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Optimal placement of viscoelastic dampers and supporting members under variable critical excitations

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Abstract. A gradient-based evolutionary optimization methodology is presented for finding the optimal design of both the added dampers and their supporting members to minimize an objective function of a linear multi-storey structure subjected to the critical ground acceleration. The objective function is taken as the sum of the stochastic interstorey drifts. A frequency-dependent viscoelastic damper and the supporting member are treated as a vibration control device. Due to the added stiffness by the supplemental viscoelastic damper, the variable critical excitation needs to be updated simultaneously within the evolutionary phase of the optimal damper placement. Two different models of the entire damper unit are investigated. The first model is a detailed model referred to as "the 3*N* model" where the relative displacement in each component (i.e., the spring and the dashpot) of the damper unit is defined. The second model is a simpler model referred to as "the *N* model" where the entire damper unit is converted into an equivalent frequency-dependent Kelvin-Voigt model. Numerical analyses for 3 and 10-storey building models are conducted to investigate the characters of the optimal design using these models and to examine the validity of the proposed technique.

Keywords: optimal damper placement; evolutionary optimization; viscoelastic damper; critical excitation; stochastic process; supporting stiffness, multi-storey buildings.

1. Introduction

In the early stage of the development in passive structural control, the installation itself of supplemental dampers was the principal objective. It appears natural that, after extensive developments of various damper systems, another target was directed to the development of smart and effective installation of such dampers.

Although the motivation was inspired and directed to smart and effective installation of dampers, research on optimal damper placement has been limited. Several studies have dealt with this subject in the early stage. De Silva (1981) presented a gradient algorithm for the optimal design of discrete dampers in the vibration control of a class of flexible systems. Constantinou and Tadjbakhsh (1983) derived the optimum damping coefficient for a damper placed on the first storey of a shear building subjected to horizontal ground motions. Gurgoze and Muller (1992) presented a numerical optimal design method for a single viscous damper in a prescribed linear multi-degree-of-freedom system. Zhang and Soong (1992) proposed a seismic design method for finding the optimal configuration of

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viscous dampers for a building with specified storey stiffnesses. Hahn and Sathiavageeswaran (1992) performed parametric studies on the effects of damper distribution on the earthquake response of buildings, and showed that, for a building with uniform storey stiffnesses, dampers should be added to the lower half of the building. Tsuji and Nakamura (1996) proposed an algorithm to find both the optimal storey stiffness and damper distribution for a shear building subjected to the spectrum-compatible ground motions.

Rather recently, Takewaki (1997) developed another optimal method for the smart damper placement with the help of the concepts of inverse problem approaches and optimal criteria-based design approaches. He solved a problem of optimal damper placement by deriving the optimality criteria and then by developing an incremental inverse problem approach. Subsequently, Takewaki and Yoshitomi (1998), Takewaki *et al.* (1999) and Takewaki (2000a) introduced a different approach based on the concept of optimal sensitivity. The optimal quantity of passive dampers is obtained automatically together with the optimal placement through this new method. The essence of these approaches is summarized in Takewaki (2009).

In the meanwhile, significant works have been developed by many researchers (Singh and Moreschi 2001, 2002, Garcia 2001, Garcia and Soong 2002, Liu *et al.* 2003, Silvestri *et al.* 2003, Xu *et al.* 2003, 2004, Uetani *et al.* 2003, Park *et al.* 2004, Wongprasert and Symans 2004, Kiu *et al.* 2004, Trombetti and Silvestri 2004, 2007, Tan *et al.* 2005, Liu *et al.* 2005, Lavan and Levy 2005, 2006a, b, Levy and Lavan 2006, Silvestri and Trombetti 2007, Marano *et al.* 2007, Aydin *et al.* 2007, Cimellaro 2007, Cimellaro and Retamales 2007, Attard 2007, Cimellaro and Retamales 2007, Viola and Guidi 2008, Wang and Dyke 2008). Most of these studies have developed new optimal design methods of supplemental dampers and proposed effective and useful approaches.

In this paper, an evolutionary method is proposed for finding the optimal design of both dampers and their supporting members to minimize an objective function of a linear multistorey structure subjected to critical resonant ground input. While many researches have been accumulated on the design of passive dampers themselves, passive dampers design including the supporting members are very limited. The objective function is taken as the sum of the mean-squares of the interstorey drifts. A frequency-dependent viscoelastic damper including the supporting unit is taken into account. Due to the added stiffness by the viscoelastic damper, the resonant variable critical excitation (Takewaki 2002, 2007) needs to be updated in the evolutionary phase of optimal damper placement. Two different models of the whole damper unit are investigated. The first model is a detailed model referred to as "the $3N$ model" where the relative displacement between each component of damper unit can be defined. The second model is a simpler model referred to as "the N model" where the whole damper unit is converted to an equivalent frequency-dependent Kelvin-Voigt model. Numerical analyses are conducted to show the accuracy of these models and to examine the validity of the proposed optimal design method.

2. Structural model with viscoelastic dampers and the supporting members

Consider an N-storey planar shear building model with frequency-dependent acrylic visco-elastic dampers (VED) and their supporting members. Let k_{Fi} and c_{Fi} denote the storey stiffness and the damping coefficient in the i-th storey of the main frame. The floor mass of the i-th storey is denoted by M_i. The dependency of VED on temperature and strain amplitude is not taken into account here. This acrylic VED is assumed to be described by a 4-elements model as shown in Fig. 1 (Lee *et al.*

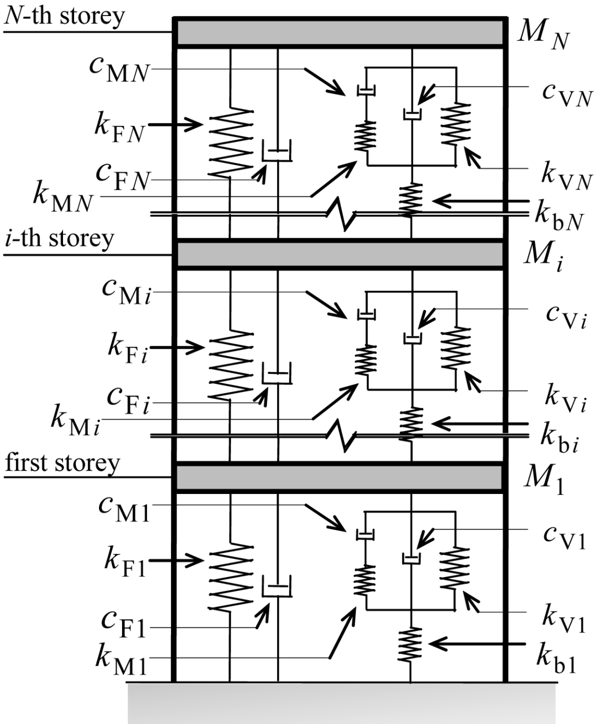


Fig. 1 Structural model with viscoelastic damper including supporting member

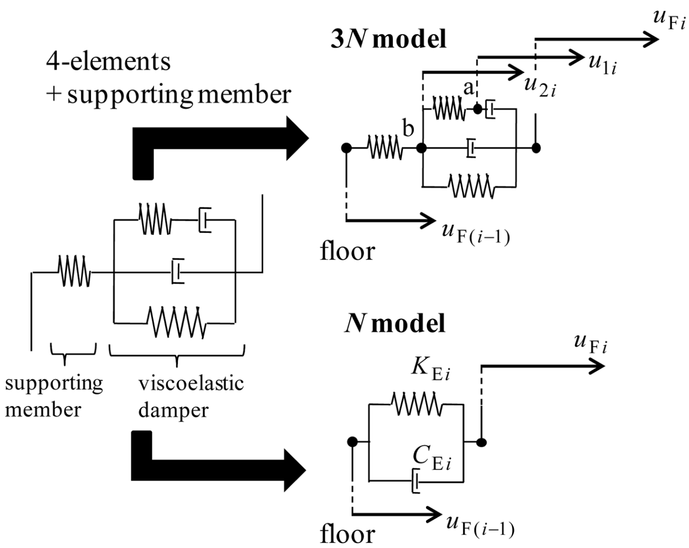


Fig. 2 Damper models simplified as "3N model" and "N model"

2002). k_{Mi}, k_{Vi}, c_{Mi} and c_{Vi} represent the spring stiffnesses and dashpot damping coefficients in VED in the i-th storey. In order to take into account the stiffness k_{bi} of the supporting member, the 4-elements VED model is connected in series with another spring k_{bi}. This entire damper unit can be converted to two different models (Fig. 2).

The first model is a detailed model, as shown in Fig. 2, allocating small lumped masses, i.e., a and b, between the components of spring and dashpot. Let u_{Fi}, u_{1i} and u_{2i} denote the i-th storey floor displacement relative to the ground and the displacements of the lumped masses, a and b, relative to the ground. This VED model has three degrees-of-freedom in each storey and the structure with this VED model is referred to as "the 3N model". The stiffness and damping matrices of this VED model are independent of frequency. The components of stiffness and damping matrices $\mathbf{K}_{\text{full}}$, $\mathbf{C}_{\text{full}}$, of the whole building model with VED are linear combinations of k_{Fi}, k_{Mi}, k_{Vi}, c_{Fi}, c_{Mi}, c_{Vi} and k_{bi}. Because of the complex connection of each structural component, the overall stiffness matrix is not a simple one (not triple-type matrix). For this reason, it may be disadvantageous to utilize this 3N model for a large-scale structure and a simpler VED model is needed.

The second VED model is referred to as "the N model". In this VED model, the whole damper unit is converted into an equivalent frequency-dependent Kelvin-Voigt model. The equivalent stiffness and damping coefficients in the i-th storey may be expressed by

$$K_{Ei}(\omega, S_{di}, k_{bi}) = \left\{(A - B\omega^2)(C - D\omega^2) + EF\omega^2\right\} / \left\{(C - D\omega^2)^2 + F^2\omega^2\right\} \tag{1}$$

$$C_{Ei}(\omega, S_{di}, k_{bi}) = \left\{(BF - DE)\omega^2 + CE - AF\right\} / \left\{(C - D\omega^2)^2 + F^2\omega^2\right\} \tag{2}$$

where the coefficients A to F are given as

$$\begin{aligned} &A = k_{bi}k_{Mi}k_{Vi}, \quad B = k_{bi}c_{Mi}c_{Vi}, \quad C = k_{bi}k_{Mi} + k_{Mi}k_{Vi} \\ &D = c_{Mi}c_{Vi}, \quad E = k_{bi}\left(k_{Vi}c_{Mi} + k_{Mi}c_{Mi} + k_{Mi}c_{Vi}\right) \\ &F = c_{Mi}\left(k_{bi} + k_{Vi}\right) + k_{Mi}\left(c_{Mi} + c_{Vi}\right) \end{aligned} \tag{3a-f}$$

The derivation of Eqs. (1) and (2) can be found in Appendix 1.

In this paper, a comparison between these two models (3N model and N model) is performed later within the scope of optimal damper placement.

3. Critical excitation for variable design

In the seismic-resistant design of important structures, time-history analysis is often used for a set of recorded ground motions. However, it is well recognized that the ground motions include various uncertainties of various levels. In order to account for these uncertainties, more reliable and robust structural design methods are needed. The critical excitation method is adopted in this paper. The critical excitation method was initiated by Drenick (1970) and much subsequent research has been accumulated (Takewaki 2007).

Takewaki (2002) introduced the concept of variable critical excitation which has a variable resonant frequency close to the fundamental natural circular frequency ω_0 of the structure with varied stiffnesses of the supplemental dampers. Based on this concept, a problem is posed here such that the optimal viscoelastic damper placement is identified together with the selection of the optimal stiffness of the supporting members.

Let $S_g(\omega)$ denotes the power spectral density (PSD) function of the input ground acceleration $\ddot{u}_g(t)$. The constraints on $S_g(\omega)$ are the power of the PSD function (i.e., the area under the PSD

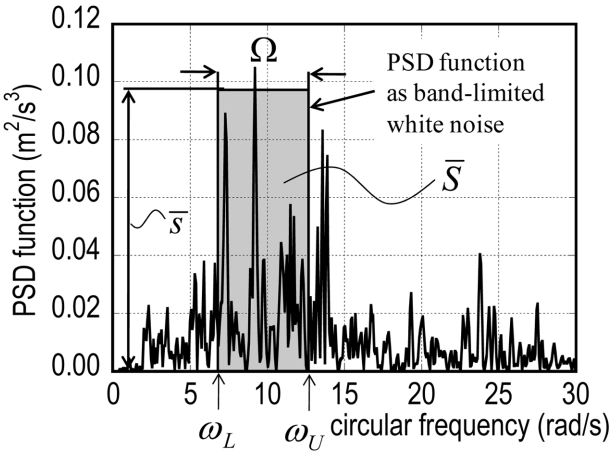


Fig. 3 Critical PSD function as band-limited white noise compared with that of El Centro NS 1940 record

function representing the input variance), described by

$$\int_{-\infty}^{\infty} S_g(\omega)d\omega \le \bar{S} \tag{4}$$

and the intensity of the PSD function (i.e., the maximum or the peak value of the PSD function), expressed by

$$\sup S_g(\omega) \le \bar{s} \tag{5}$$

$\bar{S}$ and $\bar{s}$ are the limits on the power and intensity, respectively. These parameters of the critical excitation are determined from the analysis of recorded ground motions. A shape of the PSD function as a solution of this problem is assumed to be a Dirac delta function (when $\bar{s}$ is infinity) or a band-limited white noise (when $\bar{s}$ is finite). A band-limited white noise is shown in Fig. 3 where a frequency band-width Ω and upper and lower bounds ω_U, ω_L of frequency are obtained from the given parameters $\bar{S}$ and $\bar{s}$.

In the following sections, the optimal placement of VED is investigated in which the fundamental natural frequency of the structure with different VED distributions may vary and ω_U, ω_L for critical excitation may change. This concept is the critical excitation for variable design (Takewaki 2002). A similar concept has been developed by Takewaki (2000b), but the parameters ω_U, ω_L were kept unchanged (i.e., they do not depend on the optimization variables).

4. Stochastic response evaluation in frequency domain

4.1 3N model

Let $\mathbf{M}_{\mathrm{full}}$, $\mathbf{r} = \{1,...,1\}^T$ denote the system mass matrix ($3N\times3N$) of the $3N$ model and the influence coefficient vector, respectively. The imaginary unit is denoted by $i = \sqrt{-1}$. The equations of motion of the building with VED in frequency domain can be expressed by

$$(-\omega^2\mathbf{M}_{\mathrm{full}} + \mathrm{i}\omega\mathbf{C}_{\mathrm{full}} + \mathbf{K}_{\mathrm{full}})\mathbf{U}_{\mathrm{full}}(\omega) = -\mathbf{M}_{\mathrm{full}}\mathbf{r}\ddot{U}_{\mathrm{g}}(\omega) \tag{6}$$

where $\mathbf{U}_{\mathrm{full}}(\omega)$ and $\ddot{U}_{\mathrm{g}}(\omega)$ are the Fourier transforms of the nodal displacements $\mathbf{u}_{\mathrm{full}} = \{u_{\mathrm{F}1}\ u_{11}\ u_{12}\ u_{\mathrm{F}2}\ u_{12}\ u_{22}\ \cdots\ \cdots\ \cdots\ u_{\mathrm{F}N}\ u_{1N}\ u_{2N}\}^T$ and the Fourier transform of the ground acceleration $\ddot{u}_{\mathrm{g}}(t)$. Eq. (6) can be described simply as

$$\mathbf{A}_{\mathrm{full}}\mathbf{U}_{\mathrm{full}}(\omega) = \mathbf{B}_{\mathrm{full}}\ddot{U}_{\mathrm{g}}(\omega) \tag{7}$$

where

$$\mathbf{A}_{\mathrm{full}} = -\omega^2\mathbf{M}_{\mathrm{full}} + \mathrm{i}\omega\mathbf{C}_{\mathrm{full}} + \mathbf{K}_{\mathrm{full}},\ \mathbf{B}_{\mathrm{full}} = -\mathbf{M}_{\mathrm{full}}\mathbf{r} \tag{8a,b}$$

Fourier transforms of the interstorey drifts $\mathbf{D}(\omega) = \{D_1,\cdots,D_N\}^T$ can then be derived by

$$\mathbf{D}(\omega) = \mathbf{T}_{\mathrm{full}}\mathbf{U}_{\mathrm{full}}(\omega) \tag{9}$$

where $\mathbf{T}$ is a constant transformation matrix consisting of 1, -1 and 0. By substituting Eq. (7) into Eq. (9), $\mathbf{D}(\omega)$ can be rewritten as

$$\mathbf{D}(\omega) = \mathbf{T}_{\mathrm{full}}\mathbf{A}_{\mathrm{full}}^{-1}\mathbf{B}_{\mathrm{full}}\ddot{U}_{\mathrm{g}}(\omega) \tag{10}$$

The transfer functions $\mathbf{H}_{\mathrm{D}}^{\mathrm{full}}(\omega) = \{H_{\mathrm{D}i}(\omega)\}$ of interstorey drifts can be defined as

$$\mathbf{H}_{\mathrm{D}}^{\mathrm{full}}(\omega) = \mathbf{T}_{\mathrm{full}}\mathbf{A}_{\mathrm{full}}^{-1}\mathbf{B}_{\mathrm{full}} \tag{11}$$

By using the resonant PSD function, the objective function as the sum of the mean-squares responses $\sigma_{\mathrm{D}i}^{\mathrm{full}}$ of the interstorey drifts for the $3N$ model can be evaluated by

$$f_{3N} = \sum_{i=1}^{N}\left(\sigma_{\mathrm{D}i}^{\mathrm{full}}\right)^2 = \sum_{i=1}^{N}\int_{-\infty}^{\infty} H_{\mathrm{D}\{1+3(i-1)\}}^{\mathrm{full}}(\omega) H_{\mathrm{D}\{1+3(i-1)\}}^{\mathrm{full}}{}^{*}(\omega) S_{\mathrm{g}}(\omega)\mathrm{d}\omega \tag{12}$$

where $H_{\mathrm{D}i}^{\mathrm{full}}(\omega)$ is the i-th component of the interstorey drift transfer function $\mathbf{H}_{\mathrm{D}}^{\mathrm{full}}(\omega)$ for the $3N$ model and ( )* denotes the complex conjugate.

It is well understood that the stiffness of supporting members should be strong enough to ensure the effectiveness of the damper unit. For this reason, the stiffness $\mathbf{k}_{\mathrm{b}} = \{k_{\mathrm{b}i}\}(i=1,\cdots,N)$ of each supporting member is treated as another design variable and the axial force of each supporting member is constrained to an upper limit (e.g., the yield force). In the $3N$ model, the maximum value of the axial force of the supporting member can be evaluated by

$$N_{\mathrm{b}i} = \rho k_{\mathrm{b}i}\left(\int_{-\infty}^{\infty}\left|[\mathbf{T}_{\mathrm{b}}\mathbf{H}_{\mathrm{full}}(\omega)]_{2+3(i+1)}\right|^2 S_{\mathrm{g}}(\omega)\mathrm{d}\omega\right)^{1/2} \tag{13}$$

where ρ is the peak factor for the maximum axial force of the supporting member. In order to evaluate the maximum axial force of the supporting member, the peak factor has been introduced. $\mathbf{T}_{\mathrm{b}}$ is a transformation matrix from the nodal displacements to the relative displacements between both ends of supporting members (see Appendix 2).

4.2 N model

Let **A**, **B** denote the matrix and vector for the N model corresponding to Eqs. 8(a) and 8(b). The equations of motion for the N model in frequency domain may be expressed by

$$\mathbf{A}\mathbf{U}(\omega) = \mathbf{B}\ddot{U}_g(\omega) \tag{14}$$

The objective function can be described by

$$f_N = \sum_{i=1}^{N} \int_{-\infty}^{\infty} H_{\delta i}(\omega) H_{\delta i}^{*}(\omega) S_g(\omega) d\omega \tag{15}$$

where $H_{\delta i}(\omega)$ is the i-th component of the interstorey drift transfer function vector $\mathbf{T}\mathbf{A}^{-1}\mathbf{B}$ for the N model.

The axial force of the supporting member can be evaluated in the N model as the internal force of the frequency-dependent Kelvin-Voigt model.

$$N_{bi} = \rho\left(\int_{-\infty}^{\infty} |H_{Nbi}(\omega)|^2 S_g(\omega) d\omega\right)^{1/2} \tag{16}$$

where ρ and $H_{Nbi}(\omega)$ are the peak factor and the transfer function, respectively, of the axial force. The transfer function $H_{Nbi}(\omega)$ of the axial force can be expressed by

$$H_{Nbi}(\omega) = \{K_{Ei}(\omega) + i\omega C_{Ei}(\omega)\} H_{\delta i}(\omega) \tag{17}$$

where $K_{Ei}(\omega) + i\omega C_{Ei}(\omega)$ represents the complex stiffness of the equivalent Kelvin-Voigt model of the damper unit including the supporting member.

5. Optimal design problem

The problem of optimal damper placement of passive dampers and optimal stiffness selection of supporting members for the N-storey shear building model subjected to variable critical excitation is to find the distribution of both VED shear areas $\mathbf{S}_d = \{S_{d1}, \cdots, S_{dN}\}$ and supporting members stiffnesses $\mathbf{k}_b = \{k_{b1}, \cdots, k_{bN}\}$. S_{di} is the VED shear area in the i-th storey. The fundamental natural circular frequency ω_0 of the building can vary according to the change of the damper unit (damper area and stiffness of supporting member). The property of critical excitation $S_g(\omega)$ is therefore dependent on the design variables $\mathbf{S}_d$ and $\mathbf{k}_b$. The objective function f, Eq. (15), can then be regarded as a function of ω_0, $\mathbf{S}_d$ and $\mathbf{k}_b$, i.e., $f(\omega_0, \mathbf{S}_d, \mathbf{k}_b)$.

The first constraint on damper capacity is

$$\sum_{i=1}^{N} S_{di} = \overline{W} \tag{18}$$

where $\overline{W}$ is a specified total damper area. Additional constraints on the added damper's area in each storey are also considered, specifically

$$0 \le S_{di} \le \bar{S}_{di} \quad (i=1,\cdots, N) \tag{19}$$

where $\bar{S}_{di}$ is the upper bound of damper area in the i-th storey.

The constraint on axial force of the supporting member may be expressed as

$$N_{bi}(\mathbf{S}_d, \mathbf{k}_b) \le \bar{P}_{yi}(k_{bi}) \quad (i=1,2,\cdots, N) \tag{20}$$

where $\bar{P}_{yi}$ is the yield force of the supporting member and a function of k_{bi}.

6. Optimality conditions

The generalized Lagrangian L for the optimal design problem can be defined as

$$L(\mathbf{S}_d, \mathbf{k}_b, \lambda, \boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{\kappa})$$
$$= f + \lambda\left(\sum_{i=1}^{N} S_{di} - \bar{W}\right) + \sum_{i=1}^{N}\mu_i\left(0 - S_{di}\right) + \sum_{i=1}^{N}\gamma_i\left(S_{di} - \bar{S}_{di}\right) + \sum_{i=1}^{N}\kappa_i\left(N_{bi} - \bar{P}_{yi}\right) \tag{21}$$

where λ, $\boldsymbol{\mu} = \{\mu_i\}$, $\boldsymbol{\gamma} = \{\gamma_i\}$ and $\boldsymbol{\kappa} = \{\kappa_i\}$ are the Lagrange multipliers. The principal optimality conditions for this problem without active upper and lower bound conditions on damper area and axial force of supporting member may be derived from the stationarity conditions of $L(\boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\gamma} = \mathbf{0}, \boldsymbol{\kappa} = \mathbf{0})$ with respect to $\mathbf{S}_d$ and $\mathbf{k}_b$.

$$f_{,j} + \lambda = 0 \qquad \text{for} \qquad 0 < S_{dj} < \bar{S}_{dj},\ N_{bj} < \bar{P}_{yj} \tag{22}$$

$$f^{,j} = 0 \qquad \text{for} \qquad N_{bj} < \bar{P}_{yj} \tag{23}$$

The symbols $(\ )_{,j}$ and $(\ )^{,j}$ denote the partial differentiation with respect to S_{dj} and k_{bj}, respectively.

In the process of increasing the quantity of VED in each storey, the axial force of the supporting member usually increases. When the constraint on axial force of the supporting member is active, the optimality conditions should be modified by the stationarity conditions of $L(\boldsymbol{\mu} = \mathbf{0}, \boldsymbol{\gamma} = \mathbf{0})$ as follows

$$f_{,j} + \lambda + \sum_{i=q_1}^{q_n} \kappa_i N_{bi,j} = 0 \quad \text{for} \quad 0 < S_{dj} < \bar{S}_{dj}\,(j = p_1,\cdots,p_m), N_{bi} = \bar{P}_{yi}\,(i = q_1,\cdots,q_n) \tag{24}$$

$$f^{,j} + \sum_{i=q_1}^{q_n} \kappa_i\left(N_{bi}{}^{,j} - \bar{P}_{yi}{}^{,j}\right) = 0 \quad \text{for} \quad N_{bj} = \bar{P}_{yj} \quad \left(j = q_1,\cdots,q_n\right) \tag{25}$$

where m and p_i denote the number of storeys and their locations having inactive constraint on damper area and n and q_i denote the number of storeys and their locations having active constraints on axial force of the supporting member. In Eq. (25), it is assumed that the partial differentiation of i-th storey axial force N_{bi} with respect to other storey's supporting member stiffness k_{bj} can be neglected ($N_{bi}{}^{,j} = 0$ $(i \neq j)$). This accuracy has been confirmed through numerical tests. In addition, the yield force $\bar{P}_{yj}$ of each supporting member is assumed to be a function of only the stiffness k_{bj} of that supporting member. As a result, Eq. (25) reduces to

$$f^{,j} + \kappa_j \left(N_{bj}{}^{,j} - \bar{P}_{yj}{}^{,j} \right) = 0 \tag{26}$$

It is noted that the Lagrange multiplier κ_j can be evaluated directly from Eq. (26). This assumption facilitates a simple sensitivity expression of the objective function.

When the other constraints on upper and lower bounds of damper's area are active, the optimality conditions should be modified by

$$f_{,j} + \lambda \geq 0 \qquad \text{for} \qquad S_{di} = 0 \tag{27}$$

$$f_{,j} + \lambda + \sum_{i=q_1}^{q_n} \kappa_i N_{bi,j} \leq 0 \qquad \text{for} \qquad S_{dj} = \bar{S}_{dj}, N_{bi} = \bar{P}_{yi}\,(i = q_1, \cdots, q_n) \tag{28}$$

If there is no VED for which the axial force of supporting member attains its yield force, Eq. (28) should be replaced by

$$f_{,j} + \lambda \leq 0 \qquad \text{for} \qquad S_{dj} = \bar{S}_{dj} \tag{29}$$

7. Solution procedure of optimal design

7.1 Algorithm for optimal damper placement and optimal design of supporting members

A gradient-based evolutionary solution algorithm is presented for the problem of optimal damper placement. Since it is quite beneficial to obtain the optimal damper placement for various capacities of dampers, the total damper quantity $\bar{W}$ is increased gradually. The flow chart of this solution algorithm is shown in Fig. 4. Furthermore, Fig. 5 explains the evolution of the design variables $\mathbf{S}_d$ and $\mathbf{k}_b$ during the optimal design process. The solution procedure is summarized as follows :

Step 0 Design the main frame without VED under the system dependent critical excitation.

Step 1 Calculate the undamped fundamental natural circular frequency ω_0 of the frame.

Step 2 Create the critical PSD function $S_g(\omega)$ as a band-limited white noise which has a central frequency ω_0.

Step 3 Evaluate the axial force N_{bi} of supporting member and count the number n of storeys in which N_{bi} reaches its yield axial force $\bar{P}_{yi}$.

Step 4 Identify the location of storey where the absolute value of the first-order sensitivity of the objective function f is maximized.

Step 5 Count the number m of storeys where the maximum absolute values of the first-order sensitivity of the objective function coincide.

The above global procedures can be further subdivided into 4 different domains, called A, B, C and D (see Fig. 5), depending on the values m and n. To find the optimal increment of $\mathbf{S}_d$ and $\mathbf{k}_b$, an appropriate set of optimality conditions have to be selected from Eq. (22) through Eq. (29). The

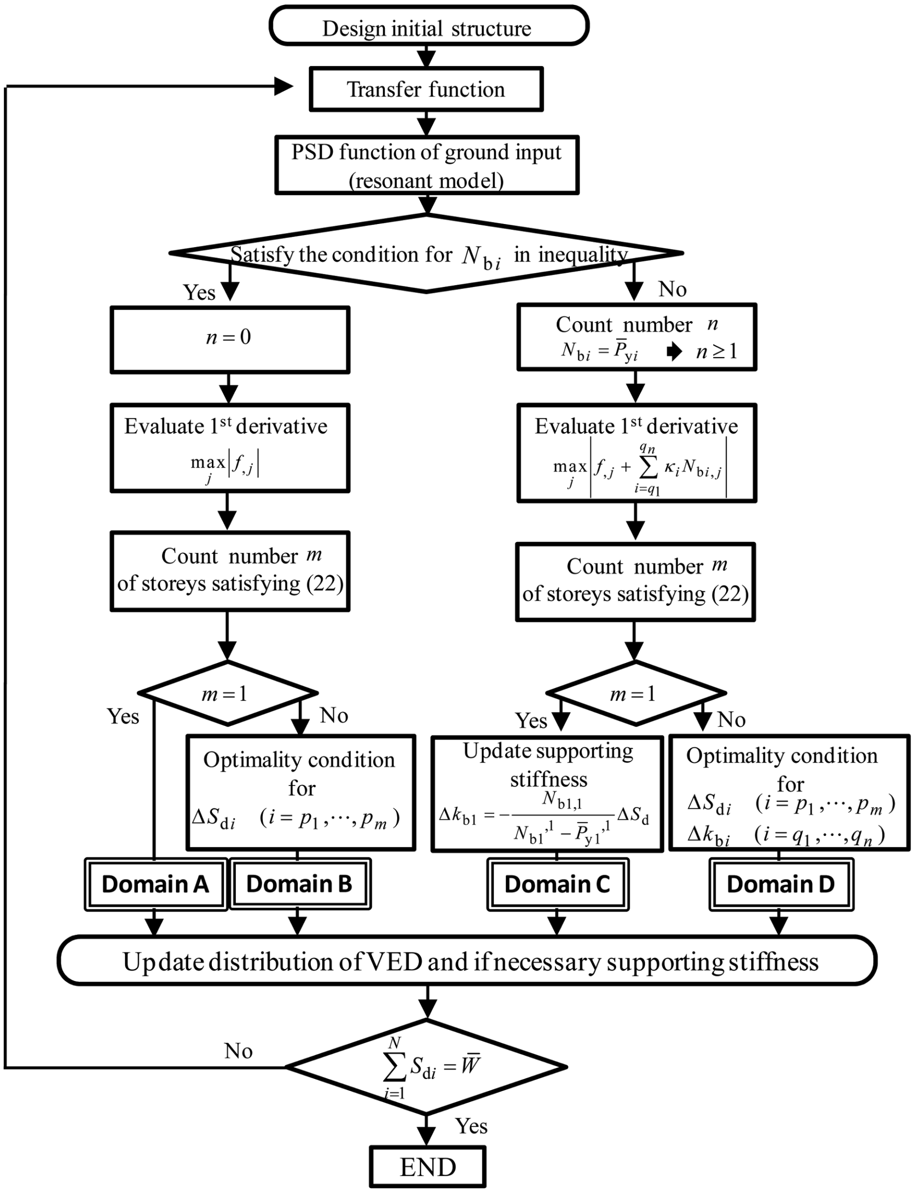


Fig. 4 Flowchart for the optimal placement of viscoelastic dampers and optimal stiffnesses of the supporting members

relationship between the optimality conditions and each domain is shown in Table 3.

Step 6A The case of $m = 1$, $n = 0$ corresponds to the domain A. The increment ΔW of VED is added only to the specific storey attaining $\max_j |f_{,j}|$.

Step 6B The case of $m \geq 2$, $n = 0$ corresponds to the domain B. When the multiple equality optimality conditions, Eq. (22), are satisfied, the optimal damper distribution S_{di} has to be computed and updated to keep the coincidence of the multiple first-order sensitivities.

Step 6C The case of $m = 1$, $n = 1$ corresponds to the domain C. The stiffness k_{bi} of the supporting

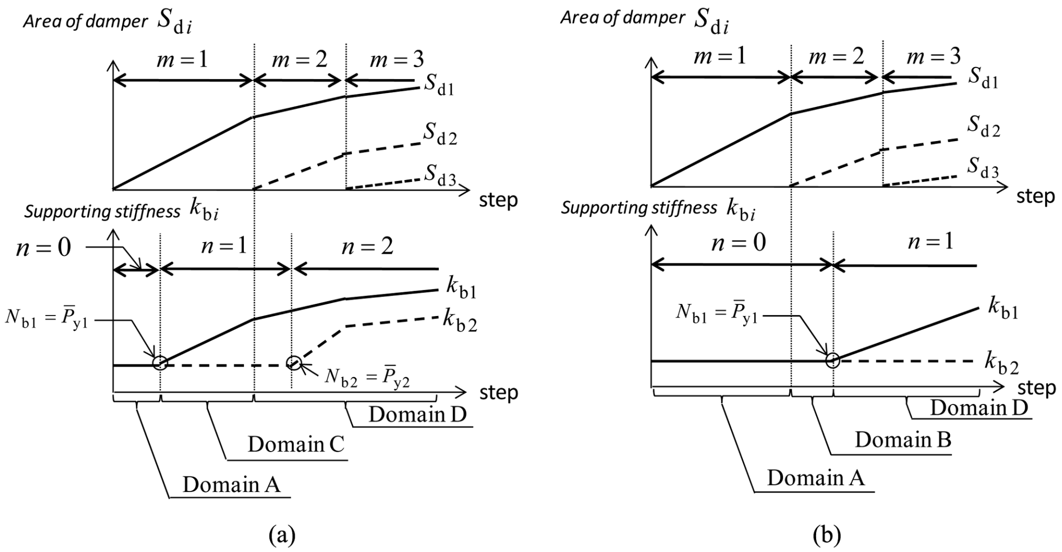


Fig. 5 Evolution of design variables in the proposed optimization process (a) Case including domains A, C, D, (b) Case including domains A, B, D

member is increased to prevent N_{bi} from exceeding the yield axial force $\bar{P}_{yi}$.

Step 6D The case of $m \geq 2$, $n \geq 1$ corresponds to the domain D. All the optimality conditions have to be satisfied. This corresponds to the conditions that the multiple first-order sensitivities coincide due to Eq. (22) and the corresponding k_{bi} is increased to satisfy $N_{bi} = \bar{P}_{yi}$.

***Step** 7* Update design variables $\mathbf{S}_d$ and $\mathbf{k}_b$ according to the optimality conditions summarized in Table 3.

***Step** 8* Repeat Step 1 through Step 7 until the constraint Eq. (18), i.e., the total area of VED, is satisfied.

The initial model is without VED, i.e., $S_{di} = 0$ ($i = 1,\cdots, N$). Additional VED is distributed via the steepest direction search algorithm (Takewaki 2009). Let $\Delta\mathbf{S}_d = \{\Delta S_{di}\}$ and ΔW denote the increment of VED area and the increment of the sum of VED areas, respectively. When ΔW is given, it is needed to find the optimal placement to decrease the objective function most effectively. For this purpose, the first and the second-order sensitivities of the objective function with respect to the design variables $\mathbf{S}_d$ and $\mathbf{k}_b$ are necessary. Those sensitivities $f_{,j}$, $f^{,j}$, $f_{,jk}$, $f_{,j}^{,k}$ and $f^{,jk}$ can be derived by differentiating Eq. (15) with respect to the design variables. The N model is used here. Detailed expressions of the first and the second-order sensitivities are shown in the next section.

For clarification, the steps 6C and 6D are explained in more detail below.

[Step 6C] When $m = 1$, VED is added only to a single specific storey. When N_{bj} attains its upper bound $\bar{P}_{yj}$, k_{bj} has to be increased so as to keep the increment ΔN_{bj} coinciding with $\Delta\bar{P}_{yj}$. This requires

$$\mathrm{d}N_{bj} = \mathrm{d}\bar{P}_{yj} \Rightarrow \sum_{i=q_1}^{q_n}\left(N_{bj,i}\Delta S_{di}\right) + N_{bj}{}^{,j}\Delta k_{bj} = \bar{P}_{yj}{}^{,j}\Delta k_{bj}\;(j = q_1,\ldots,q_n) \tag{30}$$

Herein, the assumption discussed below Eq. (25) is employed again. When $n = 1$, the increment of the stiffness Δk_{bj} of the supporting member can be derived by

$$\Delta k_{bq_1} = \frac{-N_{bq_1,q_1}}{N_{bq_1}{}^{,q_1} - \bar{P}_{yq_1}{}^{,q_1}} \Delta S_{dq_1} \tag{31}$$

[Step 6D] When $m \geq 2$, an appropriate distribution of VED to more than two storeys should be employed for a given value of ΔW. In this case, the number of unknown variables is $m + n$. From Eq. (30), n equations with respect to $\mathbf{k}_b$ can be derived. We, therefore, need more m equations.

Successive satisfaction of Eq. (24) requires that

$$\sum_{i=p_1}^{p_m} f_{,ji}\Delta S_{di} + \sum_{i=q_1}^{q_n} f_{,j}^{,i}\Delta k_{bi} + \sum_{i=p_1}^{p_m}\sum_{k=q_1}^{q_n} (\kappa_k N_{bk,j})_{,i}\Delta S_{di} + \sum_{i=q_1}^{q_n}\sum_{k=q_1}^{q_n} (\kappa_k N_{bk,j})^{,i}\Delta k_{bi} = 0$$

$$(j = q_1,\ldots, q_n) \tag{32}$$

where κ_k in Eq. (32) can be derived from Eq. (26) as

$$\kappa_k = \frac{-f^{,k}}{N_{bk}{}^{,k} - \bar{P}_{yk}{}^{,k}} \qquad (k = q_1,\cdots,q_n) \tag{33}$$

It can be mentioned that, after the multiple optimality conditions are updated, the first-order sensitivities should continue to be satisfied. To achieve this, the following equation can be derived by substituting Eq. (33) into Eq. (32)

$$\sum_{i=p_1}^{p_m}\left[\left[f_{,ji} - \sum_{k=q_1}^{q_n}\left\{f^{,k}N_{bk,j}/(N_{bk}{}^{,k} - \bar{P}_{yk}{}^{,k})\right\}_{,i}\right]\Delta S_{di}\right]$$
$$+\sum_{i=q_1}^{q_n}\left[\left[f_{,j}^{,i} - \sum_{k=q_1}^{q_n}\left\{f^{,k}N_{bk,j}/(N_{bk}{}^{,k} - \bar{P}_{yk}{}^{,k})\right\}^{,i}\right]\Delta k_{bi}\right] = \text{const.}\ (j = q_1,\ldots,q_n) \tag{34}$$

In case of using Eq. (22) in place of Eq. (24), the following equations should be employed.

$$\sum_{i=p_1}^{p_m} f_{,ji}\Delta S_{di} + \sum_{i=q_1}^{q_n} f_{,j}^{,i}\Delta k_{bi} = \text{const.} \qquad (j = p_{n+1},\ldots,p_m) \tag{35}$$

After some manipulation in Eq. (34), we can derive m–1 equations to determine the optimal solution $\mathbf{S}_d$ and $\mathbf{k}_b$.

The last condition with respect to the design variables $\mathbf{S}_d$ is

$$\sum_{i=p_1}^{p_m} \Delta S_{di} = \mathrm{d}W \tag{36}$$

From Eqs. (30), (34), (35) and (36), we can derive the following simultaneous linear equation for $\{\Delta S_{dp_1}, \ldots, \Delta S_{dp_m}, \Delta k_{bq_1}, \ldots, \Delta k_{bq_n}\}$.

$$\begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1m} & \beta_{11} & \cdots & \beta_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{m-1,1} & \cdots & \alpha_{m-1,m} & \beta_{m-1,1} & \cdots & \beta_{m-1,n} \\ N_{\mathrm{b}q_1,p_1} & \cdots & N_{\mathrm{b}q_1,p_m} & N_{\mathrm{b}q_1}{}^{,q_1} - \overline{P}_{\mathrm{y}q_1}{}^{,q_1} & \cdots & N_{\mathrm{b}q_1}{}^{,q_n} - \overline{P}_{\mathrm{y}q_1}{}^{,q_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ N_{\mathrm{b}q_n,p_1} & \cdots & N_{\mathrm{b}q_n,p_m} & N_{\mathrm{b}q_n}{}^{,q_1} - \overline{P}_{\mathrm{y}q_n}{}^{,q_1} & \cdots & N_{\mathrm{b}q_n}{}^{,q_n} - \overline{P}_{\mathrm{y}q_n}{}^{,q_n} \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix} \begin{Bmatrix} \Delta S_{\mathrm{d}p_1} \\ \vdots \\ \Delta S_{\mathrm{d}p_m} \\ \Delta k_{\mathrm{b}q_1} \\ \vdots \\ \Delta k_{\mathrm{b}q_n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \\ \mathrm{d}W \end{Bmatrix} \tag{37}$$

where α_{ij} ($i = 1,2,...,m-1$; $j = 1,2,...,m$) and β_{ij} ($i = 1,2,...,m-1$; $j = 1,2,...,n$) are described by

$$\begin{aligned} \alpha_{ij} &= f_{,q_1 p_j} - f_{,q_{i+1} p_j} \\ &\quad + \sum_{k=q_1}^{q_n} \frac{1}{N_{\mathrm{b}k}{}^{,k} - \overline{P}_{\mathrm{y}k}{}^{,k}} \begin{Bmatrix} -f^{,k}_{,p_j}(N_{\mathrm{b}k,q_1} - N_{\mathrm{b}k,q_i}) - f^{,k}(N_{\mathrm{b}k,q_1 p_j} - N_{\mathrm{b}k,q_i p_j}) \\ +f^{,k} N_{\mathrm{b}k,p_j}^{,k}(N_{\mathrm{b}k,q_1} - f^{,k} N_{\mathrm{b}k,q_i}) / (N_{\mathrm{b}k}{}^{,k} - \overline{P}_{\mathrm{y}k}{}^{,k}) \end{Bmatrix} \\ &\qquad\qquad (i = 1,\cdots,n-1 \quad j = 1,\ldots,m) \\ \alpha_{ij} &= f_{,p_n p_j} - f_{,p_{i+1} p_j} \qquad\qquad (i = n,\cdots,m-1 \quad j = 1,\ldots,m) \end{aligned} \tag{38-a,b}$$

$$\begin{aligned} \beta_{ij} &= f^{,q_j}_{,p_1} - f^{,q_j}_{,p_{i+1}} \\ &\quad + \sum_{k=q_1}^{q_n} \frac{1}{N_{\mathrm{b}k}{}^{,k} - \overline{P}_{\mathrm{y}k}{}^{,k}} \begin{Bmatrix} -f^{,kq_j}(N_{\mathrm{b}k,q_1} - N_{\mathrm{b}k,q_{i+1}}) - f^{,k}(N_{\mathrm{b}k,q_1}^{,q_j} - N_{\mathrm{b}k,q_{i+1}}^{,q_j}) \\ +N_{\mathrm{b}k}{}^{,kq_j}(f^{,k} N_{\mathrm{b}k,q_1} - f^{,k} N_{\mathrm{b}k,q_{i+1}}) / (N_{\mathrm{b}k}{}^{,k} - \overline{P}_{\mathrm{y}k}{}^{,k}) \end{Bmatrix} \\ &\qquad\qquad (i = 1,\cdots,n-1 \quad j = 1,\ldots,n) \\ \beta_{ij} &= f^{,q_j}_{,p_n} - f^{,q_j}_{,p_{i+1}} \qquad\qquad (i = n,\cdots,m-1 \quad j = 1,\ldots,n) \end{aligned} \tag{39-a,b}$$

7.2 Sensitivity with respect to damper area

The first and second-order sensitivities of the objective function with respect to design variables $\mathbf{S}_{\mathrm{d}}$ and $\mathbf{k}_{\mathrm{b}}$ are derived here. The PSD function of the variable critical excitation has the power $\bar{s}$ in the frequency band between ω_L and ω_U. The objective function in Eq. (15) can be simplified to

$$f(\mathbf{S}_{\mathrm{d}}, \mathbf{k}_{\mathrm{b}}) = \bar{s} \sum_{i=1}^{N} \{\Psi_{\delta i}(\omega_U; \mathbf{S}_{\mathrm{d}}, \mathbf{k}_{\mathrm{b}}) - \Psi_{\delta i}(\omega_L; \mathbf{S}_{\mathrm{d}}, \mathbf{k}_{\mathrm{b}})\} \tag{40}$$

where $\psi_{\delta i}$ is defined by

$$\Psi_{\delta i}(\hat{\omega}; \mathbf{S}_{\mathrm{d}}, \mathbf{k}_{\mathrm{b}}) = \int_0^{\hat{\omega}} H_{\delta i}(\omega) H_{\delta i}{}^{*}(\omega) \mathrm{d}\omega \tag{41}$$

Note that ω_L and ω_U are dependent on the natural circular frequency ω_0 of the building with VED. For this reason, the objective function in Eq. (40) can be regarded as an implicit function of ω_0.

The first and the second-order sensitivities with respect to $\mathbf{S}_\mathrm{d}$ can be derived from

$$f_{,j}=2\overline{s}\sum_{i=1}^{N}\left[\begin{array}{l}\left(\omega_0\right)_{,j}\{H_{\delta i}(\omega_U)-H_{\delta i}(\omega_L)\}\\ +\int_{\omega_L}^{\omega_U}\left[\{H_{\delta i}(\omega)\}_{,j}H_{\delta i}^{*}(\omega)+H_{\delta i}(\omega)\{H_{\delta i}^{*}(\omega)\}_{,j}\right]\mathrm{d}\omega\end{array}\right] \tag{42}$$

$$f_{,jk}=2\overline{s}\sum_{i=1}^{N}\left[\begin{array}{l}\left(\omega_0\right)_{,jk}\{H_{\delta i}(\omega_U)-H_{\delta i}(\omega_L)\}\\ +\left(\omega_0\right)_{,j}\left\{\begin{array}{l}\left(\omega_0\right)_{,k}\ \left(\partial H_{\delta i}(\omega_U)/\partial\omega_0-\partial H_{\delta i}(\omega_L)/\partial\omega_0\right)\\ +H_{\delta i}(\omega_U)_{,k}-H_{\delta i}(\omega_L)_{,k}\end{array}\right\}\\ +\left(\omega_0\right)_{,k}\{H_{\delta i}(\omega_U)_{,j}-H_{\delta i}(\omega_L)_{,j}\}\\ +\int_{\omega_L}^{\omega_U}\left[\begin{array}{l}H_{\delta i}(\omega)_{,jk}H_{\delta i}^{*}(\omega)+H_{\delta i}(\omega)_{,j}H_{\delta i}^{*}(\omega)_{,k}\\ +H_{\delta i}(\omega)_{,k}H_{\delta i}^{*}(\omega)_{,j}+H_{\delta i}(\omega)\{H_{\delta i}^{*}(\omega)\}_{,jk}\end{array}\right]\mathrm{d}\omega\end{array}\right] \tag{43}$$

where $H_{\delta i}(\omega)_{,j}$ and $H_{\delta i}(\omega)_{,jk}$ are derived as follows.

$$H_{\delta i}(\omega)_{,j}=\mathbf{T}_i\,(\mathbf{A}^{-1})_{,j}\,\mathbf{B} \tag{44}$$

$$H_{\delta i}(\omega)_{,jk}=\mathbf{T}_i\,(\mathbf{A}^{-1})_{,jk}\,\mathbf{B} \tag{45}$$

In Eq. (44), the first derivative of $\mathbf{A}^{-1}$ with respect to $\mathbf{S}_\mathrm{d}$ is given by

$$\mathbf{A}_{,j}^{-1}=-\mathbf{A}^{-1}\mathbf{A}_{,j}\mathbf{A}^{-1}=-\mathbf{A}^{-1}\left(\mathbf{K}_{,j}+\mathrm{i}\,\omega\mathbf{C}_{,j}\right)\mathbf{A}^{-1} \tag{46}$$

Furthermore $(\omega_0)_{,j}$ and $(\omega_0)_{,jk}$ are evaluated following the method of Fox and Kapoor (1968).

Referring to Eqs. (1) and (2), the first derivative of the matrices $\mathbf{K}$ and $\mathbf{C}$ with respect to $S_{\mathrm{d}j}$ can be computed by the first derivative of the equivalent stiffness $K_{\mathrm{E}j}$ and the damping coefficient $C_{\mathrm{E}j}$ described by

$$K_{\mathrm{E}i}(\omega,S_{\mathrm{d}i},k_{\mathrm{b}i})_{,j}=k_{\mathrm{b}i}^2\frac{S_{\mathrm{d}i}^2c_1c_2+2c_1c_3k_{\mathrm{b}i}S_{\mathrm{d}i}+c_2c_3k_{\mathrm{b}i}^2}{(c_1S_{\mathrm{d}i}^2+2c_2k_{\mathrm{b}i}S_{\mathrm{d}i}+c_3k_{\mathrm{b}i}^2)^2} \tag{47}$$

$$C_{\mathrm{E}i}(\omega,S_{\mathrm{d}i},k_{\mathrm{b}i})_{,j}=k_{\mathrm{b}i}^2\frac{c_4(c_1S_{\mathrm{d}i}^2+c_3k_{\mathrm{b}i}^2)}{(c_1S_{\mathrm{d}i}^2+2c_2k_{\mathrm{b}i}S_{\mathrm{d}i}+c_3k_{\mathrm{b}i}^2)^2} \tag{48}$$

Eqs. (1) and (2) have been re-written in terms of the parameters k_{dM}, c_{dM}, k_{dV}, c_{dV} without $\{S_{\mathrm{d}j}\}$. The coefficients $c_1 \cdots c_4$ are defined as follows

$$c_1 = \varepsilon_1{}^2 + \varepsilon_2{}^2 \omega^2$$
$$c_2 = k_{\mathrm{dM}} \varepsilon_1 + c_{\mathrm{dM}} \varepsilon_2 \omega^2$$
$$c_3 = k_{\mathrm{dM}}^2 + c_{\mathrm{dM}}^2 \omega^2$$
$$c_4 = -c_{\mathrm{dM}} \varepsilon_1 + k_{\mathrm{dM}} \varepsilon_2 \quad \text{(49-a,b,c,d)}$$

where $\varepsilon_1 = k_{\mathrm{dM}} k_{\mathrm{dV}} - c_{\mathrm{dM}} c_{\mathrm{dV}} \omega^2$, $\varepsilon_2 = k_{\mathrm{dM}} c_{\mathrm{dV}} + k_{\mathrm{dV}} c_{\mathrm{dM}} + k_{\mathrm{dM}} c_{\mathrm{dV}}$.

On the other hand, in Eq. (45), the second derivative of $\mathbf{A}^{-1}$ with respect to $\mathbf{S}_{\mathrm{d}}$ can be computed by differentiating Eq. (46) with respect to $S_{\mathrm{d}k}$.

$$\mathbf{A}_{,jk}^{-1} = \mathbf{A}^{-1}\left(\mathbf{A}_{,j}\mathbf{A}^{-1}\mathbf{A}_{,k} + \mathbf{A}_{,k}\mathbf{A}^{-1}\mathbf{A}_{,j}\right)\mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{A}_{,jk}\mathbf{A}^{-1} \quad (50)$$

In Eq. (50), it should be remarked that in case of using viscous damper or viscoelastic damper in the "3N model", $\mathrm{A}_{,jk} = 0$. This is because the stiffness and damping matrix consist of a linear combination of damper damping coefficients and stiffnesses, i.e., all the components of $\mathbf{A}_{,j}$ are constant values. On the other hand, in the case of using the frequency-dependent "N model", $\mathbf{A}_{,jk} \neq 0$ ($j = k$) because the first derivative of $\mathbf{A}$ contains $K_{\mathrm{E}i}$ and $C_{\mathrm{E}i}$, Eqs. (47) and (48), which are also the functions of design variables $\mathbf{S}_{\mathrm{d}}$ and $\mathbf{k}_{\mathrm{b}}$.

The second-order sensitivities of the equivalent stiffness and damping coefficient with respect to damper area are shown in Appendix 3.

7.3 Sensitivity with respect to the stiffness of the supporting members

The sensitivity of the objective function with respect to $\mathbf{k}_{\mathrm{b}}$ is also needed to determine the optimal solution for the stiffness of the supporting members. These sensitivities can be derived following the same procedures used in deriving Eqs. (42) and (43).

$$f^{,j} = 2\overline{s}\sum_{i=1}^{N}\left[\int_{\omega_L}^{\omega_U} H_{\delta i}(\omega)^{,j} H_{\delta i}^{*}(\omega)\mathrm{d}\omega + \int_{\omega_L}^{\omega_U} H_{\delta i}(\omega) H_{\delta i}^{*}(\omega)^{,j}\,\mathrm{d}\omega\right] \quad (51)$$

$$f^{,jk} = 2\overline{s}\sum_{i=1}^{N}\left[\begin{array}{l}\int_{\omega_L}^{\omega_U} H_{\delta i}(\omega)^{,jk} H_{\delta i}^{*}(\omega)\mathrm{d}\omega + \int_{\omega_L}^{\omega_U} H_{\delta i}(\omega)^{,j} H_{\delta i}^{*}(\omega)^{,k}\mathrm{d}\omega \\ + \int_{\omega_L}^{\omega_U} H_{\delta i}(\omega)^{,k} H_{\delta i}^{*}(\omega)^{,j}\,\mathrm{d}\omega + \int_{\omega_L}^{\omega_U} H_{\delta i}(\omega)\{H_{\delta i}^{*}(\omega)\}^{,jk}\,\mathrm{d}\omega\end{array}\right] \quad (52)$$

where $H_{\delta i}(\omega)^{,j} = \mathbf{T}_i\,(\mathbf{A}^{-1})^{,j}\,\mathbf{B}$ and $H_{\delta i}(\omega)^{,jk} = \mathbf{T}_i\,(\mathbf{A}^{-1})^{,jk}\,\mathbf{B}$. The first derivative of $\mathbf{A}^{-1}$ with respect to $\mathbf{k}_{\mathrm{b}}$ can be computed by replacing $\mathbf{K}_{,j}$ and $\mathbf{C}_{,j}$ with $\mathbf{K}^{,j}$ and $\mathbf{C}^{,j}$ whose components consist of $K_{\mathrm{E}i}{}^{,j}$ and $C_{\mathrm{E}i}{}^{,j}$ given as

$$K_{\mathrm{E}i}(\omega, S_{\mathrm{d}i}, k_{\mathrm{b}i})^{,i} = S_{\mathrm{d}i}{}^2\,\frac{S_{\mathrm{d}i}^2 c_1^2 + 2c_1c_2k_{\mathrm{b}i}S_{\mathrm{d}i} + (2c_2^2 - c_1c_3)k_{\mathrm{b}i}^2}{(c_1S_{\mathrm{d}i}^2 + 2c_2k_{\mathrm{b}i}S_{\mathrm{d}i} + c_3k_{\mathrm{b}i}^2)^2} \quad (53)$$

$$C_{\mathrm{E}i}(\omega, S_{\mathrm{d}i}, k_{\mathrm{b}i})^{,i} = \frac{2S_{\mathrm{d}i}^2 k_{\mathrm{b}i} c_4 (k_{\mathrm{b}i}c_2 + c_1S_{\mathrm{d}i})}{(c_1S_{\mathrm{d}i}^2 + 2c_2k_{\mathrm{b}i}S_{\mathrm{d}i} + c_3k_{\mathrm{b}i}^2)^2} \quad (54)$$

Note that the parameters K_{Ei} and C_{Ei} need to be evaluated only for $j = i$ In the same way, the second derivative of $\mathbf{A}^{-1}$ with respect to $\mathbf{k}_b$ can be evaluated in terms of $\mathbf{A}^{,j}$, $\mathbf{A}^{,jk}$ and $\mathbf{A}^{-1}$.

The second-order sensitivities of the equivalent stiffness and damping coefficient with respect to the stiffness of the supporting members are shown in Appendix 3.

7.4 Sensitivities of the axial force of the supporting members

The sensitivity of the axial force N_{bi} of the supporting member with respect to $\mathbf{S}_d$ can be derived by differentiating Eq. (16) as follows.

$$N_{bi,j} = \rho\left(2\bar{s} / \int_{\omega_L}^{\omega_U} \left|H_{Nbi}(\omega)\right|^2 d\omega\right)^{1/2} \int_{\omega_L}^{\omega_U} \mathrm{Re}\left[H_{Nbi}(\omega)_{,j} H_{Nbi}^{*}(\omega)\right] d\omega \tag{55}$$

As defined before in Eq. (17), $H_{Nbi}(\omega)$ is obtained from the complex stiffness of the equivalent "N model". In Eq. (55), the sensitivity of the axial force N_{bi} with respect to $\mathbf{S}_d$, i.e., $(H_{Nbi}(\omega))_{,j}$, can be derived from Eq. (17).

$$H_{Nbi}(\omega)_{,j} = \begin{cases} \hat{K}_{Ei} H_{\delta i}(\omega)_{,j} + \hat{K}_{Ei,j} H_{\delta i}(\omega) & (i = j) \\ \hat{K}_{Ei} H_{\delta i}(\omega)_{,j} & (i \neq j) \end{cases} \tag{56}$$

where $\hat{K}_{Ei}$ is a complex stiffness of the equivalent whole damper unit as an "N model", defined by $\hat{K}_{Ei} = K_{Ei} + i\omega C_{Ei}$.

The sensitivity of N_{bi} with respect to $\mathbf{k}_b$ can be derived by replacing $H_{Nbi}(\omega)_{,j}$ in Eq. (55) with $H_{Nbi}(\omega)^{,j}$ computed by substituting $\hat{K}_{Ei}{}^{,j}$, i.e., $K_{Ei}{}^{,j} + i\omega C_{Ei}{}^{,j}$, into Eq. (56) instead of $\hat{K}_{Ei,j}$.

8. Numerical examples

Numerical examples are presented for 3-storey and 10-storey building models to demonstrate the usefulness and validity of the proposed optimal design method. Detailed comparison between "3N model" and "N model" is also presented to demonstrate the validity and accuracy of the proposed formulations for the two models.

The structural parameters are shown in Table 1. The floor masses and frame storey stiffnesses are identical in all the storeys. The structural damping ratio of the main frame is assumed to be 0.02

Table 1 Structural parameters of main frame

	3-storey model	10-storey model
Floor mass [kg]	512×10^3	1024×10^3
Storey stiffness (N/mm)	6.02×10^8	1.20×10^9
Natural circular frequency without damper (rad/s)	15.268	5.125
Natural period without damper (s)	0.411	1.225

Table 2 Properties of acrylic viscoelastic damper per unit area

k_{dV}	2756.3[N/m^3]	k_{dM}	5120.5[N/m^3]
c_{dV}	221.7[Ns/m^3]	c_{dM}	254.8[Ns/m^3]

Table 3 Optimality conditions in each domain

Domain A (m = 1, n = 0)	p_1 storey Eq. (22) Eq. (23)	Other storeys Eq. (27)	
Domain B ($m \geq 2$, n = 0)	$p_1 \sim p_m$ storey Eq. (22) Eq. (23)	Other storey Eq. (27)	
Domain C (m = 1, n = 1)	p_1 (=q_1) storey Eq. (24) Eq. (25)	Other storey Eq. (27)	
Domain D ($m \geq 2$, $n \geq 1$)	$q_1 \sim q_n$ storey Eq. (24) Eq. (25)	$p_{n+1} \sim p_m$ storey Eq. (22) Eq. (23)	Other storey Eq. (27)

(stiffness-proportional damping). The properties of each component in four elements of VED per unit damper area (thickness is fixed at 10 mm and area is 1 m^2) are shown in Table 2 (Lee *et al.* 2002). The ratio $r_s = k_{bi} / k_{fi}$ between the stiffness k_{fi} of the building frame and that k_{bi} of the supporting member is varied between 0.5 and 3.0. The initial stiffness of the supporting member is given by selecting an appropriate ratio r_s. The total damper area $\overline{W}$ is taken as $\overline{W}$ = 51.2 m^2 for the 3-storey building model and $\overline{W}$ = 400 m^2 for the 10-storey building model. The peak factor for axial force of supporting member is given as 3.0 in both examples. The parameters $\bar{s}$ and $\bar{S}$ of the input critical acceleration are estimated as $\bar{s}$ = 0.02 m^2/s^3 and $\bar{S}$ = 0.04 m^2/s^4 from the El Centro NS 1940 with PGA = 0.32 g.

The PSD function of the critical ground motion defined as a variable critical excitation is computed to have the circular frequency resonant to the fundamental natural circular frequency of the building model. In the process of optimal damper placement where the stiffness and damping

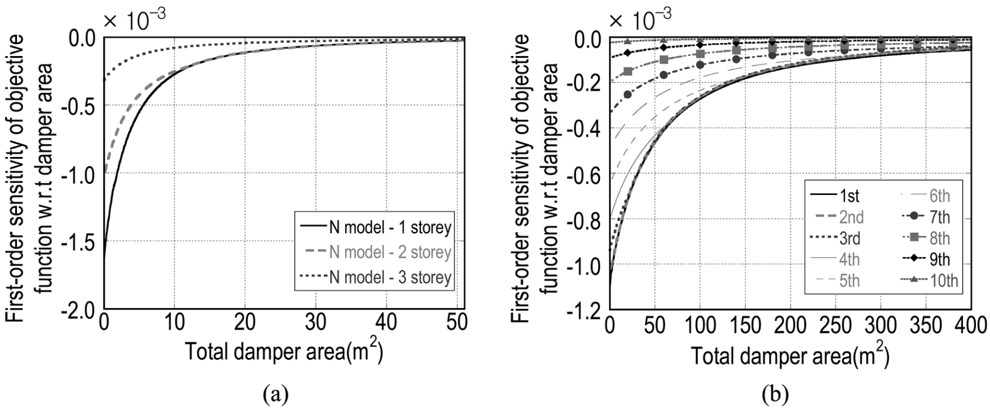
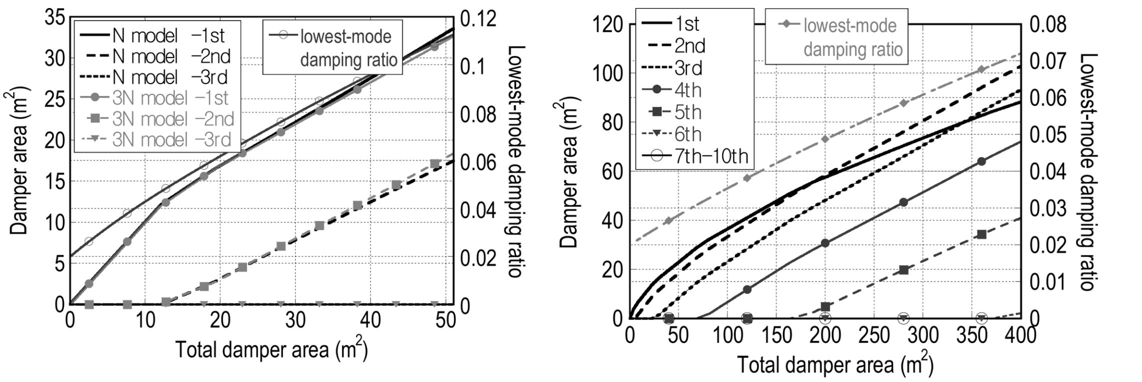


Fig. 6 First-order sensitivity of sum of mean-squares interstorey drifts with respect to VED area (a) 3-storey model, (b) 10-storey model

matrices are updated according to the available additional VED, eigenvalue analysis has to be carried out in every step. It should be remarked that the 3N model, whose components are independent of frequency, facilitates the computation of eigenvalues.

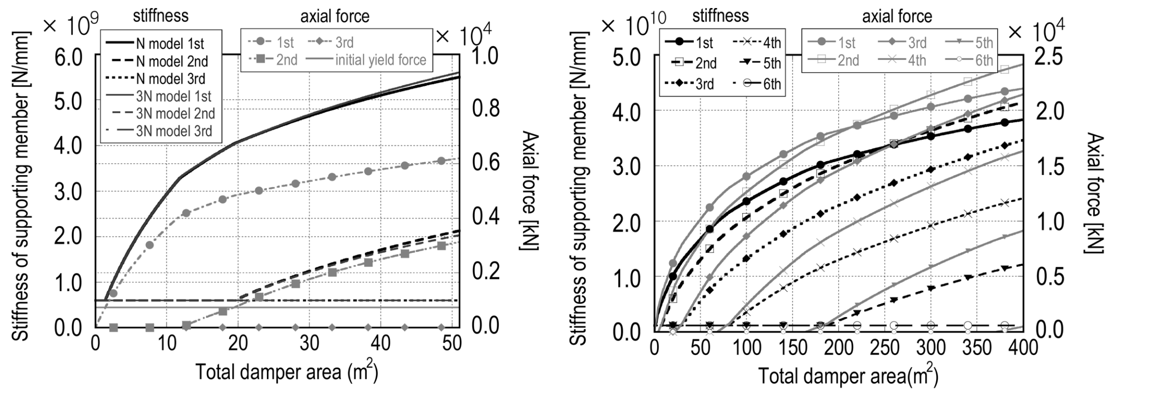
Fig. 6 shows the variation of the first-order sensitivities of the objective function with respect to $\mathbf{S}_d = \{S_{d1},..., S_{dN}\}$ for (a) 3-storey and (b) 10-storey models where $r_s = 1.0$. In the initial phase of the optimization process which corresponds to the domain A, the maximum absolute value of the first derivative is attained only in the first storey, i.e., $|f_{,1}|$. It is shown in Fig. 6 that, after the multiple coincidence of the maximum absolute value of the first derivatives in domain B, they continue to be satisfied in the optimal design process. This fact indicates a continuing satisfaction of the stationarity conditions of Lagrangian as the optimality conditions.

Fig. 7 illustrates the optimal area distribution $\mathbf{S}_d$ of VED with respect to varied total damper area and a variation of the lowest-mode damping ratio for (a) 3-storey and (b) 10-storey models. It can be observed that the passive dampers are placed optimally in the building model according to the variation of the first derivative of the objective function shown in Fig. 6. Fig. 7(a) depicts the comparison of this result by using different damper models, i.e., "N model" and "3N model" for the



(a) (b)

Fig. 7 Optimal damper placement and lowest-mode damping ratio (a) 3-storey model, (b) 10-storey model



(a) (b)

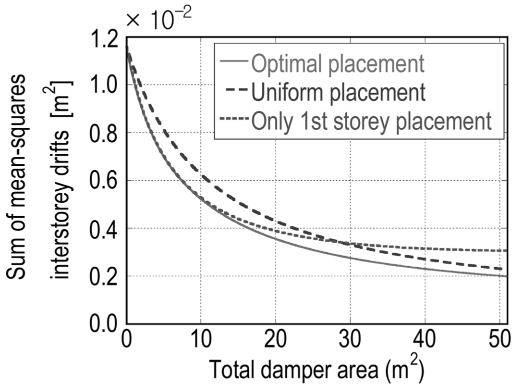
Fig. 8 Stiffness and axial force of supporting member (a) 3-storey model, (b) 10-storey model

3-storey model. It can be seen that almost identical result are obtained by using either of the two damper models. This verifies the validity of the two proposed VED models.

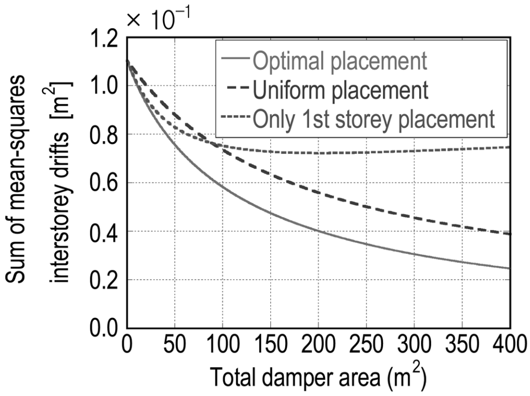
Fig. 8 shows the evolution of the stiffness and the axial force of the supporting members in the process of optimal damper placement for (a) 3-storey and (b) 10-storey models. The initial yield force of the supporting members is also shown in Fig. 8(a). It can be observed that, after the axial force N_{bi} coincides with the initial yield force, $\mathbf{k}_b$ is updated according to the optimality conditions.

Fig. 9 shows the variation of the objective function for the following three distributions in (a) 3-storey and (b) 10-storey models: Case 1) Optimal damper placement based on the proposed method, Case 2) Uniform placement, and Case 3) First-storey placement. It can be observed that the result of case 1 perfectly coincides with that of case 3 in the early stage of the optimization procedure. However, as the total damper area increases, the objective function is minimized by the optimal damper placement.

In order to investigate the effect of the stiffness of the supporting members on the optimal damper distribution, two cases of $r_s = k_{bi} / k_{Fi} = 0.5$ and 3.0 are considered for the 3-storey model. The yield

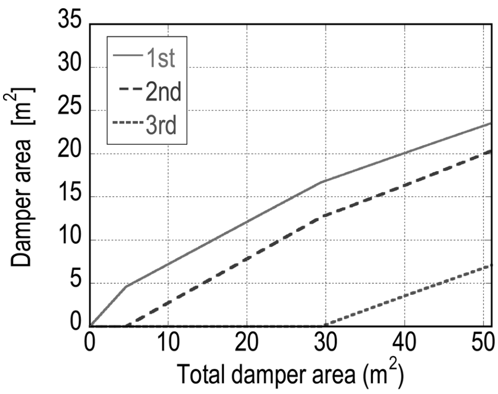


(a)

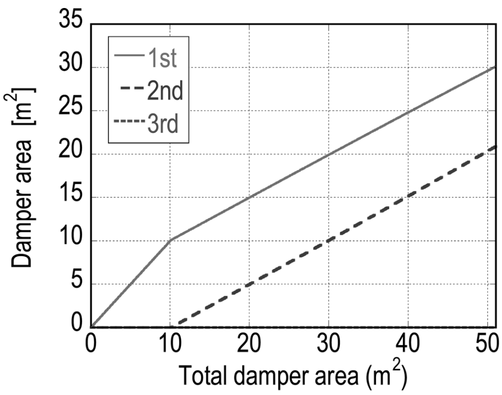


(b)

Fig. 9 Variation of objective function in optimal placement, uniform placement and first-storey placement (a) 3-storey model, (b) 10-storey model



(a)



(b)

Fig. 10 Evolution of optimal damper placement (a) $k_{bi} / k_{Fi} = 0.5$, (b) $k_{bi} / k_{Fi} = 3.0$

forces of the supporting members are set at a high level to investigate the effect of the stiffness of the supporting members. Fig. 10 shows the evolution of the optimal damper distribution for (a) $r_s = k_{bi} / k_{Fi} = 0.5$ and (b) $r_s = k_{bi} / k_{Fi} = 3.0$. It can be observed that, while the dampers are concentrated in the lower storeys for $r_s = k_{bi} / k_{Fi} = 3.0$, they are distributed in all the storeys for $r_s = k_{bi} / k_{Fi} = 0.5$. This reveals that the optimal distribution of the dampers requires the damper installation in the effective position where the interstorey drift is large.

9. Conclusions

The conclusions may be stated as follows:

(1) An evolutionary optimal placement method of viscoelastic dampers and supporting members has been proposed. The critical earthquake ground motion is defined as the resonant input to the structure with viscoelastic dampers. As the size or quantity of viscoelastic dampers becomes large, the force acting on the supporting member increases and an appropriate cross-sectional area of the supporting member is required. Simultaneous design consideration of viscoelastic dampers and supporting members is a new aspect of the theoretical development and practicality.

(2) The sum of the mean-squares of interstorey drifts under random input is taken as an objective function. The total quantity of viscoelastic dampers has been increased evolutionary while the constraint on the member force of the supporting member is satisfied.

(3) Two models are used in the modeling of the viscoelastic dampers. The first model is the four-elements model of viscoelastic dampers with a supporting member. Two masses are considered in this model. Then the $3N$ degrees-of-freedom model for structural analysis is employed in the first model. The second model is an equivalent Kelvin-Voigt model of viscoelastic dampers with a supporting member. There is no additional mass in the model of the equivalent Kelvin-Voigt model. Then the N degrees-of-freedom model for structural analysis is employed in the second model.

(4) A gradient-based evolutionary optimization technique is developed by using the Lagrange multiplier optimization technique. Simultaneous satisfaction of the optimality criterion on placement of viscoelastic dampers and the constraint on forces of the supporting members have been guaranteed which have been demonstrated through numerical examples.

In this study, the simultaneous optimal placement of viscoelastic dampers and the optimal design of the stiffnesses of the supporting members are estimated under the system-dependent variable critical input that is modeled as stationary random ground motion. Although the objective function becomes a time-dependent function for non-stationary earthquake inputs, the framework remains the same. In such case, the optimal damper and the stiffness parameters should be evaluated by minimizing the maximum response. This requires the minimization of the objective function at discrete time instants which obviously increases the computations. This aspect is of interest and will be carried out in a future work.

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Appendix 1 Equivalent stiffness and damping coefficient of damper unit including supporting member in *N*-model (Eqs. (1) and (2))

Let δ_{Fi}, δ_{1i}, δ_{2i} denote the interstorey drift, the internal nodal displacement in the Maxwell model in Fig. 2 relative to the (i-1)-th floor and the displacement of the node between the damper unit and the supporting member in the i-th storey, i.e., $\delta_{Fi} = u_{Fi} - u_{F(i-1)}$, $\delta_{1i} = u_{1i} - u_{F(i-1)}$ and $\delta_{2i} = u_{2i} - u_{F(i-1)}$. The equations of dynamic equilibrium of the $3N$ model can be derived as

$$k_{bi}\delta_{2i} = p_i(t) \tag{A1-1}$$

$$k_{Mi}(\delta_{1i} - \delta_{2i}) + k_{Vi}(\delta_{Fi} - \delta_{2i}) + c_{Vi}(\dot{\delta}_{Fi} - \dot{\delta}_{2i}) - k_{bi}\delta_{2i} = 0 \tag{A1-2}$$

$$c_{Mi}(\dot{\delta}_{Fi} - \dot{\delta}_{1i}) - k_{Mi}(\delta_{1i} - \delta_{2i}) = 0 \tag{A1-3}$$

where $p_i(t)$ denotes the internal force of the supporting member in the i-th storey. Let $\Delta_{Fi}(\omega)$, $\Delta_{1i}(\omega)$, $\Delta_{2i}(\omega)$, $P_i(\omega)$ denote the Fourier transforms of $\delta_{Fi}(t)$, $\delta_{1i}(t)$, $\delta_{2i}(t)$ and $p_i(t)$. From Eq. (A1-1), $\Delta_{2i}(\omega)$ can be described by $P_i(\omega)/k_{bi}$. By substituting this equation into Eq. (A1-3) expressed in frequency domain, we can obtain $\Delta_{1i}(\omega)$ as

$$\Delta_{1i}(\omega) = \frac{\mathrm{i}\,\omega k_{bi} c_{Mi} \Delta_{Fi}(\omega) + k_{Mi} P_i(\omega)}{k_{bi}(\mathrm{i}\,\omega c_{Mi} + k_{Mi})} \tag{A1-4}$$

Substitution of these equations for $\Delta_{1i}(\omega)$ and $\Delta_{2i}(\omega)$ into Eq. (A1-2) in frequency domain leads to the relationship between $\Delta_{Fi}(\omega)$ and $P_i(\omega)$.

$$\begin{aligned}&\left(\frac{\mathrm{i}\,\omega k_{bi} k_{Mi} c_{Mi}}{k_{Mi} + \mathrm{i}\,\omega c_{Mi}} + k_{bi}k_{Vi} + \mathrm{i}\,\omega k_{bi} c_{Vi}\right)\Delta_{Fi}(\omega) \\ &= \left\{(k_{bi} + k_{Mi} + k_{Vi} + \mathrm{i}\,\omega c_{Vi}) - \frac{k_{Mi}^2}{k_{Mi} + \mathrm{i}\,\omega c_{Mi}}\right\} P_i(\omega)\end{aligned} \tag{A1-5}$$

After some manipulations, Eq. (A1-5) can be rewritten as

$$\frac{k_{bi}k_{Vi}k_{Mi}^2 + \omega^2 k_{bi} c_{Mi}^2 (k_{Vi} + k_{Mi}) + \mathrm{i}\,\omega k_{bi}\left\{k_{Mi}^2 (c_{Vi} + c_{Mi}) + \omega^2 c_{Vi} c_{Mi}^2\right\}}{\left((k_{bi} + k_{Vi})k_{Mi}^2 + (k_{bi} + k_{Mi} + k_{Vi})\omega^2 c_{Mi}^2\right) + \mathrm{i}\,\omega\left\{k_{Mi}^2 (c_{Vi} + c_{Mi}) + \omega^2 c_{Vi} c_{Mi}^2\right\}}\Delta_{Fi}(\omega) = P_i(\omega) \tag{A1-6}$$

On the other hand, the force-displacement relation of the general Kelvin-Voigt model can be given by

$$(K_{Ei} + \mathrm{i}\,\omega C_{Ei})\Delta_{Fi}(\omega) = P_i(\omega) \tag{A1-7}$$

where K_{Ei} and C_{Ei} are the equivalent stiffness and the damping coefficient of the frequency-dependent Kelvin-Voigt model in the i-th storey defined by Eqs. (1) and (2). By comparing Eq. (A1-6) and Eq. (A1-7), Eqs. (1) and (2) can be derived.

Appendix 2 Transformation matrix from the nodal displacements to the relative displacements between both ends of supporting members

For evaluating the axial force of the supporting member, the relative displacements $\mathbf{u}_b$ between both ends of supporting members are expressed by

$$\mathbf{u}_b = \mathbf{T}_b \mathbf{u}_{full} \tag{A2-1}$$

where $\mathbf{T}_b$ denotes the transformation matrix. In the case of the 3-storey building model, $\mathbf{T}_b$ can be given by

$$\mathbf{T_b} = \left[\begin{array}{ccc:ccc:ccc} 0 & 0 & 0 & & & & & & \\ 0 & 1 & 0 & & [\mathbf{0}] & & & [\mathbf{0}] & \\ 0 & 0 & 0 & & & & & & \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & & & \\ -1 & 0 & 0 & 0 & 1 & 0 & & [\mathbf{0}] & \\ 0 & 0 & 0 & 0 & 0 & 0 & & & \\ \hdashline & & & 0 & 0 & 0 & 0 & 0 & 0 \\ & [\mathbf{0}] & & -1 & 0 & 0 & 0 & 1 & 0 \\ & & & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{A2-2}$$

Appendix 3 Second-order sensitivities of the equivalent stiffness and damping coefficient

Second-order sensitivities of the equivalent stiffness and the damping coefficient for the N model can be derived as follows

$$K_{Ei,ii} = -2k_{bi}^2 \frac{c_1{}^2 c_2 S_{di}^3 + 3c_1{}^2 c_3 k_{bi} S_{di}^2 + 3c_1 c_2 c_3 k_{bi}^2 S_{di} + c_3 (2c_2{}^2 - c_1 c_3) k_{bi}^3}{(c_1 S_{di}^2 + 2c_2 k_{bi} S_{di} + c_3 k_{bi}^2)^3}$$

$$K_{Ei}{}^{,ii} = \left(S_{di}^2 / k_{bi}^2\right) K_{Ei,jk}$$

$$K_{Ei,i}^{,i} = 2k_{bi} S_{di} \frac{c_1{}^2 c_2 S_{di}^3 + 3c_1{}^2 c_3 k_{bi} S_{di}^2 + 3c_1 c_2 c_3 k_{bi}^2 S_{bi} + \left(2c_2{}^2 c_3 - c_1 c_2{}^2\right) k_{bi}^2 S_{di}}{(c_1 S_{di}^2 + 2c_2 k_{bi} S_{di} + c_3 k_{bi}^2)^3}$$

$$C_{Ei,ii} = -2k_{bi}^2 c_5 \frac{c_1 c_3 k_{bi}^2 S_{di} + 2c_2 c_3 k_{bi}^3 + c_1{}^2 S_{di}^3}{(c_1 S_{di}^2 + 2c_2 k_{bi} S_{di} + c_3 k_{bi}^2)^3}$$

$$C_{\mathrm{E}i}{}^{,ii} = -2S_{\mathrm{d}i}^2 c_5 \frac{3c_1c_3k_{\mathrm{b}i}^2 S_{\mathrm{d}i} + 2c_2c_3k_{\mathrm{b}i}^3 - {c_1}^2 S_{\mathrm{d}i}^3}{(c_1S_{\mathrm{d}i}^2 + 2c_2k_{\mathrm{b}i}S_{\mathrm{d}i} + c_3k_{\mathrm{b}i}^2)^3}$$

$$C_{\mathrm{E}i,i}{}^{,i} = -2k_{\mathrm{b}i}S_{\mathrm{d}i}c_5 \frac{2c_2c_3k_{\mathrm{b}i}^3 + c_1c_3k_{\mathrm{b}i}^2 S_{\mathrm{d}i} + {c_1}^2 S_{\mathrm{d}i}^3}{(c_1S_{\mathrm{d}i}^2 + 2c_2k_{\mathrm{b}i}S_{\mathrm{d}i} + c_3k_{\mathrm{b}i}^2)^3}$$

(A3-1,2,3,4,5,6)

where c_1, c_2, c_3 are given by Eq. (49a-c) and $c_5 = k_{\mathrm{dM}}\,\varepsilon_2 - c_{\mathrm{dM}}\,(k_{\mathrm{dM}}\,k_{\mathrm{dV}} - c_{\mathrm{dM}}\,c_{\mathrm{dv}}\,\omega^2)$.