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Semi-active bounded optimal control of uncertain nonlinear coupling vehicle system with rotatable inclined supports and MR damper under random road excitation

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Abstract. The semi-active optimal vibration control of nonlinear torsion-bar suspension vehicle systems under random road excitations is an important research subject, and the boundedness of MR dampers and the uncertainty of vehicle systems are necessary to consider. In this paper, the differential equations of motion of the coupling torsion-bar suspension vehicle system with MR damper under random road excitation are derived and then transformed into strongly nonlinear stochastic coupling vibration equations. The dynamical programming equation is derived based on the stochastic dynamical programming principle firstly for the nonlinear stochastic system. The semi-active bounded parametric optimal control law is determined by the programming equation and MR damper dynamics. Then for the uncertain nonlinear stochastic system, the minimax dynamical programming equation is derived based on the minimax stochastic dynamical programming principle. The worst-case disturbances and corresponding semi-active bounded parametric optimal control are obtained from the programming equation under the bounded disturbance constraints and MR damper dynamics. The control strategy for the nonlinear stochastic vibration of the uncertain torsion-bar suspension vehicle system is developed. The good effectiveness of the proposed control is illustrated with numerical results. The control performances for the vehicle system with different bounds of MR damper under different vehicle speeds and random road excitations are discussed.

Keywords: stochastic nonlinear vibration; semi-active bounded optimal control; coupling vehicle system; uncertainty; random road excitation; MR damper; stochastic dynamical programming; minimax strategy

1. Introduction

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The stochastic vibration of vehicles subjected to rough road excitations as random processes is an important problem, which can degrade vehicle performance. The stochastic vibration control of vehicles under random road excitations is a significant research subject. The vehicle vibration control is performed commonly by suppressing vehicle suspension vibration using smart devices such as magneto-rheological (MR) dampers (Tseng and Hrovat 2015, Sharp and Peng 2011, Spencer *et al*. 1997, Dyke *et al*. 1996, Spencer and Nagarajaiah 2003, Casciati *et al*. 2012, Ying *et al*. 2003, Wang and Liao 2009a, b, Li *et al*. 2013, Kavianipour 2017, Raheem 2018, Karami *et al*. 2016). The suspension coupled with MR dampers by semi-active control is called the intelligent suspension. The suspension, wheel, shafts, elbows and MR dampers compose a coupling vehicle system. The vehicle system with vertically supported suspension is firstly studied. The linear active controls such as linear quadratic (LQ) control for linear vehicle suspension systems have been presented (Tseng and Hrovat 2015, Sharp and Peng 2011, Thompson 1976, Hac 1985, Ulsoy *et al*. 1994). The MR damper has been applied to vehicle suspension systems, and the semi-active controls such as sky-hook control and proportional-integral-derivative control have been presented (Tseng and Hrovat 2015, Sharp and Peng 2011, Zhang *et al*. 2006, Yu *et al*. 2009, Du *et al*. 2011, Cunha and Chavarette 2014, Gad *et al*. 2017). The certain nonlinearity has been considered for the large amplitude vibration of vehicle systems, and the active and semi-active controls such as LQ control for linearization vehicle systems have been presented (Turnip *et al*. 2008, Rao and Narayanan 2009, Balamurugan *et al*. 2014, Khiavi *et al*. 2014). The uncertainty of vehicle systems with MR dampers has been considered, and the adaptive control, fuzzy logic control and neural network control for uncertain vehicle systems have been presented, which controls differ from the optimal controls (Choi *et al*. 2002, Guo *et al*. 2004, Du *et al*. 2005, Nilkhamhang *et al*. 2008, Nguyen *et al*. 2015, Majdoub *et al*. 2015, Phu *et al*. 2016).

However, those researches on the vehicle vibration control were based on the vertically supported suspension, and then the coupling relation between vehicle suspension and supports is not the large-motion nonlinearity. For special kinds of vehicles, the suspension is supported by rotatable inclined elbows and MR dampers due to the spatial limitation, simple and compact structure, which is called the torsion bar suspension. As a result of the elbow or bar rotation, the coupling relation between vehicle suspension and supports is the essential large-motion nonlinearity. The strong nonlinear vibration of vehicle systems with the torsion bar suspension and then the nonlinear stochastic vibration control of the vehicles are very different from that of ordinary vehicles. Only the structure optimization of the torsion bar suspension has been studied by the multi-body dynamics analysis (Fichera *et al*. 2004, Mun *et al*. 2010). Therefore, the nonlinear stochastic vibration control of the torsion-bar suspension vehicle systems is necessary to study.

In fact, the control force produced by MR dampers is bounded due to magnetic saturation and MR fluid properties, and then the control boundedness needs to be considered (Ying *et al*. 2007, 2015). On the other hand, there are always the uncertainties of actual control system parameters and the difference between actual system parameters and corresponding model parameters, which can make the control performance degenerated (Debbarma and Chakraborty 2015). The control designed for uncertain systems is frequently referred to as robust control. The minimax control based on the differential game theory is an optimal robust control, and it determines the optimal control according to the worst system to achieve robustness (Ying 2010, Basar and Bernhard 1995). Therefore, the bounded minimax control for the nonlinear stochastic vibration of the uncertain torsion-bar suspension vehicle systems needs to be studied further.

In this paper, the semi-active bounded minimax control for the nonlinear stochastic vibration of the uncertain torsion-bar suspension vehicle system with MR damper is studied. The nonlinear coupling vehicle system with the torsion bar suspension and MR damper under random road excitation is considered. First, the vehicle system is modelled as a two-degree-of-freedom nonlinear stochastic system with the coupling vertical motion of vehicle body and rotation of inclined elbow. The differential equations of motion of the torsion-bar suspension vehicle system with MR damper under random road excitation are established and then transformed into strongly nonlinear stochastic coupling vibration equations. The optimal parametric control problem of the nonlinear stochastic vibration of the vehicle system is given. Second, for the optimal parametric control problem of the nonlinear stochastic vibration of the coupling torsion-bar suspension vehicle system with MR damper under random road excitation, the dynamical programming equation is derived based on the stochastic dynamical programming principle. The bounded optimal vibration control law is determined by the dynamical programming equation and the bounded constraint of MR damper. Then the semi-active bounded optimal control is determined based on the MR damper dynamics. Third, the uncertainty of system parameters including stiffness and damping is considered further. The optimal parametric control problem of the nonlinear stochastic vibration of the uncertain vehicle system is given and solved using the stochastic dynamical programming principle, minimax stochastic control based on the differential game theory and semi-active bang-bang control strategy. For the optimal parametric control problem of the nonlinear stochastic vibration of the uncertain coupling torsion-bar suspension vehicle system with MR damper under random road excitation, the dynamical programming equation is derived based on the minimax stochastic dynamical programming principle. Under the bounded disturbance constraints and MR damper dynamics, the worst-case disturbances and corresponding semi-active bounded optimal control are obtained by the maximization and minimization of the dynamical programming equation. The random road excitation is produced by filtering Gaussian white noise. The semi-actively and passively controlled system responses are obtained using numerical algorithm. The control effectiveness of the proposed strategy is evaluated using the relative response reduction. Finally, numerical results are given to illustrate the effectiveness of the semi-active bounded optimal control for the nonlinear stochastic vibration of the uncertain torsion-bar suspension vehicle system with MR damper under random road excitation.

2. Optimal vibration control equations of torsion-bar suspension vehicle system

The coupling torsion-bar suspension vehicle system with rotatable inclined elbow and MR damper is simplified as a two-degree-of-freedom (y_c and θ_z) dynamic system with control as shown in Fig. 1. The vehicle body and suspension are modelled by mass m_c, and the wheels are modelled by mass m_w. The suspension is supported by the rotatable inclined elbow and controlled by MR damper. The vertical motion of the vehicle body is considered which is described by the absolute coordinate y_c, and the rotation is neglected. The rotation of the inclined elbow with length l_z is considered, which is represented by the angle coordinate θ_z. The wheel has vertical and horizontal coupling motions which are determined by coordinates y_c and θ_z. The elbow has the torsion stiffness k_r and the pre-set angle θ_{z0}. The wheel has the support stiffness k_s and the original length is r_{w0}. The horizontal and vertical coordinates of the MR damper are x_{1d} and y_c+y_{1d}, respectively, which determine the inclined angle θ_d and the distance l_d between two ends. The rough road has the horizontal baseline represented by axis x. The road roughness is described by the coordinate $y_r(x)$ as a function of x. If the vehicle with wheel moves at speed v, the $y_r(vt)$ becomes the time function or random road excitation. According to the Lagrangian equations, the differential equations of motion of the coupling torsion-bar suspension vehicle system with MR damper are obtained as

$$(m_c+m_w)\ddot{y}_c - m_w l_z \cos\theta_z \ddot{\theta}_z + m_w l_z \sin\theta_z \dot{\theta}_z^{\,2} + (m_c+m_w)g + k_s(y_c - l_z\sin\theta_z - r_{w0}) = k_s y_r \tag{1}$$

$$\begin{aligned} & m_w l_z^{\,2}\ddot{\theta}_z - m_w l_z\cos\theta_z \ddot{y}_c + k_r(\theta_z-\theta_{z0}) - m_w g l_z\cos\theta_z - k_s l_z(y_c - l_z\sin\theta_z - r_{w0})\cos\theta_z \\ & - l_z F_d \sin(\theta_d-\theta_z) = -k_s l_z\cos\theta_z y_r \end{aligned} \tag{2}$$

where g is the acceleration of gravity, and F_d is the force produced by the MR damper. Based on the Bingham model, F_d is expressed as

$$F_d = -C_0\dot{l}_d(\theta_z) - f_y\,\mathrm{sgn}[\dot{l}_d(\theta_z)] = -C_0\dot{l}_d(\theta_z) - U \tag{3}$$

where C_0 is the viscous damping coefficient, f_y is the yield force, and U is the semi-active control force. The force f_y or U can be adjusted by applied external voltages. However, its value is bounded due to magnetic saturation, for example, $f_y\in[0, U_a]$, where U_a is a constant. The distance between two ends of the damper is

$$\begin{aligned} l_d(\theta_z) &= \sqrt{(x_{1d} - l_z\cos\theta_z)^2 + (y_{1d} + l_z\sin\theta_z)^2} \\ &= \sqrt{2l_z(y_{1d}\sin\theta_z - x_{1d}\cos\theta_z) + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}} \end{aligned} \tag{4}$$

There is the geometric relation

$$\sin(\theta_d - \theta_z) = \frac{y_{1d}\cos\theta_z + x_{1d}\sin\theta_z}{l_d(\theta_z)} \tag{5}$$

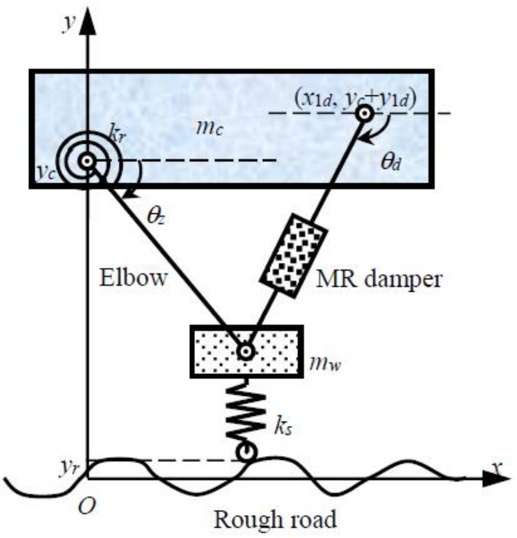

Fig. 1 Simplified model of torsion-bar suspension vehicle system

By uncoupling the second-order derivative terms and using Eqs. (3)-(5), Eqs. (1) and (2) are converted into the differential equations for coordinate y_c and θ_z

$$\begin{aligned}
&\ddot{y}_c + \frac{m_w l_z \sin\theta_z}{m_c + m_w \sin^2\theta_z}\dot{\theta}_z^{\,2} + g + \frac{k_s(y_c - l_z\sin\theta_z - r_{w0})\sin^2\theta_z}{m_c + m_w\sin^2\theta_z} + \frac{k_r(\theta_z - \theta_{z0})\cos\theta_z}{(m_c + m_w\sin^2\theta_z)l_z} \\
&\quad + \frac{C_0\cos\theta_z l_z(y_{1d}\cos\theta_z + x_{1d}\sin\theta_z)^2}{(m_c + m_w\sin^2\theta_z)[2l_z(y_{1d}\sin\theta_z - x_{1d}\cos\theta_z) + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}]}\dot{\theta}_z \\
&\quad + \frac{\cos\theta_z(y_{1d}\cos\theta_z + x_{1d}\sin\theta_z)}{(m_c + m_w\sin^2\theta_z)\sqrt{2l_z(y_{1d}\sin\theta_z - x_{1d}\cos\theta_z) + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}}}U \\
&\quad = \frac{k_s\sin^2\theta_z}{m_c + m_w\sin^2\theta_z}y_r(t)
\end{aligned} \tag{6}$$

$$\begin{aligned}
&\ddot{\theta}_z + \frac{m_w\cos\theta_z\sin\theta_z}{m_c + m_w\sin^2\theta_z}\dot{\theta}_z^{\,2} + \frac{k_r(\theta_z - \theta_{z0})(m_c + m_w)}{(m_c + m_w\sin^2\theta_z)m_w l_z^{\,2}} - \frac{k_s(y_c - l_z\sin\theta_z - r_{w0})m_c\cos\theta_z}{(m_c + m_w\sin^2\theta_z)m_w l_z} \\
&\quad + \frac{C_0(m_c + m_w)(y_{1d}\cos\theta_z + x_{1d}\sin\theta_z)^2}{(m_c m_w + m_w^{\,2}\sin^2\theta_z)[2l_z(y_{1d}\sin\theta_z - x_{1d}\cos\theta_z) + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}]}\dot{\theta}_z \\
&\quad + \frac{(m_c + m_w)(y_{1d}\cos\theta_z + x_{1d}\sin\theta_z)}{(m_c + m_w\sin^2\theta_z)m_w l_z\sqrt{2L_z(y_{1d}\sin\theta_z - x_{1d}\cos\theta_z) + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}}}U \\
&\quad = -\frac{k_s m_c\cos\theta_z}{(m_c + m_w\sin^2\theta_z)m_w l_z}y_r(t)
\end{aligned} \tag{7}$$

The equilibrium equations can be obtained from Eqs. (6) and (7). By eliminating the equilibrium relation, Eqs. (6) and (7) are transformed into the vibration equations

$$\begin{aligned}
&\ddot{u} + \frac{m_w l_z\sin(\theta_0 + \alpha)}{m_c + m_w\sin^2(\theta_0 + \alpha)}\dot{\alpha}^2 + \frac{k_s[u + y_0 - l_z\sin(\theta_0 + \alpha) - r_{w0}]\sin^2(\theta_0 + \alpha)}{m_c + m_w\sin^2(\theta_0 + \alpha)} \\
&\quad - \frac{k_s(y_0 - l_z\sin\theta_0 - r_{w0})\sin^2\theta_0}{m_c + m_w\sin^2\theta_0} + \frac{k_r(\theta_0 + \alpha - \theta_{z0})\cos(\theta_0 + \alpha)}{[m_c + m_w\sin^2(\theta_0 + \alpha)]l_z} - \frac{k_r(\theta_0 - \theta_{z0})\cos\theta_0}{(m_c + m_w\sin^2\theta_0)l_z} \\
&\quad + \frac{C_0 l_z\cos(\theta_0 + \alpha)[y_{1d}\cos(\theta_0 + \alpha) + x_{1d}\sin(\theta_0 + \alpha)]^2}{[m_c + m_w\sin^2(\theta_0 + \alpha)]\{2l_z[y_{1d}\sin(\theta_0 + \alpha) - x_{1d}\cos(\theta_0 + \alpha)] + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}\}}\dot{\alpha} \\
&\quad + \frac{\cos(\theta_0 + \alpha)[y_{1d}\cos(\theta_0 + \alpha) + x_{1d}\sin(\theta_0 + \alpha)]}{[m_c + m_w\sin^2(\theta_0 + \alpha)]\sqrt{2l_z[y_{1d}\sin(\theta_0 + \alpha) - x_{1d}\cos(\theta_0 + \alpha)] + x_{1d}^{\,2} + y_{1d}^{\,2} + l_z^{\,2}}}U \\
&\quad = \frac{k_s\sin^2(\theta_0 + \alpha)}{m_c + m_w\sin^2(\theta_0 + \alpha)}y_r(t)
\end{aligned} \tag{8}$$

$$
\begin{aligned}
&\ddot{\alpha} + \frac{m_w \cos(\theta_0 + \alpha)\sin(\theta_0 + \alpha)}{m_c + m_w \sin^2(\theta_0 + \alpha)} \dot{\alpha}^2 + \frac{k_r (m_c + m_w)}{[m_c + m_w \sin^2(\theta_0 + \alpha)] m_w l_z^2} \alpha \\
&\quad - \frac{k_s m_c [u + y_0 - l_z \sin(\theta_0 + \alpha) - r_{w0}] \cos(\theta_0 + \alpha) - k_s m_c (y_0 - l_z \sin\theta_0 - r_{w0}) \cos\theta_0}{[m_c + m_w \sin^2(\theta_0 + \alpha)] m_w l_z} \\
&\quad + \frac{C_0 (m_c + m_w)[y_{1d} \cos(\theta_0 + \alpha) + x_{1d} \sin(\theta_0 + \alpha)]^2}{[m_c m_w + m_w^2 \sin^2(\theta_0 + \alpha)]\{2 l_z [y_{1d} \sin(\theta_0 + \alpha) - x_{1d} \cos(\theta_0 + \alpha)] + x_{1d}^2 + y_{1d}^2 + l_z^2\}} \dot{\alpha} \\
&\quad + \frac{(m_c + m_w)[y_{1d} \cos(\theta_0 + \alpha) + x_{1d} \sin(\theta_0 + \alpha)]}{[m_c + m_w \sin^2(\theta_0 + \alpha)] m_w l_z \sqrt{2 l_z [y_{1d} \sin(\theta_0 + \alpha) - x_{1d} \cos(\theta_0 + \alpha)] + x_{1d}^2 + y_{1d}^2 + l_z^2}} U \\
&\quad = - \frac{k_s m_c \cos(\theta_0 + \alpha)}{[m_c + m_w \sin^2(\theta_0 + \alpha)] m_w l_z} y_r(t)
\end{aligned} \tag{9}
$$

where the line displacement $u=y_c-y_0$, the angular displacement $\alpha=\theta_z-\theta_0$, y_0 and θ_0 are the line and angular coordinates in static equilibrium, respectively. Eqs. (8) and (9) are the strong nonlinear coupling differential equations, which describe the nonlinear vibration of the coupling torsion-bar suspension vehicle system with MR damper under random road excitation. The semi-active vibration control is performed by the MR damper with U.

To determine the control U, introduce the system state vector $\mathbf{Z} = [u, \alpha, \dot{u}, \dot{\alpha}]^{\mathrm{T}}$. Eqs. (8) and (9) are rewritten as

$$\dot{\mathbf{Z}} = \mathbf{A}(\mathbf{Z}) + \mathbf{B}(\mathbf{Z})U + \mathbf{F} y_r(t) \tag{10}$$

where vectors $\mathbf{A}(\mathbf{Z})$, $\mathbf{B}(\mathbf{Z})$ and $\mathbf{F}$ are determined by Eqs. (8) and (9). The control U is a parameter coupled with the system state $\mathbf{Z}$, and then system (10) determines the parametric control problem. The control aim is to minimize system response $\mathbf{Z}$. The performance index of the stochastic optimal control is expressed as

$$J(U, \mathbf{Z}) = E[\int_0^{t_f} g_c\{\mathbf{Z}(t), U\} \mathrm{d}t + \psi\{\mathbf{Z}(t_f)\}] \rightarrow \min_U \tag{11}$$

where $E[\cdot]$ is the expectation operator, $g_c(\mathbf{Z},U)\geq 0$, t_f is the terminal time, and ψ is the terminal value of the control. Eqs. (10) and (11) construct the optimal parametric control problem of the nonlinear stochastic vibration of the vehicle system.

3. Bounded vibration control law

For the optimal parametric control problem of the nonlinear stochastic vibration of the coupling torsion-bar suspension vehicle system with MR damper under random road excitation, the vibration control law can be determined based on the stochastic dynamical programming principle and semi-active bang-bang strategy (Ying *et al*. 2007, 2015). According to the stochastic dynamical programming principle, the dynamical programming equation for system (10) and index (11) is obtained as

$$\frac{\partial V}{\partial t}+\min_{U}\{\frac{1}{2}tr(D_y\mathbf{F}\mathbf{F}^{\mathrm{T}}\frac{\partial^2 V}{\partial \mathbf{Z}^2})+[\mathbf{A}(\mathbf{Z})+\mathbf{B}(\mathbf{Z})U]^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}}+g_c(\mathbf{Z},U)\}=0 \tag{12}$$

where $V(\mathbf{Z},t)$ is the value function, D_y is the intensity of random excitation y_r as filtering Gaussian white noise, and $tr(\cdot)$ is the trace operator. The control force U produced by the MR damper is bounded due to magnetic saturation, and the bounded constraint is expressed as

$$|U|\le U_a \tag{13}$$

where U_a is the control bound. For the bounded optimal control, the function g_c can be chosen as independent of U. The bounded optimal vibration control law is determined by minimizing the second term on the left side of Eq. (12) under constraint (13) as

$$U^*=-U_a\,\mathrm{sgn}(\mathbf{B}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}}) \tag{14}$$

Substituting the optimal control (14) into Eq. (12) yields the value function equation

$$\frac{\partial V}{\partial t}+\frac{1}{2}tr(D_y\mathbf{F}\mathbf{F}^{\mathrm{T}}\frac{\partial^2 V}{\partial \mathbf{Z}^2})+\mathbf{A}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}}-U_a\left|\mathbf{B}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}}\right|+g_c(\mathbf{Z})=0 \tag{15}$$

The value function V is obtained by solving Eq. (15). Then the optimal control force U^* can be obtained by substituting V into Eq. (14). However, the semi-active optimal control force has to conform with the dynamics of the MR damper as given by Eq. (3). Thus, the semi-active bounded optimal control is further determined as

$$U^*=\begin{cases}-U_a\,\mathrm{sgn}(\mathbf{B}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}}) & \text{for}\quad (\mathbf{B}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}})\dot{l}_d<0\\ 0 & \text{for}\quad (\mathbf{B}^{\mathrm{T}}\frac{\partial V}{\partial \mathbf{Z}})\dot{l}_d\ge 0\end{cases} \tag{16}$$

4. Minimax optimal control law for uncertain system

The uncertainty of actual control system parameters can make the above control performance degenerated. Then the uncertain system needs to be considered and the minimax control based on the differential game theory can be used. For the torsion-bar suspension vehicle system with MR damper, the stiffness and damping (k_r, k_s, C_0) are considered as uncertain parameters. Based on Eqs. (8) and (9), the nonlinear vibration equations of the uncertain torsion-bar suspension vehicle system with MR damper under random road excitation are expressed as

$$\begin{aligned}
&\ddot{u}+\frac{m_w l_z\sin(\theta_0+\alpha)}{m_c+m_w\sin^2(\theta_0+\alpha)}\dot{\alpha}^2+\frac{(\bar{k}_s+\tilde{k}_s)[u+y_0-l_z\sin(\theta_0+\alpha)-r_{w0}]\sin^2(\theta_0+\alpha)}{m_c+m_w\sin^2(\theta_0+\alpha)}\\
&-\frac{(\bar{k}_s+\tilde{k}_s)(y_0-l_z\sin\theta_0-r_{w0})\sin^2\theta_0}{m_c+m_w\sin^2\theta_0}+\frac{(\bar{k}_r+\tilde{k}_r)(\theta_0+\alpha-\theta_{z0})\cos(\theta_0+\alpha)}{[m_c+m_w\sin^2(\theta_0+\alpha)]l_z}\\
&+\frac{(\bar{C}_0+\tilde{C}_0)l_z\cos(\theta_0+\alpha)[y_{1d}\cos(\theta_0+\alpha)+x_{1d}\sin(\theta_0+\alpha)]^2\dot{\alpha}}{[m_c+m_w\sin^2(\theta_0+\alpha)]\{2l_z[y_{1d}\sin(\theta_0+\alpha)-x_{1d}\cos(\theta_0+\alpha)]+x_{1d}{}^2+y_{1d}{}^2+l_z{}^2\}}\\
&+\frac{\cos(\theta_0+\alpha)[y_{1d}\cos(\theta_0+\alpha)+x_{1d}\sin(\theta_0+\alpha)]U}{[m_c+m_w\sin^2(\theta_0+\alpha)]\sqrt{2l_z[y_{1d}\sin(\theta_0+\alpha)-x_{1d}\cos(\theta_0+\alpha)]+x_{1d}{}^2+y_{1d}{}^2+l_z{}^2}}\\
&-\frac{(\bar{k}_r+\tilde{k}_r)(\theta_0-\theta_{z0})\cos\theta_0}{(m_c+m_w\sin^2\theta_0)l_z}=\frac{(\bar{k}_s+\tilde{k}_s)\sin^2(\theta_0+\alpha)}{m_c+m_w\sin^2(\theta_0+\alpha)}y_r(t)
\end{aligned}\tag{17}$$

$$\begin{aligned}
&\ddot{\alpha}+\frac{m_w\cos(\theta_0+\alpha)\sin(\theta_0+\alpha)}{m_c+m_w\sin^2(\theta_0+\alpha)}\dot{\alpha}^2+\frac{(\bar{k}_r+\tilde{k}_r)(m_c+m_w)}{[m_c+m_w\sin^2(\theta_0+\alpha)]m_w l_z{}^2}\alpha\\
&-\frac{(\bar{k}_s+\tilde{k}_s)m_c\{[u+y_0-l_z\sin(\theta_0+\alpha)-r_{w0}]\cos(\theta_0+\alpha)-(y_0-l_z\sin\theta_0-r_{w0})\cos\theta_0\}}{[m_c+m_w\sin^2(\theta_0+\alpha)]m_w l_z}\\
&+\frac{(\bar{C}_0+\tilde{C}_0)(m_c+m_w)[y_{1d}\cos(\theta_0+\alpha)+x_{1d}\sin(\theta_0+\alpha)]^2}{[m_c m_w+m_w{}^2\sin^2(\theta_0+\alpha)]\left\{2l_z[y_{1d}\sin(\theta_0+\alpha)-x_{1d}\cos(\theta_0+\alpha)]+x_{1d}{}^2+y_{1d}{}^2+l_z{}^2\right\}}\dot{\alpha}\\
&+\frac{(m_c+m_w)[y_{1d}\cos(\theta_0+\alpha)+x_{1d}\sin(\theta_0+\alpha)]U}{[m_c+m_w\sin^2(\theta_0+\alpha)]m_w l_z\sqrt{2l_z[y_{1d}\sin(\theta_0+\alpha)-x_{1d}\cos(\theta_0+\alpha)]+x_{1d}{}^2+y_{1d}{}^2+l_z{}^2}}\\
&=-\frac{(\bar{k}_s+\tilde{k}_s)m_c\cos(\theta_0+\alpha)}{[m_c+m_w\sin^2(\theta_0+\alpha)]m_w l_z}y_r(t)
\end{aligned}\tag{18}$$

where $\bar{k}_r$, $\bar{k}_s$, $\bar{C}_0$ are respectively the nominal torsion stiffness, support stiffness and viscous damping, and $\tilde{k}_r$, $\tilde{k}_s$, $\tilde{C}_0$ are the corresponding parameter disturbances. It is assumed that the uncertain parameter disturbances are bounded and expressed as $\tilde{k}_r\in[-k_r{}^0,k_r{}^0]$, $\tilde{k}_s\in[-k_s{}^0,k_s{}^0]$ and $\tilde{C}_0\in[-C_0{}^0,C_0{}^0]$, where k_r^0, k_s^0 and C_0^0 are the disturbance bounds. By introducing the disturbance vector $\boldsymbol{\zeta}=[\tilde{C}_0,\tilde{k}_s,\tilde{k}_r]^{\mathrm{T}}$, the disturbance boundedness is rewritten as

$$|\zeta_i|\le b_{pi}\qquad i=1,2,3\tag{19}$$

Eqs. (17) and (18) are converted into the system state equation

$$\dot{\mathbf{Z}}=\bar{\mathbf{A}}(\mathbf{Z})+\bar{\mathbf{B}}(\mathbf{Z})U+\bar{\mathbf{F}}y_r(t)+\mathbf{f}(\mathbf{Z})\boldsymbol{\zeta}\tag{20}$$

where vectors $\bar{\mathbf{A}}(\mathbf{Z})$, $\bar{\mathbf{B}}(\mathbf{Z})$, $\bar{\mathbf{F}}$ and matrix $\mathbf{f}(\mathbf{Z})$ are determined by Eqs. (17) and (18). System (20) determines the uncertain parametric control problem. The performance index of the stochastic minimax optimal control is expressed as

$$J(U,\zeta,\mathbf{Z}) = E[\int_0^{t_f} g_c\{\mathbf{Z}(t),U,\zeta\}\mathrm{d}t + \psi\{\mathbf{Z}(t_f)\}] \to \min_U \max_\zeta \tag{21}$$

Eqs. (20) and (21) construct the minimax optimal parametric control problem of the nonlinear stochastic vibration of the coupling torsion-bar suspension vehicle system with MR damper under random road excitation. The vibration control law can be determined based on the minimax stochastic dynamical programming principle and semi-active bang-bang strategy (Ying 2010, Basar and Bernhard 1995). According to the minimax stochastic dynamical programming principle, the dynamical programming equation for system (20) and index (21) is obtained as

$$\frac{\partial V}{\partial t} + \min_U \max_\zeta \{\frac{1}{2} tr(D_y \bar{\mathbf{F}}\bar{\mathbf{F}}^{\mathrm{T}} \frac{\partial^2 V}{\partial \mathbf{Z}^2}) + [\bar{\mathbf{A}}(\mathbf{Z}) + \bar{\mathbf{B}}(\mathbf{Z})U + \mathbf{f}(\mathbf{Z})\zeta]^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} + g_c(\mathbf{Z},U)\} = 0 \tag{22}$$

The bounded control constraint of the MR damper is given by Eq. (13), and the bounded disturbance constraints are given by Eq. (19). The worst-case disturbances are determined by maximizing the left side of Eq. (22) under constraints (19) as

$$\zeta_i^* = b_{pi}\,\mathrm{sgn}(\mathbf{f}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}})_i \tag{23}$$

Submitting the worst-case disturbances (23) into Eq. (22) yields the dynamical programming equation for the worst case

$$\frac{\partial V}{\partial t} + \min_U \{\frac{1}{2} tr(D_y \bar{\mathbf{F}}\bar{\mathbf{F}}^{\mathrm{T}} \frac{\partial^2 V}{\partial \mathbf{Z}^2}) + (\bar{\mathbf{A}} + \bar{\mathbf{B}}U)^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} + \sum_{i=1}^{3} b_{pi} \left| (\mathbf{f}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}})_i \right| + g_c\} = 0 \tag{24}$$

The bounded optimal vibration control law is determined by minimizing the left side of Eq. (24) under constraint (13) as

$$U^* = -U_a\,\mathrm{sgn}(\bar{\mathbf{B}}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}}) \tag{25}$$

Substituting the optimal control (25) into Eq. (24) yields the value function equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} tr(D_y \bar{\mathbf{F}}\bar{\mathbf{F}}^{\mathrm{T}} \frac{\partial^2 V}{\partial \mathbf{Z}^2}) + \bar{\mathbf{A}}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} - U_a \left| \bar{\mathbf{B}}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}} \right| + \sum_{i=1}^{3} b_{pi} \left| (\mathbf{f}^{\mathrm{T}} \frac{\partial V}{\partial \mathbf{Z}})_i \right| + g_c = 0 \tag{26}$$

The value function V is obtained by solving Eq. (26). Then the worst-case disturbances $\boldsymbol{\zeta}^*$ and the corresponding optimal control force U^* can be obtained by substituting V into Eqs. (23) and (25), respectively. Further, the semi-active bounded optimal control for uncertain torsion-bar suspension vehicle system with MR damper is determined based on the MR damper dynamics (3) as

$$U^* = \begin{cases} -U_a \operatorname{sgn}(\overline{\mathbf{B}}^{\mathrm{T}} \dfrac{\partial V}{\partial \mathbf{Z}}) & \text{for} \quad (\overline{\mathbf{B}}^{\mathrm{T}} \dfrac{\partial V}{\partial \mathbf{Z}}) \dot{l}_d < 0 \\ 0 & \text{for} \quad (\overline{\mathbf{B}}^{\mathrm{T}} \dfrac{\partial V}{\partial \mathbf{Z}}) \dot{l}_d \geq 0 \end{cases} \tag{27}$$

The optimally controlled system with uncertain parameters is determined by substituting the control (27) and disturbances (23) into Eq. (20), and the controlled system responses are obtained by solving the equation. According to the minimax strategy, the response of the optimally controlled system with non-worst-case disturbances is smaller than that of the optimally controlled system with the worst-case disturbances. Thus the minimax semi-active bounded control for the uncertain vehicle system is robust.

5. Numerical results and discussions

To illustrate the application and effectiveness of the proposed semi-active bounded minimax control strategy, consider the uncertain torsion-bar suspension vehicle system with parameters m_c=3000 kg, m_w=105 kg, l_z=0.35 m, k_r=14.5 kN·m/rad, $k_s=2\times10^4$ kN/m, θ_{z0}=1 rad, r_{w0}=0.295 m, x_{1d}=0.621 m, y_{1d}=0.312 m, C_0=1500 kg/s, k_r^0=500 N·m/rad, $k_s^0=6\times10^2$ kN/m, C_0^0=50 kg/s and U_a=15 kN. The random road excitation $y_r(t)$ is modelled by filtering Gaussian white noise, and its differential equation is

$$\dot{y}_r + 2\pi n_{00} v y_r = 2\pi n_0 \sqrt{G_q(n_0) v} W(t) \tag{28}$$

where $n_{00}=0.01$ m^{-1} is the spatial cut-off frequency, $n_0=0.1$ m^{-1} is the reference space frequency, v is the vehicle speed, $W(t)$ is the Gaussian white noise with unit intensity, and $G_q(n_0)$ is the power spectral density of road surface under the reference space frequency n_0, which is called the road roughness coefficient. According to different values of $G_q(n_0)$, the road roughness can be classified into different grades such as grade C. The semi-actively (U^*) and passively (U^*=0) controlled system responses are obtained by solving Eq. (10) or (20) using the Runge-Kutta algorithm. The semi-active bounded optimal control effectiveness is evaluated by the relative response reduction

$$K = \frac{\sigma_{\text{passive}} - \sigma_{\text{semi-active}}}{\sigma_{\text{passive}}} \times 100\% \tag{29}$$

where $\sigma_{\text{semi-active}}$ and σ_{passive} are the standard deviations of the semi-actively and passively controlled system responses, respectively. A larger value of K indicates a better control effectiveness.

5.1 Bounded vibration control

A field road is considered usually as grade C for the vehicle vibration analysis. Fig. 2 shows a sample of the displacement y_r of C-grade road under the vehicle speed of v=50 km/h, the power spectral density of which is shown in Fig. 3. Fig. 4 shows the vehicle body displacement (u) by using the proposed semi-active bounded optimal control for the C-grade road and vehicle speed of

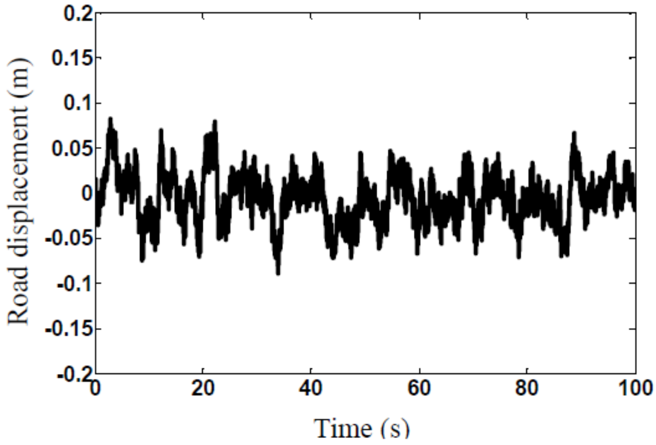

Fig. 2 Displacement excitation (y_r) of C-grade road under vehicle speed 50 km/h

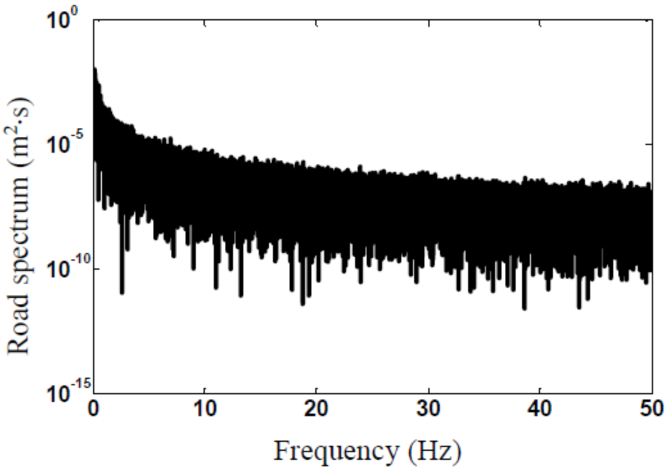

Fig. 3 Power spectral density of C-grade road under vehicle speed 50 km/h

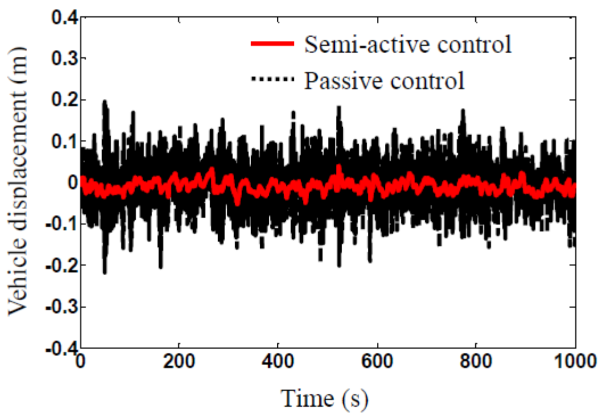

Fig. 4 Semi-actively and passively controlled vehicle body displacements for C-grade road and vehicle speed 50 km/h

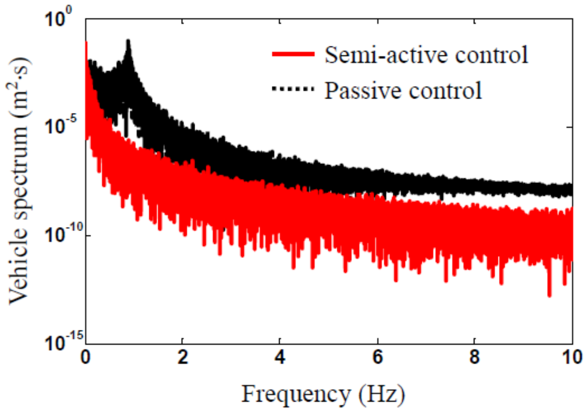

Fig. 5 Power spectral densities of semi-actively and passively controlled vehicle body displacements for *C*-grade road and vehicle speed 50 km/h

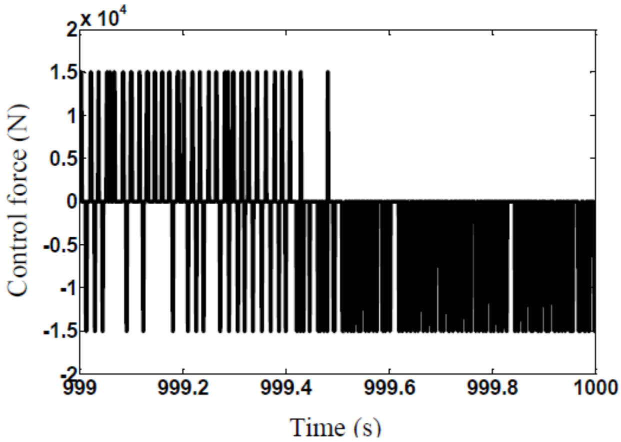

Fig. 6 Semi-active control force for vehicle under *C*-grade road and vehicle speed 50 km/h

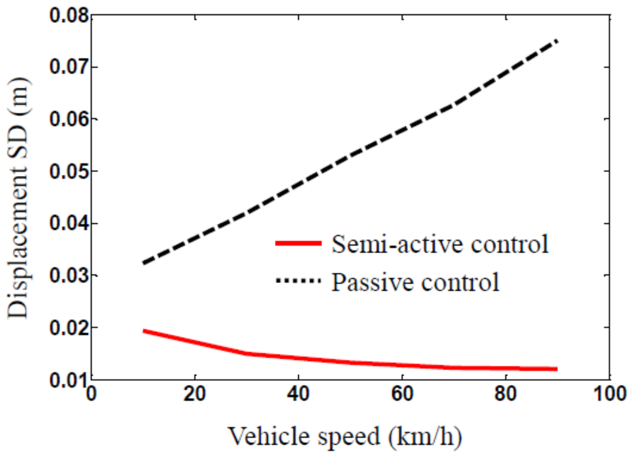

Fig. 7 Standard deviations (SD) of semi-actively and passively controlled vehicle body displacements under *C*-grade road for different vehicle speeds

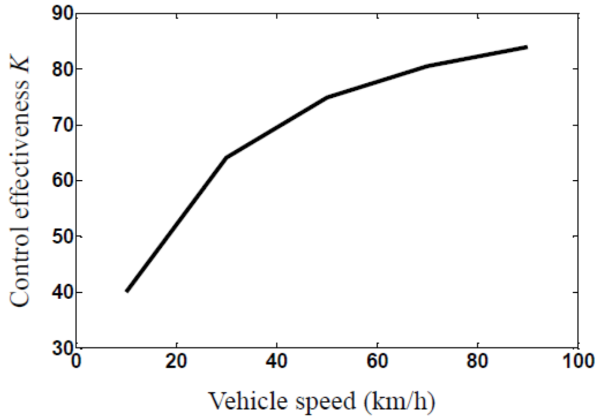

Fig. 8 Effectiveness of semi-actively controlled vehicle body displacement under *C*-grade road for different vehicle speeds

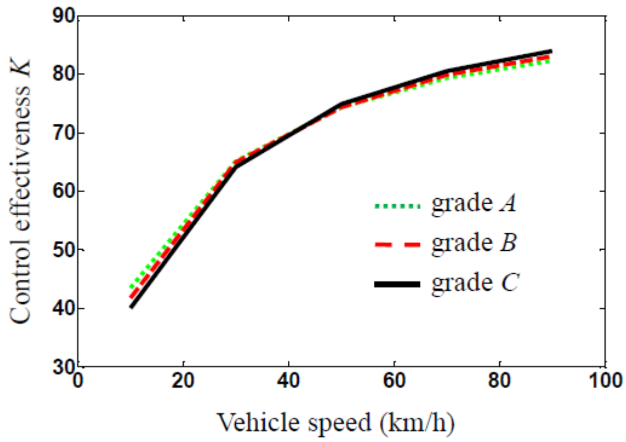

Fig. 9 Effectiveness of semi-actively controlled vehicle body displacement for different road grades and vehicle speeds

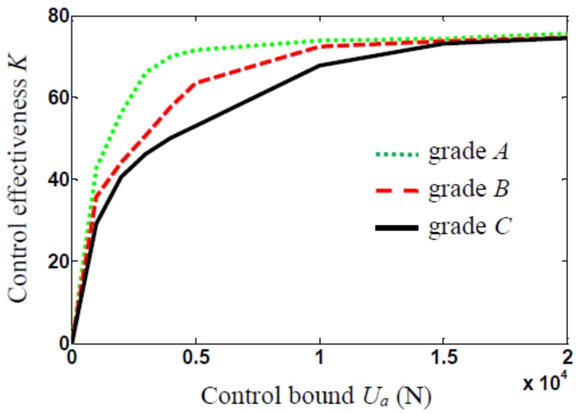

Fig. 10 Effectiveness of semi-actively controlled vehicle body displacement for different road grades and control bounds

50 km/h, compared with the passively controlled vehicle body displacement. The passive control is for the MR damper with viscous damping force ($C_0\dot{l}_d$) and semi-active control force U=0. The corresponding power spectral densities of the semi-actively and passively controlled vehicle body displacements are shown in Fig. 5. The semi-active bounded optimal control force is shown in Fig. 6. The standard deviation of the vehicle body displacement is reduced from passively controlled 0.053 m to semi-actively controlled 0.013 m. The control effectiveness (K) or the relative response reduction of the displacement standard deviation is 74.9%. Thus, the vehicle body vibration is suppressed largely by using the proposed semi-active bounded optimal control.

Fig. 7 shows the standard deviation of the semi-actively controlled vehicle body displacement (u) under the C-grade road for different vehicle speeds (v), compared with the standard deviation of the passively controlled vehicle body displacement. The control effectiveness (K) or the relative response reduction of the displacement standard deviation is shown in Fig. 8. It is seen that the control effectiveness of the vehicle body vibration increases with the vehicle speed. For road grades A, B and C, Fig. 9 shows the control effectiveness (K) or the relative response reductions of the standard deviations of the semi-actively controlled vehicle body displacements (u) for different vehicle speeds (v). The control effectiveness of the vehicle body vibration is insensitive to the road grade. Fig. 10 shows the control effectiveness (K) or the relative response reductions of the standard deviations of the semi-actively controlled vehicle body displacements (u) under the vehicle speed of 50 km/h for different control force bounds (U_a). The control effectiveness of the vehicle body vibration increases with the control force bound. However, the control effectiveness is improved significantly by increasing smaller control bound, and the improvement of the control effectiveness is limited for larger control bound (for example, U_a>150 kN). It is obtained that the proposed semi-active bounded control strategy can achieve the good control effectiveness for the nonlinear stochastic vibration of the torsion-bar suspension vehicle system.

5.2 Minimax bounded vibration control for uncertain system

Further, consider the uncertain torsion-bar suspension vehicle system with the uncertain stiffness and damping (k_r, k_s, C_0). The nominal torsion stiffness, support stiffness and viscous damping are $\bar{k}_r$=14.5 kN·m/rad, $\bar{k}_s$=2×10^4 kN/m, $\bar{C}_0$=1500 kg/s, and the disturbance bounds of the uncertain parameters (k_r^0, k_s^0, C_0^0) are given as above. The uncertainty of the torsion-bar suspension vehicle system can make the control (16) performance degenerated. Then the semi-active minimax bounded optimal control (27) for the worst-case disturbances is used for the uncertain torsion-bar suspension vehicle system under random road excitation. Fig. 11 shows the vehicle body displacement (u) with the worst-case disturbances by using the semi-active minimax bounded optimal control for the C-grade road and vehicle speed of 50 km/h, which is compared with the passively controlled vehicle body displacement. Fig. 12 shows the corresponding power spectral densities of the semi-actively minimax-controlled and passively controlled vehicle body displacements. The semi-active minimax bounded optimal control force is shown in Fig. 13. The standard deviation of the vehicle body displacement is reduced from passively controlled 0.053 m to semi-actively minimax-controlled 0.016 m. The control effectiveness (K) or the relative response reduction of the displacement standard deviation is 69.1%. The vehicle body vibration is also suppressed largely by using the proposed semi-active minimax bounded optimal control.

Fig. 14 shows the standard deviation of the semi-actively minimax-controlled vehicle body displacement (u) under the C-grade road for different vehicle speeds (v), which is compared with

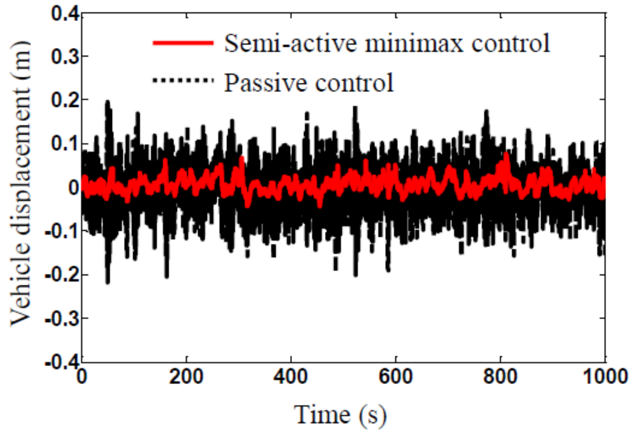

Fig. 11 Semi-actively minimax-controlled and passively controlled vehicle body displacements for *C*-grade road and vehicle speed 50 km/h

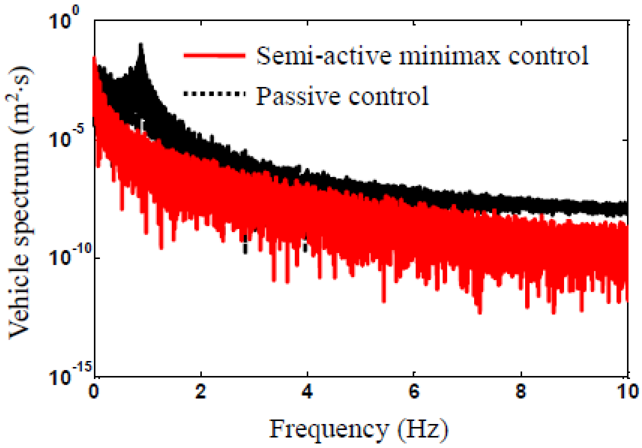

Fig. 12 Power spectral densities of semi-actively minimax-controlled and passively controlled vehicle body displacements for *C*-grade road and vehicle speed 50 km/h

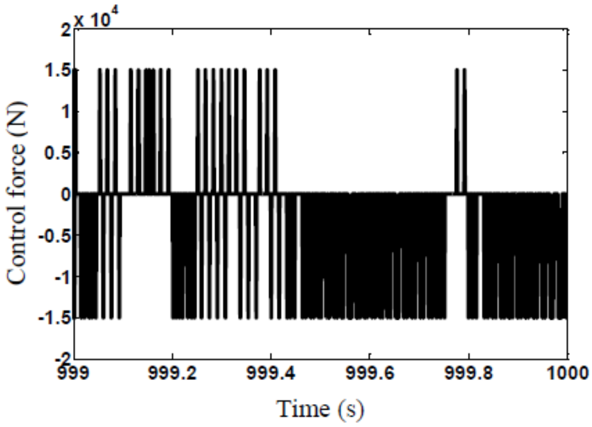

Fig. 13 Semi-active minimax control force for vehicle under *C*-grade road and vehicle speed 50 km/h

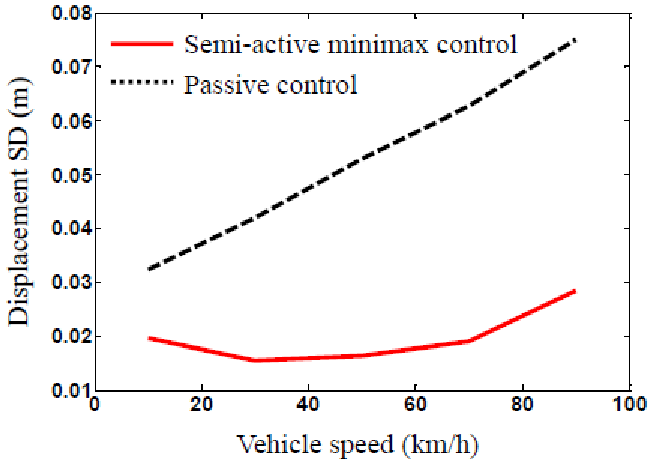

Fig. 14 Standard deviations (SD) of semi-actively minimax-controlled and passively controlled vehicle body displacements under *C*-grade road for different vehicle speeds

the standard deviation of the passively controlled vehicle body displacement. Fig. 15 illustrates the control effectiveness (K) or the relative response reduction of the displacement standard deviation. The control effectiveness of the uncertain vehicle body vibration increases nonlinearly with the vehicle speed, but decreases for larger vehicle speed, because the minimax control has a complicated dependence on random road excitation or vehicle speed and the passively controlled displacement increases correspondingly. For road grades *A*, *B* and *C*, Fig. 16 shows the control effectiveness (K) or the relative response reductions of the standard deviations of the semi-actively minimax-controlled vehicle body displacements (u) for different vehicle speeds (v). The control effectiveness of the uncertain vehicle body vibration for grade *A* or *B* is better than that for grade *C*, in particular, for larger vehicle speed. Fig. 17 shows the control effectiveness (K) or the relative response reductions of the standard deviations of the semi-actively minimax-controlled vehicle body displacements (u) under the vehicle speed of 50km/h for different control force bounds (U_a). The control effectiveness of the uncertain vehicle body vibration increases with the control force bound. Similarly, the control effectiveness is improved remarkably by increasing smaller control bound, and the improvement of the control effectiveness is limited for larger control bound.

Figs. 18 and 19 show the control effectiveness (K) or the relative response reductions of the standard deviations of the semi-actively minimax-controlled vehicle body displacements (u) under the *C*-grade road and vehicle speed of 50km/h for different damping disturbance bounds (C_0^0) and torsion-stiffness disturbance bounds (k_r^0), respectively. It is seen that the control effectiveness of the uncertain vehicle body vibration is insensitive to the damping and torsion-stiffness disturbance bounds as given. Fig. 20 illustrates that the control effectiveness (K) or the relative response reduction of the standard deviations of the semi-actively minimax-controlled vehicle body displacement (u) decreases nonlinearly as the support-stiffness disturbance bound (k_s^0) increases. Thus, the support stiffness needs to be determined accurately or the support-stiffness disturbance bound needs to be reduced to obtain better semi-active minimax control effectiveness for the uncertain vehicle body vibration. However, the proposed semi-active minimax bounded control strategy can also achieve the good control effectiveness for the nonlinear stochastic vibration of the uncertain torsion-bar suspension vehicle system.

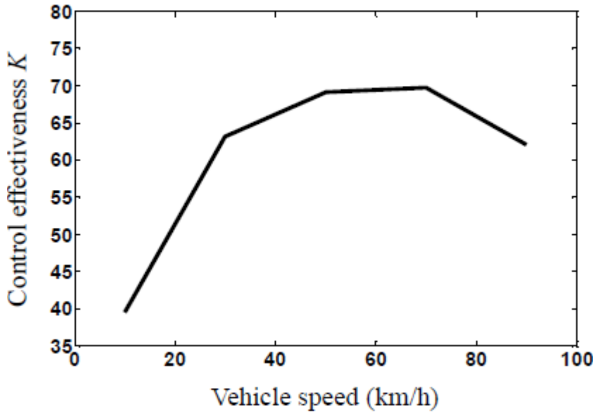

Fig. 15 Effectiveness of semi-actively minimax-controlled vehicle body displacement under *C*-grade road for different vehicle speeds

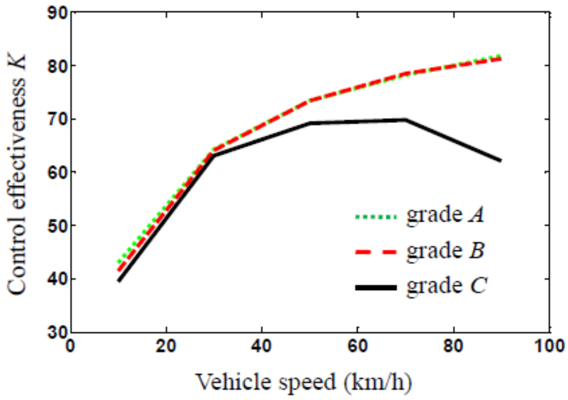

Fig. 16 Effectiveness of semi-actively minimax-controlled vehicle body displacement for different road grades and vehicle speeds

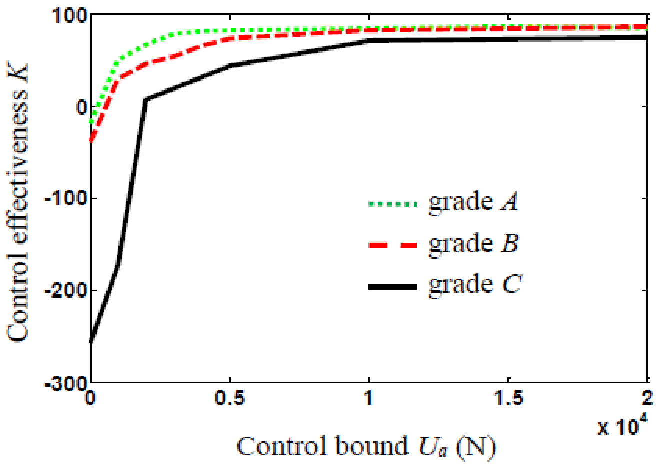

Fig. 17 Effectiveness of semi-actively minimax-controlled vehicle body displacement for different road grades and control bounds

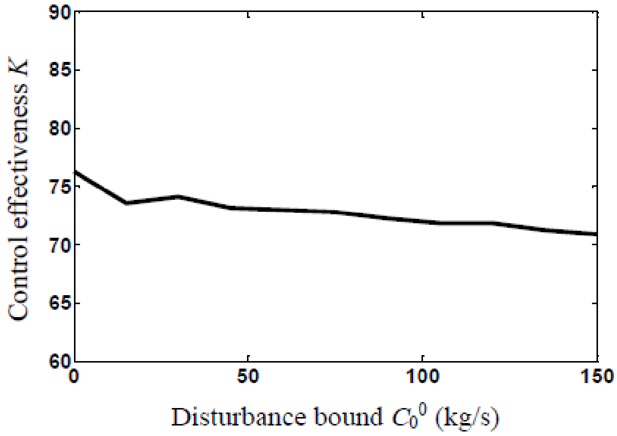

Fig. 18 Effectiveness of semi-actively minimax-controlled vehicle body displacement for different damping disturbance bounds

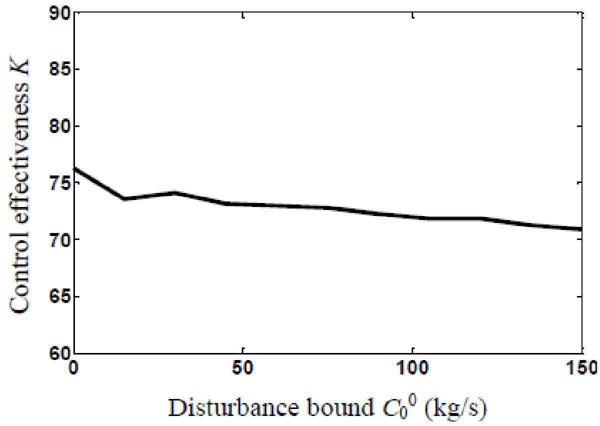

Fig. 19 Effectiveness of semi-actively minimax-controlled vehicle body displacement for different torsion-stiffness disturbance bounds

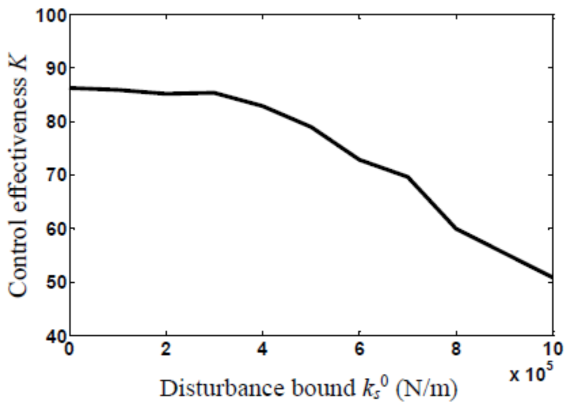

Fig. 20 Effectiveness of semi-actively minimax-controlled vehicle body displacement for different support-stiffness disturbance bounds

6. Conclusions

The stochastic optimal semi-active control problem of the strongly nonlinear vibration of the uncertain coupling vehicle system with rotatable inclined supports and MR damper under random road excitation has been studied. The two-degree-of-freedom nonlinear stochastic system with the coupling vertical motion of vehicle body and rotation of inclined elbow has been modelled for the torsion-bar suspension vehicle system. The differential equations of motion of the nonlinear torsion-bar suspension vehicle system with MR damper under random road excitation have been derived. For the optimal parametric control problem of the nonlinear stochastic vibration of the uncertain torsion-bar suspension vehicle system with MR damper under random road excitation, the dynamical programming equation has been obtained based on the minimax stochastic dynamical programming principle. The semi-active bounded optimal vibration control law has been determined by the dynamical programming equation and MR damper dynamics. The control effectiveness has been evaluated using the relative response reduction of the semi-actively optimal-controlled vehicle body vibration compared with the passively controlled vehicle body vibration.

Numerical results illustrate that: (1) the proposed semi-active bounded optimal control strategy has the good control effectiveness for the nonlinear stochastic vibration of the torsion-bar suspension vehicle system; (2) the semi-active optimal control effectiveness of the nonlinear stochastic vehicle system increases with the control force bound of MR damper, especially for small control force; (3) the semi-active optimal control effectiveness of the nonlinear stochastic vehicle system increases with the vehicle speed, but it can decrease for the uncertain vehicle system and higher road grade; (4) the semi-active optimal control effectiveness of the uncertain nonlinear stochastic vehicle system is insensitive to the damping and torsion-stiffness disturbance bounds and then is robust, but it decreases nonlinearly as the support-stiffness disturbance bound increases and then the support stiffness needs to be determined accurately. In a word, the proposed semi-active bounded optimal control can effectively suppress the nonlinear stochastic vibration of the uncertain vehicle system with MR damper under random road excitation.

Acknowledgments

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CC

Nomenclature

$\mathbf{A}$	system vector
$\overline{\mathbf{A}}$	uncertain system vector
$\mathbf{B}$	parameter vector related to control
$\overline{\mathbf{B}}$	uncertain system parameter vector related to control
b_{pi}	parameter disturbance bound of ζ_i
C_0	viscous damping coefficient of MR damper
$\overline{C}_0$	nominal value of C_0

$\tilde{C}_0$	disturbance of C_0
C_0^0	disturbance bound of C_0
D_y	intensity of random excitation
$\mathbf{f}$	parameter matrix related to disturbance
$\mathbf{F}$	parameter vector related to excitation
$\bar{\mathbf{F}}$	uncertain system parameter vector related to excitation
F_d	force produced by MR damper
f_y	yield force of MR damper
g	acceleration of gravity
g_c	function in performance index
G_q	power spectral density of road surface
k_r	torsion stiffness
$\bar{k}_r$	nominal value of k_r
$\tilde{k}_r$	disturbance of k_r
k_r^0	disturbance bound of k_r
k_s	support stiffness
$\bar{k}_s$	nominal value of k_s
$\tilde{k}_s$	disturbance of k_s
k_s^0	disturbance bound of k_s
l_d	distance between two ends of MR damper
l_z	support elbow length
m_c	vehicle body mass

m_w	wheel mass
n_0	reference space frequency of random road
n_{00}	spatial cut-off frequency of random road
r_{w0}	original length of wheel radius
t_f	control terminal time
u	displacement corresponding to y_c
U	semi-active control force of MR damper
U_a	upper bound of U
v	vehicle speed
V	value function of control
W	Gaussian white noise
x_{1d}	horizontal coordinate of upper MR damper end
y_{1d}	height difference of two connected points of vehicle body
y_c	vertical coordinate of vehicle body
y_r	vertical coordinate of rough road or excitation
$\mathbf{Z}$	system state vector
α	displacement corresponding to θ_z
K	control effectiveness or relative response reduction
σ	standard deviation of system response
θ_d	inclined angle of MR damper
θ_z	angle coordinate of support elbow
θ_{z0}	pre-set angle of θ_z
$\boldsymbol{\zeta}$	parameter disturbance vector