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Non-stationary mixed problem of elasticity for a semi-strip

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Abstract. This study is dedicated to the dynamic elasticity problem for a semi-strip. The semi-strip is loaded by the dynamic load at the center of its short edge. The conditions of fixing are given on the lateral sides of the semi-strip. The initial problem is reduced to one-dimensional problem with the help of Laplace's and Fourier's integral transforms. The one-dimensional boundary problem is formulated as the vector boundary problem in the transform's domain. Its solution is constructed as the superposition of the general solution for the homogeneous vector equation and the partial solution for the inhomogeneous vector equation. The matrix differential calculation is used for the deriving of the general solution. The partial solution is constructed with the help of Green's matrix-function, which is searched as the bilinear expansion. The case of steady-state oscillations is considered. The problem is reduced to the solving of the singular integral equation. The orthogonalization method is applied for the calculations. The stress state of the semi-strip is investigated for the different values of the frequency.

Keywords: semi-strip; dynamic problem; steady-state oscillation; singular integral equation; Green matrix-function

1. Introduction

The plane problems of elasticity for a semi-strip in a static statement were investigated by many authors. However many unresolved issues remain especially for a dynamic statement of the problem. As for the static statements, for example, the problem for a symmetrically loaded semi-strip fixed by its short edge was reduced to the Fredholm integral equation of the first kind in Vorovich and Kopasenko (1966). The static problem for an elastic semi-strip loaded by its short edge in three configurations was solved in Menshykov *et al*. (2018). The first basic odd-symmetric boundary value problem in the theory of elasticity in a half-strip with free longitudinal sides was solved in Kovalenko *et al*. (2018) by the use of Papkovich-Fadle eigenfunctions.

The solving of the dynamic problems is usually done with the help of Laplace's transformation. However, the inversion of this transformation is enough complicated, so some authors use a numerical inversion or an asymptotic analysis of the derived solution in the transformation's domain. The Laplace's transform was used for the stress state evaluation of an elastic half-strip under a nonstationary load applied to its boundary and the solution is expanded into a Fourier

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series in Kubenko and Yanchevskii (2015). Dynamic stress in an infinite elastic strip, containing two circular cylindrical cavities, of equal radii, were explored under the assumption of plane strain in Itou (1994). In the Laplace's transform domain, boundary conditions at the plane surfaces and those at the circular cavity were satisfied with the Fourier transformation and the Schmidt method respectively. Edge-vibration, and associated resonance phenomena, was investigated in respect of a semi-infinite strip composed of pre-stressed incompressible elastic material in Kaplunov *et al.* (2004).

The motion equations for gain control systems were deduced in Tartakovsky (1957). The solutions for these problems are constructed by the method of generalized power series. The investigation of cracks and rigid inclusions in the dynamic statement was conducted in Mykhaskiv and Khay (2009). The convergence of series was analyzed and the method of separation of singularities for the method of homogeneous solutions was proved on the example of the mixed problem for a semi-strip in Gomilko *et al.* (1990). The problems for a functionally graded piezoelectric materials were investigated in Wünsche *et al.* (2017).

Two cases, where boundary conditions and solutions of the well-known integrable equations on a semi-strip are uniquely determined by the initial conditions, are studied in Sakhnovich (2016). The elasticity operator in a semi-strip subject to free boundary conditions was studied in Roitberg *et al.* (1997). Using the method of complex analysis and through constructing appropriate conformal mapping, the plane elasticity problem of dynamic cracks in finite-width single-edged cracked strips was analyzed in Guan (2015). The efficient method of approximate factorization of matrix functions was proposed in Babeshko (1979). The inverse problem for a quadratic pencil of Sturm-Liouville operators with periodic potential was solved in Babajanov *et al.* (2005).

The dynamics of the oscillating moving ring load acting in the interior of the hollow circular cylinder surrounded by an elastic medium was studied in Akbarov and Mehdiyev (2018). A two-dimensional thermoelastic problem of thick circular plate of finite thickness under fractional order theory of thermoelastic diffusion has been considered in frequency domain in Lata (2019). The effect of frequency in the axisymmetric thick circular plate has been depicted there. The thermo-mechanical vibration characteristics of functionally graded nanobeams subjected to three types of thermal loading including uniform, linear and non-linear temperature change are investigated in the framework of third-order shear deformation beam theory which captures both the microstructural and shear deformation effects in Ebrahimi and Barati (2017).

In the proposed work the new approach for the solving of the dynamic problem for an elastic semi-strip is proposed. It is based on the apparatuses of matrix differential calculation and matrix Green function. The analytical solution is derived in Laplace's transform domain. The case of steady-state oscillations is investigated.

2. The statement of the problem

The plane elastic semi-strip (Fig. 1) (G is a share module, μ is a Poisson ratio) occupying an area $0<x<a$, $0<y<\infty$ is loaded by its short edge by a non-stationary load

$$\begin{gathered} \sigma_y\big|_{y=0} = p(x,t), \qquad \tau_{xy}\big|_{y=0} = 0, \qquad 0 < a_0 < x < a_1 < a,\ t > 0, \\ v\big|_{y=0} = 0, \qquad \tau_{xy}\big|_{y=0} = 0, \qquad 0 < x < a_0, a_1 < x < a,\ t > 0. \end{gathered} \tag{1}$$

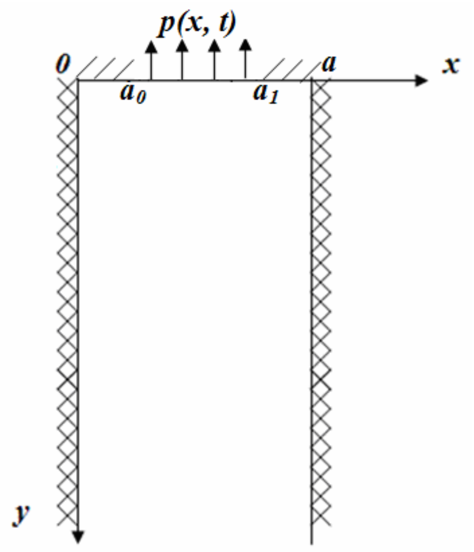

Fig. 1 The geometry and coordinate system of the semi-strip

The lateral sides of the semi-strip are fixed

$$u(0,y,t)=0,\quad v(0,y,t)=0,\quad 0<y<\infty, t>0\,, \tag{2}$$

$$u(a,y,t)=0,\quad v(a,y,t)=0,\quad 0<y<\infty, t>0\,. \tag{3}$$

Here displacement's functions are denoted as $u_x(x,y,t)=u(x,y,t),\ u_y(x,y,t)=v(x,y,t)$.

The motion equations have the following form

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\kappa-1}{\kappa+1}\dfrac{\partial^2 u}{\partial y^2}+\dfrac{2}{\kappa+1}\dfrac{\partial^2 v}{\partial x\partial y}-\dfrac{\rho}{G}\dfrac{\kappa-1}{\kappa+1}\dfrac{\partial^2 u}{\partial t^2}=0, \\ \dfrac{\partial^2 v}{\partial x^2}+\dfrac{\kappa+1}{\kappa-1}\dfrac{\partial^2 v}{\partial y^2}+\dfrac{2}{\kappa-1}\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\rho}{G}\dfrac{\partial^2 v}{\partial t^2}=0, \end{cases} \tag{4}$$

where ρ is semi-strip's density, $\kappa=3-4\mu$ is Muskhelishvili's constant. It is supposed that initial conditions of this problem are null.

It is necessary to evaluate the wave field of the semi-strip, to derive the analytical formulas for the displacements and stresses, and investigate them depending on the strip's geometrical parameters, size of segment where the load is applied and load's behavior.

3. Application of integral transforms

To reduce the stated boundary initial problem to one dimensional boundary value problem the Laplace's transform is applied to the correspondences (1)-(4)

$$\begin{cases} \dfrac{\partial^2 u_s}{\partial x^2} + \dfrac{\kappa-1}{\kappa+1}\dfrac{\partial^2 u_s}{\partial y^2} + \dfrac{2}{\kappa+1}\dfrac{\partial^2 v_s}{\partial x \partial y} - q^2\dfrac{\kappa-1}{\kappa+1}u_s = 0, \\ \dfrac{\partial^2 v_s}{\partial x^2} + \dfrac{\kappa+1}{\kappa-1}\dfrac{\partial^2 v_s}{\partial y^2} + \dfrac{2}{\kappa-1}\dfrac{\partial^2 u_s}{\partial x \partial y} - q^2 v_s = 0, \\ u_s(0,y) = 0, \quad v_s(0,y) = 0, \quad 0 < y < \infty, \\ u_s(a,y) = 0, \quad v_s(a,y) = 0, \quad 0 < y < \infty, \\ \dfrac{2G}{1-2\mu}\left[\mu\dfrac{\partial u_s}{\partial x}(x,0) + (1-\mu)\dfrac{\partial v_s}{\partial y}(x,0)\right] = p_s(x), a_0 < x < a_1, \\ v_s(x,0) = 0, \quad 0 < x < a_0, a_1 < x < a, \\ \dfrac{\partial u_s}{\partial y}(x,0) + \dfrac{\partial v_s}{\partial x}(x,0) = 0, \quad 0 < x < a, \end{cases} \tag{5}$$

here $q^2 = \rho / G \cdot s^2$, s is the parameter of Laplace's transform.

The semi-infinite sin-, cos- integral Fourier transformation is applied to the boundary problem (5) with respect to variable y by the scheme

$$\begin{bmatrix} u_{s\beta}(x) \\ v_{s\beta}(x) \end{bmatrix} = \int_0^\infty \begin{bmatrix} u_s(x,y) \\ v_s(x,y) \end{bmatrix} \begin{bmatrix} \cos\beta y \\ \sin\beta y \end{bmatrix} dy$$

The one-dimensional boundary problem in transformations' domain is formulated as the vector boundary problem Vaysfeld and Zhuravlova (2015)

$$\begin{cases} L_2 \vec{y}_{s\beta}(x) = \vec{f}_s(x), \\ \vec{y}_{s\beta}(0) = 0, \vec{y}_{s\beta}(a) = 0 \end{cases} \tag{6}$$

where $L_2 \vec{y}_{s\beta}(x) = I\vec{y}_{s\beta}''(x) + 2\beta Q\vec{y}_{s\beta}'(x) - P_{s\beta}\vec{y}_{s\beta}(x)$, I is a unit matrix, $Q = \begin{pmatrix} 0 & \dfrac{1}{\kappa+1} \\ -\dfrac{1}{\kappa-1} & 0 \end{pmatrix}$, $P_{s\beta} = \begin{pmatrix} (\beta^2+q^2)\dfrac{\kappa-1}{\kappa+1} & 0 \\ 0 & \beta^2\dfrac{\kappa+1}{\kappa-1}+q^2 \end{pmatrix}$, $\vec{f}_s(x) = \begin{pmatrix} \dfrac{3-\kappa}{\kappa+1}\chi_s'(x) \\ -\beta\dfrac{\kappa+1}{\kappa-1}\chi_s(x) \end{pmatrix}$,

$\chi(x,t) = v\big|_{y=0}, \chi'(x,t) = v'\big|_{y=0}$, $\vec{y}_{s\beta}(x) = \begin{pmatrix} u_{s\beta}(x) \\ v_{s\beta}(x) \end{pmatrix}$.

The solution of inhomogeneous equation in the vector boundary problem (6) is constructed as the superposition Vaisfel'd and Zhuravlova (2018)

$$\vec{y}_{s\beta}(x) = \vec{y}^{\,0}_{s\beta}(x) + \vec{y}^{\,1}_{s\beta}(x) \tag{7}$$

here $\vec{y}^{\,0}_{s\beta}(x)$ is the general solution of the vector homogeneous Eq. (6) and $\vec{y}^{\,1}_{s\beta}(x)$ is the partial solution of the vector inhomogeneous equation.

4. The general solution of the homogeneous vector equation

The general solution is constructed with the help of matrix differential calculation. Accordingly to it the corresponding matrix equation is considered $L_2 Y_{s\beta}(x) = 0$. The matrix $Y_{s\beta}(x)$ is chosen in the form $Y_{s\beta}(x) = e^{\xi x} I$ and substituted into the matrix equation. As the result, the equality $L_2 e^{\xi x} I = M(\xi) e^{\xi x}$ is derived, where

$$M(\xi) = I\xi^2 + 2\beta Q\xi - P_{s\beta} = \begin{pmatrix} \xi^2 - \left(q^2+\beta^2\right)\dfrac{\kappa-1}{\kappa+1} & \dfrac{2\beta\xi}{\kappa+1} \\ -\dfrac{2\beta\xi}{\kappa-1} & \xi^2 - q^2 - \beta^2\dfrac{\kappa+1}{\kappa-1} \end{pmatrix}.$$

The solution of the matrix homogeneous equation is constructed as the following

$$Y(x) = \frac{1}{2\pi i}\oint_C e^{\xi x} M^{-1}(\xi)\, d\xi,$$

here

$$M^{-1}(\xi) = \frac{1}{\left(\xi^2-\beta^2-q^2\right)\left(\xi^2-\beta^2-q^2\dfrac{\kappa-1}{\kappa+1}\right)} \begin{pmatrix} \xi^2 - q^2 - \beta^2\dfrac{\kappa+1}{\kappa-1} & -\dfrac{2\beta\xi}{\kappa+1} \\ \dfrac{2\beta\xi}{\kappa-1} & \xi^2 - \left(q^2+\beta^2\right)\dfrac{\kappa-1}{\kappa+1} \end{pmatrix}$$

The determinant of the matrix $M(\xi)$ has four different roots $\xi_{1,2} = \pm\sqrt{\beta^2+q^2}, \xi_{3,4} = \pm\sqrt{\beta^2+q^2\dfrac{\kappa-1}{\kappa+1}}$, and the system of fundamental matrix solutions has the following form

$$Y^{+}_{1,2}(x) = e^{\pm x\sqrt{\beta^2+q^2}} \begin{pmatrix} \mp\dfrac{\beta^2(\kappa+1)}{2q^2\sqrt{\beta^2+q^2}\,(\kappa-1)} & -\dfrac{\beta}{2q^2} \\ \dfrac{\beta(\kappa+1)}{2q^2(\kappa-1)} & \pm\dfrac{\sqrt{\beta^2+q^2}}{2q^2} \end{pmatrix},$$

$$Y^{+}_{3,4}(x) = e^{\pm x\sqrt{\beta^2+q^2\frac{\kappa-1}{\kappa+1}}} \begin{pmatrix} \pm\dfrac{\beta^2(\kappa+1)+q^2(\kappa-1)}{2q^2\sqrt{\beta^2+q^2\dfrac{\kappa-1}{\kappa+1}}\,(\kappa-1)} & \dfrac{\beta}{2q^2} \\ -\dfrac{\beta(\kappa+1)}{2q^2(\kappa-1)} & \mp\dfrac{\beta^2}{2q^2\sqrt{\beta^2+q^2\dfrac{\kappa-1}{\kappa+1}}} \end{pmatrix}.$$

In the case of steady-state oscillations $q^2<0$, so two cases of the expressions under the root sign should be considered.

When $\beta^2+q^2>0$ matrices $Y_{1,2}^{+}(x)$ are real.

When $\beta^2+q^2<0$

$$Y_{1,2}^{-}(x)=e^{\pm xi\sqrt{-\beta^2-q^2}}\begin{pmatrix} \mp\dfrac{\beta^2(\kappa+1)}{2q^2 i\sqrt{-\beta^2-q^2}(\kappa-1)} & -\dfrac{\beta}{2q^2} \\ \dfrac{\beta(\kappa+1)}{2q^2(\kappa-1)} & \pm\dfrac{i\sqrt{-\beta^2-q^2}}{2q^2} \end{pmatrix}=$$

$$=c_1\cdot A_1\mp s_1\cdot B_1+i(c_1\cdot B_1\pm s_1\cdot A_1),$$

where $c_1=\cos\left(x\sqrt{-\beta^2-q^2}\right)$, $s_1=\sin\left(x\sqrt{-\beta^2-q^2}\right)$, $A_1=\begin{pmatrix} 0 & -\dfrac{\beta}{2q^2} \\ \dfrac{\beta(\kappa+1)}{2q^2(\kappa-1)} & 0 \end{pmatrix}$,

$$B_1=\begin{pmatrix} \pm\dfrac{\beta^2(\kappa+1)}{2q^2\sqrt{-\beta^2-q^2}(\kappa-1)} & 0 \\ 0 & \pm\dfrac{\sqrt{-\beta^2-q^2}}{2q^2} \end{pmatrix}.$$

To avoid the complex values the property of differential equation's solutions is used, and the matrices are chosen in the form $Y_{1,2}^{-}(x)=(c_1\pm s_1)\cdot A_1+(c_1\mp s_1)\cdot B_1$.

When $\beta^2+q^2\dfrac{\kappa-1}{\kappa+1}>0$ matrices $Y_{3,4}^{+}(x)$ are real.

When $\beta^2+q^2\dfrac{\kappa-1}{\kappa+1}<0$

$$Y_{3,4}^{-}(x)=e^{\pm xi\sqrt{-\beta^2-q^2\frac{\kappa-1}{\kappa+1}}}\begin{pmatrix} \pm\dfrac{\beta^2(\kappa+1)+q^2(\kappa-1)}{2q^2 i\sqrt{-\beta^2-q^2\dfrac{\kappa-1}{\kappa+1}}(\kappa-1)} & \dfrac{\beta}{2q^2} \\ -\dfrac{\beta(\kappa+1)}{2q^2(\kappa-1)} & \mp\dfrac{\beta^2}{2q^2 i\sqrt{-\beta^2-q^2\dfrac{\kappa-1}{\kappa+1}}} \end{pmatrix}.$$

Analogically to $Y_{1,2}^{-}(x)$ the matrices $Y_{3,4}^{-}(x)$ are chosen in the following form $Y_{3,4}^{-}(x)=(c_2\pm s_2)\cdot A_2+(c_2\mp s_2)\cdot B_2$, where $c_2=\cos\left(x\sqrt{-\beta^2-q^2\dfrac{\kappa-1}{\kappa+1}}\right)$,

$$s_2 = \sin\left(x\sqrt{-\beta^2 - q^2\frac{\kappa-1}{\kappa+1}}\right), \qquad A_2 = \begin{pmatrix} 0 & \frac{\beta}{2q^2} \\ -\frac{\beta(\kappa+1)}{2q^2(\kappa-1)} & 0 \end{pmatrix},$$

$$B_2 = \begin{pmatrix} \mp\frac{\beta^2(\kappa+1)+q^2(\kappa-1)}{2q^2\sqrt{-\beta^2-q^2\frac{\kappa-1}{\kappa+1}}(\kappa-1)} & 0 \\ 0 & \pm\frac{\beta^2}{2q^2\sqrt{-\beta^2-q^2\frac{\kappa-1}{\kappa+1}}} \end{pmatrix}.$$

So, the matrices for the general solution are chosen with regard to the value of β by the following scheme:

1) when $\beta^2 > -q^2$ and $\beta^2 > -q^2\frac{\kappa-1}{\kappa+1}$ $Y_{1,2}(x) = Y_{1,2}^{+}(x)$, $Y_{3,4}(x) = Y_{3,4}^{+}(x)$;

2) when $\beta^2 < -q^2$ and $\beta^2 > -q^2\frac{\kappa-1}{\kappa+1}$ $Y_{1,2}(x) = Y_{1,2}^{-}(x)$, $Y_{3,4}(x) = Y_{3,4}^{+}(x)$;

3) when $\beta^2 < -q^2$ and $\beta^2 < -q^2\frac{\kappa-1}{\kappa+1}$ $Y_{1,2}(x) = Y_{1,2}^{-}(x)$, $Y_{3,4}(x) = Y_{3,4}^{-}(x)$.

Notice that the case when $\beta^2 > -q^2$ and $\beta^2 < -q^2\frac{\kappa-1}{\kappa+1}$ would never happen since $-q^2\frac{\kappa-1}{\kappa+1} < -q^2$.

The general solution of the vector equation in (6) has the following form

$$\vec{y}_{s\beta}^{\,0}(x) = \left(Y_1(x) + Y_3(x)\right)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \left(Y_2(x) + Y_4(x)\right)\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} \tag{8}$$

here constants $c_i, i = \overline{1,4}$ are found from the boundary conditions on the semi-infinite sides.

5. The partial solution of the inhomogeneous vector equation

The partial solution is derived with the help of Green's matrix function $G_s(x, \xi)$. The Green's matrix function is constructed for the problem of the following structure Vaysfel'd *et al.* (2016)

$$\begin{cases} L_2\vec{y}_{s\beta}(x) = \vec{f}_s(x), \\ V_i\left[\vec{y}_{s\beta}(x)\right] = 0, i = 0,1 \end{cases} \tag{9}$$

$$V_0\left[\vec{y}_{s\beta}(x)\right]=\alpha_n\begin{pmatrix}1&0\\0&0\end{pmatrix}\vec{y}_{s\beta}(0)-\begin{pmatrix}0&0\\0&1\end{pmatrix}\vec{y}\,'_{s\beta}(0),$$

where

$$V_1\left[\vec{y}_{s\beta}(x)\right]=\alpha_n\begin{pmatrix}1&0\\0&0\end{pmatrix}\vec{y}_{s\beta}(a)-\begin{pmatrix}0&0\\0&1\end{pmatrix}\vec{y}\,'_{s\beta}(a).$$

The matrix sin-, cos- integral Fourier's transformation method with the kernel $H(x,\alpha_n)=\begin{pmatrix}\sin\alpha_n x & 0\\ 0 & \cos\alpha_n x\end{pmatrix}, \alpha_n=\dfrac{n\pi}{a}, n=0,1,2...$ was applied to the problem (9).

Green's matrix function is derived in the bilinear expansion form Popov *et al.* (1999)

$$G_s(x,\xi)=\frac{2}{a}\sum_{n=0}^{\infty}{}'\,H(x,\alpha_n)\Omega_{s\beta}^{-1}(\alpha_n)H(\xi,\alpha_n),$$

here $\Omega_{s\beta}(\alpha_n)=-I\alpha_n^2-2\beta\alpha_n\tilde{Q}-P_{s\beta}$, $\tilde{Q}=\begin{pmatrix}0&\dfrac{1}{\kappa+1}\\ \dfrac{1}{\kappa-1}&0\end{pmatrix}$, stroke means that the zeroth member is multiplied by ½.

The elements of Green's matrix function are calculated by formulae from Gradshtein, Rizhik (1963). The partial solution of the vector equation in (6) can be written as

$$\vec{y}_{s\beta}^{\,1}(x)=\int_0^a G_s(x,\xi)\,\vec{f}_s(\xi)\,d\xi \tag{10}$$

After substitution of the found general (8) and partial (10) solutions into the formula (7), the solution of the vector boundary problem (6) has the following form

$$\vec{y}_{s\beta}(x)=\left(Y_1(x)+Y_3(x)\right)\begin{pmatrix}c_1\\c_2\end{pmatrix}+\left(Y_2(x)+Y_4(x)\right)\begin{pmatrix}c_3\\c_4\end{pmatrix}+\int_0^a G_s(x,\xi)\,\vec{f}_s(\xi)\,d\xi \tag{11}$$

The inverse Fourier transformation is applied to (11) and the solution of the stated problem is constructed in Laplace's transform domain in an analytical form as the following

$$\begin{aligned}\begin{bmatrix}u_s(x,y)\\v_s(x,y)\end{bmatrix}=&\frac{2}{\pi}\int_0^{\sqrt{-q^2\frac{\kappa-1}{\kappa+1}}}\left[\left(Y_1^-(x)+Y_3^-(x)\right)\begin{pmatrix}c_1^{(3)}\\c_2^{(3)}\end{pmatrix}+\left(Y_2^-(x)+Y_4^-(x)\right)\begin{pmatrix}c_3^{(3)}\\c_4^{(3)}\end{pmatrix}\right]\begin{bmatrix}\cos\beta y\\\sin\beta y\end{bmatrix}d\beta+\\&+\frac{2}{\pi}\int_{\sqrt{-q^2\frac{\kappa-1}{\kappa+1}}}^{\sqrt{-q^2}}\left[\left(Y_1^-(x)+Y_3^+(x)\right)\begin{pmatrix}c_1^{(2)}\\c_2^{(2)}\end{pmatrix}+\left(Y_2^-(x)+Y_4^+(x)\right)\begin{pmatrix}c_3^{(2)}\\c_4^{(2)}\end{pmatrix}\right]\begin{bmatrix}\cos\beta y\\\sin\beta y\end{bmatrix}d\beta+\\&+\frac{2}{\pi}\int_{\sqrt{-q^2}}^{\infty}\left[\left(Y_1^+(x)+Y_3^+(x)\right)\begin{pmatrix}c_1^{(1)}\\c_2^{(1)}\end{pmatrix}+\left(Y_2^+(x)+Y_4^+(x)\right)\begin{pmatrix}c_3^{(1)}\\c_4^{(1)}\end{pmatrix}\right]\begin{bmatrix}\cos\beta y\\\sin\beta y\end{bmatrix}d\beta+\\&+\frac{2}{\pi}\int_0^{\infty}\int_0^a G_s(x,\xi)\,\vec{f}_s(\xi)\,d\xi\begin{bmatrix}\cos\beta y\\\sin\beta y\end{bmatrix}d\beta\end{aligned} \tag{12}$$

This representation contains unknown function $\chi(x,t)$. It can be derived from the first condition in (1). The substitution of the expressions for the displacements into the condition $\sigma_y\big|_{y=0} = p(x,t)$ reduces the solving of the problem to the solving of the singular integral equation regarding to the unknown function $\chi(x,t)$.

6. The solving of the singular integral equation

The detalization of the initial problem was done for the subcase of the steady-state load applied to a short edge of a semi-strip. In this case $q^2=\rho/G(i\omega)^2$ and the singular integral equation has the following structure

$$\int_{a_0}^{a_1} \chi(\xi,\omega)\, f(\xi,x,\omega)\, d\xi = r(x,\omega), \quad a_0 < x < a_1, \tag{13}$$

where $r(x,\omega), f(\xi,x,\omega)$ are known functions, which are regular when $a_0 \neq 0, a_1 \neq a$.

The Eq. (13) is rewritten in the form

$$\int_{-1}^{1} \tilde{\chi}(\xi,\omega)\, \tilde{f}(\xi,x,\omega)\, d\xi = \tilde{r}(x,\omega), \quad -1 < x < 1, \tag{14}$$

here $\tilde{\chi}(\xi,\omega) = \chi\left(\frac{(a_1-a_0)\xi+(a_0+a_1)}{2},\omega\right)$, $\tilde{r}(x,\omega) = \frac{(a_1-a_0)}{2} r\left(\frac{(a_1-a_0)\xi+(a_0+a_1)}{2},\omega\right)$,

$\tilde{f}(\xi,x,\omega) = \frac{(a_1-a_0)^2}{4} f\left(\frac{(a_1-a_0)\xi+(a_0+a_1)}{2},\frac{(a_1-a_0)x+(a_0+a_1)}{2},\omega\right)$.

The unknown function is searched as

$$\tilde{\chi}(\xi,\omega) = \sum_{n=0}^{\infty} s_n(\omega)\sqrt{1-\xi^2}\,U_n(\xi), \quad \xi \in [-1;1]. \tag{15}$$

The expression (15) is substituted into Eq. (14). The solving of (14) is reduced to the solving of the infinite system of linear algebraic equations

$$\sum_{n=0}^{\infty} s_n(\omega) D_{m,n}(\omega) = f_m(\omega), \; m = 0,1,2..., \tag{16}$$

where $D_{m,n}(\omega) = \int_{-1}^{1} dx \int_{-1}^{1} U_n(\xi) U_m(x) \sqrt{1-\xi^2}\sqrt{1-x^2}\, \tilde{f}(\xi,x,\omega)\, d\xi,$

$f_m(\omega) = \int_{-1}^{1} \tilde{r}(x,\omega) U_m(x)\sqrt{1-x^2}\, dx$.

The solving of (16) and the following substitution of $s_n(\omega)$ into (15) and (12) complete the construction of the solution.

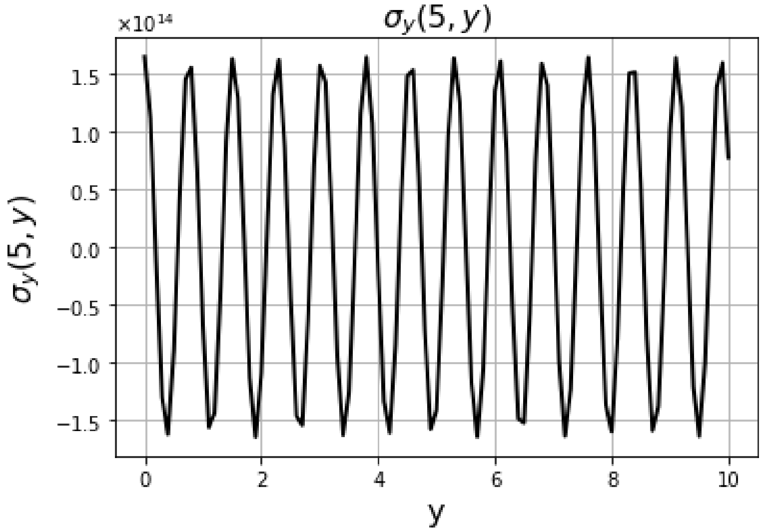

Fig. 2 The stresses σ_y $(a/2, y)$ when ω=1

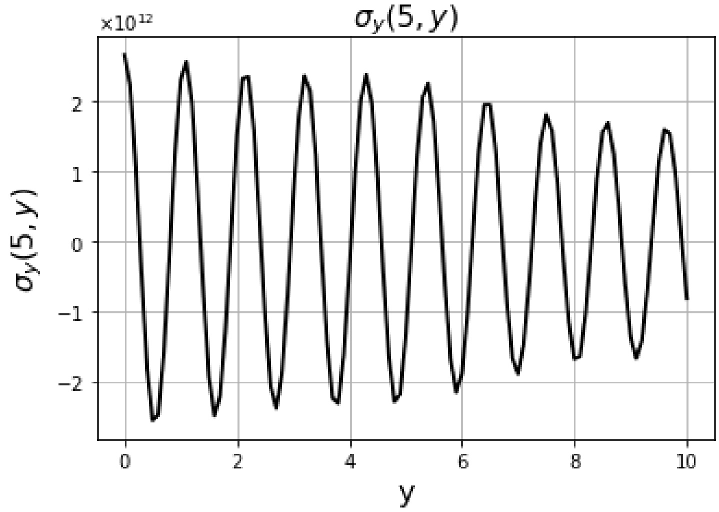

Fig. 3 The stresses σ_y $(a/2, y)$ when ω=0.5

7. The numerical results and discussion

The calculations were provided for the elastic semi-strip with the parameters G=61.2781955 GPa, μ=0.33, ρ=8850 kg/m^3, $p(x,t)=e^{i\omega t}$ GPa, a=10 m, $a_0=a-a_1=a/10$.

The normal stresses σ_y $(a/2,y)$ are presented on Figs. 2-4 when ω=1, ω=0.5 and ω=0.1 correspondingly. The normal stresses are periodic, and the period is 0.8 when ω=1, 1.1 when ω=0.5 and 6.2 when ω=0.1. As it can be seen, the amplitude of normal stresses with frequency ω=1 is bigger than the amplitude when ω=0.5 and is much bigger than in the case with frequency ω=0.1.

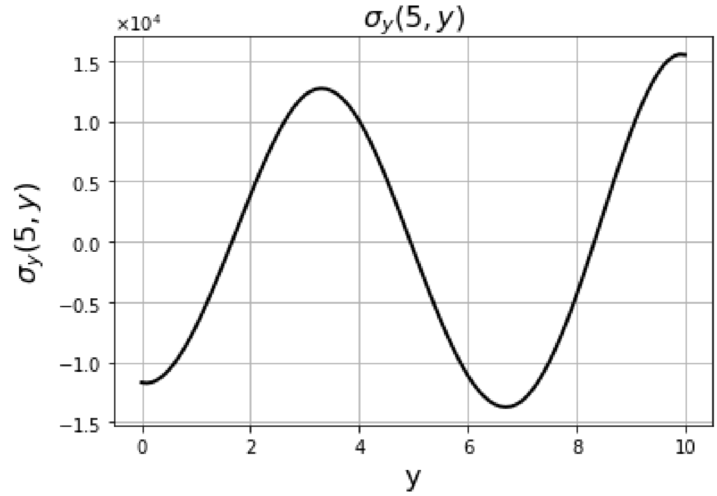

Fig. 4 The stresses σ_y $(a/2, y)$ when ω=0.1

When the frequency $\varpi \geq 3$, the asymptotic expansions for the displacement and stress functions should be considered instead of the direct formulae.

The graphics for the displacements have the analogical character. The values of the tangential stresses are smaller than the values of the normal stresses.

The calculations are stable when the distances $a_0 = a - a_1 \geq a/100$. If these distances are smaller, there are fixed singularities in the kernel of singular integral Eq. (14). These fixed singularities should be taken into consideration.

8. Conclusions

The proposed approach allows to construct the analytical solution of the problem in the Laplace's transform domain. However, it is necessary to inverse the mutual Fourier-Laplace's transform to derive the final solution, which is enough complicated mathematic problem. The derived solution in the Laplace's transform domain is used to get the solution to the subcase of the steady state oscillations: the substitution of $i\varpi$ instead of Laplace's parameter s allows immediately to consider the problem for the steady-state oscillations and to investigate the semi-strip's stress state in regard of the oscillation frequency. For the high values of the frequency the asymptotic formulae should be used for the calculations.

The calculations are stable when the load is far enough from the angular points of the semi-strip. The fixed singularities at the kernel of the singular integral equation should be considered when the load is distributed on the whole semi-strip's short edge.

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