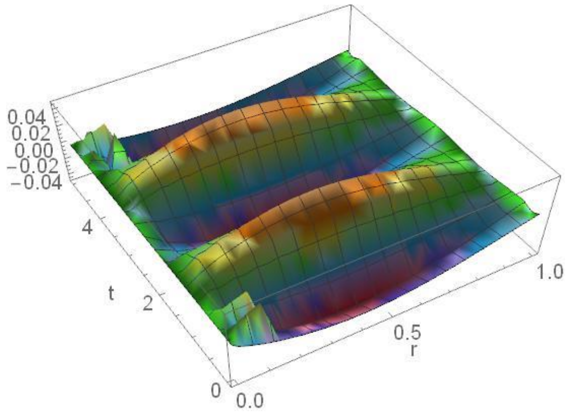
$$\xi(t)=\sqrt{t}\left(J_{\frac{1}{\alpha}}\left(\frac{2v\chi_n}{\alpha}t^{\frac{\alpha}{2}}\right)+Y_{\frac{1}{\alpha}}\left(\frac{2v\chi_n}{\alpha}t^{\frac{\alpha}{2}}\right)\right). \tag{65}$$

Substituting Eqs. (64) and (65) into Eq. (60) gives

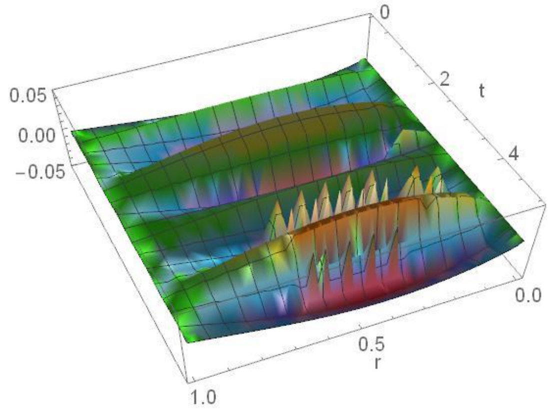
$$\varphi(r,t)=\sqrt{t}J_m(\chi_n r)\left(J_{\frac{1}{\alpha}}\left(\frac{2v\chi_n}{\alpha}t^{\frac{\alpha}{2}}\right)+Y_{\frac{1}{\alpha}}\left(\frac{2v\chi_n}{\alpha}t^{\frac{\alpha}{2}}\right)\right) \tag{66}$$

4. Graphical illustration

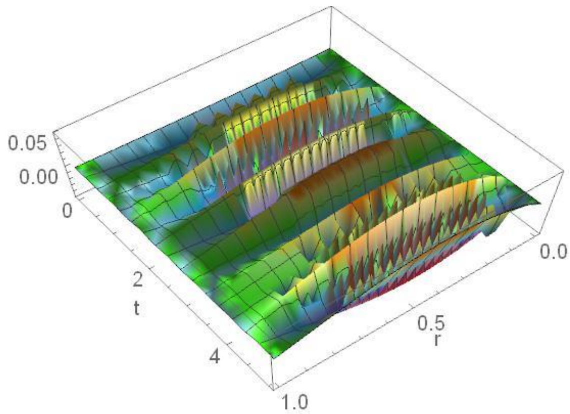
For purpose of graphical illustration, we set the effective material constants $\lambda_0=2.0\times10^6$ N/m^2, $\lambda_1=3\times10^6$ N/m^2, $\lambda_2=3.2\times10^6$ N/m^2, and radius a=1.0 m. It is observed that in amenable elastic materials, the diffusive-wave are characterized by the presence of peakons (characterizing the presence of non-smooth/non-differentiable waves) the appearance of which fade in density as the order of the time fractional derivative approaches 2.



Wave patterns when order of derivative (α) is 2



Wave patterns when order of derivative (α) is 1.7



Wave patterns when order of derivative (α) is 1.1

5. Conclusions

This research presents the derivation of a generalized time fractional wave equation governing radially vibrating non-classical cylindrical medium which exhibits frequency-dependent anomalous behavior. In the special case of parameter τ=1, the solution of the obtained wave equation is expressed in terms of product of Bessel functions. In addition, the law relating the non-zero components of Cauchy stress tensor in the considered non-classical cylinder generalizes the hypothesis of plane strain in infinitesimal deformation theory of elasticity. Finally, the results in this work can be employed in the analysis of continuous-random work, anomalous wave propagation during earthquakes, super-diffusive systems, and diffusive-wave propagation.

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