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High conservative nonlinear vibration equations by means of energy balance method

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Abstract. This paper presents He's Energy Balance Method (EBM) for solving nonlinear oscillatory differential equations. Three strong nonlinear cases have been studied analytically. Analytical results of the EBM are compared with numerical solutions using Runge-Kutta's algorithm. The effects of different important parameters on the nonlinear response of the systems are studied. The results show the presented method is potentially to solve high nonlinear vibration equations.

Keywords: Energy Balance Method (EBM); Runge- Kutta's Method (RKM); nonlinear vibrations

1. Introduction

One of the most important problems in nonlinear vibrations is to have an exact solution for them. To have better understanding of the effects of the different parameters on the response of the systems, it's better to try to have analytical response of them. The traditional analytical method has lots of shortcoming such as perturbation method. Recently, some approaches have been developed such as: Homotopy perturbation method (Bayat *et al.* 2013a, 2014a),Hamiltonian approach (He 2010, Xu 2010, Bayat *et al*. 2014b, c, d, e, f, 2013b, Bayat and Pakar 2013c), Energy balance method (He 2002, Jamshidi *et al*. 2010, Bayat *et al*. 2014g, Mehdipour 2010), Variational iteration method (Dehghan 2008), Amplitude frequency formulation (He 2008), Max-Min approach (Shen *et al*. 2009, Zeng *et al*. 2009), Variational approach (He 2007, Bayat and Pakar 2012a, Bayat *et al*. 2012b, Bayat and Pakar 2013a, Bayat *et al*. 2013b, Shahidi *et al*. 2011, Pakar and Bayat 2013a, b), and the other analytical and numerical (Bayat and Abdollahzade 2011a, b, Pakar *et al*. 2014a, b, 2011, Xu 2009, Bor-Lih *et al*. 2009, Wu 2011, Odibat *et al*. 2008, Zhifeng *et al*. 2013, Rajasekaran 2013, Akgoz and Civalek 2011, Atmane *et al*. 2011, Cunedioglu and Beylergil 2014, Radomirovic *et al*. 2015, Filobello-Nino *et al.* 2015, Xu *et al*. 2015, Filippov 1999, Evakin 2001, Grigolyuk 1987, Han 1965, Andrianov 2004, Cveticanin 2015, 2012). Among of the mentioned papers and approaches, the Energy Balance Method (EBM) is considered to solve the nonlinear vibration equations in this paper. The paper has been collocated as follows:

In section 2, the basic idea of the He's Energy Balance Method (EBM) is presented in detail.

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The basic idea of the Runge- Kutta algorithm is in section 3. Section 4 is the application of the energy balance method in three different cases for high nonlinear vibratory systems. The validation of the approach and also the discussion on the nonlinear parameters of the systems and the comparison with the numerical results are studied in section 5. Finally, it has been demonstrated that the energy balance method can be a precise periodic solution for nonlinear vibration equations.

2. Basic idea of Energy Balance Method (EBM)

In the present paper, we consider a general nonlinear oscillator in the Form (He 2002)

$$\ddot{u}+f\,(u(t))=0 \tag{1}$$

In which u and t are generalized dimensionless displacement and time variables, respectively. Its variational principle can be easily obtained

$$J(u)=\int_0^t(-\frac{1}{2}\dot{u}^2+F(u))\,dt \tag{2}$$

Where T=2π/ω is period of the nonlinear oscillator, $F(u)=\int f\,(u)du$.

Its Hamiltonian, therefore, can be written in the form

$$H=\frac{1}{2}\dot{u}^2+F(u)+F(A) \tag{3}$$

Or

$$R(t)=-\frac{1}{2}\dot{u}^2+F(u)-F(A)=0 \tag{4}$$

Oscillatory systems contain two important physical parameters, i.e.,

The frequency ω and the amplitude of oscillation. A. So let us consider such initial conditions

$$u(0)=A, \qquad \dot{u}(0)=0 \tag{5}$$

We use the following trial function to determine the angular frequency ω

$$u(t)=A\cos(\omega t) \tag{6}$$

Substituting Eq. (6) into u term of (4), yield

$$R(t)=\frac{1}{2}\omega^2A^2\sin^2(\omega t)+F\left(A\cos(\omega t)\right)-F\left(A\right)=0 \tag{7}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (6) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at ωt=π/4 gives

$$\omega=\sqrt{\frac{2\left(F\left(A\right)\right)-F\left(A\cos(\omega t)\right)}{A^2\sin^2(\omega t)}} \tag{8}$$

3. Basic idea of Runge-Kutta's Method (RKM)

The Runge-Kutta method is an important iterative method for the approximation solutions of ordinary differential equations. These methods were developed by the German mathematician Runge and Kutta around 1900. For simplicity, we explain one of the important methods of Runge-Kutta methods, called forth-order Runge-Kutta method.

Consider an initial value problem be specified as follows

$$\dot{u} = f\left(t, u\right), \quad u\left(t_0\right) = u_0 \tag{9}$$

u is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} u_{n+1} &= u_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{10}$$

for n=0, 1, 2, 3,..., using

$$\begin{aligned} k_1 &= f\left(t_n, u_n\right), \\ k_2 &= f\left(t_n + \frac{1}{2}h,\, u_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \frac{1}{2}h,\, u_n + \frac{1}{2}hk_3\right), \\ k_4 &= f\left(t_n + h,\, u_n + hk_3\right). \end{aligned} \tag{11}$$

Where u_{n+1} is the RK4 approximation of $u(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.

4. Applications of energy balance method

In order to achieve the accuracy and the applicability of Energy Balance Method for solving nonlinear vibration equations, we will consider the following examples.

4.1 Example 1

For the first example we considered governing equation of the oscillation as follow

$$\left(\frac{1}{\xi} - \cos\left(u\right)\right)\left(2\ddot{u}\right) + \sin\left(u\right)\dot{u}^2 + g\,\xi\sin\left(u\right) = 0 \tag{12}$$

With the boundary condition of

$$u\left(0\right) = A, \quad \dot{u}\left(0\right) = 0. \tag{13}$$

Variational of the Eq. (12) can be readily obtained as

$$J(u)=\int_0^t\left(\frac{1}{\xi}\dot{u}^2-\dot{u}^2\cos(u)-g\,\xi\cos(u)\right)dt\,. \tag{14}$$

Its Hamiltonian, therefore, can be written in the form

$$H=\left(\frac{1}{\xi}\dot{u}^2-\dot{u}^2\cos(u)-g\,\xi\cos(u)\right) \tag{15}$$

And

$$H_{t=0}=-g\,\xi\cos(A), \tag{16}$$

$$H_t-H_{t=0}=\left(\frac{1}{\xi}\dot{u}^2-\dot{u}^2\cos(u)-g\,\xi\cos(u)\right)-\left(-g\,\xi\cos(A)\right), \tag{17}$$

We will use the trial function to determine the angular frequency ω, i.e.

$$u(t)=A\cos(\omega t) \tag{18}$$

If we substitute (18) into (17), it results the following residual equation

$$\begin{aligned}&\left(\frac{1}{\lambda}\left(A^2\omega^2\sin^2(\omega t)\right)-\left(\left(A^2\omega^2\sin^2(\omega t)\right)\cos\left(A\cos(\omega t)\right)\right)-g\,\lambda\cos\left(A\cos(\omega t)\right)\right)\\&-\left(-g\,\lambda\cos(A)\right)=0\end{aligned} \tag{19}$$

If we collocate at $\omega t=\pi/4$ we obtain

$$\frac{1}{2}\frac{A^2\omega^2}{\xi}-\frac{1}{2}A^2\omega^2\cos\left(\frac{\sqrt{2}}{2}A\right)-g\,\lambda\cos\left(\frac{\sqrt{2}}{2}A\right)+g\,\xi\cos(A)=0 \tag{20}$$

This leads to the following result

$$\omega=\frac{\sqrt{\left(2-2\cos\left(\frac{\sqrt{2}}{2}A\right)\xi\right)g\left(\cos\left(\frac{\sqrt{2}}{2}A\right)-\cos(A)^2\right)\xi}}{\left(1-\xi\cos\left(\frac{\sqrt{2}}{2}A\right)\right)A} \tag{21}$$

According to Eqs. (18) and (21), we can obtain the following approximate solution

$$u(t)=A\cos\left(\frac{\sqrt{\left(2-2\cos\left(\frac{\sqrt{2}}{2}A\right)\xi\right)g\left(\cos\left(\frac{\sqrt{2}}{2}A\right)-\cos(A)^2\right)\xi}}{\left(1-\xi\cos\left(\frac{\sqrt{2}}{2}A\right)\right)A}t\right) \tag{22}$$

4.2 Example 2

For the second example the following governing equation of the oscillation is considered (Nayfeh 1973)

$$\left(m_1+\frac{m_2u^2}{l^2-u^2}\right)\ddot{u}+\frac{m_2l^2u\dot{u}^2}{\left(l^2-u^2\right)^2}+k\,u+m_2\,g\,\frac{u}{\sqrt{l^2-u^2}}=0 \tag{23}$$

With the boundary condition of

$$u(0)=A \qquad \dot{u}(0)=0 \tag{24}$$

Variational and Hamiltonian formulations of the Eq. (23) can be readily obtained as

$$J(u)=\int_0^t\left(-\frac{1}{2}\left(m_1\dot{u}^2-\frac{m_2l^2\dot{u}^2u^4}{\left(-l^2+u^2\right)^2}+\frac{m_2\dot{u}^2u^2}{\left(-l^2+u^2\right)^2}\right)+\frac{1}{2}k\,u^2-m_2\,g\sqrt{l^2-u^2}\right)dt \tag{25}$$

$$\begin{aligned}H&=\frac{1}{2}\left(m_1\dot{u}^2-\frac{m_2l^2\dot{u}^2u^4}{\left(-l^2+u^2\right)^2}+\frac{m_2\dot{u}^2u^2}{\left(-l^2+u^2\right)^2}\right)+\frac{1}{2}k\,u^2-m_2\,g\sqrt{l^2-u^2}\\&=\frac{1}{2}k\,A^2-m_2\,g\sqrt{l^2-A^2}\end{aligned} \tag{26}$$

Or

$$\begin{aligned}R(t)=&\frac{1}{2}\left(m_1\dot{u}^2-\frac{m_2l^2\dot{u}^2u^4}{\left(-l^2+u^2\right)^2}+\frac{m_2\dot{u}^2u^2}{\left(-l^2+u^2\right)^2}\right)+\frac{1}{2}k\,u^2-m_2\,g\sqrt{l^2-u^2}\\&-\frac{1}{2}k\,A^2+m_2\,g\sqrt{l^2-A^2}=0\end{aligned} \tag{27}$$

Choosing the trial function $u(t)=A\cos(\omega t)$, we obtain the following residual equation.

Which trigger the following results

$$\begin{aligned}&\frac{1}{2}\left(m_1A^2\omega^2\sin^2(\omega t)+\frac{m_2A^4\omega^2\sin^2(\omega t)\cos^2(\omega t)}{\left(l^2-A^2\cos^2(\omega t)\right)^2}-\frac{m_2A^6\omega^2\sin^2(\omega t)\cos^4(\omega t)}{\left(l^2-A^2\cos^2(\omega t)\right)^2}\right)\\&+\frac{1}{2}k\,A^2\cos^2(\omega t)-m_2\,g\sqrt{l^2-A^2\cos^2(\omega t)}-\frac{1}{2}k\,A^2+m_2\,g\sqrt{l^2-A^2}=0\end{aligned} \tag{28}$$

If we collocate at $\omega=\pi/4$, we obtain the following result

$$\omega=\frac{\sqrt{\left(A^2-2l^2\right)\left((m_1-m_2)A^2-2m_1l^2\right)\left(kA^2+2\sqrt{4l^2-2A^2}\,g\,m_2-4\sqrt{l^2-A^2}\right)g\,m_2}}{\left(2m_1l^2-(m_1-m_2)A^2\right)A} \tag{29}$$

We can obtain the following approximate solution

$$u(t)=A\cos\left(\frac{\sqrt{\left(A^2-2l^2\right)\left((m_1-m_2)A^2-2m_1l^2\right)\left(kA^2+2\sqrt{4l^2-2A^2}\,g\,m_2-4\sqrt{l^2-A^2}\right)g\,m_2}}{\left(2m_1l^2-(m_1-m_2)A^2\right)A}t\right) \tag{30}$$

4.3 Example 3

For the third example we have considered the following governing equation

$$\ddot{u}+\alpha u\dot{u}^2+u\ddot{u}+\beta u+\lambda u^3+\varepsilon u^5=0, \tag{31}$$

with the boundary condition of

$$u(0)=A,\quad \dot{u}(0)=0, \tag{32}$$

Variational and Hamiltonian formulations of the Eq. (31) can be readily obtained as

$$\begin{aligned} J(u) &= \int_0^t\left(-\frac{1}{2}\dot{u}^2+\frac{1}{2}\alpha\ u^2\dot{u}^2+\frac{1}{2}\beta u^2+\frac{1}{4}\lambda u^4+\frac{1}{6}\varepsilon u^6\right)dt \\ H &= \frac{1}{2}\dot{u}^2+\frac{1}{2}\alpha u^2\dot{u}^2+\frac{1}{2}\beta u^2+\frac{1}{4}\lambda u^4+\frac{1}{6}\varepsilon u^6 \\ &= \frac{1}{2}\beta A^2+\frac{1}{4}\lambda A^4+\frac{1}{6}\varepsilon A^6 \end{aligned} \tag{33}$$

Or

$$\begin{aligned} R(t) &= \frac{1}{2}\dot{u}^2+\frac{1}{2}\alpha u^2\dot{u}^2+\frac{1}{2}\beta u^2+\frac{1}{4}\lambda u^4+\frac{1}{6}\varepsilon u^6 \\ &-\frac{1}{2}\beta A^2-\frac{1}{4}\lambda A^4-\frac{1}{6}\varepsilon A^6 \end{aligned} \tag{34}$$

The first guess for trial function is $u(t)=A\cos(\omega t)$ and substituting to the Eq. (33)

$$\begin{aligned} R(t) &= \frac{1}{2}A^2\omega^2\sin^2(\omega t)+\frac{1}{2}\alpha A^4\omega^2\cos^2(\omega t)\sin^2(\omega t)+\frac{1}{2}\beta A^2\cos^2(\omega t) \\ &+\frac{1}{4}\lambda A^4\cos^4(\omega t)+\frac{1}{6}\varepsilon A^6\cos^6(\omega t)-\frac{1}{2}\beta A^2-\frac{1}{4}\lambda A^4-\frac{1}{6}\varepsilon A^6 \end{aligned} \tag{35}$$

If we collocate at $\omega t=\pi/4$, we have

$$\omega=\frac{\sqrt{6}}{6}\frac{\sqrt{\left(2+\alpha A^2\right)\left(12\beta+9\lambda A^2+7\varepsilon A^4\right)}}{2+\alpha A^2}, \tag{36}$$

We can obtain the following approximate solution

$$u(t)=A\cos\left(\frac{\sqrt{6}}{6}\frac{\sqrt{\left(2+\alpha A^2\right)\left(12\beta+9\lambda A^2+7\varepsilon A^4\right)}}{2+\alpha A^2}\ t\right), \tag{37}$$

5. Results and discussions

In this section, for better understanding the full comparison of the presented approach and numerical solutions are presented in detail to validate the results of the energy balance method.

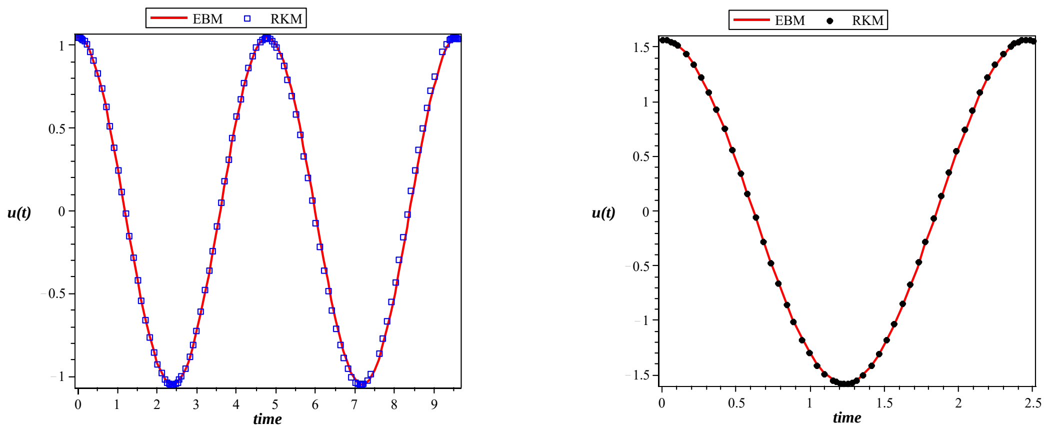
Figs. 1 to 3 show the comparison of time history response of the EBM solution with the numerical solution for two different cases

Example 1: (I): $A=\pi/3,\ \xi=0.5,\ g=10$ (II): $A=\pi/2,\ \xi=1,\ g=10$

Example 2: (I): $A=1.5,\ l=2,\ k=10,\ m_1=5,\ m_2=2, g=10$

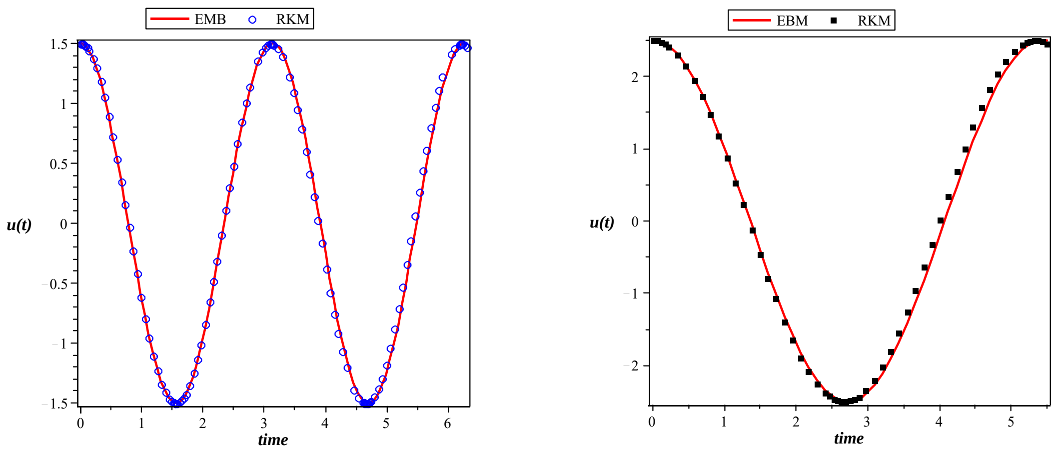
(II): $A=2.5,\ l=3,\ k=5,\ m_1=10,\ m_2=2, g=10$

Example 3: (I): $A=0.5,\ \alpha=0.5,\ \beta=1,\ \lambda=1, \varepsilon=1$ (II): $A=0.5,\ \alpha=0.5,\ \beta=1,\ \lambda=1, \varepsilon=1$



(I) (II)

Fig. 1 (Ex1) Comparison of time history response of the EBM solution with the numerical solution for (I): $A=\pi/3,\ \xi=0.5,\ g=10$ (II): $A=\pi/2,\ \xi=1,\ g=10$



(I) (II)

Fig. 2 (Ex2) Comparison of time history response of the EBM solution with the numerical solution for (I): $A=1.5,\ l=2,\ k=10,\ m_1=5,\ m_2=2, g=10$ (II): $A=2.5,\ l=3,\ k=5,\ m_1=10,\ m_2=2, g=10$

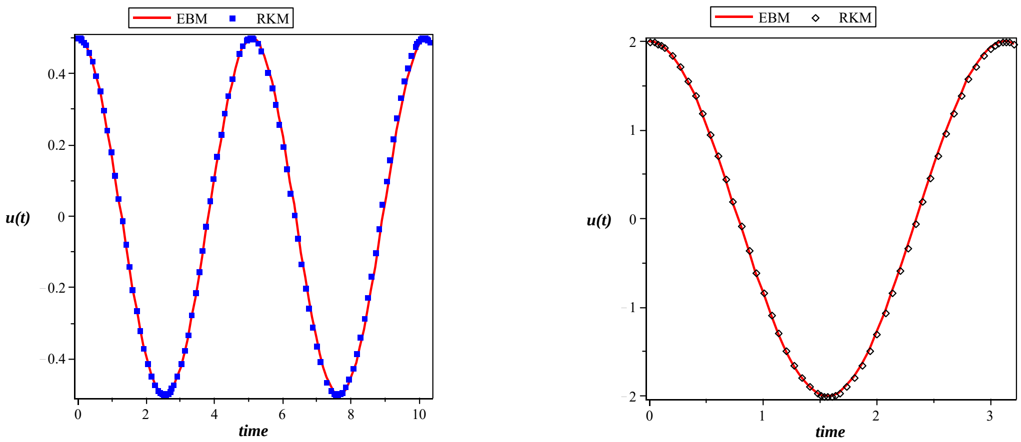


Fig. 3 (Ex3)Comparison of time history response of the EBM solution with the numerical solution for (I): $A = 0.5,\ \alpha = 0.5,\ \beta = 1,\ \lambda = 1,\ \varepsilon = 1$ (II): $A = 0.5,\ \alpha = 0.5,\ \beta = 1,\ \lambda = 1,\ \varepsilon = 1$

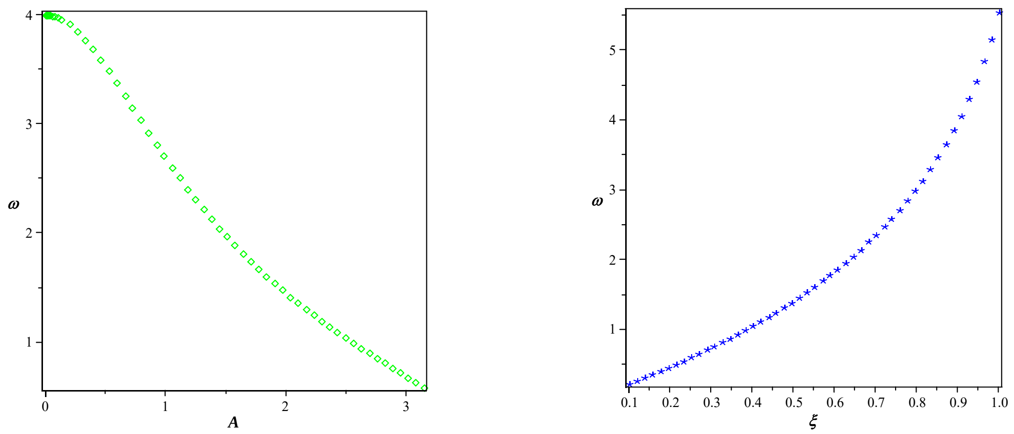


Fig. 4 (Ex1) Effect of amplitude and parameter ξ on nonlinear frequency of oscillation

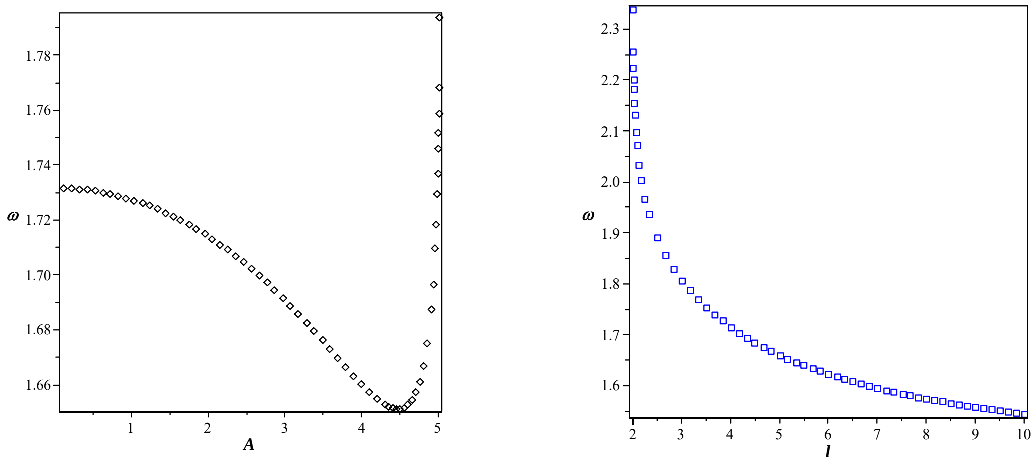


Fig. 5 (Ex2) Effect of amplitude and parameter l on nonlinear frequency of oscillation

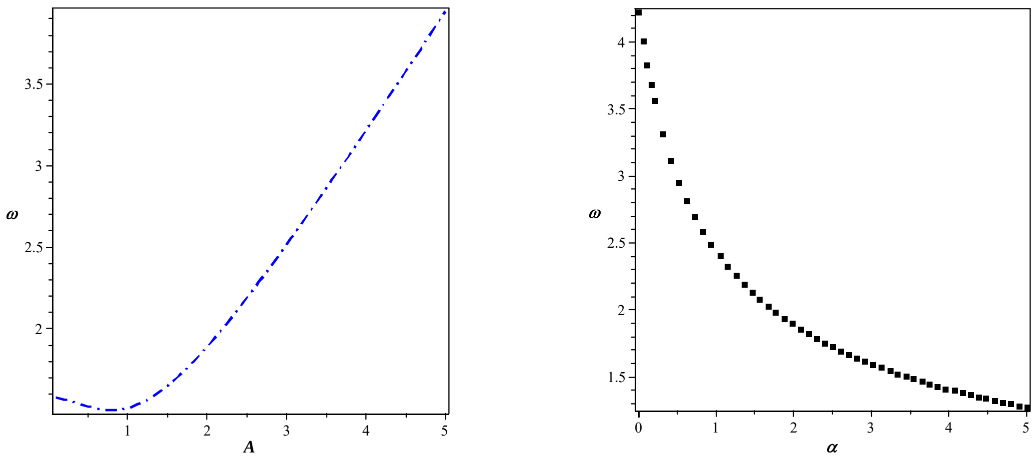


Fig. 6 (Ex3) Effect of amplitude and parameter α on nonlinear frequency of oscillation

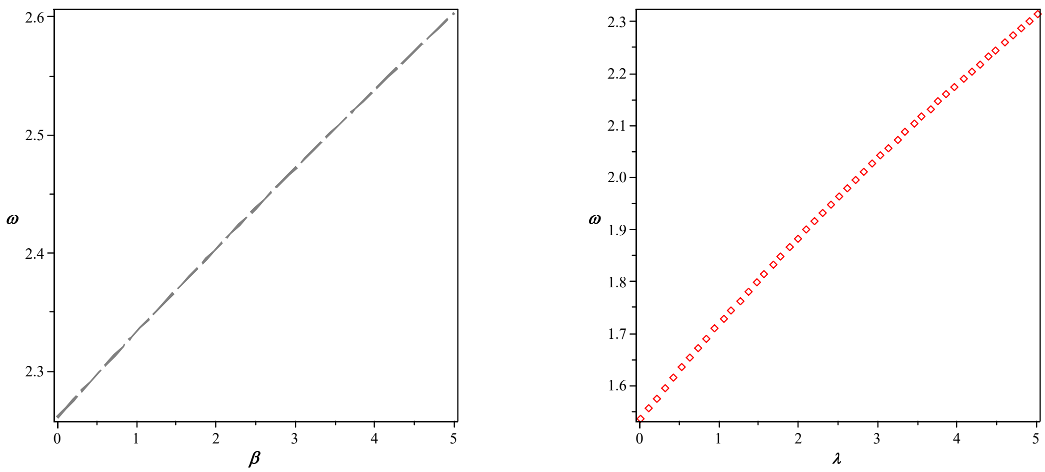


Fig. 7 (Ex3) Effect of parameters β and λ on nonlinear frequency of oscillation

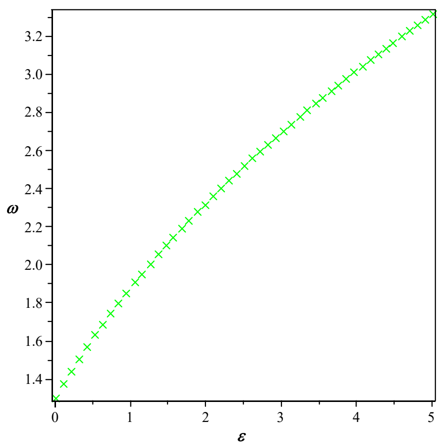


Fig. 8 (Ex3) Effect of parameter ε on nonlinear frequency of oscillation

It's obvious for the results that the motions of the systems are periodic and the results show an excellent agreement between the analytical and numerical solution.

Two different parameters of the example one has been studied in Fig. 4. The effects of amplitude and parameter ξ on nonlinear frequency of oscillation shows the by increasing the amplitude we have decrease on the nonlinear frequency and with parameter ξ, an increase has been occurred on the nonlinear frequency of the system. The effects of amplitude and parameter l on nonlinear frequency of oscillation for example 2 are studied in Fig. 5.

For example 3, more parameters are studied to show the sensitivity of the response to them, in Fig. 6 the effect of amplitude and parameter α on nonlinear frequency of oscillation are shown.

Fig. 7 is the effect of parameters β and λ on nonlinear frequency of oscillation and the final sensitive analysis is for the effect of parameter ε on nonlinear frequency of oscillation in Fig. 8. The results demonstrate the accuracy of the presented approach and also the high capability of it to see the effects of different important parameters on the nonlinear frequency of the system.

6. Conclusions

In this study the main objective is to present a new analytical solution, Energy Balance Method (EBM), for high nonlinear vibration equations. Three different nonlinear cases were studied completely to show the accuracy and application of the presented approach. The effects of the different important parameters on the nonlinear response of the systems were studied. The results compared with Runge-Kutta algorithm. The first iteration of the solution leads us to a reasonable solution.

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